



**HAL**  
open science

## On Mori cone of Bott towers

Narasimha Chary Bonala

► **To cite this version:**

| Narasimha Chary Bonala. On Mori cone of Bott towers. 2017. hal-01544844v1

**HAL Id: hal-01544844**

**<https://hal.science/hal-01544844v1>**

Preprint submitted on 22 Jun 2017 (v1), last revised 16 Oct 2017 (v2)

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# ON MORI CONE OF BOTT TOWERS

B. NARASIMHA CHARY

ABSTRACT. A Bott tower of height  $r$  is a sequence of projective bundles

$$X_r \xrightarrow{\pi_r} X_{r-1} \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_2} X_1 = \mathbb{P}^1 \xrightarrow{\pi_1} X_0 = \{pt\},$$

where  $X_i = \mathbb{P}(\mathcal{O}_{X_{i-1}} \oplus \mathcal{L}_{i-1})$  for a line bundle  $\mathcal{L}_{i-1}$  over  $X_{i-1}$  for all  $1 \leq i \leq r$  and  $\mathbb{P}(-)$  denotes the projectivization. These are smooth projective toric varieties and we refer to the top object  $X_r$  also as a Bott tower. In this article, we study the Mori cone and numerically effective (nef) cone of Bott towers, and we classify Fano, weak Fano and log Fano Bott towers. We prove some vanishing theorems for the cohomology of tangent bundle of Bott towers. We give some applications to Bott-Samelson-Demazure-Hansen (BSDH) varieties, by using the degeneration of a BSDH variety to a Bott tower. We also recover the results in [PK16], by toric methods.

**Keywords:** Bott towers, Bott-Samelson-Demazure-Hansen varieties, Mori cone, primitive relations and toric varieties.

## 1. INTRODUCTION

In [BS58], R. Bott and H. Samelson introduced a family of (smooth differentiable) manifolds which may be viewed as the total spaces of iterated  $\mathbb{P}^1$ -bundles over a point  $\{pt\}$ , where each  $\mathbb{P}^1$ -bundle is the projectivization of a rank 2 decomposable vector bundle. In [GK94], M. Grossberg and Y. Karshon proved (in complex geometry setting) that these manifolds have a natural action of a compact torus and also obtained some applications to representation theory and symplectic geometry. In [Civ05], Y. Civan proved that these are smooth projective toric varieties. These are called Bott towers, we denote them by  $\{(X_i, \pi_i) : 1 \leq i \leq r\}$ , where

$$X_r \xrightarrow{\pi_r} X_{r-1} \xrightarrow{\pi_{r-1}} \cdots \xrightarrow{\pi_2} X_1 = \mathbb{P}^1 \xrightarrow{\pi_1} \{pt\},$$

$X_i = \mathbb{P}(\mathcal{O}_{X_{i-1}} \oplus \mathcal{L}_{i-1})$  for a line bundle  $\mathcal{L}_{i-1}$  over  $X_{i-1}$  for all  $1 \leq i \leq r$  and  $r$  is the dimension of  $X_r$ . In [CS11], [CMS10] and [Ish12], the authors studied “cohomological rigidity” properties of Bott towers. These also play an important role in algebraic topology and K-theory (see [CR05], [DJ91] and references therein). In this article we refer to  $X_r$  also as a Bott tower (it is also called Bott manifold).

On the other hand, Bott-Samelson-Demazure-Hansen (for short, BSDH) varieties are also iterated projective line bundles, but where each projective bundle is the projectivization of certain rank 2 vector bundle (not necessarily decomposable). The BSDH varieties were algebraically constructed by M. Demazure and H.C. Hansen independently

---

2010 *Mathematics Subject Classification.* 14F17, 14M15.

The author is supported by AGIR Pole MSTIC project run by the University of Grenoble Alpes, France.

by adapting a differential geometric approach from the paper of R. Bott and H. Samelson (see [BS58], [Dem74] and [Han73]). These are natural desingularizations of Schubert varieties by factoring the Schubert variety using Bruhat decomposition, and these are dependent on the given expression of the Weyl group element corresponding to the Schubert variety (see for instance [CKP15, Page 32]). We also see in this paper some properties of these varieties which depend on the given expression.

In [GK94], M. Grossberg and Y. Karshon constructed degenerations of BSDH varieties to Bott towers, by complex geometric methods. In [Pas10] B. Pasquier and in [PK16] A.J. Parameswaran and P. Karuppuchamy constructed these toric degenerations algebraically. B. Pasquier used these degenerations to study the cohomology of line bundles on BSDH varieties (see [Pas10]). In [PK16], the authors studied the limiting toric variety for a simple simply connected algebraic group by geometric methods.

In this paper we study the geometry of Bott towers in more detail by methods of toric geometry and we prove some applications to BSDH varieties. We also recover the results in [PK16] and extend them to the Kac-Moody setting.

We work over the field  $\mathbb{C}$  of complex numbers. We study the **Mori cone** of  $X_r$  and prove that the class of curves corresponding to ‘primitive relations  $r(P_i)$ ’ forms a basis of the real vector space of numerical classes of one-cycles in  $X_r$  (see Theorem 5.7 and Corollary 5.8). An extremal ray  $R$  in the Mori cone is called **Mori ray** if  $R \cdot K_{X_r} < 0$ , where  $K_{X_r}$  is the canonical divisor in  $X_r$ . We describe extremal rays and Mori rays of the Mori cone of  $X_r$  (see Theorem 9.1). We characterize the ampleness and numerically effectiveness of line bundles on  $X_r$  (see Lemma 6.1) and describe the generators of the *nef* cone of  $X_r$  (see Theorem 6.7).

Recall that a smooth projective variety  $X$  is called **Fano** (respectively, **weak Fano**) if its anti-canonical divisor  $-K_X$  is ample (respectively, *nef* and *big*). Following [AS14], we say that a pair  $(X, D)$  of a normal projective variety  $X$  and an effective  $\mathbb{Q}$ -divisor  $D$  is **log Fano** if it is Kawamata log terminal and  $-(K_X + D)$  is ample (see Section 8 for more details). We study the Fano, weak Fano and the log Fano (of the pair  $(X_r, D)$  for a suitably chosen divisor  $D$  in  $X_r$ ) properties of the Bott tower  $X_r$ .

To describe these results we need some notation. It is known that a Bott tower  $\{(X_i, \pi_i) : 1 \leq i \leq r\}$  is uniquely determined by an upper triangular matrix  $M_r$  with integer entries, defined via the first Chern class of the line bundles  $\mathcal{L}_{i-1}$  on  $X_{i-1}$ , where  $X_i = \mathbb{P}(\mathcal{O}_{X_{i-1}} \oplus \mathcal{L}_{i-1})$  for  $1 \leq i \leq r$  (see [GK94, Section 2.3], [Civ05] and [VT15, Section 7.8]). For more details see Section 2. Let

$$M_r := \begin{bmatrix} 1 & \beta_{12} & \beta_{13} & \dots & \beta_{1r} \\ 0 & 1 & \beta_{23} & \dots & \beta_{2r} \\ 0 & 0 & 1 & \dots & \beta_{3r} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & & 1 \end{bmatrix}_{r \times r},$$

where  $\beta_{ij}$ ’s are integers. Define for  $1 \leq i \leq r$ ,

$$\eta_i^+ := \{r \geq j > i : \beta_{ij} > 0\}$$

and

$$\eta_i^- := \{r \geq j > i : \beta_{ij} < 0\}.$$

If  $|\eta_i^+| = 1$  (respectively,  $|\eta_i^+| = 2$ ), then let  $\eta_i^+ = \{m\}$  (respectively,  $\eta_i^+ = \{m_1, m_2\}$ ). If  $|\eta_i^-| = 1$  (respectively,  $|\eta_i^-| = 2$ ), then set  $\eta_i^- = \{l\}$  (respectively,  $\eta_i^- = \{l_1, l_2\}$ ).

The following can be viewed as a condition on  $i^{\text{th}}$  row of the matrix  $M_r$ :

- $N_i^1$  is the condition that  $|\eta_i^+| = 0$  and  $|\eta_i^-| = 0$ .
- $N_i^2$  is the condition that
  - (i)  $|\eta_i^+| = 0$ ,  $|\eta_i^-| \leq 1$ , and if  $|\eta_i^-| = 1$  then  $\beta_{il} = -1$ ; or
  - (ii)  $|\eta_i^-| = 0$ ,  $|\eta_i^+| \leq 1$ , and if  $|\eta_i^+| = 1$  then  $\beta_{im} = 1$  and  $\beta_{mk} = 0$  for all  $k > m$ .
- $N_i^3$  is the condition that
  - Case 1: Assume that  $|\eta_i^+| = 0$ . Then  $|\eta_i^-| \leq 2$ , and if  $|\eta_i^-| = 1$  (respectively, 2) then  $\beta_{li} = -1$  or  $-2$  (respectively,  $\beta_{il_1} = -1 = \beta_{il_2}$ ).
  - Case 2: If  $|\eta_i^-| = 1 = |\eta_i^+|$  and  $l < m$ , then  $\beta_{il} = -1$ ,  $\beta_{im} = 1$  and  $\beta_{mk} = 0$  for all  $k > m$ .
  - Case 3: Assume that  $|\eta_i^+| = 1$ . Then  $\beta_{im} = 1$  and either it satisfies
    - (i) Case 2; or
    - (ii)  $|\eta_i^-| = 0$  and  $\beta_{mk} = 0$  for all  $k > m$ ; or
    - (iii) there exists unique  $r \geq s > m$  such that
      - $\beta_{ms} - \beta_{is} = 1$  and  $\beta_{mk} - \beta_{ik} = 0$  for all  $k > s$ , or
      - $\beta_{ms} - \beta_{is} = -1$  and  $\beta_{is} - \beta_{ms} - \beta_{sk} = 0$  for all  $k > s$ .

**Definition 1.1.** We say  $X_r$  satisfies condition I, condition II or condition III if  $N_i^1$ ,  $N_i^2$  or  $N_i^3$  holds for all  $1 \leq i \leq r$  respectively.

**Remark 1.2.**

- Let  $1 \leq i \leq r$ . We have  $N_i^1 \implies N_i^2 \implies N_i^3$ .
- If  $X_r$  satisfies condition I, then it also satisfies conditions II and III.
- If  $X_r$  satisfies condition II, then it also satisfies condition III.

**1.1. Results for Bott towers.** We prove,

**Theorem** (See Theorem 7.2).

- (1)  $X_r$  is Fano if and only if it satisfies II.
- (2)  $X_r$  is weak Fano if and only if it satisfies III.

As a consequence we get some vanishing results for the cohomology of tangent bundle of Bott towers and hence local rigidity results. Let  $T_{X_r}$  denote the tangent bundle of  $X_r$ .

**Corollary** (see Corollary 7.3 and Corollary 7.4). *If  $X_r$  satisfies II, then  $H^i(X_r, T_{X_r}) = 0$  for all  $i \geq 1$ . In particular,  $X_r$  is locally rigid.*

For  $1 \leq i \leq r$ , we define some constants  $k_i$  which again depend on the given matrix  $M_r$  corresponding to the Bott tower  $X_r$  (for more details see Section 8). We prove,

**Theorem** (see Theorem 8.1). *The pair  $(X_r, D)$  is log Fano if and only if  $k_i < 0$  for all  $1 \leq i \leq r$ .*

**1.2. Results for BSDH varieties.** Let  $G$  be a Kac-Moody group (for the definition see [Kum12]). Let  $B$  be a Borel subgroup containing a fixed maximal torus  $T$ . Let  $W$  be the Weyl group corresponding to the pair  $(G, B, T)$  and let  $w \in W$ . Let  $\tilde{w} := s_{\beta_1} \cdots s_{\beta_n}$  be an

expression (possibly non-reduced) of  $w$  in simple reflections and let  $Z(\tilde{w})$  be the BSDH variety corresponding to  $\tilde{w}$  (see Section 2). Let  $X_{\tilde{w}}$  be the toric limit of  $Z(\tilde{w})$  constructed as in [Pas10] and [PK16] (see Section 3, do not confuse with the notation,  $X_{\tilde{w}}$  does NOT denote a Schubert variety). Note that  $X_{\tilde{w}}$  is a Bott tower (see Corollary 3.5). We prove that the ample cone  $\text{Amp}(X_{\tilde{w}})$  of  $X_{\tilde{w}}$  can be identified with a subcone of the ample cone  $\text{Amp}(Z(\tilde{w}))$  of  $Z(\tilde{w})$  (see Corollary 10.1).

When  $G$  is a simple algebraic group and the expression  $\tilde{w}$  is reduced, Fanoness and weak Fanoness of the BSDH variety  $Z(\tilde{w})$  are considered in [Cha17]. Here we have the following results: Let  $\tilde{w} = s_{\beta_1} \cdots s_{\beta_i} \cdots s_{\beta_j} \cdots s_{\beta_r}$  be an expression (remember that  $\beta_k$ 's are simple roots). Let  $\beta_{ij} := \langle \beta_j, \tilde{\beta}_i \rangle$ , where  $\tilde{\beta}_i$  is the co-root of  $\beta_i$ . We use the same terminology for BSDH varieties as in Bott tower case above.

**Corollary** (See Corollary 10.2).

- (1) *If  $Z(\tilde{w})$  satisfies II, then  $Z(\tilde{w})$  is Fano.*
- (2) *If  $Z(\tilde{w})$  satisfies III, then  $Z(\tilde{w})$  is weak Fano.*

In [AS14], D. Anderson and A. Stapledon studied the log Fanoness of Schubert varieties, and in [And14], log Fanoness of BSDH varieties is studied for chosen divisors. Let  $D'$  be a divisor in  $Z(\tilde{w})$  with support in the boundary. For  $1 \leq i \leq r$ , we define some constants  $f_i$  which depend on the given expression  $\tilde{w}$  (for more details see Section 10).

**Theorem** (see Theorem 10.5). *The pair  $(Z(\tilde{w}), D')$  is log Fano if  $f_i > 0$  for all  $1 \leq i \leq r$ .*

When  $G$  is a simple algebraic group in [CKP15] and [CK17], we obtained some vanishing results of the cohomology of tangent bundle of  $Z(\tilde{w})$  when  $\tilde{w}$  is reduced, and in [CKP] the case  $\tilde{w}$  is non-reduced was considered. Here we get some vanishing results for  $Z(\tilde{w})$  in our case and hence some rigidity results, by using its toric limit as a Bott tower. Let  $T_{Z(\tilde{w})}$  denote the tangent bundle of  $Z(\tilde{w})$ .

**Corollary** (see Corollary 10.6). *If  $Z(\tilde{w})$  satisfies II, then  $H^i(Z(\tilde{w}), T_{Z(\tilde{w})}) = 0$  for all  $i \geq 1$ . In particular,  $Z(\tilde{w})$  is locally rigid.*

The paper is organized as follows:

In Section 2, we discuss preliminaries on Bott towers, BSDH varieties and toric varieties. In Section 3, we recall the algebraic construction of toric degeneration of BSDH variety. We describe the limiting toric variety as an iterated  $\mathbb{P}^1$ -bundle, a Bott tower. In Section 4, we discuss the Picard group of the Bott tower and compute the relative tangent bundle.

Section 5 contains detailed study of primitive collections and primitive relations of the Bott tower and we also describe the Mori cone. In Section 6 we describe ample and *nef* line bundles on the Bott tower, and we also find the generators of the *nef* cone.

In Section 7 and 8, we study Fano, weak Fano and log Fan properties for Bott towers. We observe that these properties depend on the given matrix corresponding to the Bott tower and we also see some vanishing results. In Section 9, we describe extremal rays and Mori rays for the Bott tower and we see some applications. Section 10 contains some applications to the BSDH-varieties and their toric limits. In Section 11, we give another proof of the results in [PK16] by toric methods.

## 2. PRELIMINARIES

In this section we recall toric varieties (see [CLS11]), Bott towers (see [Civ05] and [VT15]) and Bott-Samelson-Demazure-Hansen varieties (see [Kum12]). We work throughout the article over the field  $\mathbb{C}$  of complex numbers. We expect that the proofs work for algebraically closed fields of arbitrary characteristic, but did not find appropriate references in that generality.

**2.1. Toric varieties.** We briefly recall the structure of toric varieties from [CLS11] (see also [Ful93] and [Oda88]).

**Definition 2.1.** *A normal variety  $X$  is called a toric variety (of dimension  $n$ ) if it contains an  $n$ -dimensional torus  $T$  (i.e.  $T = (\mathbb{C}^*)^n$ ) as a Zariski open subset such that the action of the torus on itself by multiplication extends to an action of the torus on  $X$ .*

Toric varieties are completely described by the combinatorics of the corresponding fans. We briefly recall here, let  $N$  be the lattice of one-parameter subgroups of  $T$  and let  $M$  be the lattice of characters of  $T$ . Let  $M_{\mathbb{R}} := M \otimes \mathbb{R}$  and  $N_{\mathbb{R}} := N \otimes \mathbb{R}$ . Then we have a natural bilinear pairing

$$\langle -, - \rangle : M_{\mathbb{R}} \times N_{\mathbb{R}} \rightarrow \mathbb{R}.$$

A fan  $\Sigma$  in  $N_{\mathbb{R}}$  is a collection of convex polyhedral cones that is closed under intersections and cone faces. Let  $\check{\sigma}$  be the dual cone of  $\sigma \in \Sigma$  in  $M_{\mathbb{R}}$ . For  $\sigma \in \Sigma$ , the semigroup algebra  $\mathbb{C}[\check{\sigma} \cap M]$  is a normal domain and finitely generated  $\mathbb{C}$ -algebra. Then the scheme  $\text{Spec}(\mathbb{C}[\check{\sigma} \cap M])$  is called the affine toric variety corresponding to  $\sigma$ . For a given fan  $\Sigma$ , we can define a toric variety  $X_{\Sigma}$  by gluing the affine toric varieties  $\text{Spec}(\mathbb{C}[\check{\sigma} \cap M])$  as  $\sigma$  varies in  $\Sigma$ . For all  $1 \leq s \leq n$ ,

$$\Sigma(s) := \{\sigma \in \Sigma : \dim(\sigma) = s\}.$$

For each  $\rho \in \Sigma(1)$ , we denote  $u_{\rho}$ , the generator of  $\rho \cap N$ . For  $\sigma \in \Sigma$ ,

$$\sigma(1) := \Sigma(1) \cap \sigma.$$

There is a bijective correspondence between the cones in  $\Sigma$  and the  $T$ -orbits in  $X_{\Sigma}$ . For each  $\sigma \in \Sigma$ , the dimension  $\dim(O(\sigma))$  of the  $T$ -orbit  $O(\sigma)$  corresponding to  $\sigma$  is  $n - \dim(\sigma)$ . Let  $\tau, \sigma \in \Sigma$ , then  $\tau$  is a face of  $\sigma$  if and only if  $O(\sigma) \subset \overline{O(\tau)}$ , where  $\overline{O(\sigma)}$  is the closure of  $T$ -orbit  $O(\sigma)$ . We denote  $V(\sigma) = \overline{O(\sigma)}$  and it is a toric variety with the corresponding fan being  $\text{Star}(\sigma)$ , the star of  $\sigma$  which is the set of cones in  $\Sigma$  which have  $\sigma$  as a face. Let  $D_{\rho} = \overline{O(\rho)}$  be the torus-invariant prime divisor in  $X_{\Sigma}$  corresponding to  $\rho \in \Sigma(1)$ . The group  $T\text{Div}(X_{\Sigma})$  of  $T$ -invariant divisors in  $X_{\Sigma}$  is given by

$$T\text{Div}(X_{\Sigma}) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_{\rho}.$$

For each  $m \in M$ , the character  $\chi^m$  of  $T$  is a rational function on  $X_{\Sigma}$  and the corresponding divisor is given by

$$\text{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_{\rho} \rangle D_{\rho}.$$

**2.2. Bott towers.** In this section we recall some basic definitions and results on Bott towers. Let  $\mathcal{L}_0$  be a trivial line bundle over a single point  $X_0 := \{pt\}$ , and let  $X_1 := \mathbb{P}(\mathcal{O}_{X_0} \oplus \mathcal{L}_0)$ , where  $\mathbb{P}(-)$  denotes the projectivization. Let  $\mathcal{L}_1$  be a line bundle on  $X_1$ , then define  $X_2 := \mathbb{P}(\mathcal{O}_{X_1} \oplus \mathcal{L}_1)$ , which is a  $\mathbb{P}^1$ -bundle over  $X_1$ . Repeat this process  $r$ -times, so that each  $X_i$  is a  $\mathbb{P}^1$ -bundle over  $X_{i-1}$  for  $1 \leq i \leq r$ . We get the following:

$$\begin{array}{c}
X_r = \mathbb{P}(\mathcal{O}_{X_{r-1}} \oplus \mathcal{L}_{r-1}) \\
\downarrow \pi_r \\
X_{r-1} = \mathbb{P}(\mathcal{O}_{X_{r-2}} \oplus \mathcal{L}_{r-2}) \\
\downarrow \pi_{r-1} \\
\vdots \\
\downarrow \pi_2 \\
X_1 = \mathbb{P}(\mathcal{O}_{X_0} \oplus \mathcal{L}_0) \\
\downarrow \pi_1 \\
X_0 = \{pt\}
\end{array}$$

For each  $1 \leq i \leq r$ ,  $X_i$  is a smooth projective toric variety (see [Civ05, Theorem 22]). Consider the points  $[1 : 0]$  and  $[0 : 1]$  in  $\mathbb{P}^1$ , we call them the south pole and the north pole respectively. The zero section of  $\mathcal{L}_{i-1}$  gives a section  $s_i^0 : X_{i-1} \rightarrow X_i$ , the south pole section; similarly, the north pole section  $s_i^1 : X_{i-1} \rightarrow X_i$  by letting the first coordinate in  $\mathbb{P}(\mathcal{O}_{X_{i-1}} \oplus \mathcal{L}_{i-1})$  to vanish.

Let  $1 \leq i \leq r$ . Since  $\pi_i : X_i \rightarrow X_{i-1}$  is a projective bundle, by a standard result on the cohomology ring of projective bundles we have the following (see [Har77, Page 429] for instance):

**Theorem 2.2.** *The cohomology ring  $H^*(X_i, \mathbb{Z})$  of  $X_i$  is a free module over  $H^*(X_{i-1}, \mathbb{Z})$  on generators 1 and  $u_i$ , which have degree 0 and 2 respectively, that is*

$$H^*(X_i, \mathbb{Z}) = H^*(X_{i-1}, \mathbb{Z})1 \oplus H^*(X_{i-1}, \mathbb{Z})u_i.$$

*The ring structure is determined by the single relation*

$$u_i^2 = c_1(\mathcal{L}_{i-1})u_i,$$

*where  $c_1(-)$  denotes the first Chern class and the restriction of  $u_i$  to the fibre  $\mathbb{P}^1 \subset X_i$  is the first Chern class of the canonical line bundle over  $\mathbb{P}^1$ . Hence we have*

$$H^*(X_i, \mathbb{Z}) = H^*(X_{i-1}, \mathbb{Z})[u_i]/J_i,$$

*where  $J_i$  is the ideal generated by  $u_i^2 - c_1(\mathcal{L}_{i-1})u_i$ .*

Consider the exponential sequence (see [Har77, Page 446]):

$$0 \rightarrow \mathbb{Z} \rightarrow \mathcal{O}_{X_{i-1}} \rightarrow \mathcal{O}_{X_{i-1}}^* \rightarrow 0.$$

Then we get the following exact sequence:

$$0 \longrightarrow H^1(X_{i-1}, \mathbb{Z}) \longrightarrow H^1(X_{i-1}, \mathcal{O}_{X_{i-1}}) \longrightarrow H^1(X_{i-1}, \mathcal{O}_{X_{i-1}}^*) \xrightarrow{c_1(-)} H^2(X_{i-1}, \mathbb{Z}) \longrightarrow H^2(X_{i-1}, \mathcal{O}_{X_{i-1}}) \longrightarrow \cdots$$

Since  $X_{i-1}$  is toric, we have  $H^j(X_{i-1}, \mathcal{O}_{X_{i-1}}) = 0$  for all  $j > 0$  (see [Oda88, Corollary 2.8]). As  $H^1(X_{i-1}, \mathcal{O}_{X_{i-1}}^*) = \text{Pic}(X_{i-1})$ , we get  $c_1(-) : \text{Pic}(X_{i-1}) \xrightarrow{\sim} H^2(X_{i-1}, \mathbb{Z})$ . Then we have the following:

**Theorem 2.3.** *Each line bundle  $\mathcal{L}_{i-1}$  on  $X_{i-1}$  is determined (up to algebraic isomorphism) by its first Chern class, which can be written as a linear combination*

$$c_1(\mathcal{L}_{i-1}) = - \sum_{k=1}^{i-1} \beta_{ki} u_k \in H^2(X_{i-1}, \mathbb{Z}),$$

where  $\beta_{ik}$ 's are integers for  $1 \leq k \leq i-1$ .

Then by Theorem 2.2 and 2.3, by iteration, we get the following:

**Corollary 2.4.** *We have*

$$H^*(X_r, \mathbb{Z}) = \mathbb{Z}[u_1, \dots, u_r]/J,$$

where  $J$  is the ideal generated by  $\{u_j^2 + \sum_{i < j} \beta_{ij} u_i u_j : 1 \leq j \leq r\}$  and the integers  $\beta_{ij}$ 's are as in Theorem 2.3.

Write  $\{\beta_{ij} : 1 \leq i < j \leq r\}$ , the collection of  $r(r-1)/2$  integers, as an upper triangular  $r \times r$  matrix

$$M_r := \begin{bmatrix} 1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1r} \\ 0 & 1 & \beta_{23} & \cdots & \beta_{2r} \\ 0 & 0 & 1 & \cdots & \beta_{3r} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & & 1 \end{bmatrix}_{r \times r} \quad (2.1)$$

Then we get the following result (see for instance [GK94, Lemma 2.15] and also [Civ05, Section 3]).

**Corollary 2.5.** *There is a bijective correspondence between  $\{\text{Bott towers of height } r\}$  and  $\{r \times r \text{ upper triangular matrices with integer entries as in (2.1)}\}$ .*

Two Bott towers  $\{(X_i, \pi_i) : 1 \leq i \leq r\}$  and  $\{(X'_i, \pi'_i) : 1 \leq i \leq r\}$  are isomorphic if there exists a collection of isomorphisms  $\{\phi_i : X_i \rightarrow X'_i : 1 \leq i \leq r\}$  such that the following diagram is commutative:

$$\begin{array}{ccccccc} X_r & \xrightarrow{\pi_r} & X_{r-1} & \xrightarrow{\pi_{r-1}} & \cdots & \xrightarrow{\pi_2} & X_1 & \xrightarrow{\pi_1} & X_0 \\ \downarrow \phi_r & & \downarrow \phi_{r-1} & & & & \downarrow \phi_1 & & \downarrow \phi_0 \\ X'_r & \xrightarrow{\pi'_r} & X'_{r-1} & \xrightarrow{\pi'_{r-1}} & \cdots & \xrightarrow{\pi'_2} & X'_1 & \xrightarrow{\pi'_1} & X'_0 \end{array}$$



2.2.1. *Toric structure on Bott tower.* Let  $\{e_1^+, \dots, e_r^+\}$  be the standard basis of the lattice  $\mathbb{Z}^r$ . Define, for all  $i \in \{1, \dots, r\}$ ,

$$e_i^- := -e_i^+ - \sum_{j>i} \beta_{ij} e_j^+, \quad (2.2)$$

where  $\beta_{ij}$ 's are integers as above. Then we have the following theorem (see [Civ05, Section 3 and Theorem 22] and for algebraic topology setting see [VT15, Theorem 7.8.7]):

**Theorem 2.6.** *The Bott tower  $\{(X_i, \pi_i) : 1 \leq i \leq r\}$  corresponding to a matrix  $M_r$  as in (2.1) is isomorphic to  $\{(X_{\Sigma_i}, \pi_{\Sigma_i}) : 1 \leq i \leq r\}$ , the collection of smooth projective toric varieties corresponding to the fan  $\Sigma_i$  with the  $2^i$  maximal cones generated by the set of vectors*

$$\{e_j^\epsilon : 1 \leq j \leq i \text{ and } \epsilon \in \{+, -\}\},$$

and where  $\pi_{\Sigma_i} : X_{\Sigma_i} \rightarrow X_{\Sigma_{i-1}}$  is the toric morphism induced by the projection  $\overline{\pi_{\Sigma_i}} : \mathbb{Z}^i \rightarrow \mathbb{Z}^{i-1}$  for all  $1 \leq i \leq r$ .

Note that by Theorem 2.6,  $\Sigma_i$  has  $2i$  one-dimensional cones generated by the vectors

$$\{e_j^+, e_j^- : 1 \leq j \leq i\},$$

and by (2.2), we can see that the divisors  $D_{\rho_j^+}$  corresponding to  $e_j^+$  for  $1 \leq j \leq i$  form a basis of the Picard group of  $X_i$  (see Section 4 for more details).

2.3. **BSDH varieties.** Let  $A = (a_{ij})_{1 \leq i, j \leq n}$  be a generalized Cartan matrix. Let  $G$  be the Kac-Moody group associated to  $A$  (see [Kum12, Chapter IV]). Fix a maximal torus  $T$  and a Borel subgroup  $B$  containing  $T$ . Let  $S := \{\alpha_1, \dots, \alpha_n\}$  be the set of all simple roots of  $(G, B, T)$ . We denote  $s_{\alpha_i}$  the simple reflection corresponding to  $\alpha_i$ . Note that the Weyl group  $W$  of  $G$  is generated by

$$\{s_{\alpha_i} : 1 \leq i \leq n\}.$$

Let  $w \in W$ , an expression  $\tilde{w}$  of  $w$  is a sequence  $(s_{\beta_1}, \dots, s_{\beta_r})$  of simple reflections  $s_{\beta_1}, \dots, s_{\beta_r}$  such that  $w = s_{\beta_1} \cdots s_{\beta_r}$ . An expression  $\tilde{w}$  of  $w$  is said to be reduced if the number  $r$  of simple reflections is minimal. In such case we call  $r$  the length of  $w$ . By abuse of notation, we also denote the expression  $\tilde{w}$  by  $\tilde{w} = s_{\beta_1} \cdots s_{\beta_r}$ . For  $\alpha \in S$ , we denote  $P_\alpha$ , the minimal parabolic subgroup of  $G$  generated by  $B$  and a representative of  $s_\alpha$ .

**Definition 2.7.** *Let  $w \in W$  and  $\tilde{w} := s_{\beta_1} \cdots s_{\beta_r}$  be an expression (not necessarily reduced) of  $w$ . The Bott-Samelson-Demazure-Hansen (for short, BSDH) variety corresponding to  $\tilde{w}$  is*

$$Z(\tilde{w}) := P_{\beta_1} \times \cdots \times P_{\beta_r} / B^r,$$

where the action of  $B^r$  on  $P_{\beta_1} \times \cdots \times P_{\beta_r}$  is defined by

$$(p_1, \dots, p_r) \cdot (b_1, \dots, b_r) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{r-1}^{-1} p_r b_r) \text{ for all } p_i \in P_{\beta_i}, b_i \in B.$$

These are smooth projective varieties of dimension  $r$ . There is a natural morphism  $\phi_{\tilde{w}} : Z(\tilde{w}) \rightarrow G/B$  defined by

$$[(p_1, \dots, p_r)] \mapsto p_1 \cdots p_r B.$$

If  $\tilde{w}$  is reduced, the BSDH variety  $Z(\tilde{w})$  is a natural desingularization of the Schubert variety, the  $B$ -orbit closure of  $wB/B$  in  $G/B$  (see [Dem74], [Han73] and [Kum12, Chapter VIII]).

We can also construct the BSDH variety by iterated  $\mathbb{P}^1$ -bundles. Let  $\tilde{w}' := s_{\beta_1} \cdots s_{\beta_{r-1}}$ . Let  $f : G/B \rightarrow G/P_{\beta_r}$  be the map given by  $gB \mapsto gP_{\beta_r}$  and let  $p : Z(\tilde{w}') \rightarrow G/P_{\beta_r}$  be the map given by  $[(p_1, \dots, p_{r-1})] \mapsto p_1 \cdots p_{r-1} P_{\beta_r}$ . Then we have the following cartesian diagram (see [BK07, Page 66] and [Kum12, Chapter VII]):

$$\begin{array}{ccc} Z(\tilde{w}) = Z(\tilde{w}') \times_{G/P_{\beta_r}} G/B & \xrightarrow{\phi_{\tilde{w}}} & G/B \\ \downarrow f_{\tilde{w}} & & \downarrow f \\ Z(\tilde{w}') & \xrightarrow{p} & G/P_{\beta_r} \end{array}$$

Note that  $f_{\tilde{w}}$  is a  $\mathbb{P}^1$ -fibration and the relative tangent bundle  $T_{f_{\tilde{w}}}$  of  $f_{\tilde{w}}$  is  $\phi_{\tilde{w}}^*(\mathcal{L}_{\beta_r})$ , where  $\mathcal{L}_{\beta_r}$  is the homogeneous line bundle on  $G/B$  corresponding to  $\beta_r$ . Using the cohomology of the relative tangent bundle  $T_{f_{\tilde{w}}}$  we studied the cohomology of the tangent bundle of  $Z(\tilde{w})$ , when  $G$  is finite dimensional and  $\tilde{w}$  is a reduced expression (see [CKP15] and [CK17]). In Section 3, we compute the relative tangent bundle of the corresponding toric limits, in the more general setting of Bott towers. The fibration  $f_{\tilde{w}}$  comes with a natural section  $\sigma_{\tilde{w}} : Z(\tilde{w}') \rightarrow Z(\tilde{w})$  induced by the projection

$$P_{\beta_1} \times \cdots \times P_{\beta_r} \rightarrow P_{\beta_1} \times \cdots \times P_{\beta_{r-1}}.$$

For the toric limits we get two natural sections, as will be explained in Section 3. For all  $i \in \{1, \dots, r\}$ , we denote  $Z_i$ , the divisor in  $Z(\tilde{w})$  defined by

$$\{[(p_1, \dots, p_r)] \in Z(\tilde{w}) : p_i \in B\}.$$

In [LT04], N. Lauritzen and J.F. Thomsen proved that  $Z'_i$ s forms a basis of the Picard group of  $Z(\tilde{w})$  and they also proved that if  $\tilde{w}$  is a reduced expression these form a basis of the monoid of effective divisors (see [LT04, Proposition 3.5]). Recently, the effective divisors of  $Z(\tilde{w})$  for  $\tilde{w}$  non-reduced case have been considered in [And14].

### 3. TORIC DEGENERATION OF A BSDH VARIETY

In [GK94], toric degenerations of BSDH varieties to Bott towers, were constructed by complex geometric methods. In [Pas10] and [PK16] they have given an algebraic construction for toric degeneration of a BSDH variety. We recall the algebraic construction here.

Note that the simple roots are linearly independent elements in the character group of  $G$ . We can choose a positive integer  $q$  and a injective morphism  $\lambda : \mathbb{G}_m \rightarrow T$  (i.e.  $\lambda \in N$  and  $\lambda$  is injective) such that for all  $1 \leq i \leq n$  and  $u \in \mathbb{G}_m$ ,  $\alpha_i(\lambda(u)) = u^q$  (see [Pas10, Page 2836]). When  $G$  is finite dimensional, for each one-parameter subgroup  $\lambda \in N$ , define

$$P(\lambda) := \{g \in G : \lim_{u \rightarrow 0} \lambda(u)g\lambda(u)^{-1} \text{ exists in } G\}.$$

The set  $P(\lambda)$  is a parabolic subgroup and the unipotent radical  $R_u(P(\lambda))$  of  $P(\lambda)$  is given by

$$R_u(P(\lambda)) = \{g \in G : \lim_{u \rightarrow 0} \lambda(u)g\lambda(u)^{-1} \text{ is identity in } G\}.$$

Any parabolic subgroup of  $G$  is of this form (see [Spr10, Proposition 8.4.5]). Choose a one-parameter subgroup  $\lambda \in N$  such that the corresponding parabolic subgroup is  $B$ .

Let us define an endomorphism of  $G$  for all  $u \in \mathbb{G}_m$  by

$$\tilde{\Psi}_u : G \rightarrow G, \quad g \mapsto \lambda(u)g\lambda(u)^{-1}.$$

Let  $\mathcal{B}$  be the set of all endomorphisms of  $B$ . Now define a morphism

$$\Psi : \mathbb{G}_m \rightarrow \mathcal{B} \text{ by } u \mapsto \tilde{\Psi}_u|_B.$$

This map can be extended to 0 and for all  $x \in U$ ,  $\Psi_u|_B(x)$  goes to identity when  $u$  goes to zero. Let  $\mathbb{A}^1 := \text{Spec} \mathbb{C}[t]$  be the affine line over  $\mathbb{C}$ . We denote for all  $u \in \mathbb{A}^1$ ,  $\Psi_u$  the image of  $u$  in  $\mathcal{B}$ . Note that  $\Psi_u$  is the identity on  $T$  and  $\Psi_0$  is the projection from  $B$  to  $T$ .

Let  $\tilde{w} = s_{\beta_1} \cdots s_{\beta_r}$  be an expression.

**Definition 3.1.**

(i) Let  $\mathcal{X}$  be the variety defined by

$$\mathcal{X} := \mathbb{A}^1 \times P_{\beta_1} \times \cdots \times P_{\beta_r} / B^r,$$

where the action of  $B^r$  on  $\mathbb{A}^1 \times P_{\beta_1} \times \cdots \times P_{\beta_r}$  is given by

$$(u, p_1, \dots, p_r) \cdot (b_1, \dots, b_r) = (u, p_1 b_1, \Psi_u(b_1)^{-1} p_2 b_2, \dots, \Psi_u(b_{r-1})^{-1} p_r b_r).$$

(ii) For all  $i \in \{1, \dots, r\}$ , we denote  $\mathcal{Z}_i$  the divisor in  $\mathcal{X}$  defined by

$$\{(u, p_1, \dots, p_r) \in \mathcal{X} : p_i \in B\}.$$

Note that  $\mathcal{X}$  and  $\mathcal{Z}_i$ 's are integral. Let  $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$  be the projection onto the first factor. Then we have the following theorem (see [Pas10, Proposition 1.3 and 1.4] and [PK16, Theorem 9]).

**Theorem 3.2.**

- (1)  $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$  is a smooth projective morphism.
- (2) For all  $u \in \mathbb{A}^1 \setminus \{0\}$ , the fiber  $\pi^{-1}(u)$  is isomorphic to the BSDH variety  $Z(\tilde{w})$  such that  $\pi^{-1}(u) \cap \mathcal{Z}_i$  corresponds to the divisor  $Z_i$  in  $Z(\tilde{w})$ .
- (3)  $\pi^{-1}(0)$  is a smooth projective toric variety.

Now onwards, we denote  $\mathcal{X}_u := \pi^{-1}(u)$  for  $u \in \mathbb{A}^1$  and the limiting toric variety  $\mathcal{X}_0 = \pi^{-1}(0)$  by  $X_{\tilde{w}}$  (do not confuse with the Schubert variety).

Recall that  $\{e_1^+, \dots, e_r^+\}$  is the standard basis of the lattice  $\mathbb{Z}^r$ . Define, for all  $i \in \{1, \dots, r\}$ ,

$$e_i^- := -e_i^+ - \sum_{j>i} \beta_{ij} e_j^+, \tag{3.1}$$

here we take  $\beta_{ij} := \langle \beta_j, \check{\beta}_i \rangle$ . The following proposition will give the description of the fan of the toric variety  $X_{\tilde{w}}$  (see [Pas10, Proposition 1.4]).

**Proposition 3.3.**

- (1) The fan  $\Sigma$  of the smooth toric variety  $X_{\tilde{w}}$  consists of the cones generated by subsets of

$$\{e_1^+, \dots, e_r^+, e_1^-, \dots, e_r^-\}$$

and containing no subset of the form  $\{e_i^+, e_i^-\}$ .

- (2) For all  $i \in \{1, \dots, r\}$ ,  $\mathcal{Z}_i^0$  is the irreducible  $(\mathbb{C}^*)^r$ -stable divisor in  $X_{\tilde{w}}$  corresponding to the one-dimensional cone of  $\Sigma$  generated by  $e_i^+$  and these form a basis of the divisor class group of  $X_{\tilde{w}}$ .

Note that the maximal cones of  $\Sigma$  are generated by  $\{e_i^\epsilon : 1 \leq i \leq r, \epsilon \in \{+, -\}\}$ .

In [Pas10], B. Pasquier used the combinatorics of the toric variety  $X_{\tilde{w}}$  to obtain vanishing theorems for the cohomology of line bundles on BSDH variety  $Z(\tilde{w})$ .

Now we describe the toric limit  $X_{\tilde{w}}$  as an iterated  $\mathbb{P}^1$ -bundle. Let  $\tilde{w}' := s_{\beta_1} \cdots s_{\beta_{r-1}}$ . Then we get a toric morphism

$$f_r : X_{\tilde{w}} \rightarrow X_{\tilde{w}'}$$

induced by the lattice map

$$\bar{f}_r : \mathbb{Z}^r \rightarrow \mathbb{Z}^{r-1},$$

the projection onto the first  $r - 1$  coordinates.

We prove,

**Lemma 3.4.**

- (1)  $f_r : X_{\tilde{w}} \rightarrow X_{\tilde{w}'}$  is a toric  $\mathbb{P}^1$ -fibration with two disjoint toric sections.  
(2)  $X_{\tilde{w}} \simeq \mathbb{P}(\mathcal{O}_{X_{\tilde{w}'}} \oplus \mathcal{L})$  for some unique line bundle  $\mathcal{L}$  on  $X_{\tilde{w}'}$ .

*Proof.* Let  $\Sigma'$  be the fan corresponding to the toric variety  $X_{\tilde{w}'}$ . From the above proposition, we can see that  $\Sigma$  has a splitting by  $\Sigma'$  and  $\{e_r^+, 0, e_r^-\}$ . Then by [CLS11, Theorem 3.3.19],

$$f_r : X_{\tilde{w}} \rightarrow X_{\tilde{w}'}$$

is a locally trivial fibration with the fan  $\Sigma_F$  of the fiber being  $\{e_r^+, 0, e_r^-\}$ . Since  $\Sigma_F$  is the fan of the projective line  $\mathbb{P}^1$ , we conclude  $f_r$  is a toric  $\mathbb{P}^1$ -fibration. As toric sections of the toric fibration correspond to the maximal cones in  $\Sigma_F$ , we get two disjoint toric sections for  $f_r$ . This proves (1).

Proof of (2): Since  $f_r : X_{\tilde{w}} \rightarrow X_{\tilde{w}'}$  is  $\mathbb{P}^1$ -fibration with a section, we see  $X_{\tilde{w}}$  is a projective bundle  $\mathbb{P}(\mathcal{E})$  over  $X_{\tilde{w}'}$  corresponding to a rank 2 vector bundle  $\mathcal{E}$  on  $X_{\tilde{w}'}$  (see for example [Har77, Chapter V, Proposition 2.2, page 370]).

Recall that the sections of projective bundle  $X_{\tilde{w}} = \mathbb{P}(\mathcal{E})$  correspond to the quotient line bundles of  $\mathcal{E}$  (see [Har77, Proposition 7.12]). Since  $X_{\tilde{w}} = \mathbb{P}(\mathcal{E})$  is projective line bundle on  $X_{\tilde{w}'}$  with two disjoint sections, we see  $\mathcal{E}$  is decomposable as a direct sum of line bundles on  $X_{\tilde{w}'}$ .

As

$$\mathbb{P}(\mathcal{E}) \simeq \mathbb{P}(\mathcal{L}' \otimes \mathcal{E})$$

for any line bundle  $\mathcal{L}'$  on  $X_{\tilde{w}'}$  (see [Har77, Lemma 7.9]), we can assume without loss of generality

$$\mathcal{E} = \mathcal{O}_{X_{\tilde{w}'}} \oplus \mathcal{L}$$

for some unique line bundle  $\mathcal{L}$  on  $X_{\tilde{w}'}$ . Hence  $X_{\tilde{w}} \simeq \mathbb{P}(\mathcal{O}_{X_{\tilde{w}'}} \oplus \mathcal{L})$  and this completes the proof of the lemma.  $\square$

Then by definition of Bott tower we get:

**Corollary 3.5.** *The toric limit  $X_{\tilde{w}}$  is a Bott tower.*

We have the following situation:

$$\begin{array}{ccc}
 \mathbb{P}^1 & & \mathbb{P}^1 \\
 \downarrow & & \downarrow \\
 Z(\tilde{w}) & \rightsquigarrow & X_{\tilde{w}} \\
 \downarrow f_{\tilde{w}} & & \downarrow f_r \\
 Z(\tilde{w}') & \rightsquigarrow & X_{\tilde{w}'}
 \end{array}$$

(Red arrows indicate maps from  $Z(\tilde{w})$  to  $Z(\tilde{w}')$  and from  $X_{\tilde{w}}$  to  $X_{\tilde{w}'}$ .)

#### 4. ON PICARD GROUP OF A BOTT TOWER

Now we describe a basis of the Picard group  $Pic(X_r)$  of  $X_r$ .

Let  $\epsilon \in \{+, -\}$  and for  $1 \leq i \leq r$ , let  $\rho_i^\epsilon$  be the one-dimensional cone generated by  $e_i^\epsilon$ . For all  $1 \leq i \leq r$ , we define  $D_{\rho_i^\epsilon}$  to be the toric divisor corresponding to the one-dimensional cone  $\rho_i^\epsilon$ .

We prove,

**Lemma 4.1.** *The set  $\{D_{\rho_i^\epsilon} : 1 \leq i \leq r \text{ and } \epsilon \in \{+, -\}\}$  forms a basis of  $Pic(X_r)$ .*

*Proof.* By Theorem 2.6, using the description of the one-dimensional cones we have the following decomposition of  $\Sigma(1)$ :

$$\Sigma(1) = \{\rho_i^+ : 1 \leq i \leq r\} \cup \{\rho_i^- : 1 \leq i \leq r\}. \quad (4.1)$$

Again by Theorem 2.6,  $\{D_{\rho_i^+} : 1 \leq i \leq r\}$  forms a basis of the Picard group  $Pic(X_r)$  of  $X_r$ . Since

$$0 \sim \text{div}(\chi^{e_i^+}) = \sum_{\rho \in \Sigma(1)} \langle u_\rho, e_i^+ \rangle D_\rho,$$

by (2.2) we can see that  $\{D_{\rho_i^-} : 1 \leq i \leq r\}$  also forms a basis of  $Pic(X_r)$ .

In general, let  $\sigma \in \Sigma$  be the maximal cone generated by  $\{e_i^\epsilon : 1 \leq i \leq r\}$ . Take the torus-fixed point  $x^\epsilon$  in  $X_r$  corresponding to the maximal cone  $\sigma$ . Let  $U$  be the torus-invariant open affine neighbourhood of  $x^\epsilon$  in  $X_r$ . Then  $U$  is an affine space of dimension  $r$ ; in particular,  $Pic(U) = 0$ . Therefore, we get

$$X_r \setminus U = \cup_{i=1}^r D_{\rho_i^\epsilon}$$

and  $Pic(X_r)$  is generated by

$$\{D_{\rho_i^\epsilon} : 1 \leq i \leq r\}$$

(see [Har70, Chapter II, Proposition 3.1, page 66]). Since  $\{D_{\rho_i^\epsilon} : 1 \leq i \leq r\}$  is linearly independent and the rank of  $Pic(X_r)$  is  $r$ , this set  $\{D_{\rho_i^\epsilon} : 1 \leq i \leq r\}$  forms a basis of  $Pic(X_r)$ .  $\square$

**Remark 4.2.** In Section 12, for  $1 \leq i \leq r$ , we express  $D_{\rho_i^-}$  in terms of  $D_{\rho_j^+}$  ( $1 \leq j \leq r$ ).

Let  $1 \leq i \leq r$ . We prove the following.

**Lemma 4.3.** The relative tangent bundle  $T_{\pi_i}$  of  $\pi_i : X_i \rightarrow X_{i-1}$  is given by

$$T_{\pi_i} \simeq \mathcal{O}_{X_i}(D_{\rho_i^+} + D_{\rho_i^-}) \simeq \mathcal{O}_{X_i}\left(\sum_{j=1}^{i-1} \beta_{ij} D_{\rho_j^-} + 2D_{\rho_i^-}\right).$$

*Proof.* By definition of Bott tower,  $\pi_i$  is a  $\mathbb{P}^1$ -fibration. Then the relative canonical bundle  $K_{\pi_i}$  is given by

$$K_{\pi_i} = \mathcal{O}_{X_i}(K_{X_i}) \otimes \pi_i^*(\mathcal{O}_{X_{i-1}}(-K_{X_{i-1}}))$$

(see [Kle80, Corollary 24, page 56]). By [CLS11, Theorem 8.2.3] (see also [Ful93, Page 74]), we have

$$K_{X_\Sigma} = - \sum_{\rho \in \Sigma(1)} D_\rho.$$

Then

$$K_{\pi_i} = \mathcal{O}_{X_i}\left(- \sum_{\rho \in \Sigma(1)} D_\rho\right) \otimes \pi_i^*\left(\mathcal{O}_{X_{i-1}}\left(\sum_{\rho' \in \Sigma'(1)} D_{\rho'}\right)\right),$$

where  $\Sigma'$  is the fan of  $X_{i-1}$ . Since  $X_{i-1}$  is smooth, any divisor of the form  $D = \sum_{\rho' \in \Sigma'(1)} a_{\rho'} D_{\rho'}$  with  $a_{\rho'} \in \mathbb{Z}$ , in  $X_{i-1}$  is Cartier. Hence the pullback  $\pi_i^*(D)$  is defined and given by

$$\pi_i^*(D) = \pi_i^*\left(\sum_{\rho' \in \Sigma'(1)} a_{\rho'} D_{\rho'}\right) = \sum_{\rho \in \Sigma(1)} -\varphi_D(\bar{\pi}_i(u_\rho)) D_\rho,$$

where  $\varphi_D$  is the support function corresponding to the divisor  $D$  (see [CLS11, Theorem 4.2.12] for the correspondence between support functions and Cartier divisors).

Since the lattice map  $\bar{\pi}_i : \mathbb{Z}^i \rightarrow \mathbb{Z}^{i-1}$  is the projection onto the first  $i-1$  factors (see page 8), by definition of  $u_\rho$  and  $e_j^-$  (see (2.2)), for  $\epsilon \in \{+, -\}$  we have

$$\bar{\pi}_i(u_{\rho_j^\epsilon}) = \begin{cases} u_{\rho_j^\epsilon} & \text{if } 1 \leq j \leq i-1. \\ 0 & \text{if } j = i. \end{cases}$$

Hence

$$-\varphi_D(\bar{\pi}_i(u_{\rho_j^\epsilon})) = \begin{cases} a_{\rho_j^\epsilon} & \text{if } 1 \leq j \leq i-1. \\ 0 & \text{if } j = i. \end{cases}$$

Thus we have,

$$\pi_i^*\left(\sum_{\rho' \in \Sigma'(1)} D_{\rho'}\right) = \sum_{\rho \in \Sigma(1) \setminus \{\rho_i^+, \rho_i^-\}} D_\rho.$$

Therefore, we see that

$$K_{\pi_i} = \mathcal{O}_{X_i}(-D_{\rho_i^+} - D_{\rho_i^-}). \quad (4.2)$$

By (2.2), we note that

$$0 \sim \operatorname{div}(\chi^{e_i^+}) = D_{\rho_i^+} - D_{\rho_i^-} - \sum_{j=1}^{i-1} \beta_{ij} D_{\rho_j^-}. \quad (4.3)$$

Since  $\check{K}_{\pi_i} = \det T_{\pi_i}$ , we get  $\check{K}_{\pi_i} = T_{\pi_i}$  as  $\pi_i$  is a  $\mathbb{P}^1$ -fibration. Therefore, the result follows from (4.2) and (4.3).  $\square$

The following is well known and proved here for completeness.

**Lemma 4.4.** *Let  $X$  and  $Y$  be smooth varieties. Let  $f : X \rightarrow Y$  be a fibration with a section  $\sigma$  and denote by  $\sigma(Y)$  its image in  $X$ . Then the restriction of the relative tangent bundle  $T_f$  to  $\sigma(Y)$  is isomorphic to the normal bundle  $\mathcal{N}_{\sigma(Y)/X}$  of  $\sigma(Y)$  in  $X$ .*

*Proof.* Consider the normal bundle short exact sequence

$$0 \rightarrow T_{\sigma(Y)} \rightarrow T_X|_{\sigma(Y)} \rightarrow \mathcal{N}_{\sigma(Y)/X} \rightarrow 0, \quad (4.4)$$

where  $T_{\sigma(Y)}$  and  $T_X$  are the tangent bundles of  $\sigma(Y)$  and  $X$  respectively.

Also consider the following short exact sequence

$$0 \rightarrow T_f \rightarrow T_X \rightarrow f^*T_Y \rightarrow 0. \quad (4.5)$$

By restricting (4.5) to  $\sigma(Y)$ , since  $\sigma$  is a section of  $f$ , we get the following short exact sequence

$$0 \rightarrow T_f|_{\sigma(Y)} \rightarrow T_X|_{\sigma(Y)} \rightarrow T_{\sigma(Y)} \rightarrow 0. \quad (4.6)$$

By using (4.4) and (4.6), we see  $T_f|_{\sigma(Y)}$  is isomorphic to  $\mathcal{N}_{\sigma(Y)/X}$ . This completes the proof.  $\square$

We prove,

**Lemma 4.5.** *Let  $1 \leq i \leq r$ . The normal bundle  $\mathcal{N}_{X_i/X_{i-1}}$  of  $X_{i-1}$  in  $X_i$  is  $\check{\mathcal{L}}_{i-1}$ , where  $\mathcal{L}_{i-1}$  is as in the definition of Bott tower and  $\check{\mathcal{L}}_{i-1}$  is denotes the dual of  $\mathcal{L}_{i-1}$ .*

*Proof.* Fix  $1 \leq i \leq r$  and let  $\mathcal{L} := \mathcal{L}_{i-1}$ . Recall that  $\mathbb{P}(\mathcal{E})$  is by definition  $\operatorname{Proj}(S(\mathcal{E}))$ ,  $S(\mathcal{E})$  is symmetric algebra of  $\mathcal{E}$  (see [Har77, Page 162]). Let  $V(\mathcal{L}) = \operatorname{Spec}(S(\mathcal{L}))$ , the geometric vector bundle associated to the locally free sheaf (line bundle)  $\mathcal{L}$  (see [Har77, Exercise 5.18, Page 128]). Then,  $V(\mathcal{L})$  is an open subvariety in  $\mathbb{P}(\mathcal{E})$  and we have the following commutative diagram

$$\begin{array}{ccc} V(\mathcal{L}) & \hookrightarrow & \mathbb{P}(\mathcal{E}) = X_i \\ & \searrow \pi & \swarrow \pi_r \\ & & X_{i-1} \end{array}$$

Also note that the the section  $s_i^0(X_{i-1})$  of  $\pi_i$  corresponding to the projection  $\mathcal{E} \rightarrow \mathcal{O}_{X_i}$  is same as the zero section  $\sigma_\pi(X_{i-1})$  of  $\pi$ .

Now consider the following short exact sequence

$$0 \longrightarrow T_\pi \longrightarrow T_{V(\mathcal{L})} \longrightarrow \pi^* T_{X_{i-1}} \longrightarrow 0. \quad (4.7)$$

Since the restriction  $T_{\pi|_{\sigma_\pi(X_{i-1})}}$  of  $T_\pi$  to  $\sigma_\pi(X_{i-1})$  is  $\check{\mathcal{L}}$ , by Lemma 4.4 and by above short exact sequence (4.7) we see that

$$\mathcal{N}_{\sigma_\pi(X_{i-1})/V(\mathcal{L})} \simeq \check{\mathcal{L}}.$$

Hence we conclude that

$$\mathcal{N}_{X_{i-1}/X_i} \simeq \check{\mathcal{L}}$$

(here we are identifying  $X_{i-1}$  with the section corresponding to the projection

$$\mathcal{E} = \mathcal{O}_{X_{i-1}} \oplus \mathcal{L} \rightarrow \mathcal{O}_{X_{i-1}} \quad ).$$

This completes the proof of the lemma.  $\square$

Let  $1 \leq i \leq r$ . We prove,

**Lemma 4.6.**

- (1) *The toric sections of  $\pi_i$  are given by  $D_{\rho_i^\epsilon}$ ,  $\epsilon \in \{+, -\}$ .*
- (2) *The normal bundle  $\mathcal{N}_{X_{i-1}/X_i}$  of  $X_{i-1}$  in  $X_i$  is given by*

$$\mathcal{N}_{X_{i-1}/X_i} = \check{\mathcal{L}}_{i-1} = \mathcal{O}_{X_i}(D_{\rho_i^+}),$$

where the line bundle  $\check{\mathcal{L}}_{i-1}$  is as in the definition of the Bott tower  $X_i$ .

*Proof.* Proof of (1): Recall that  $\pi_i$  is a  $\mathbb{P}^1$ -fibration induced by the projection  $\bar{\pi}_i : \mathbb{Z}^i \rightarrow \mathbb{Z}^{i-1}$ . For each cone  $\sigma \in \Sigma_F$  of dimension 1 (which is a maximal cone in  $\Sigma_F$ ), the subvariety  $V(\sigma)$  is an invariant section of  $\pi_i$ , which is an invariant divisor in  $X_i$ . Hence we get two invariant divisors  $V(\rho_i^+) = D_{\rho_i^+}$  and  $V(\rho_i^-) = D_{\rho_i^-}$ .

Proof of (2): By Lemma 4.5, we have  $\mathcal{N}_{X_{i-1}/X_i} = \check{\mathcal{L}}_{i-1}$  and the section  $X_{i-1}$  is given by the projection  $\mathcal{E} = \mathcal{O}_{X_{i-1}} \oplus \mathcal{L}_{i-1} \rightarrow \mathcal{O}_{X_{i-1}}$ . Hence (2) follows from (1).  $\square$

## 5. PRIMITIVE RELATIONS OF THE BOTT TOWER

**5.1. Primitive collections and primitive relations.** First recall the notion of primitive collections and primitive relations of a fan  $\Sigma$ , which are basic tools for the classification of Fano toric varieties due to Batyrev (see [Bat91]).

**Definition 5.1.** *We say  $P \subset \Sigma(1)$  is a **primitive collection** if  $P$  is not contained in  $\sigma(1)$  for some  $\sigma \in \Sigma$  but any proper subset is. Note that if  $\Sigma$  is simplicial, primitive collection means that  $P$  does not generate a cone in  $\Sigma$  but every proper subset does.*

**Definition 5.2.** *Let  $P = \{\rho_1, \dots, \rho_k\}$  be a primitive collection in a complete simplicial fan  $\Sigma$ . Recall  $u_\rho$  be the primitive vector of the ray  $\rho \in \Sigma$ . Then  $\sum_{i=1}^k u_{\rho_i}$  is in the relative interior of a cone  $\gamma_P$  in  $\Sigma$  with a unique expression*

$$\sum_{i=1}^k u_{\rho_i} = \sum_{\rho \in \gamma_P(1)} c_\rho u_\rho, \quad c_\rho \in \mathbb{Q}_{>0}.$$



Hence

$$\sum_{i=1}^k u_{\rho_i} - \left( \sum_{\rho \in \gamma_P(1)} c_\rho u_\rho \right) = 0. \quad (5.1)$$

Then we call (5.1) the **primitive relation** of  $X_\Sigma$  corresponding to  $P$ .

Recall that  $TDiv(X_\Sigma)$  denote the group of torus-invariant divisors in  $X_\Sigma$  (see Page 8). Since the fan  $\Sigma$  of  $X_r$  is full dimensional, we have the following short exact sequence

$$0 \longrightarrow M \xrightarrow{\varphi_1} TDiv(X_r) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z}D_\rho \xrightarrow{\varphi_2} Pic(X_r) \longrightarrow 0, \quad (5.2)$$

where the maps are given by  $\varphi_1 : m \mapsto div(\chi^m)$  and  $\varphi_2 : D \mapsto \mathcal{O}_{X_r}(D)$  (see [CLS11, Theorem 4.2.1]).

Now we recall some standard notations: Let  $X$  be a smooth projective variety, we define

$$N_1(X)_\mathbb{Z} := \left\{ \sum_{\text{finite}} a_i C_i : a_i \in \mathbb{Z}, C_i \text{ irreducible curve in } X \right\} / \equiv$$

where  $\equiv$  is the numerical equivalence, i.e.  $Z \equiv Z'$  if and only if  $D \cdot Z = D \cdot Z'$  for all divisors  $D$  in  $X$ . We denote by  $[C]$  the class of  $C$  in  $N_1(X)_\mathbb{Z}$ . Let  $N_1(X) := N_1(X)_\mathbb{Z} \otimes \mathbb{R}$ . It is a well known fact that  $N_1(X)$  is a finite dimensional real vector space (see [Kle66, Proposition 4, §1, Chapter IV]). In the case where  $X$  is a (smooth projective) toric variety,  $N_1(X)_\mathbb{Z}$  is dual to  $Pic(X)$  via the natural pairing (see [CLS11, Proposition 6.3.15]).

In our case  $X = X_r$ , there are dual exact sequences:

$$0 \longrightarrow M \xrightarrow{\varphi_1} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\varphi_2} Pic(X_r) \longrightarrow 0$$

and

$$0 \longrightarrow N_1(X_r)_\mathbb{Z} \xrightarrow{\varphi_2^*} \mathbb{Z}^{\Sigma(1)} \xrightarrow{\varphi_1^*} N \longrightarrow 0, \quad (5.3)$$

where

$$\varphi_2^*([C]) = (D_\rho \cdot C)_{\rho \in \Sigma(1)}, \quad C \text{ is an irreducible complete curve in } X_r$$

and

$$\varphi_1^*(e_\rho) = u_\rho, \quad e_\rho \text{ is a standard basis vector of } \mathbb{R}^{\Sigma(1)}$$

(see [CLS11, Proposition 6.4.1]).

Let  $P$  be a primitive collection in  $\Sigma$ . Note that since  $X_r$  is smooth projective,

$$P \cap \gamma_P(1) = \emptyset$$

and

$$c_\rho \in \mathbb{Z}_{>0} \text{ for all } \rho \in \gamma_P(1) \quad (5.4)$$

(see [CLS11, Proposition 7.3.6]).

As an element in  $\mathbb{Z}^{\Sigma(1)}$ , we write  $r(P) = (r_\rho)_{\rho \in \Sigma(1)}$ , where

$$r_\rho = \begin{cases} 1 & \text{if } \rho \in P \\ -c_\rho & \text{if } \rho \in \gamma_P(1) \\ 0 & \text{otherwise} \end{cases} \quad (5.5)$$

Then by (5.1) we see that

$$\sum_{\rho \in \Sigma(1)} r_\rho u_\rho = 0.$$

Hence by the exact sequence (5.3) and by (5.4), we observe that  $r(P)$  gives an element in  $N_1(X_r)_\mathbb{Z}$  (see [CLS11, Page 305]).

We prove,

**Lemma 5.3.** *Let  $P_i := \{\rho_i^+, \rho_i^-\}$ ,  $1 \leq i \leq r$ . Then*

$$\{P_i : 1 \leq i \leq r\}$$

*is the set of all primitive collections of the fan  $\Sigma$  of  $X_r$ .*

*Proof.* By Theorem 2.6, the cones in the fan  $\Sigma$  of  $X_r$  are generated by subsets of

$$\{e_1^+, \dots, e_r^+, e_1^-, \dots, e_r^-\}$$

and containing no subset of the form  $\{e_i^+, e_i^-\}$ . Then by Definition 5.1, it is clear that  $P_i = \{\rho_i^+, \rho_i^-\}$  is a primitive collection for all  $i$ . Also note that again by description of the cones in  $\Sigma$ , any primitive collection must contain a  $P_i$  for some  $1 \leq i \leq r$ .

Fix  $1 \leq i \leq r$ . Let  $Q$  be a collection of one-dimensional cones such that it properly contains  $P_i$ , i.e. there exists  $1 \leq j \leq r$  and  $j \neq i$  such that  $\rho_j^\epsilon \in Q \supset P_i$ ,  $\epsilon \in \{+, -\}$ . Assume that  $Q$  is a primitive collection. Then by Definition 5.1,  $\{\rho_i^+, \rho_i^-\} \subset Q$  generates a cone in  $\Sigma$ . This is a contradiction to the description of the cones in  $\Sigma$ . Therefore, we conclude that

$$\{P_i : 1 \leq i \leq r\}$$

is the set of all primitive collections. □

Now we define the **Contractible classes** from [Cas03]: Let  $X$  be a smooth projective toric variety. We define  $NE(X)_\mathbb{Z}$  in  $N_1(X)$  by

$$NE(X)_\mathbb{Z} := \left\{ \sum_{finite} a_i C_i : a_i \in \mathbb{Z}_{\geq 0} \text{ and } C_i \text{ irreducible curve in } X \right\}.$$

Let  $\gamma \in NE(X)_\mathbb{Z}$  be primitive (i.e. the generator of  $\mathbb{Z}_{\geq 0}\gamma$ ) and such that there exists some irreducible curve in  $X$  having numerical class in  $\mathbb{Q}_{\geq 0}\gamma$ . Then

**Definition 5.4.** (see [Cas03, Definition 2.3]) *The above class  $\gamma$  is called **contractible** if there exists a toric variety  $X_\gamma$  and an equivariant morphism  $\phi_\gamma : X \rightarrow X_\gamma$ , surjective with connected fibers, such that for every irreducible curve  $C$  in  $X$ ,*

$$\phi_\gamma(C) = \{pt\} \text{ if and only if } [C] \in \mathbb{Q}_{\geq 0}\gamma.$$

**Remark 5.5.** *Note that any contractible class is always a class of some invariant curve and also a primitive relation (see [Cas03, Theorem 2.2] and [Sca09, Page 74]).*

Recall the following result from [Cas03, Proposition 3.4].

**Proposition 5.6.** *Let  $P = \{\rho_1, \dots, \rho_k\}$  be a primitive collection in  $\Sigma$ , with the primitive relation  $r(P)$ :*

$$\sum_{i=1}^k u_{\rho_i} - \sum_{\rho \in \gamma_P(1)} c_\rho u_\rho = 0.$$

Then  $r(P)$  is contractible if and only if for every primitive collection  $Q$  of  $\Sigma$  such that  $P \cap Q \neq \emptyset$  and  $P \neq Q$ , the set  $(Q \setminus P) \cup \gamma_P(1)$  contains a primitive collection.

**5.2. Mori cone.** We use the notation as above. Let  $X$  be a smooth projective variety. We define  $NE(X)$  the real convex cone in  $N_1(X)$  generated by classes of irreducible curves. The **Mori cone**  $\overline{NE}(X)$  is the closure of  $NE(X)$  in  $N_1(X)$  and it is a strongly convex cone of maximal dimension.

If  $X$  is a (smooth projective) toric variety, it is known that  $NE(X)_{\mathbb{Z}}$  is generated by the finitely many torus-invariant irreducible curves in  $X$  and hence  $NE(X)_{\mathbb{Z}}$  is a finitely generated monoid. Hence the cone  $NE(X) = \overline{NE}(X)$  is a rational polyhedral cone and we have

$$\overline{NE}(X) = \sum_{\tau \in \Sigma(r-1)} \mathbb{R}_{\geq 0}[V(\tau)],$$

where  $r = \dim(X)$  and  $[V(\tau)] \in N_1(X)_{\mathbb{Z}}$  is the class of the toric curve  $V(\tau)$ . This is called the Toric Cone Theorem (see [CLS11, Theorem 6.3.20]). Let  $\tau \in \Sigma(r-1)$  be a wall, that is  $\tau = \sigma \cap \sigma'$  for some  $\sigma, \sigma' \in \Sigma(r)$ . Let  $\sigma$  (respectively,  $\sigma'$ ) be generated by  $\{u_{\rho_1}, u_{\rho_2}, \dots, u_{\rho_r}\}$  (respectively, by  $\{u_{\rho_2}, \dots, u_{\rho_{r+1}}\}$ ) and let  $\tau$  be generated by  $\{u_{\rho_2}, \dots, u_{\rho_r}\}$ . Then we get a linear relation,

$$u_{\rho_1} + \sum_{i=2}^r b_i u_{\rho_i} + u_{\rho_{r+1}} = 0 \quad (5.6)$$

The relation (5.6) called **wall relation** and we have

$$D_{\rho} \cdot V(\tau) = \begin{cases} b_i & \text{if } \rho = \rho_i \text{ and } i \in \{2, 3, \dots, r\} \\ 1 & \text{if } \rho = \rho_i \text{ and } i \in \{1, r+1\} \\ 0 & \text{otherwise} \end{cases}$$

(see [CLS11, Proposition 6.4.4 and eq. (6.4.6) page 303]).

Now we describe the Mori cone  $\overline{NE}(X_r)$  of  $X_r$  in terms of the primitive relations of  $X_r$ .

**Theorem 5.7.**  $\overline{NE}(X_r)_{\mathbb{Z}} = \sum_{i=1}^r \mathbb{Z}_{\geq 0} r(P_i)$ .

*Proof.* We have

$$\overline{NE}(X_r) = \sum_{P \in \mathcal{P}} \mathbb{R}_{\geq 0} r(P),$$

where  $\mathcal{P}$  is the set of all primitive collections in  $X_r$  (see [CLS11, Theorem 6.4.11]). By Lemma 5.3,  $\{P_i : 1 \leq i \leq r\}$  is the set of all primitive collections of  $X_r$ . Therefore, we get

$$\overline{NE}(X_r) = \sum_{i=1}^r \mathbb{R}_{\geq 0} r(P_i).$$

By [Cas03, Theorem 4.1], we have

$$\overline{NE}(X_r)_{\mathbb{Z}} = \sum_{\gamma \in \mathcal{C}} \mathbb{Z}_{\geq 0} \gamma,$$

where  $\mathcal{C}$  is the set of all contractible classes in  $X_r$ .

By Proposition 5.6, we can see that the primitive relations  $r(P_i)$  are contractible classes for  $1 \leq i \leq r$ . Since any contractible class is a primitive relation, we get

$$\mathcal{C} = \{r(P_i) : 1 \leq i \leq r\}.$$

Hence we conclude that

$$\overline{NE}(X_r)_{\mathbb{Z}} = \sum_{i=1}^r \mathbb{Z}_{\geq 0} r(P_i).$$

This completes the proof of the theorem.  $\square$

We have

**Corollary 5.8.** *The set  $\{r(P_i) : 1 \leq i \leq r\}$  forms a basis of  $N_1(X_r)_{\mathbb{Z}}$ .*

*Proof.* By Theorem 5.7,  $\{r(P_i) : 1 \leq i \leq r\}$  generates the monoid  $\overline{NE}(X_r)_{\mathbb{Z}}$  and the cone  $\overline{NE}(X_r)$  is of dimension  $r$ . So  $r(P_i)$  for  $1 \leq i \leq r$  are linearly independent. Also the group  $N_1(X_r)_{\mathbb{Z}}$  is generated by  $\overline{NE}(X_r)_{\mathbb{Z}}$ , hence by  $r(P_i)$  for  $1 \leq i \leq r$ . Hence these form a basis of  $N_1(X_r)_{\mathbb{Z}}$ .  $\square$

Next we describe the primitive relation  $r(P_i)$  explicitly by finding the cone  $\gamma_{P_i}$  in (5.1) for  $1 \leq i \leq r$ . We also observe that these cones depend on the given matrix corresponding to the Bott tower. We need some notation to state the result.

Recall the matrix  $M_r$  corresponding to the Bott tower  $X_r$  is

$$M_r = \begin{bmatrix} 1 & \beta_{12} & \beta_{13} & \cdots & \beta_{1r} \\ 0 & 1 & \beta_{23} & \cdots & \beta_{2r} \\ 0 & 0 & 1 & \cdots & \beta_{3r} \\ \vdots & \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & & 1 \end{bmatrix}_{r \times r}$$

(see Section 2).

Fix  $1 \leq i \leq r$ . Define:

- (1) Let  $r \geq j > j_1 = i \geq 1$  and define  $a_{1,j} := \beta_{j_1 j}$ .
- (2) Let  $r \geq j_2 > j_1$  be the least integer such that  $a_{1,j} > 0$ , then define for  $j > j_2$

$$a_{2,j} := \beta_{i j_2} \beta_{j_2 j} - \beta_{i j}.$$

- (3) Let  $k > 2$  and let  $r \geq j_k > j_{k-1}$  be the least integer such that  $a_{k-1,j} < 0$ , then inductively, define for  $j > j_k$

$$a_{k,j} := -a_{k-1,j_k} \beta_{j_k j} + a_{k-1,j}.$$

- (4) For  $j \leq i$ ,  $b_j := 0$ , and for  $j > i$  define

$$b_j := a_{l,j} \text{ if } j_{l+1} \geq j > j_l, l \geq 1. \quad (5.7)$$

Note that we have

$$b_j = \begin{cases} 0 & \text{for } j \leq i \\ < 0 & \text{for } j \in \{j_3, \dots, j_m\} \\ \geq 0 & \text{otherwise.} \end{cases}$$

- (5) Let  $I_i := \{j_1, \dots, j_m\}$ .

**Example 5.9.** *Let*

$$M_7 = \begin{bmatrix} 1 & -1 & -1 & -1 & 2 & -1 & 2 \\ 0 & 1 & 0 & 2 & -1 & 2 & -1 \\ 0 & 0 & 1 & 0 & -1 & -1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{7 \times 7}$$

Let  $G = SL(5, \mathbb{C})$  and let

$$\tilde{w} = s_{\beta_1} \cdots s_{\beta_7} = s_{\alpha_2} s_{\alpha_1} s_{\alpha_3} s_{\alpha_1} s_{\alpha_2} s_{\alpha_1} s_{\alpha_2}.$$

Note that the matrix corresponding to the toric limit  $X_{\tilde{w}}$  is  $M_7$ .

Let  $i = 1$  then  $j_1 = 1$  and

- (1)  $a_{1,2} = \beta_{12} = \langle \beta_2, \check{\beta}_1 \rangle = \langle \alpha_1, \check{\alpha}_2 \rangle = -1$  ;
- (2)  $a_{1,3} = \beta_{13} = \langle \beta_3, \check{\beta}_1 \rangle = \langle \alpha_3, \check{\alpha}_2 \rangle = -1$  ;
- (3)  $a_{1,4} = \beta_{14} = \langle \beta_4, \check{\beta}_1 \rangle = \langle \alpha_1, \check{\alpha}_2 \rangle = -1$  ;
- (4)  $a_{1,5} = \beta_{15} = \langle \beta_5, \check{\beta}_1 \rangle = \langle \alpha_2, \check{\alpha}_2 \rangle = 2$  ;
- (5)  $a_{1,6} = \beta_{16} = \langle \beta_6, \check{\beta}_1 \rangle = \langle \alpha_1, \check{\alpha}_2 \rangle = -1$  ;
- (6)  $a_{1,7} = \beta_{17} = \langle \beta_7, \check{\beta}_1 \rangle = \langle \alpha_2, \check{\alpha}_2 \rangle = 2$  .

Then  $j_2 = 5$  and

- (1)  $a_{2,6} = \beta_{15}\beta_{56} - \beta_{16} = \langle \beta_5, \check{\beta}_1 \rangle \langle \beta_6, \check{\beta}_5 \rangle - \langle \beta_6, \check{\beta}_1 \rangle = \langle \alpha_1, \check{\alpha}_2 \rangle = -1$  ;
- (2)  $a_{2,7} = \beta_{15}\beta_{57} - \beta_{17} = \langle \alpha_2, \check{\alpha}_2 \rangle = 2$  .

Then  $j_3 = 6$  and

- (1) 
$$\begin{aligned} a_{3,7} &= -a_{2,6}\beta_{67} + a_{2,7} \\ &= -(\langle \beta_6, \check{\beta}_5 \rangle)(\langle \beta_7, \check{\beta}_6 \rangle) + (\langle \beta_7, \check{\beta}_5 \rangle) \\ &= -(-1)(-1) + (2) \\ &= 1. \end{aligned}$$

Therefore,  $I_1 = \{1, 5, 6\}$  .

Let  $1 \leq i \leq r$ . Let  $\mathcal{A}_i := \{e_j^{\epsilon_j} : 1 \leq j \leq r, b_j \neq 0 \text{ and}$

$$\epsilon_j = \begin{cases} + & \text{for } j \notin I_i \\ - & \text{for } j \in I_i \end{cases} \}.$$

**Remark 5.10.** Note that as  $b_j = 0$  for  $j \leq i$ , we can take  $i < j \leq r$  in the definition of  $\mathcal{A}_i$ .

Now we prove,

**Proposition 5.11.** Let  $1 \leq i \leq r$ . The cone  $\gamma_{P_i}$  in the primitive relation of  $X_r$  corresponding to  $P_i$  is generated by  $\mathcal{A}_i$ .

Before going to the proof we see an example.

**Example 5.12.** We use same setting as in Example 5.9. By Lemma 5.3, we have  $P_i = \{\rho_i^+, \rho_i^-\}$  for all  $1 \leq i \leq 7$ .

By definition of  $e_i^-$  (see (2.2)), we have

- (i)  $e_1^- + e_1^+ = e_2^+ + e_3^+ + e_4^+ - 2e_5^+ + e_6^+ - 2e_7^+$ .
- (ii)  $e_2^- + e_2^+ = -2e_4^+ + e_5^+ - 2e_6^+ + e_7^+$ .
- (iii)  $e_3^- + e_3^+ = e_5^+ + e_7^+$ .
- (iv)  $e_4^- + e_4^+ = e_5^+ - 2e_6^+ + e_7^+$
- (v)  $e_5^- + e_5^+ = e_6^+ - 2e_7^+$
- (vi)  $e_6^- + e_6^+ = e_7^+$ .
- (vii)  $e_7^- + e_7^+ = 0$ .

Now we describe the cone  $\gamma_{P_1}$ .

Observe that in (i) coefficient of  $e_5^+$  is negative. By (v), we can see

$$e_1^- + e_1^+ = e_2^+ + e_3^+ + e_4^+ + 2(e_5^- - e_6^+ + 2e_7^+) + e_6^+ - 2e_7^+.$$

Then

$$e_1^- + e_1^+ = e_2^+ + e_3^+ + e_4^+ + 2e_5^- - e_6^+ + 2e_7^+.$$

By (vi),

$$e_1^- + e_1^+ = e_2^+ + e_3^+ + e_4^+ + 2e_5^- + e_6^- + e_7^+. \quad (5.8)$$

In this case,  $I_1 = \{1, 5, 6\}$  (see Example 5.9) and the cone  $\gamma_{P_1}$  is generated by

$$\{e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\}.$$

Now we prove Proposition 5.11:

*Proof.* By (2.2), for all  $1 \leq i \leq r$ , we have

$$e_i^- + e_i^+ = - \sum_{j>i} \beta_{ij} e_j^+. \quad (5.9)$$

If for all  $j > i$ ,  $\beta_{ij} \leq 0$ , then the cone  $\gamma_{P_i}$  is generated by

$$\{e_j^+ : j > i, \beta_{ij} < 0\}.$$

If not, choose the least integer  $j_2 > i$  such that  $\beta_{ij_2} > 0$ . Now write

$$e_i^- + e_i^+ = -\left(\sum_{j_2 > j > i} \beta_{ij} e_j^+\right) + \beta_{ij_2}(-e_{j_2}^+) - \left(\sum_{j > j_2} \beta_{ij} e_j^+\right).$$

Again by using (5.9), we have

$$e_i^- + e_i^+ = -\left(\sum_{j_2 > j > i} \beta_{ij} e_j^+\right) + \beta_{ij_2}(e_{j_2}^- + \sum_{j > j_2} \beta_{j_2 j} e_j^+) - \left(\sum_{j > j_2} \beta_{ij} e_j^+\right).$$

Then

$$e_i^- + e_i^+ = -\left(\sum_{j_2 > j > i} \beta_{ij} e_j^+\right) + \beta_{ij_2} e_{j_2}^- + \sum_{j > j_2} (\beta_{ij_2} \beta_{j_2 j} - \beta_{ij}) e_j^+.$$

By definition  $a_{2,j} = \beta_{ij_2} \beta_{j_2 j} - \beta_{ij}$ , then we have

$$e_i^- + e_i^+ = -\left(\sum_{j_2 > j > i} \beta_{ij} e_j^+\right) + \beta_{ij_2} e_{j_2}^- + \left(\sum_{j > j_2} a_{2,j} e_j^+\right).$$

If  $a_{2,j} \geq 0$  for all  $j > j_2$ , then  $\gamma_{P_i}$  is generated by

$$\{e_j^{\epsilon_j} : j > i, \epsilon_j = +\forall j \neq j_2, \text{ and } \epsilon_j = - \text{ for } j = j_2\}.$$

Otherwise, choose the least integer  $j_3 > j_2$  such that  $a_{2,j_3} < 0$ . By substituting  $-e_{j_3}^+$  from (5.9), we get

$$e_i^- + e_i^+ = -\left(\sum_{j_2 > j > i} a_{1,j} e_j^+\right) + \beta_{ij_2} e_{j_2}^- + \left(\sum_{j_3 > j > j_2} a_{2,j} e_j^+\right) - a_{2,j_3} (e_{j_3}^- + \sum_{j > j_3} \beta_{j_3 j} e_j^+) + \left(\sum_{j > j_3} a_{2,j} e_j^+\right).$$

Then,

$$e_i^- + e_i^+ = -\left(\sum_{j_2 > j > i} a_{1,j} e_j^+\right) + \beta_{ij_2} e_{j_2}^- + \left(\sum_{j_3 > j > j_2} a_{2,j} e_j^+\right) - a_{2,j_3} e_{j_3}^- + \sum_{j > j_3} (-a_{2,j_3} \beta_{j_3 j} + a_{2,j}) e_j^+.$$

By definition of  $a_{3,j} = -a_{2,j_3} \beta_{j_3 j} + a_{2,j}$ , then we have

$$e_i^- + e_i^+ = -\left(\sum_{j_2 > j > i} a_{1,j} e_j^+\right) + 2e_{j_2}^- + \left(\sum_{j_3 > j > j_2} a_{2,j} e_j^+\right) - a_{2,j_3} e_{j_3}^- + \left(\sum_{j > j_3} a_{3,j} e_j^+\right).$$

By repeating this process, we get the cone  $\gamma_{P_i}$  as we required.  $\square$

Let  $1 \leq i \leq r$ . Recall  $I_i = \{i = j_1, \dots, j_m\}$  as in page 19. Define for  $1 \leq j \leq r$ ,

$$c_j := \begin{cases} -b_j & \text{if } j \in I_i \setminus \{j_1, j_2\} \\ b_j & \text{otherwise} \end{cases}$$

Set  $\gamma_{P_i}(1) := \{\gamma_1, \dots, \gamma_l\}$ . Then we have

**Corollary 5.13.** For  $1 \leq i \leq r$ , the primitive relation  $r(P_i)(= (r_\rho)_{\rho \in \Sigma(1)})$  of  $X_r$  given by

$$r_\rho = \begin{cases} 1 & \text{for } \rho = \rho_i^+ \text{ or } \rho_i^- \\ -c_j & \text{for } \rho = \gamma_j \in \gamma_{P_i}(1) \\ 0 & \text{otherwise} \end{cases}$$

**Example 5.14.** We use Example 5.12. The following can be seen easily from (5.8).

(1)  $\gamma_{P_1}(1) = \{\rho_2^+, \rho_3^+, \rho_4^+, \rho_5^-, \rho_6^-, \rho_7^+\}$ .

(2) The primitive relation  $r(P_1) = (r_\rho)_{\rho \in \Sigma(1)}$  is given by

$$r_\rho = \begin{cases} 1 & \text{for } \rho = \rho_1^+ \text{ or } \rho_1^- \\ -1 & \text{for } \rho = \rho_k^+, k \in \{2, 3, 4, 7\} \text{ and } \rho = \rho_6^- \\ -2 & \text{for } \rho = \rho_5^- \\ 0 & \text{otherwise} \end{cases}$$

Now we describe the primitive relations  $r(P_i)$  in terms of intersection of two maximal cones in the fan of  $X_r$ . Let  $1 \leq i \leq r$ . Let  $\mathcal{C}'_i := \{e_j^{\epsilon_j} : 1 \leq j \leq r \text{ and}$

$$\epsilon_j = \begin{cases} + & \text{if } j \notin I_i \setminus \{j_1\} \\ - & \text{if } j \in I_i \end{cases} \}.$$

Let  $\mathcal{C}''_i := \{e_j^{\epsilon_j} : 1 \leq j \leq r \text{ and}$

$$\epsilon_j = \begin{cases} + & \text{if } j \notin I_i \\ - & \text{if } j \in I_i \end{cases} \}.$$

**Example 5.15.** We use Example 5.12, for  $i = 1$ , we have  $I_1 = \{1, 5, 6\}$ . Then

$$\mathcal{C}'_1 = \{e_1^+, e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\} \text{ and } \mathcal{C}''_1 = \{e_1^-, e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\}.$$

We prove the following by using *wall relation* (see page 16).

**Proposition 5.16.** Fix  $1 \leq i \leq r$ . The class of curve  $r(P_i)$  is given by

$$r(P_i) = [V(\tau_i)],$$

where  $\tau_i = \sigma \cap \sigma'$  and  $\sigma$  (respectively,  $\sigma'$ ) is the cone generated by  $\mathcal{C}'_i$  (respectively, by  $\mathcal{C}''_i$ ).

*Proof.* From Corollary 5.13, we have the following.

$$e_i^+ + e_i^- - \sum_{j>i} c_j e_j^{\epsilon_j} = 0, \tag{5.10}$$

where  $\epsilon_j$  is as in the statement.

First we show that the set  $Q := \{\rho \in \Sigma(1) : D_\rho \cdot V(\tau_i) > 0\}$  is not contained in  $\sigma(1)$  for any  $\sigma \in \Sigma$  (we adapt the arguments of [CLS11, Proof of Theorem 6.4.11, page 306], here



we are not assuming the curve  $V(\tau_i)$  is extremal). Indeed, suppose  $Q \subseteq \sigma(1)$  for some  $\sigma \in \Sigma$ . Let  $D$  be an ample divisor in  $X_r$ . Then, we can assume that  $D$  is of the form

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho, \quad a_\rho = 0 \text{ for all } \rho \in \sigma(1) \text{ and } a_\rho \geq 0 \text{ for all } \rho \notin \sigma(1)$$

(see [CLS11, (6.4.10), page 306]). Then we can see

$$D \cdot V(\tau_i) = \sum_{\rho \notin \sigma(1)} a_\rho D_\rho \cdot V(\tau_i).$$

As  $Q \subseteq \sigma(1)$ , by definition of  $Q$ ,  $D_\rho \cdot V(\tau_i) \leq 0$  for  $\rho \notin \sigma(1)$ . Since  $a_\rho \geq 0$  for  $\rho \notin \sigma(1)$ , we get  $D \cdot V(\tau_i) \leq 0$ , which is a contradiction as  $D$  is ample. Therefore,  $Q$  is not contained in  $\sigma(1)$  for any  $\sigma \in \Sigma$ .

Hence to prove the proposition it is enough to prove

$$P_i = Q( := \{ \rho \in \Sigma(1) : D_\rho \cdot V(\tau_i) > 0 \} )$$

(see again [CLS11, Proof of Theorem 6.4.11, page 306]).

From (5.10) and by using *wall relation*, we can see that

$$D_\rho \cdot V(\tau_i) = \begin{cases} 1 & \text{if } \rho = \rho_i^+ \text{ or } \rho_i^-. \\ -c_j & \text{if } \rho = \rho_j^{\epsilon_j} \text{ and } j \in I_i \setminus \{j_1\}. \\ 0 & \text{otherwise.} \end{cases}$$

Since  $c_j$ 's are all positive integers (see (5.4)), by Lemma 5.3 we conclude that

$$P_i = \{ \rho \in \Sigma(1) : D_\rho \cdot V(\tau_i) > 0 \}$$

and hence  $r(P_i) = [V(\tau_i)]$ . This completes the proof of the proposition.  $\square$

**Example 5.17.** In Example 5.12, the curve  $r(P_1) = [V(\tau_1)]$  with  $\tau_1 = \sigma \cap \sigma'$  where  $\sigma$  is the cone generated by

$$\mathcal{C}'_1 = \{e_1^+, e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\}$$

and  $\sigma'$  is the cone generated by

$$\mathcal{C}''_1 = \{e_1^-, e_2^+, e_3^+, e_4^+, e_5^-, e_6^-, e_7^+\}.$$

## 6. AMPLE AND *nef* LINE BUNDLES ON THE BOTT TOWER

Let  $X$  be a smooth projective variety. Recall  $N^1(X)$  is the real finite dimensional vector space of numerical classes of real divisors in  $X$  (see [Kle66, §1, Chapter IV]). In  $N^1(X)$ , we define the *nef* cone  $Nef(X)$  to be the cone generated by classes of numerically effective divisors and it is a strongly convex closed cone in  $N^1(X)$ . The ample cone  $Amp(X)$  of  $X$  is the cone in  $N^1(X)$  generated by classes of ample divisors. Note that the ample cone  $Amp(X)$  is interior of the *nef* cone  $Nef(X)$  (see [Kle66, Theorem 1, §2, Chapter IV]). Recall that the *nef* cone  $Nef(X)$  and the Mori cone  $\overline{NE}(X)$  are closed convex cones and are dual to each other (see [Kle66, §2, Chapter IV]).

In our case, we have  $Pic(X_r)_\mathbb{R} = N^1(X_r)$ , as the numerical equivalence and linear equivalence coincide (see [CLS11, Proposition 6.3.15]).

In this section, we characterize the ampleness and numerical effectiveness of line bundles on  $X_r$  and we study the generators of the *nef* cone of  $X_r$ . We use the notation as in Section 5. Let  $D = \sum a_\rho D_\rho$  be a toric divisor in  $X_r$  and for  $1 \leq i \leq r$ , define

$$d_i := (a_{\rho_i^+} + a_{\rho_i^-} - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j a_{\gamma_j}).$$

Then we prove,

**Lemma 6.1.**

- (1) *The divisor  $D$  is ample if and only if  $d_i > 0$  for all  $1 \leq i \leq r$ .*
- (2) *The divisor  $D$  is numerically effective (nef) if and only if  $d_i \geq 0$  for all  $1 \leq i \leq r$ .*

*Proof.* Proof of (2): Recall that the primitive relation  $r(P_i)$  is given by

$$r(P_i) = (r_\rho)_{\rho \in \Sigma(1)}$$

(see page 15). First observe that we have the following

$$D \cdot r(P_i) = \sum_{\rho \in \Sigma(1)} a_\rho (D_\rho \cdot r(P_i)) = \sum_{\rho \in \Sigma(1)} a_\rho r_\rho$$

(see [CLS11, Proposition 6.4.1, page 299]).

Then by (5.5), we get

$$D \cdot r(P_i) = \sum_{\rho \in P_i} a_\rho - \sum_{\rho \in \gamma_{P_i}(1)} r_\rho a_\rho.$$

By Lemma 5.3, we have  $P_i = \{\rho_i^+, \rho_i^-\}$ . Then by Corollary 5.13, we get

$$D \cdot r(P_i) = (a_{\rho_i^+} + a_{\rho_i^-} - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j a_{\gamma_j}) =: d_i. \quad (6.1)$$

Since the *nef* cone  $Nef(X_r)$  and the Mori cone  $\overline{NE}(X_r)$  are dual to each other, the divisor  $D$  is *nef* if and only if  $D \cdot C \geq 0$  for all torus-invariant irreducible curves  $C$  in  $X_r$ . By Theorem 5.7, we have

$$\overline{NE}(X_r) = \sum_{i=1}^r \mathbb{R}_{\geq 0} r(P_i).$$

Hence  $D$  is *nef* if and only if  $D \cdot r(P_i) \geq 0$  for all  $1 \leq i \leq r$ . Therefore, by (6.1), we conclude that the divisor  $D$  is *nef* if and only if  $d_i \geq 0$  for all  $1 \leq i \leq r$ . This completes the proof of (2).

Proof of (1): Recall that the divisor  $D$  is ample if and only if its class in  $Pic(X_r)_\mathbb{R}$  lies in the interior of the *nef* cone  $Nef(X_r)$ . Hence by using similar arguments as in the proof of (2) and the toric Kleiman criterion for ampleness [CLS11, Theorem 6.3.13], we can see that  $D$  is ample if and only if  $d_i > 0$  for all  $1 \leq i \leq r$ .  $\square$

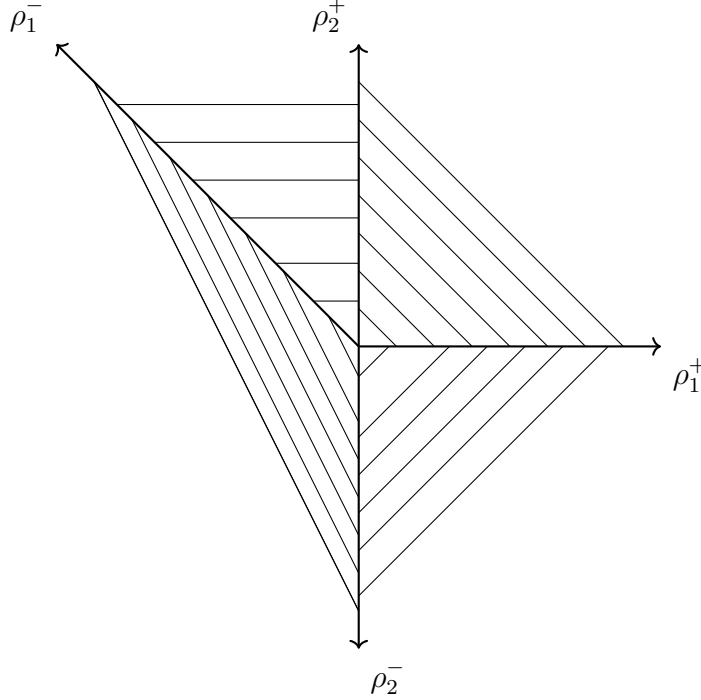
Next we describe the generators of the *nef* cone  $Nef(X_r)$  of  $X_r$ .

**Example 6.2.** Let  $G = SL(3, \mathbb{C})$  and  $\tilde{w} = s_{\alpha_1} s_{\alpha_2}$ . Note that the corresponding matrix for the Bott tower  $X_{\tilde{w}}$  is

$$A_2 = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}_{2 \times 2}$$

Then  $X_{\tilde{w}} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(1))$ , the Hirzebruch surface  $\mathcal{H}_1$  and the rays  $(\rho_1^+, \rho_1^-, \rho_2^+$  and  $\rho_2^-)$  of the fan (shown below) of  $X_{\tilde{w}}$  are generated by

$$e_1^+, e_1^- = -e_1^+ + e_2^+, e_2^+ \text{ and } -e_2^+.$$



**Figure.** Fan of Hirzebruch surface  $\mathcal{H}_1$ .

The primitive relations  $r(P_1)$  and  $r(P_2)$  are given by

$$r(P_1) : e_1^+ + e_1^- = e_2 \text{ and } r(P_2) : e_2^+ + e_2^- = 0.$$

By wall relation, we observe that

$$(1) D_{\rho_1^+} \cdot r(P_1) = 1 \text{ and } D_{\rho_1^+} \cdot r(P_2) = 0.$$

$$(2) D_{\rho_2^-} \cdot r(P_1) = 0 \text{ and } D_{\rho_2^-} \cdot r(P_2) = 1.$$

Then the dual basis of  $\{r(P_1), r(P_2)\}$  is  $\{D_{\rho_1^+}, D_{\rho_2^-}\}$ . Hence the generators of the nef cone  $\text{Nef}(\mathcal{H}_1)$  are  $D_{\rho_1^+}$  and  $D_{\rho_2^-}$ . Note that by Lemma 4.1,  $\text{Pic}(\mathcal{H}_1)$  is generated by  $\{D_{\rho_1^+}, D_{\rho_2^-}\}$ . Let  $D = aD_{\rho_1^+} + bD_{\rho_2^-} \in \text{Pic}(\mathcal{H}_1)$ . Then

$$D \text{ is ample if and only if } a > 0 \text{ and } b > 0$$

(this gives back [CLS11, Example (6.1.16), page 273]).

Now we prove the similar results for  $X_r$ . For  $1 \leq m \leq r$ , define

$$J_m := \{1 \leq i < m : \{\rho_m^+\} \cap \gamma_{P_i}(1) \neq \emptyset\}.$$

**Remark 6.3.** Note that the set  $J_m$  is the collection of indices  $i < m$  for which  $u_{\rho_m^+}$  appear in the  $\gamma_{P_i}$  part of the expression (5.1) for the primitive relation  $r(P_i)$ .

We set  $D_1 := D_{\rho_1^+}$ , and for  $m > 1$  define inductively

$$D_m := \begin{cases} D_{\rho_m^+} & \text{if } J_m = \emptyset \\ (\sum_{k \in J_m} c_{\rho_m^+}^{\gamma_{P_k}} D_k) + D_{\rho_m^+} & \text{if } J_m \neq \emptyset, \end{cases}$$

where  $-c_{\rho_m^+}^{\gamma_{P_k}}$  is the coefficient of  $e_m^+$  in the primitive relation  $r(P_k)$ .

**Example 6.4.** In Example 6.2,  $D_1 = D_{\rho_1^+}$ ,  $J_2 = \{1\}$  and  $D_2 = D_1 + D_{\rho_2^+}$ . By using (2.2), we see that

$$0 \sim \text{div}(\chi^{e_1^+}) \sim D_{\rho_1^+} - D_{\rho_1^-}$$

and

$$0 \sim \text{div}(\chi^{e_2^+}) \sim D_{\rho_2^+} - D_{\rho_2^-} + D_{\rho_1^-}.$$

Hence  $D_2 = D_1 + D_{\rho_2^+} = D_{\rho_2^-}$ .

**Example 6.5.** In Example 5.12,

(1) Recall by (5.8), we have

$$e_1^- + e_1^+ = e_2^+ + e_3^+ + e_4^+ + 2e_5^- + e_6^- + e_7^+.$$

Then,  $\gamma_{P_1}(1) = \{\rho_2^+, \rho_3^+, \rho_4^+, \rho_5^-, \rho_6^-, \rho_7^+\}$ .

(2)  $\gamma_{P_2}(1) = \{\rho_4^-, \rho_5^-, \rho_6^+, \rho_7^+\}$  (since  $e_2^+ + e_2^- = 2e_4^- + e_5^- + e_6^+ + e_7^+$ ).

(3)  $\gamma_{P_3}(1) = \{\rho_5^+, \rho_7^+\}$  (since  $e_3^+ + e_3^- = e_5^+ + e_7^+$ ).

(4)  $\gamma_{P_4}(1) = \{\rho_5^+, \rho_6^-, \rho_7^-\}$  (since  $e_4^+ + e_4^- = e_5^+ + 2e_6^- + e_7^-$ ).

(5)  $\gamma_{P_5}(1) = \{\rho_6^+, \rho_7^-\}$  (since  $e_5^+ + e_5^- = e_6^+ + 2e_7^-$ ).

(6)  $\gamma_{P_6}(1) = \{\rho_7^+\}$  (since  $e_6^+ + e_6^- = e_7^+$ ).

(7)  $\gamma_{P_7}(1) = \emptyset$ . (since  $e_7^+ + e_7^- = 0$ ).

Then,

(1) If  $m = 1$ , then  $D_1 = D_{\rho_1^+}$ .

(2) If  $m = 2$ , then  $J_2 = \{1\}$  and  $c_{\rho_2^+}^{\gamma_{P_1}} = 1$ . Hence  $D_2 = D_1 + D_{\rho_2^+}$ .

(3) If  $m = 3$ , then  $J_3 = \{1\}$  and  $c_{\rho_3^+}^{\gamma_{P_1}} = 1$ . Hence  $D_3 = D_1 + D_{\rho_3^+}$ .

(4) If  $m = 4$ , then  $J_4 = \{1\}$  and  $c_{\rho_4^+}^{\gamma_{P_1}} = 1$ . Hence  $D_4 = D_1 + D_{\rho_4^+}$ .

(5) If  $m = 5$ , then  $J_5 = \{3, 4\}$  and  $c_{\rho_5^+}^{\gamma_{P_3}} = 1$ ;  $c_{\rho_5^+}^{\gamma_{P_4}} = 1$ . Hence

$$D_5 = D_3 + D_4 + D_{\rho_5^+}.$$

(6) If  $m = 6$ , then  $J_6 = \{2, 5\}$  and  $c_{\rho_6^+}^{\gamma_{P_2}} = 1$  ;  $c_{\rho_6^+}^{\gamma_{P_5}} = 1$ . Hence

$$D_6 = D_2 + D_5 + D_{\rho_6^+}.$$

(7) If  $m = 7$ , then  $J_7 = \{1, 2, 3, 6\}$  and

$c_{\rho_7^+}^{\gamma_{P_1}} = 1$  ;  $c_{\rho_7^+}^{\gamma_{P_2}} = 1$  ;  $c_{\rho_7^+}^{\gamma_{P_3}} = 1$  ; and  $c_{\rho_7^+}^{\gamma_{P_6}} = 1$  . Hence

$$D_7 = D_1 + D_2 + D_3 + D_6 + D_{\rho_7^+}.$$

We prove,

**Proposition 6.6.** *The set  $\{D_i : 1 \leq i \leq r\}$  is dual basis of  $\{r(P_i) : 1 \leq i \leq r\}$ .*

*Proof.* Fix  $1 \leq i \leq r$ . By Proposition 5.16, the class of curve corresponding to the primitive relation  $r(P_i)$  is given by

$$r(P_i) = [V(\tau_i)]$$

(where  $\tau_i$  is described as in Proposition 5.16). From Corollary 5.13, the primitive relation  $r(P_i)(= [V(\tau_i)])$  is

$$e_i^+ + e_i^- - \sum_{j>i} c_j e_j^{\epsilon_j} = 0, \quad (6.2)$$

where  $\epsilon_j$  is as in Proposition 5.16. Note that this is the *wall relation* for the torus-invariant curve  $V(\tau_i)$ . We prove

$$D_m \cdot r(P_i) = D_m \cdot V(\tau_i) = \begin{cases} 1 & \text{if } i = m. \\ 0 & \text{if } i \neq m. \end{cases} \quad (6.3)$$

By (6.2) and by *wall relation*, we have

$$D_{\rho_m^+} \cdot V(\tau_i) = \begin{cases} 1 & \text{for } m = i \\ 0 & \text{for } m < i \\ -c_{\rho_m^+}^{\gamma_{P_i}} & \text{for } m > i \text{ and } i \in J_m \\ 0 & \text{for } m > i \text{ and } i \notin J_m \end{cases} \quad (6.4)$$

Hence by definition of  $D_m$ , it is clear that

$$D_m \cdot V(\tau_i) = \begin{cases} 1 & \text{for } m = i \\ 0 & \text{for } m < i \end{cases} \quad (6.5)$$

Now we claim  $D_m \cdot V(\tau_i) = 0$  for all  $m > i$ . Assume that  $m > i$  and write  $m = i + j$ , where  $1 \leq j \leq r - i$ . We prove the claim by induction on  $j$ . If  $j = 1$ , then  $D_m = D_{i+1}$ .

Case 1: If  $J_{i+1} = \emptyset$ , then  $D_{i+1} = D_{\rho_{i+1}^+}$ . By (6.4), we see that

$$D_{i+1} \cdot V(\tau_i) = 0.$$

Case 2: Assume that  $J_{i+1} \neq \emptyset$ .

Subcase 1: If  $i \notin J_{i+1}$ , then by (6.4) and (6.5), we can see that

$$D_{i+1} \cdot V(\tau_i) = 0.$$

Subcase 2: If  $i \in J_{i+1}$ , then by (6.5), we have

$$D_{i+1} \cdot V(\tau_i) = c_{\rho_{i+1}^+}^{\gamma P_i} + (D_{\rho_{i+1}^+} \cdot V(\tau_i)).$$

By (6.4),  $D_{\rho_{i+1}^+} \cdot V(\tau_i) = -c_{\rho_{i+1}^+}^{\gamma P_i}$  and hence  $D_{i+1} \cdot V(\tau_i) = 0$ . This proves the claim for  $j = 1$ .

Now assume that  $j > 1$ .

Case 1: If  $J_m = \emptyset$ , then by (6.4) and (6.5), we see that

$$D_m \cdot V(\tau_i) = 0.$$

Case 2: Assume that  $J_m \neq \emptyset$ .

Subcase 1: If  $i \notin J_m$ , then by (6.4) and (6.5), we can see that

$$D_m \cdot V(\tau_i) = \left( \sum_{k \in J_m, k > i} c_{\rho_m^+}^{\gamma P_k} D_k \right) \cdot V(\tau_i) + (D_{\rho_m^+} \cdot V(\tau_i)).$$

By induction on  $j$ ,  $D_k \cdot V(\tau_i) = 0$  for all  $i < k < m$ . By (6.4), as  $m > i$  and  $m \notin J_m$ , we have

$$D_{\rho_m^+} \cdot V(\tau_i) = 0.$$

Hence we conclude that

$$D_m \cdot V(\tau_i) = 0.$$

This completes the proof of the proposition.  $\square$

We have,

**Theorem 6.7.**

- (1) The nef cone  $Nef(X_r)$  of  $X_r$  is generated by  $\{D_i : 1 \leq i \leq r\}$ .
- (2) The divisor  $D = \sum_i a_i D_i$  is ample if and only if  $a_i > 0$  for all  $1 \leq i \leq r$ .

*Proof.* Since the nef cone  $Nef(X_r)$  is dual of the Mori cone  $\overline{NE}(X_r)$ , (1) follows from Proposition 6.6.

Proof of (2): This follows from (1) as the ample cone  $Amp(X_r)$  is interior of the nef cone  $Nef(X_r)$ .  $\square$

## 7. FANONESS AND WEAK FANONESS OF BOTT TOWER

In this section we describe the matrices  $M_r$  such that the corresponding to Bott tower  $X_r$  is Fano or weak Fano. First recall the *Itaka dimension* of a Cartier divisor  $D$  in a normal projective variety  $X$ . Let

$$N(D) := \{m \geq 0 : H^0(X, \mathcal{L}(mD)) \neq 0\},$$

where  $\mathcal{L}(mD)$  is the line bundle associated to  $mD$ . For  $m \in N(D)$ , we have a rational map

$$\phi_m : X \dashrightarrow \mathbb{P}(H^0(X, \mathcal{L}(mD))^*).$$

If  $N(D)$  is empty we define the *Itaka dimension*  $\kappa(D)$  of  $D$  as  $-\infty$ . Otherwise we define

$$\kappa(D) := \max_{m \in N(D)} \{ \dim(\phi_m(X)) \}.$$

Observe that  $\kappa(D) \in \{-\infty, 0, 1, \dots, \dim(X)\}$ . We say  $D$  is *big* if  $\kappa(D) = \dim(X)$  (see [Laz04, Section 2.2, page 139]). Note that an ample divisor is *big*.

**Lemma 7.1.** *Let  $X$  be a smooth projective variety, let  $U$  be an open affine subset of  $X$ . Let  $D$  be an effective divisor with support  $X \setminus U$ . Then  $D$  is big.*

*Proof.* It suffices to show that there exists an effective divisor  $E$  with support  $X \setminus U$  such that  $E$  is *big*. Indeed, we then have  $mD = E + F$  for some  $m \geq 0$  and for some effective divisor  $F$ . Then  $E + F$  is big and hence so is  $D$ .

There exists  $f_1, \dots, f_n \in \mathcal{O}_X(U)$  algebraically independent over  $\mathbb{C}$ , where  $n = \dim(X)$ . View  $f_1, \dots, f_n$  as rational functions on  $X$ , then  $f_1, \dots, f_n \in H^0(X, \mathcal{O}_X(E))$  for some effective divisor  $E$  with support  $X \setminus U$  (since  $\text{div}(f_i)$  is an effective divisor with support in  $X \setminus U$  for  $1 \leq i \leq r$ ). Thus, the monomials in  $f_1, \dots, f_n$  of any degree  $m$  are linearly independent elements of  $H^0(X, \mathcal{O}_X(mE))$ . So  $\dim(H^0(X, \mathcal{O}_X(mE)))$  grows like  $m^n$  as  $m \rightarrow \infty$ . Hence  $E$  is *big* (see [Laz04, Corollary 2.1.38 and Lemma 2.2.3]) and this completes the proof.  $\square$

We get the following as a variant of Lemma 7.1.

Let  $X$  be a smooth projective variety and  $D$  be an effective divisor. Let  $\text{supp}(D)$  denotes the support of  $D$ . If  $X \setminus \text{supp}(D)$  is affine, then  $D$  is *big*.

A smooth projective variety  $X$  is called Fano (respectively, weak Fano) if its anti-canonical line bundle  $-K_X$  is ample (respectively, *nef* and *big*). To describe our results we use the notation and terminology from Section 1 (see page 3).

We prove,

**Theorem 7.2.**

- (1)  $X_r$  is Fano if and only if it satisfies II.
- (2)  $X_r$  is weak Fano if and only if it satisfies III.

*Proof.* Proof of (2): We have

$$K_{X_r} = - \sum_{\rho \in \Sigma(1)} D_\rho \tag{7.1}$$

(see [CLS11, Theorem 8.2.3] or [Ful93, Page 74]). The anti-canonical line bundle of any projective toric variety is *big*, since we have

$$\text{supp}(-K_{X_r}) = X_r \setminus (\mathbb{C}^*)^r,$$

$(\mathbb{C}^*)^r$  is an affine open subset of  $X_r$ , by Lemma 7.1,  $-K_{X_r}$  is *big*.

By using Lemma 6.1, we prove that  $-K_{X_r}$  is *nef* if and only if  $X_r$  satisfies III.

Let  $D = -K_{X_r}$ . By (7.1) and by definition of  $d_i$  for  $D$  (see Lemma 6.1), we have

$$d_i = 2 - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j.$$

Then by Lemma 6.1(2),  $-K_{X_r}$  is nef if and only if  $\sum_{\gamma_j \in \gamma_{P_i}(1)} c_j \leq 2$  for all  $1 \leq i \leq r$ .

First assume that  $-K_{X_r}$  is nef. Fix  $1 \leq i \leq r$ . By above discussion, we have

$$\sum_{\gamma_j \in \gamma_{P_i}(1)} c_j \leq 2. \quad (7.2)$$

Since  $c_j$ 's are positive integers (see (5.4)), we get the following situation:

$$|\gamma_{P_i}(1)| = 0 \text{ or } |\gamma_{P_i}(1)| = 1, \text{ or } |\gamma_{P_i}(1)| = 2.$$

Case 1: If  $|\gamma_{P_i}(1)| = 0$ , then by definition of  $\gamma_{P_i}$  (see Proposition 5.11), we have

$$r(P_i) : e_i^+ + e_i^- = 0.$$

Hence we see  $X_r$  satisfies the condition  $N_i^1$ .

Case 2: If  $|\gamma_{P_i}(1)| = 1$ , then there exists unique  $r \geq j > i$ , such that  $\gamma_j \in \gamma_{P_i}(1)$  and the primitive relation is either

$$r(P_i) : e_i^+ + e_i^- = c_j e_j^+ \quad (7.3)$$

or

$$r(P_i) : e_i^+ + e_i^- = c_j e_j^- \quad (7.4)$$

By (7.2), we get  $c_j = 1$  or  $2$ .

Subcase (i): Assume that  $c_j = 1$ . If the primitive relation is (7.3), then we can see that  $|\eta_i^+| = 0$  and  $c_j = -\beta_{ij} = 1$ . Then  $\beta_{ij} = -1$  and hence  $X_r$  satisfies the condition  $N_i^2$ .

If the primitive relation is (7.4), then by (2.2)  $|\eta_i^-| = 0$  and  $|\eta_i^+| = 1$ . Hence  $c_j = \beta_{ij} = 1$  and  $\beta_{jk} = 0$  for all  $k > j$ .

Subcase (ii): Assume that  $c_j = 2$ . If the primitive relation  $r(P_i)$  is (7.3), then  $|\eta_i^+| = 0$  and  $|\eta_i^-| = 1$ . So by (2.2), we have  $c_j = -\beta_{ij}$ . If the primitive relation  $r(P_i)$  is (7.4), then  $|\eta_i^+| = 1$ ,  $|\eta_i^-| = 0$  and  $\beta_{jk} = 0$  for all  $k > j$ . Again by (2.2), we have  $c_j = \beta_{ij}$ . Thus,

$$\text{either } \beta_{ij} = -2 \text{ or } \beta_{ij} = 2.$$

Hence  $X_r$  satisfies the condition  $N_i^3$ .

Case 3: If  $|\gamma_{P_i}(1)| = 2$ , then there exists  $r \geq s_1 > s_2 > i$  with  $\gamma_{s_1}, \gamma_{s_2} \in \gamma_{P_i}(1)$  such that the primitive relation  $r(P_i)$  is

$$r(P_i) : e_i^+ + e_i^- = c_{s_1} e_{s_1}^\pm + c_{s_2} e_{s_2}^\pm \quad (7.5)$$

Subcase (i): If the primitive relation is  $r(P_i) : e_i^+ + e_i^- = c_{s_1} e_{s_1}^+ + c_{s_2} e_{s_2}^+$ , by (2.2) we see  $|\eta_i^+| = 0$  and  $|\eta_i^-| = 2$ . By (7.2) and (5.4) ( $c_i$ 's are positive integers), we get

$$c_{s_1} = 1, c_{s_2} = 1 \text{ and } \beta_{is_1} = \beta_{is_2} = -1.$$

Hence  $X_r$  satisfies the condition  $N_i^3$ .

Subcase (ii): If the primitive relation is  $r(P_i) : e_i^+ + e_i^- = c_{s_1} e_{s_1}^+ + c_{s_2} e_{s_2}^-$ , by (2.2) we see  $|\eta_i^+| = 1 = |\eta_i^-|$ . Then  $\beta_{is_1} = -1$ ,  $\beta_{is_2} = 1$  and  $\beta_{s_2 k} = 0$  for all  $k > s_2$ .

Subcase (iii): If the primitive relation is  $r(P_i) : e_i^+ + e_i^- = c_{s_1} e_{s_1}^- + c_{s_2} e_{s_2}^+$ , by (2.2) we see  $|\eta_i^+| = 1$  and  $\beta_{is_1} = 1$ . Then  $\beta_{s_1 s_2} - \beta_{is_2} = 1$  and  $\beta_{s_1 k} - \beta_{ik} = 0$  for all  $k > s_2$ .



Subcase (iv): If the primitive relation is  $r(P_i) : e_i^+ + e_i^- = c_{s_1} e_{s_1}^- + c_{s_2} e_{s_2}^-$ , by (2.2) we see  $|\eta_i^+| = 1$  and  $\beta_{is_1} = 1$ . Then  $\beta_{s_1 s_2} - \beta_{is_2} = -1$  and  $\beta_{is_2} - \beta_{s_1 s_2} - \beta_{s_2 k} = 0$  for all  $k > s_2$ .

Hence  $X_r$  satisfies the condition  $N_i^3$ . Therefore, we conclude that if  $X_r$  is weak Fano then  $X_r$  satisfies the condition *III*. Similarly, we can prove by using Lemma 6.1(2), if  $X_r$  satisfies *III* then  $X_r$  is weak Fano. This completes the proof of (2).

Proof of (1): This follows by using similar arguments as in the proof of (2) and Lemma 6.1(1).  $\square$

**7.1. Local rigidity of Bott towers.** Now we prove some vanishing results for the cohomology of tangent bundle of the Bott tower  $X_r$  and we get some local rigidity results. Let  $T_{X_r}$  denotes the tangent bundle of  $X_r$ . Then we have

**Corollary 7.3.** *If  $X_{\bar{w}}$  satisfies *II*, then  $H^i(X_{\bar{w}}, T_{X_{\bar{w}}}) = 0$  for all  $i \geq 1$ .*

*Proof.* If  $X_r$  satisfies *II*, then by Theorem 7.2,  $X_r$  is Fano variety. By [BB96, Proposition 4.2], since  $X_r$  is a smooth Fano toric variety, we get  $H^i(X_r, T_{X_r}) = 0$  for all  $i \geq 1$ .  $\square$

It is well known that by Kodaira-Spencer theory, the vanishing of  $H^1(X, T_X)$  implies that  $X$  is locally rigid, i.e. admits no local deformations (see [Huy06, Proposition 6.2.10, page 272]).

Then by above result we have

**Corollary 7.4.** *The Bott tower  $X_r$  is locally rigid if it satisfies *II*.*

## 8. LOG FANONESS OF BOTT TOWERS

Recall that a pair  $(X, D)$  of a normal projective variety  $X$  and an effective  $\mathbb{Q}$ -divisor  $D$  is **Kawamata log terminal (klt)** if  $K_X + D$  is  $\mathbb{Q}$ -Cartier, and for all proper birational maps  $f : Y \rightarrow X$ , the pull back  $f^*(K_X + D) = K_Y + D'$  satisfies  $f_* K_Y = K_X$  and  $[D'] \leq 0$ , where  $[\sum_i a_i D_i] = \sum_i [a_i] D_i$ ,  $[x]$  is the greatest integer  $\leq x$ .

The pair  $(X, D)$  is called **log Fano** if it is klt and  $-(K_X + D)$  is ample.

We recall here, a condition for the anti-canonical line bundle to be *big* (see [CG13]). Let  $X$  be a  $\mathbb{Q}$ -Gorenstein projective normal variety over  $\mathbb{C}$ . If  $X$  admits a divisor  $D$  with the pair  $(X, D)$  being log Fano then  $-K_X$  is big (In [CG13] there is a necessary and sufficient condition that  $X$  is log Fano (or ‘‘Fano type’’) variety, see [CG13, Theorem 1.1] for more details on this ).

If  $X$  is smooth and  $D$  is a normal crossing divisor, the pair  $(X, D)$  is log Fano if and only if  $[D] = 0$  and  $-(K_X + D)$  is ample (see [KM08, Lemma 2.30, Corollary 2.31 and Definition 2.34]). In case of toric variety  $X$  see also [CLS11, Definition 11.4.23 and Proposition 11.4.24, page 558]. We use notation as in Lemma 6.1. Let  $D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho$  be a toric divisor in  $X_r$ , with  $a'_\rho$ s in  $\mathbb{Q}_{\geq 0}$  and  $[D] = 0$ . For  $1 \leq i \leq r$ , define

$$k_i := d_i - 2 + \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j.$$

Then we prove,

**Theorem 8.1.** *The pair  $(X_r, D)$  is log Fano if and only if  $k_i < 0$  for all  $1 \leq i \leq r$ .*

*Proof.* From the above discussion by the condition on  $D$ , the pair  $(X_r, D)$  is log Fano if and only if  $-(K_{X_r} + D)$  is ample. Note that as  $-K_{X_r} = \sum_{\rho \in \Sigma(1)} D_\rho$ , we get

$$-(K_{X_r} + D) = \sum_{\rho \in \Sigma(1)} (1 - a_\rho) D_\rho.$$

By Lemma 6.1,  $-(K_{X_r} + D)$  is ample if and only if

$$((1 - a_{\rho_i^+}) + (1 - a_{\rho_i^-}) - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j (1 - a_{\gamma_j})) > 0 \text{ for all } 1 \leq i \leq r. \quad (8.1)$$

Recall the definition of  $d_i$  for  $D$ ,

$$d_i = a_{\rho_i^+} + a_{\rho_i^-} - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j a_{\gamma_j}.$$

Then we have

$$((1 - a_{\rho_i^+}) + (1 - a_{\rho_i^-}) - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j (1 - a_{\gamma_j})) = -(d_i - 2 + \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j).$$

Hence in (8.1)

$$((1 - a_{\rho_i^+}) + (1 - a_{\rho_i^-}) - \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j (1 - a_{\gamma_j})) = -k_i \text{ for all } 1 \leq i \leq r$$

and we conclude that  $-(K_{X_r} + D)$  is ample if and only if  $k_i < 0$  for all  $1 \leq i \leq r$ . This completes the proof of the theorem.  $\square$

## 9. EXTREMAL RAYS AND MORI RAYS OF THE BOTT TOWER

In this section we study the extremal rays and Mori rays of Mori cone of  $X_r$ . First we recall some definitions. Let  $V$  be a finite dimensional vector space over  $\mathbb{R}$  and let  $K$  be a (closed) cone in  $V$ . A subcone  $Q$  in  $K$  is called extremal if  $u, v \in K, u + v \in Q$  then  $u, v \in Q$ . A face of  $K$  is an extremal subcone. A one-dimensional face is called an extremal ray. Note that an extremal ray is contained in the boundary of  $K$ .

Let  $X$  be a smooth projective variety. An extremal ray  $R$  in  $\overline{NE}(X) \subset N_1(X)$  is called Mori if  $R \cdot K_X < 0$ , where  $K_X$  is the canonical divisor in  $X$ . Recall that  $\overline{NE}(X_r)$  is a strongly convex rational polyhedral cone of maximal dimension in  $N_1(X_r)$ .

We prove,

**Theorem 9.1.**

- (1) *The class of curves  $r(P_i)$  for  $1 \leq i \leq r$  are all extremal rays in the Mori cone  $\overline{NE}(X_r)$  of  $X_r$ .*
- (2) *Fix  $1 \leq i \leq r$ , the class of curve  $r(P_i)$  is Mori ray if and only if either  $|\gamma_{P_i}(1)| = 0$ , or  $|\gamma_{P_i}(1)| = 1$  with  $c_j = 1$  for  $\gamma_j \in \gamma_{P_i}(1)$ .*

*Proof.* Proof of (1): This follows from Theorem 5.7 and Corollary 5.8.

Proof of (2): By (1),  $r(P_i)$   $1 \leq i \leq r$  are all extremal rays in  $\overline{NE}(X_r)$ . Hence for  $1 \leq i \leq r$ ,  $r(P_i)$  is Mori if  $K_{X_r} \cdot r(P_i) < 0$ . Since  $K_{X_r} = -\sum_{\rho \in \Sigma(1)} D_\rho$ , we can see by Corollary 5.13 and by similar arguments as in the proof of Lemma 6.1

$$K_{X_r} \cdot r(P_i) = -2 + \sum_{\gamma_j \in \gamma_{P_i}(1)} c_j. \quad (9.1)$$

Thus if  $K_{X_r} \cdot r(P_i) < 0$ , then

$$\sum_{\gamma_j \in \gamma_{P_i}(1)} c_j < 2.$$

As  $c_j$  are all positive integers ( see 5.4), we get either  $|\gamma_{P_i}(1)| = 0$ , or  $|\gamma_{P_i}(1)| = 1$  and  $c_j = 1$  for  $\gamma_j \in \gamma_{P_i}(1)$ .

Similarly, by using (9.1) we can prove the converse. This completes the proof of the theorem.  $\square$

Now we prove a general result for smooth projective toric varieties,

**Lemma 9.2.** *Let  $X$  be a smooth projective toric variety of dimension  $r$ . Then  $X$  is Fano if and only if every extremal ray is Mori.*

*Proof.* By [CLS11, Theorem 6.3.20](Toric Cone Theorem), we have

$$\overline{NE}(X) = \sum_{\tau \in \Sigma(r-1)} [V(\tau)]. \quad (9.2)$$

If  $X$  is Fano, then by definition,  $-K_X$  is ample. By toric Kleiman criterion for ampleness [CLS11, Theorem 6.3.13], we can see that  $-K_X \cdot V(\tau) > 0$  for all  $\tau \in \Sigma(r-1)$ . Then  $K_X \cdot V(\tau) < 0$  for all  $\tau \in \Sigma(r-1)$ . In particular, every extremal ray is Mori.

Conversely, let  $\mathbb{R}_{\geq 0}[V(\tau)]$  be an extremal ray, by assumption it is a Mori ray. Then by definition of a Mori ray, we have  $K_X \cdot V(\tau) < 0$ . This implies  $-K_X \cdot V(\tau) > 0$ . By (9.2),  $\overline{NE}(X)$  is a polyhedral cone and hence the extremal rays generate the cone  $\overline{NE}(X)$ . Hence we see that  $-K_X \cdot C > 0$  for all classes of curves  $[C]$  in  $\overline{NE}(X)$ . Again by toric Kleiman criterion for ampleness, we conclude that  $-K_X$  is ample and hence  $X$  is Fano.  $\square$

Then we have the following:

**Corollary 9.3.** *The Bott tower  $X_r$  is Fano if and only if every extremal ray in  $\overline{NE}(X_r)$  is Mori.*

## 10. APPLICATIONS TO BSDH VARIETIES AND ITS TORIC LIMITS

In this section we see some applications to BSDH-varieties, by using their toric degeneration as a Bott tower.

**10.1. Ample cone of the toric limit of BSDH variety.** In [LT04], the ampleness of line bundles on BSDH variety  $Z(\tilde{w})$  is studied. Now we compare the ample cone of the toric limit  $X_{\tilde{w}}$  with that of the BSDH-variety  $Z(\tilde{w})$  as a consequence of Theorem 3.2.

**Corollary 10.1.** *The ample cone  $\text{Amp}(X_{\tilde{w}})$  of  $X_{\tilde{w}}$  can be identified with a subcone of the ample cone  $\text{Amp}(Z(\tilde{w}))$  of  $Z(\tilde{w})$ .*

*Proof.* By Theorem 3.2,  $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$  is a smooth projective morphism with  $\mathcal{X}_0 = X_{\tilde{w}}$  and  $\mathcal{X}_u = Z(\tilde{w})$  for  $u \neq 0$ .

Let  $\mathcal{L} = \{\mathcal{L}_u : u \in \mathbb{A}^1\}$  be a line bundle on  $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$  with  $\mathcal{L}_0$  is an ample line bundle on  $X_{\tilde{w}}$ . Note that the ampleness of line bundle is an open condition for the proper morphism  $\pi$ , i.e. there exists an open subset  $U$  in  $\mathbb{A}^1$  containing 0 such that  $\mathcal{L}_u$  is ample line bundle on  $\mathcal{X}_u$  for all  $u \in U$  (see [Laz04, Theorem 1.2.17]). Hence we can identify  $\text{Amp}(X_{\tilde{w}})$  with a subcone of  $\text{Amp}(Z(\tilde{w}))$ .  $\square$

**10.2. Fano and weak Fano properties.** In this section, we observe that Fano and weak Fano properties for BSDH variety  $Z(\tilde{w})$  depend on the given expression  $\tilde{w}$ .

When  $G$  is finite dimensional and the expression  $\tilde{w}$  is reduced, the classification of the expressions  $\tilde{w}$  for which  $Z(\tilde{w})$  is Fano or weak Fano is considered in [Cha17]. Here we get similar results when  $G$  is a Kac-Moody group.

First we discuss the conditions *I*, *II* and *III* with some examples. We use the ordering of simple roots as in [Hum72, Page 58].

The condition *I*:

This condition means that the expression  $\tilde{w}$  is fully commutative without repeating the simple reflections. For example if  $G = SL(n, \mathbb{C})$  and  $\tilde{w} = s_{\alpha_1} s_{\alpha_3} \cdots s_{\alpha_r}$ ,  $1 < r \leq n-1$  and  $r$  is odd, then  $|\eta_i^+| = 0$  and  $|\eta_i^-| = 0$  for all  $i$ .

- $X_{\tilde{w}}$  (or  $Z(\tilde{w})$ ) satisfies the condition *I* and also observe that in this case we have

$$X_{\tilde{w}} \simeq Z(\tilde{w}) \simeq \mathbb{P}^1 \times \cdots \times \mathbb{P}^1 \quad (\dim(Z(\tilde{w})) \text{ times}).$$

The condition *II*:

Let  $G = SL(n, \mathbb{C})$  and fix  $1 \leq j < r \leq n-1$  such that  $j$  is even and  $r$  is odd.

Let  $\tilde{w} = s_{\alpha_1} s_{\alpha_3} \cdots s_{\alpha_{j-3}} s_{\alpha_{j-1}} s_{\alpha_j} s_{\alpha_{j+1}} s_{\alpha_{j+3}} \cdots s_{\alpha_r}$ .

Note that  $s_{\alpha_j}$  appears only once in the expression  $\tilde{w}$  and  $|\eta_i^+| = 0$  for all  $i$ . Let  $p$  be the 'position of  $s_{\alpha_j}$ ' in the expression  $\tilde{w}$ , then  $|\eta_i^-| = 0$  for all  $i \neq p, p-1$  and  $|\eta_{p-1}^-| = 1 = |\eta_p^-|$  with  $\beta_{p-1p} = -1 = \beta_{pp+1}$ .

- $X_{\tilde{w}}$  (or  $Z(\tilde{w})$ ) satisfies condition *II* but not *I*.

The condition *III*:

Again, let  $G = SL(n, \mathbb{C})$  and fix  $1 \leq j < r \leq n-1$  such that  $j$  is even and  $r$  is odd.

Let  $\tilde{w} = s_{\alpha_1} s_{\alpha_3} \cdots s_{\alpha_{j-3}} s_{\alpha_j} s_{\alpha_{j-1}} s_{\alpha_{j+1}} s_{\alpha_{j+3}} \cdots s_{\alpha_r}$  (observe that we interchanged  $s_{\alpha_j}$  and  $s_{\alpha_{j-1}}$  in the example of condition *II*). Then  $|\eta_i^+| = 0$  and  $|\eta_i^-| \leq 2$  for all  $i$ . Let  $p$  be

the ‘position of  $s_{\alpha_j}$ ’ in the expression  $\tilde{w}$ , then  $|\eta_i^-| = 0$  for all  $i \neq p$  and  $|\eta_p^-| = 2$  with  $\beta_{pp+1} = -1 = \beta_{pp+1}$ .

- $X_{\tilde{w}}$  (or  $Z(\tilde{w})$ ) satisfies the condition *III* but not *I* and *II*.

Let  $\tilde{w} = s_{\alpha_1} s_{\alpha_3} s_{\alpha_1}$ . Then  $|\eta_1^+| = 1$  with  $\beta_{13} = 2$ , and  $|\eta_1^-| = |\eta_2^+| = |\eta_2^-| = 0$ .

- $X_{\tilde{w}_1}$  (or  $Z(\tilde{w}_1)$ ) satisfies *III* but not *I* and *II*.

Observe that the condition  $|\eta_i^-| = 1$  and  $\beta_{il} = -2$ , happens only in non-simply laced cases. Let  $G = SO(5, k)$  (i.e.  $G$  is of type  $B_2$ ), let  $\tilde{w}_1 = s_{\alpha_2} s_{\alpha_1}$  and  $\tilde{w}_2 = s_{\alpha_1} s_{\alpha_2}$ . Recall that we have  $\langle \alpha_1, \alpha_2 \rangle = -2$  and  $\langle \alpha_2, \alpha_1 \rangle = -1$ . Then

- $X_{\tilde{w}_1}$  (or  $Z(\tilde{w}_1)$ ) satisfies *III* but not *I* and *II*.
- $X_{\tilde{w}_2}$  (or  $Z(\tilde{w}_2)$ ) satisfies *II* but not *I*.

Let  $G$  be of type  $G_2$  (with  $\langle \alpha_1, \alpha_2 \rangle = -1$  and  $\langle \alpha_2, \alpha_1 \rangle = -3$ ). Let  $\tilde{w}_1 = s_{\alpha_2} s_{\alpha_1}$  and  $\tilde{w}_2 = s_{\alpha_1} s_{\alpha_2}$ . Then

- $X_{\tilde{w}_1}$  (or  $Z(\tilde{w}_1)$ ) satisfies *II* but not *I*.
- $X_{\tilde{w}_2}$  (or  $Z(\tilde{w}_2)$ ) does not satisfy any of the conditions *I*, *II* or *III*.

Recall that the canonical line bundle  $\mathcal{O}_{Z(\tilde{w})}(K_{Z(\tilde{w})})$  of  $Z(\tilde{w})$  is given by

$$\mathcal{O}_{Z(\tilde{w})}(K_{Z(\tilde{w})}) = \mathcal{O}_{Z(\tilde{w})}(-\partial Z(\tilde{w})) \otimes \mathcal{L}(-\delta),$$

where  $\partial Z(\tilde{w})$  is the boundary divisor of  $Z(\tilde{w})$  and  $\delta \in N$  such that  $\langle \delta, \check{\alpha} \rangle = 1$  for all  $\alpha \in S$ , where  $\check{\alpha}$  is the co-root of  $\alpha$  (see [Kum12, Proposition 8.1.2] and also [Ram85, Proposition 2]). Note that if  $G$  is finite dimensional,  $\delta$  is half sum of the positive roots.

Recall that by Lemma 3.5, the toric limit  $X_{\tilde{w}}$  is a Bott tower. Now we have the following result:

**Corollary 10.2.**

- (1) If  $Z(\tilde{w})$  satisfies *II*, then  $Z(\tilde{w})$  is Fano.
- (2) If  $Z(\tilde{w})$  satisfies *III*, then  $Z(\tilde{w})$  is weak Fano.

*Proof.* By Theorem 3.2,  $\phi : \mathcal{X} \rightarrow \mathbb{A}^1$  is a smooth projective morphism with  $\mathcal{X}_0 = X_{\tilde{w}}$  and  $\mathcal{X}_u = Z(\tilde{w})$  for  $u \in \mathbb{A}^1$ ,  $u \neq 0$ .

Proof of (1): By [Laz04, Theorem 1.2.17], if  $-K_{\mathcal{X}_0}$  is ample then  $-K_{\mathcal{X}_u}$  is ample for  $u \neq 0$ . By Theorem 7.2,  $-K_{\mathcal{X}_u}$  is ample if and only if  $X_{\tilde{w}}$  satisfies *II*. Hence we conclude that if  $Z(\tilde{w})$  satisfies *II*, then  $Z(\tilde{w})$  is Fano.

Proof of (2): First we prove  $-K_{Z(\tilde{w})}$  is *big*. Let

$$Z_0 := Z(\tilde{w}) \setminus \partial Z(\tilde{w}).$$

Note that  $Z_0$  is an open affine subset of  $Z(\tilde{w})$ . Then by Lemma 7.1,  $\partial Z(\tilde{w})$  is *big*. Since

$$\mathcal{O}(-K_{Z(\tilde{w})}) = \mathcal{O}(\partial Z(\tilde{w})) \otimes \mathcal{L}(\delta)$$

and  $\mathcal{L}(\delta)$  is *nef*, we conclude  $-K_{Z(\tilde{w})}$  is *big*, as tensor product of a *big* and a *nef* line bundles is again a *big* line bundle.

By [Laz04, Theorem 1.4.14] and  $\mathcal{X}_u = Z(\tilde{w})$  for  $u \neq 0$ , we can see that if  $-K_{\mathcal{X}_0}$  is nef then  $-K_{\mathcal{X}_u}$  is also nef for  $u \neq 0$ . Therefore, (2) follows from Theorem 7.2(2).  $\square$

There exists expressions  $\tilde{w}$  such that the BSDH variety  $Z(\tilde{w})$  Fano (respectively, weak Fano) but the toric limit  $X_{\tilde{w}}$  is not Fano (respectively, not weak Fano).

**Example 10.3.** Let  $G = SL(4, \mathbb{C})$ .

- (1) Let  $\tilde{w} = s_{\alpha_1} s_{\alpha_1}$ . Then  $Z(\tilde{w}) \simeq \mathbb{P}^1 \times \mathbb{P}^1$ , which is Fano. The toric limit  $X_{\tilde{w}} \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(2))$ . Since  $X_{\tilde{w}}$  does not satisfy II, then by Theorem 7.2,  $X_{\tilde{w}}$  is not Fano.
- (2) Let  $\tilde{w} = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$ . Then it can be seen  $Z(\tilde{w})$  is Fano (see [Cha17, Example 5.3]). By Theorem 7.2, the toric limit  $X_{\tilde{w}}$  is weak Fano but not Fano.

**Example 10.4.** Let  $G = SO(5, k)$ , i.e.  $G$  is of type  $B_2$ . Let  $\tilde{w} = s_{\alpha_1} s_{\alpha_2} s_{\alpha_1}$ . By Theorem 7.2, the toric limit  $X_{\tilde{w}}$  is not weak Fano. Also we can see  $Z(\tilde{w})$  is weak Fano but not Fano (see [Cha17, Theorem 5.2]).

**10.3. Log Fano BSDH varieties and its toric limits.** In [And14] and [AS14] log Fanoness of Schubert varieties and BSDH varieties were studied respectively. Now we characterize the (suitably chosen)  $\mathbb{Q}$ -divisors  $D$  in  $X_{\tilde{w}}$  (respectively,  $D'$  in  $Z(\tilde{w})$ ) for which  $(X_{\tilde{w}}, D)$  (respectively,  $(Z(\tilde{w}), D')$ ) is log Fano.

Let  $\gamma_i = s_{\beta_r} \cdots s_{\beta_{i+1}}(\beta_i)$  for  $1 \leq i \leq r$ . Then,

$$\mathcal{L}(\delta) = \sum_{i=1}^r b_i Z_i, \text{ with}$$

$$b_i = \langle \delta, \tilde{\gamma}_i \rangle = ht(\gamma_i), \quad (10.1)$$

where  $\delta$  is as in Section 10.2 (see page 36),  $\mathcal{L}(\delta)$  is the homogeneous line bundle on  $Z(\tilde{w})$  corresponding to  $\delta$  and  $ht(\beta)$  for a root  $\beta = \sum_{i=1}^n n_i \alpha_i$ , is the height defined by  $ht(\beta) = \sum_{i=1}^n n_i$  (see [MR85, Proof of Proposition 10]). When  $\tilde{w}$  is reduced,  $\gamma_i$  is a positive root and we can see the relation (10.1) from the Chevalley formula for intersection of Schubert variety by a divisor (see [AS14, Page 410] or [Che94]). It is known that

$$-K_{Z(\tilde{w})} = \sum_{i=1}^r (b_i + 1) Z_i \quad (10.2)$$

(see [MR85, Proposition 4]). Let  $D' = \sum_{i=1}^r a_i Z_i$  be a effective  $\mathbb{Q}$ -divisor in  $Z(\tilde{w})$ , with  $[D'] = 0$ . Then by (10.2), we get

$$-(K_{Z(\tilde{w})} + D') = \sum_{i=1}^r (b_i + 1 + a_i) Z_i.$$

For  $1 \leq i \leq r$ , define

$$f_i := (b_i + 1 + a_i) - \sum_{\gamma_j \in \gamma_{P_i}(1)^+} c_j (b_j + 1 + a_j),$$

where  $\gamma_{P_i}(1)^+ := \gamma_{P_i}(1) \cap \{\rho_l^+ : 1 \leq l \leq r\}$ .

We prove,

**Theorem 10.5.** *The pair  $(Z(\tilde{w}), D')$  is log Fano if  $f_i > 0$  for all  $1 \leq i \leq r$ .*

*Proof.* By definition of  $D'$ , the pair  $(Z(\tilde{w}), D')$  is log Fano if and only if  $-(K_{Z(\tilde{w})} + D')$  is ample. Now we prove  $-(K_{Z(\tilde{w})} + D')$  is ample if  $f_i > 0$  for all  $1 \leq i \leq r$ . Recall that  $Z_i = \{[(p_1, \dots, p_r)] \in Z(\tilde{w}) : p_i \in B\}$  is a divisor in  $Z(\tilde{w})$ ,  $D_{\rho_i^+}$  is the divisor corresponding to  $\rho_i^+ \in \Sigma(1)$  and  $\mathcal{Z}_i^x = \pi^{-1}(x) \cap \mathcal{Z}_i$  for  $x \in k$  (see Section 2 and Section 3).

By Theorem 3.2, we have

$$\mathcal{Z}_i^x = Z_i \text{ for } x \neq 0 \text{ and } \mathcal{Z}_i^0 = D_{\rho_i^+}. \quad (10.3)$$

Assume that  $f_i > 0$  for all  $1 \leq i \leq r$ . By (10.3) and by [Laz04, Theorem 1.2.7] to prove  $(Z(\tilde{w}), D')$  is log Fano it is enough to prove

$$\sum_{i=1}^r (b_i + 1 + a_i) D_{\rho_i^+} \text{ is ample .}$$

By Lemma 6.1, we see that  $\sum_{i=1}^r (b_i + 1 + a_i) D_{\rho_i^+}$  is ample if and only if

$$f_i = ((b_i + 1 + a_i) - \sum_{\gamma_j \in \gamma_{P_i}(1)^+} c_j (b_j + 1 + a_j)) > 0 \text{ for all } 1 \leq i \leq r.$$

Hence we conclude that  $(Z(\tilde{w}), D')$  is log Fano.  $\square$

**10.4. Local rigidity of BSDH varieties and its toric limits.** In this section we obtain some vanishing results for the cohomology of tangent bundle of the toric limit  $X_{\tilde{w}}$  and  $Z(\tilde{w})$ .

In [CKP15] and [CK17], we have obtained some vanishing results for the cohomology of tangent bundle of  $Z(\tilde{w})$ , when  $G$  is finite dimensional and  $\tilde{w}$  is reduced (see [CKP15, Section 3] and [CK17, Theorem 8.1]). The case  $\tilde{w}$  is non-reduced is considered in [CKP].

Now we prove some vanishing results for the cohomology of tangent bundle of BSDH variety  $Z(\tilde{w})$  when  $G$  is Kac-Moody group and  $\tilde{w}$  not necessarily reduced. Let  $T_X$  denotes the tangent bundle of  $X$ , where  $X = X_{\tilde{w}}$  or  $Z(\tilde{w})$ . Then we have

**Corollary 10.6.**

- (1) *If  $X_{\tilde{w}}$  satisfies II, then  $H^i(X_{\tilde{w}}, T_{X_{\tilde{w}}}) = 0$  for all  $i \geq 1$ . In particular,  $X_{\tilde{w}}$  is locally rigid.*
- (2) *If  $Z(\tilde{w})$  satisfies II, then  $H^i(Z(\tilde{w}), T_{Z(\tilde{w})}) = 0$  for all  $i \geq 1$ . In particular,  $Z(\tilde{w})$  is locally rigid.*

*Proof.* Proof of (1): This follows from Corollary 7.3 as  $X_{\tilde{w}}$  is a Bott tower.

Proof of (2): From Theorem 3.2,  $\pi : \mathcal{X} \rightarrow \mathbb{A}^1$  is a smooth projective morphism with  $\mathcal{X}_0 = X_{\tilde{w}}$  and  $\mathcal{X}_u = Z(\tilde{w})$  for  $u \in \mathbb{A}^1, u \neq 0$ . Hence (2) follows from (1) by semi-continuity theorem (see [Har77, Theorem 12.8]).  $\square$

## 11. AJP AND PK RESULTS USING TORIC GEOMETRY

In this section we are going to recover the results of [PK16] by using methods of toric geometry. In [PK16], they have assumed that  $G$  is a simple algebraic group. In our situation  $G$  is a Kac-Moody group.

Recall the following:

- (1)  $\tilde{w} = s_{\beta_1} \cdots s_{\beta_r}$  and  $\tilde{w}' = s_{\beta_1} \cdots s_{\beta_{r-1}}$ .
- (2) The toric morphism

$$f_r : X_{\tilde{w}} \rightarrow X_{\tilde{w}'}$$

induced by the lattice map

$$\bar{f}_r : \mathbb{Z}^r \rightarrow \mathbb{Z}^{r-1},$$

the projection onto the first  $r - 1$  coordinates.

As we discussed in Section 3, there are two disjoint toric sections for the  $\mathbb{P}^1$ -fibration

$$f_r : X_{\tilde{w}} \rightarrow X_{\tilde{w}'}$$

(see Lemma 3.4).

**Definition 11.1.**

- (1) **Schubert and non-Schubert sections:** We call the section corresponding to the maximal cone  $\rho_r^+$  (respectively,  $\rho_r^-$ ) in  $\Sigma_F$  (the fan of the fiber of  $f_r$ ) by ‘Schubert section  $\sigma_{r-1}^0$ ’ (respectively, ‘non-Schubert section  $\sigma_{r-1}^1$ ’).
- (2) **Schubert point:** Let  $\sigma \in \Sigma$  be the maximal cone generated by

$$\{e_1^+, \dots, e_r^+\}.$$

We call the point in  $X_{\tilde{w}}$  corresponding to the maximal cone  $\sigma$  by ‘Schubert point’.

- (3) **Schubert line:** We call the fiber of  $f_r$  over the Schubert point by ‘Schubert line  $L_r$ ’.

Note that these definitions agree with that of in [PK16].

Now onwards we denote  $\tilde{w} = (1, \dots, r)$  (respectively,  $\tilde{w}' = (1, \dots, r - 1)$ ) for the expression  $\tilde{w} = s_{\beta_1} \cdots s_{\beta_r}$  (respectively,  $\tilde{w}' = s_{\beta_1} \cdots s_{\beta_{r-1}}$ ).

Let  $I_{i_1} = (i_1, \dots, i_m)$  be a subsequence of  $\tilde{w}$ . Inductively we define the curve  $L_{I_{i_1}}$  corresponding to  $I_{i_1}$ . Let  $L_{I'_{i_1}}$  be the curve in  $X_{\tilde{w}'}$  corresponding to the subsequence  $I'_{i_1} = (i_1, \dots, i_{m-1})$  of  $I_{i_1}$ . Then define

$$L_{I_{i_1}} := \sigma_{r-1}^1(L_{I'_{i_1}}).$$

and

$$\sigma_{r-1}^0(L_{I'_{i_1}}) = L_{I'_{i_1}}.$$

We have the following result:

**Lemma 11.2.** *The classes of Schubert lines  $L_j$ ,  $1 \leq j \leq r$  form a basis of  $N_1(X_{\tilde{w}})$ .*



*Proof.* Proof is by induction on  $r$ . Assume that the result is true for  $r - 1$ . Since  $X_{\tilde{w}}$  is a projective bundle over  $X_{\tilde{w}'}$  (see Lemma 3.4), then by [Bar71, Lemma 1.1],

$$L_r \text{ and } \sigma_{r-1}^0(L_j) \text{ for } 1 \leq j \leq r$$

(the image of  $L_j$  by the Schubert section in  $X_{\tilde{w}}$ ) form a basis of  $N_1(X_{\tilde{w}})$ . By definition of  $L_I$ , we have

$$\sigma_{r-1}^0(L_j) = L_j \text{ for } 1 \leq j \leq r - 1$$

and hence the result follows.  $\square$

Let  $1 \leq j \leq r$ .

Let  $\mathcal{D} := \{e_l^{\epsilon_l} : 1 \leq l \leq r \text{ and } \epsilon_l = + \text{ for all } l\}$ .

Let  $\mathcal{D}'_j := \{e_l^{\epsilon_l} : 1 \leq l \leq r \text{ and } \epsilon_l = + \text{ for all } l \neq j; \epsilon_j = -\}$ .

**Lemma 11.3.** *Fix  $1 \leq j \leq r$ . Then the Schubert line  $L_j$  is given by*

$$L_j = V(\tau_j), \text{ with } \tau_j = \sigma \cap \sigma'_j,$$

*intersection of the maximal cones in  $\Sigma$ , where  $\sigma$  (respectively,  $\sigma'_j$ ) is generated by  $\mathcal{D}$  (respectively,  $\mathcal{D}'_j$ ).*

*Proof.* Let us consider the expression  $\tilde{w}_j = s_{\beta_1} \cdots s_{\beta_j}$  for  $1 \leq j < r$ . Let  $\Sigma_j$  be the fan of the toric variety  $X_{\tilde{w}_j}$ . By Lemma 3.4,

$$f_j : X_{\tilde{w}_j} \rightarrow X_{\tilde{w}_{j-1}}$$

is a  $\mathbb{P}^1$ -fibration induced by  $\bar{f}_j : \mathbb{Z}^j \rightarrow \mathbb{Z}^{j-1}$  the projection onto the first  $j - 1$  factors. Also note that the Schubert point in  $X_{\tilde{w}_{j-1}}$  corresponds to the maximal cone generated by

$$\{e_l^+ : 1 \leq l \leq j - 1\}$$

and the fan of the fiber is given by  $\{e_j^+, 0, e_j^-\}$ . Let  $\sigma_j$  (respectively,  $\sigma'_j$ ) be the cone generated by

$$\{e_l^+ : 1 \leq l \leq j\}$$

(respectively,

$$\{e_l^+ : 1 \leq l \leq j - 1\} \cup \{e_j^-\} \text{ ).}$$

Then by definition of Schubert line  $L_j$ , we can see that  $L_j$  is the curve in  $X_{\tilde{w}_j}$  given by

$$L_j = V(\tau_j), \text{ where } \tau_j \in \Sigma_j \text{ and } \tau_j = \sigma_j \cap \sigma'_j.$$

Since the Schubert section of  $f_k$  for  $(j \leq k \leq r)$  corresponds to  $e_k^+$ , we see

$$\sigma_r^0 \circ \cdots \circ \sigma_{j+1}^0(L_j),$$

by abuse of notation we also denote it again by  $L_j$  in  $X_{\tilde{w}}$ , is given by

$$L_j = V(\tau_j) \text{ with } \tau = \sigma \cap \sigma'_j,$$

where  $\sigma$  and  $\sigma'_j$  are as described in the statement. This completes the proof of the lemma.  $\square$

Let  $\mathcal{D}''_{i_1} := \{e_l^{\epsilon_l} : 1 \leq l \leq r \text{ and}$

$$\epsilon_l = \begin{cases} + & \text{if } l \notin I_{i_1} \setminus \{i_1\} \\ - & \text{if } l \in I_{i_1} \end{cases} \quad \}.$$

Let  $\mathcal{D}'''_{i_1} := \{e_l^{\epsilon_l} : 1 \leq l \leq r \text{ and}$

$$\epsilon_l = \begin{cases} + & \text{if } l \notin I_{i_1} \\ - & \text{if } l \in I_{i_1} \end{cases} \quad \}.$$

**Proposition 11.4.** *The curve  $L_{I_{i_1}}$  is given by*

$$L_{I_{i_1}} = V(\tau_{i_1}) \text{ with } \tau_{i_1} = \sigma_{i_1} \cap \sigma'_{i_1},$$

where  $\sigma_{i_1}$  (respectively,  $\sigma'_{i_1}$ ) is the cone generated by  $\mathcal{D}''_{i_1}$  (respectively,  $\mathcal{D}'''_{i_1}$ ).

*Proof.* As in the proof of Lemma 11.3, we start with  $j = i_1$  and  $L_{i_1}$  is the Schubert line in  $X_{\tilde{w}_{i_1}}$ . By Lemma 11.3, we have

$$L_{i_1} = V(\tau_{i_1}) \text{ with } \tau_{i_1} = \sigma_{i_1} \cap \sigma'_{i_1}.$$

By definition of  $L_I$ , we have

$$\sigma_{i_2-1}^0 \circ \cdots \circ \sigma_{i_1+1}^0(L_{i_1}) = L_{i_1} \text{ in } X_{\tilde{w}_{i_2-1}}$$

and

$$\sigma_{i_2}^1 \circ \sigma_{i_2-1}^0 \circ \cdots \circ \sigma_{i_1+1}^0(L_{i_1}) = L_{\{i_1, i_2\}} \text{ in } X_{\tilde{w}_{i_2}}.$$

By repeating the process we conclude that

$$L_{I_{i_1}} = V(\tau_{i_1}) \text{ with } \tau_{i_1} = \sigma_{i_1} \cap \sigma'_{i_1},$$

where  $\sigma_{i_1}$  and  $\sigma'_{i_1}$  are as described in the statement. This completes the proof of the proposition.  $\square$

Fix  $1 \leq i \leq r$ . Let

$$I_i := \{i = j_1, j_2, \dots, j_m\}$$

as in Proposition 5.11.

**Theorem 11.5.** *The class of curve  $L_{I_i}$  is  $r(P_i)$  in  $N_1(X_{\tilde{w}})_{\mathbb{Z}}$ .*

*Proof.* This follows from Proposition 5.16 and Proposition 11.4.  $\square$

Now we get [PK16, Theorem 22] as a corollary.

**Corollary 11.6.** *The set  $\{L_{I_i} : 1 \leq i \leq r\}$  of classes of curves forms a basis of  $N_1(X_{\tilde{w}})$  and every torus invariant curve in  $N_1(X_{\tilde{w}})$  lie in the cone generated by  $\{L_{I_i} : 1 \leq i \leq r\}$ .*

*Proof.* First assertion follows from Corollary 5.8 and Theorem 11.5. Second assertion follows from Theorem 5.7.  $\square$

We have the following result (see [PK16, Theorem 30])

**Corollary 11.7.** *The extremal rays of the toric limit  $X_{\tilde{w}}$  are precisely the curves  $L_{I_i}$  for  $1 \leq i \leq r$ .*

*Proof.* This follows from Theorem 9.1 and Theorem 11.5.  $\square$

We prove the following:

**Proposition 11.8.** *Let  $1 \leq j \leq r$  and let  $L_j$  be the Schubert curve in  $X_{\tilde{w}}$ . Then,*

$$K_{X_{\tilde{w}}} \cdot L_j = -2 - \sum_{k>j} \beta_{kj}.$$

*Proof.* By definition of  $e_j^-$ , we have

$$e_j^+ + e_j^- + \sum_{k>j} \beta_{kj} e_k^+ = 0. \quad (11.1)$$

By Lemma 11.3, we have  $L_j = V(\tau)$ , with  $\tau = \sigma \cap \sigma'$  where  $\sigma$  (respectively,  $\sigma'$ ) is generated by

$$\{e_l^{\epsilon_l} : 1 \leq l \leq r, \epsilon_l = + \text{ for all } l\}$$

(respectively,

$$\{e_l^{\epsilon_l} : \epsilon_l = + \text{ for } 1 \leq l \leq r \text{ and } l \neq j, \epsilon_j = -\} \quad ).$$

Hence (11.1) is the *wall relation* for the curve  $L_j$ . Then by *wall relation*, we see that

$$D_\rho \cdot L_j = \begin{cases} 1 & \text{if } \rho = \rho_j^+ \text{ or } \rho_j^- \\ \beta_{kj} & \text{if } \rho = \rho_k^+ \text{ and } k > j. \\ 0 & \text{otherwise.} \end{cases}$$

Since  $K_{X_{\tilde{w}}} = -\sum_{\rho \in \Sigma(1)} D_\rho$ , we get

$$K_{X_{\tilde{w}}} \cdot L_j = -2 - \sum_{k>j} \beta_{kj}.$$

This completes the proof of the proposition.  $\square$

Similarly, the results [PK16, Theorem 35] and [PK16, Corollary 36] follow from Theorem 9.1 and Corollary 9.3.

**Remark 11.9.**

- (1) *In Theorem 7.3, we have seen the vanishing results of the tangent bundle of Bott tower  $X_r$  for some special cases. It is an interesting problem to study the cohomology of tangent bundle of  $X_r$ , which gives local rigidity results for  $X_r$ , and hence for the BSDH variety  $Z(\tilde{w})$ .*
- (2) *It also interesting to find the automorphism group of the Bott tower  $X_r$ . We are working in this direction.*

**Acknowledgements:** I would like to thank Michel Brion for valuable discussions, many critical comments and for encouragement throughout the preparation of this article.

## 12. APPENDIX

By Lemma 4.1, the set  $\{D_{\rho_i^+} : 1 \leq i \leq r\}$  forms a basis of  $Pic(X_r)$ . Now we describe  $D_{\rho_i^-}$  in terms of  $D_{\rho_j^+}$ 's ( $1 \leq j \leq r$ ).

Let  $1 \leq i \leq r$ , define  $h_i^{i-1} := -\beta_{i,i-1}$  and

$$h_i^j := \begin{cases} 0 & \text{for } j > i. \\ 1 & \text{for } j = i. \\ -\sum_{k=j}^{i-1} \beta_{i,k}(h_k^j) & \text{for } j < i. \end{cases}$$

Then we prove,

**Lemma 12.1.** *Let  $1 \leq i \leq r$ . The coefficient of  $D_{\rho_j^+}$  in  $D_{\rho_i^-}$  is  $h_i^j$ .*

*Proof.* Proof is by induction on  $i$  and by using

$$0 \sim \text{div}(\chi^{e_i^+}) = \sum_{\rho \in \Sigma(1)} \langle u_\rho, e_i^+ \rangle D_\rho. \quad (12.1)$$

Recall the equation (2.2),

$$e_i^- = -e_i^+ - \sum_{j>i} \beta_{ij} e_j^+ \text{ for all } i.$$

If  $i = 1$ , by (12.1), we see

$$0 \sim \text{div}(\chi^{e_1^+}) = D_{\rho_1^+} - D_{\rho_1^-}.$$

Then

$$D_{\rho_1^-} \sim D_{\rho_1^+}. \quad (12.2)$$

If  $i = 2$ , by (12.1) and (2.2), we see

$$0 \sim \text{div}(\chi^{e_2^+}) = D_{\rho_2^+} - D_{\rho_2^-} - \beta_{21} D_{\rho_1^-}.$$

By (12.2), we get

$$D_{\rho_2^-} \sim D_{\rho_2^+} - \beta_{21} D_{\rho_1^+} = h_2^2 D_{\rho_2^+} + h_2^1 D_{\rho_1^+}.$$

By induction assume that

$$D_{\rho_k^-} \sim \sum_{j=1}^r h_k^j D_{\rho_j^+} \text{ for all } k < i.$$

Again by (12.1) and (2.2), we see

$$0 \sim \text{div}(\chi^{e_i^+}) = D_{\rho_i^+} - D_{\rho_i^-} - \sum_{k<i} \beta_{ik} D_{\rho_k^-}.$$

Then

$$D_{\rho_i^-} \sim D_{\rho_i^+} - \sum_{k<i} \beta_{ik} D_{\rho_k^-}.$$

Hence

$$D_{\rho_i^-} \sim D_{\rho_i^+} - \sum_{k < i} \beta_{ik} \left( \sum_{j=1}^r h_k^j D_{\rho_j^+} \right).$$

Since  $h_k^j = 0$  for  $k < j$ , we get

$$D_{\rho_i^-} \sim D_{\rho_i^+} - \sum_{k < i} \beta_{ik} \left( \sum_{j=1}^{i-1} h_k^j D_{\rho_j^+} \right).$$

Then

$$D_{\rho_i^-} \sim D_{\rho_i^+} + \sum_{j=1}^{i-1} \left( - \sum_{k=j}^{i-1} \beta_{ik} h_k^j \right) D_{\rho_j^+}.$$

Therefore, we conclude that  $D_{\rho_i^-} \sim D_{\rho_i^+} + \sum_{j=1}^{i-1} h_i^j D_{\rho_j^+}$ . This completes the proof of the lemma.  $\square$

Let  $\epsilon \in \{+, -\}$ . Define

$$\Sigma(1)^\epsilon := \{\rho_i^\epsilon : 1 \leq i \leq r\}.$$

Then

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho = \sum_{\rho \in \Sigma(1)^+} a_\rho D_\rho + \sum_{\rho \in \Sigma(1)^-} a_\rho D_\rho.$$

For  $1 \leq i \leq r$ , let

$$g_i := a_{\rho_i^+} + \sum_{j=i}^r a_{\rho_j^-} h_j^i.$$

Then we have

**Corollary 12.2.**

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho \sim \sum_{i=1}^r g_i D_{\rho_i^+}.$$

*Proof.* We have

$$D = \sum_{\rho \in \Sigma(1)} a_\rho D_\rho = \sum_{i=1}^r a_{\rho_i^+} D_{\rho_i^+} + \sum_{i=1}^r a_{\rho_i^-} D_{\rho_i^-}.$$

By Lemma 12.1, we can see that

$$\sum_{i=1}^r a_{\rho_i^-} D_{\rho_i^-} \sim \sum_{i=1}^r a_{\rho_i^-} \left( \sum_{j=1}^i h_i^j D_{\rho_j^+} \right).$$

Then

$$\sum_{i=1}^r a_{\rho_i^-} D_{\rho_i^-} \sim \sum_{i=1}^r \left( \sum_{j=i}^r a_{\rho_j^-} h_j^i \right) D_{\rho_i^+}.$$

Hence we have

$$D \sim \sum_{i=1}^r (a_{\rho_i^+} + \sum_{j=i}^r a_{\rho_j^-} h_j^i) D_{\rho_i^+}.$$

Thus,  $D \sim \sum_{i=1}^r g_i D_{\rho_i^+}$  and this completes the proof.  $\square$

In [Pas10]), B. Pasquier obtained vanishing theorems for the cohomology of certain line bundles on BSDH varieties, by using cohomology of line bundles on the toric limit and by semi-continuity theorem (see [Pas10, Theorem 0.1]). By Corollary 12.2, we can also see some vanishing results for the cohomology of certain line bundles on BSDH varieties. For example the following result:

Let  $D = \sum_{i=1}^r g_i Z_i$  be a divisor in BSDH variety  $Z(\tilde{w})$ , where  $g_i$  is as above for  $1 \leq i \leq r$ . We use the notation as in Section 6.

**Corollary 12.3.** *If  $d_i \geq 0$  for all  $1 \leq i \leq r$ , then  $H^j(Z(\tilde{w}), D) = 0$  for all  $j > 0$ .*

*Proof.* Since  $d_i \geq 0$  for all  $1 \leq i \leq r$ , by Lemma 6.1,  $\sum_{\rho \in \Sigma(1)} a_{\rho} D_{\rho}$  is a *nef* divisor in  $X_{\tilde{w}}$ . Then we have

$$H^j(X_{\tilde{w}}, \sum_{\rho \in \Sigma(1)} a_{\rho} D_{\rho}) = 0 \text{ for all } j > 0$$

(see [CLS11, Theorem 9.2.3, page 410] or [Oda88, Theorem 2.7, page 77]). Recall that by Theorem 3.2, we have

$$\mathcal{Z}_i^x = Z_i \text{ for } 0 \neq x \in k \text{ and } \mathcal{Z}_i^0 = D_{\rho_i^+}.$$

Hence by Theorem 3.2, Corollary 12.2 and by semi-continuity theorem (see [Har77, Theorem 12.8]), we get

$$H^j(Z(\tilde{w}), D) = 0 \text{ for all } j > 0.$$

$\square$

## REFERENCES

- [And14] D. Anderson, *Effective divisors on Bott-Samelson varieties*, arXiv preprint arXiv:1501.00034 (2014).
- [AS14] D. Anderson and A. Stapledon, *Schubert varieties are log Fano over the integers*, Proceedings of the American Mathematical Society **142** (2014), no. 2, 409–411.
- [Bar71] C.M. Barton, *Tensor products of ample vector bundles in characteristic  $p$* , American Journal of Mathematics **93** (1971), no. 2, 429–438.
- [Bat91] V.V. Batyrev, *On the classification of smooth projective toric varieties*, Tohoku Mathematical Journal, Second Series **43** (1991), no. 4, 569–585.
- [BB96] F. Bien and M. Brion, *Automorphisms and local rigidity of regular varieties*, Compositio Mathematica **104** (1996), no. 1, 1–26 (eng).
- [BS58] R. Bott and H. Samelson, *Applications of the theory of Morse to symmetric spaces*, American Journal of Mathematics (1958), 964–1029.
- [BK07] M. Brion and S. Kumar, *Frobenius splitting methods in geometry and representation theory*, vol. 231, Springer Science & Business Media, 2007.
- [Cas03] C. Casagrande, *Contractible classes in toric varieties*, Mathematische Zeitschrift **243** (2003), no. 1, 99–126.
- [CG13] P. Cascini and Y. Gongyo, *On the anti-canonical ring and varieties of Fano type*, Saitama Math. J **30** (2013), 27–38.

- [Cha17] B.N. Chary, *On Fano and weak Fano of a Bott-Samelson-Demazure-Hansen variety*, in preparation.
- [CK17] B.N. Chary and S.S. Kannan, *Rigidity of a Bott-Samelson-Demazure-Hansen variety for  $PSp(2n, \mathbb{C})$* , Journal of Lie Theory **27** (2017), 435–468.
- [CKP] B.N. Chary, S.S. Kannan, and A.J. Parameswaran, *Automorphism group of a Bott-Samelson-Demazure-Hansen variety for non reduced case*, in preparation.
- [CKP15] ———, *Automorphism group of a Bott-Samelson-Demazure-Hansen variety*, Transformation Groups **20** (2015), no. 3, 665–698.
- [Che94] C. Chevalley, *Sur les décompositions cellulaires des espaces  $G/B$* , Algebraic Groups and their Generalizations: Classical Methods, American Mathematical Society (1994), 1–23.
- [CMS10] S. Choi, M. Masuda, and D. Suh, *Topological classification of generalized Bott towers*, Transactions of the American Mathematical Society **362** (2010), no. 2, 1097–1112.
- [CS11] S. Choi and D.Y. Suh, *Properties of Bott manifolds and cohomological rigidity*, Algebraic & Geometric Topology **11** (2011), no. 2, 1053–1076.
- [Civ05] Y. Civan, *Bott towers, crosspolytopes and torus actions*, Geometriae Dedicata **113** (2005), no. 1, 55–74.
- [CR05] Y. Civan and N. Ray, *Homotopy decompositions and  $k$ -theory of bott towers*, K-theory **34** (2005), no. 1, 1–33.
- [CLS11] D.A. Cox, J.B. Little, and H.K. Schenck, *Toric varieties*, Graduate studies in mathematics, American Mathematical Soc., 2011.
- [DJ91] M.W. Davis, T. Januszkiewicz, et al., *Convex polytopes, coxeter orbifolds and torus actions*, Duke Math. J **62** (1991), no. 2, 417–451.
- [Dem74] M. Demazure, *Désingularisation des variétés de Schubert généralisées*, Annales scientifiques de l'École Normale Supérieure, vol. 7, Société mathématique de France, 1974, pp. 53–88.
- [Ful93] W. Fulton, *Introduction to toric varieties*, no. 131, Princeton University Press, 1993.
- [GK94] M. Grossberg, Y. Karshon, et al., *Bott towers, complete integrability, and the extended character of representations*, Duke Mathematical Journal **76** (1994), no. 1, 23–58.
- [Han73] H.C. Hansen, *On cycles in flag manifolds.*, Mathematica Scandinavica **33** (1973), 269–274.
- [Har70] R. Hartshorne, *Ample subvarieties of algebraic varieties*, Lecture Notes in Mathematics, vol. 156, Springer, 1970.
- [Har77] ———, *Algebraic geometry*, vol. 52, Springer Science & Business Media, 1977.
- [Hum72] J.E. Humphreys, *Introduction to Lie algebras and representation theory*, vol. 9, Springer Science & Business Media, 1972.
- [Huy06] D. Huybrechts, *Complex geometry: an introduction*, Springer Science & Business Media, 2006.
- [Ish12] H. Ishida, *Filtered cohomological rigidity of Bott towers*, Osaka Journal of Mathematics **49** (2012), no. 2, 515–522.
- [Kle66] S.L. Kleiman, *Toward a numerical theory of ampleness*, Annals of Mathematics (1966), 293–344.
- [Kle80] ———, *Relative duality for quasi-coherent sheaves*, Compositio Mathematica **41** (1980), no. 1, 39–60.
- [KM08] J. Kollár and S. Mori, *Birational geometry of algebraic varieties*, vol. 134, Cambridge university press, 2008.
- [Kum12] S. Kumar, *Kac-Moody groups, their flag varieties and representation theory*, vol. 204, Springer Science & Business Media, 2012.
- [LT04] N. Lauritzen, J.F. Thomsen, et al., *Line bundles on Bott-Samelson varieties*, Journal of Algebraic Geometry **13** (2004), 461–473.
- [Laz04] R.K. Lazarsfeld, *Positivity in algebraic geometry I: Classical setting: line bundles and linear series*, vol. 48, Springer Science & Business Media, 2004.
- [MR85] V.B. Mehta and A. Ramanathan, *Frobenius splitting and cohomology vanishing for Schubert varieties*, Annals of Mathematics, (2) **122** (1985), no. 1, 27–40.
- [Oda88] T. Oda, *Convex bodies and algebraic geometry- An introduction to the theory of toric varieties*, vol. (3)15, Springer-Verlag, Berlin Heidelberg NewYork, 1988.
- [PK16] A.J. Parameswaran and P. Karupuchamy, *Toric degeneration of Bott-Samelson-Demazure-Hansen varieties*, arXiv preprint arXiv:1604.01998 (2016).

- [Pas10] B. Pasquier, *Vanishing theorem for the cohomology of line bundles on Bott-Samelson varieties*, Journal of Algebra **323** (2010), no. 10, 2834–2847.
- [Ram85] A. Ramanathan, *Schubert varieties are arithmetically Cohen-Macaulay*, Inventiones mathematicae **80** (1985), no. 2, 283–294.
- [Sca09] A. Scaramuzza, *Algorithms for projectivity and extremal classes of a smooth toric variety*, Experimental Mathematics **18** (2009), no. 1, 71–84.
- [Spr10] T.A. Springer, *Linear algebraic groups*, Springer Science & Business Media, 2010.
- [VT15] B.M. Victor and P.E. Taras, *Toric topology*, vol. 204, Providence: American Mathematical Society, 2015.

B. NARASIMHA CHARY, INSTITUT FOURIER, UMR 5582 DU CNRS, UNIVERSITÉ DE GRENOBLE ALPES, CS 40700, 38058, GRENOBLE CEDEX 09, FRANCE., EMAIL: NARASIMHA-CHARY.BONALA@UNIV-GRENOBLE-APLES.FR