Determination of singular time-dependent coefficients for wave equations from full and partial data

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DETERMINATION OF SINGULAR TIME-DEPENDENT COEFFICIENTS FOR WAVE EQUATIONS FROM FULL AND PARTIAL DATA

GUANGHUI HU AND YAVAR KIAN

Abstract. We study the problem of determining uniquely a time-dependent singular potential q, appearing in the wave equation $\partial_t^2 u-\Delta_x u+q(t,x)u=0$ in $Q=(0,T)\times\Omega$ with $T>0$ and Ω a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^n$, $n\geqslant 2$. We start by considering the unique determination of some singular time-dependent coefficients from observations on ∂Q. Then, by weakening the singularities of the set of admissible coefficients, we manage to reduce the set of data that still guaranties unique recovery of such a coefficient. To our best knowledge, this paper is the first claiming unique determination of unbounded time-dependent coefficients, which is motivated by the problem of determining general nonlinear terms appearing in nonlinear wave equations.

Keywords: Inverse problems, wave equation, time dependent coefficient, singular coefficients, Carleman estimate.

Mathematics subject classification 2010 : 35R30, 35L05.

1. Introduction

1.1. **Statement of the problem.** Let Ω be a $\mathcal{C}^2$ bounded domain of $\mathbb{R}^n$, $n\geqslant 2$, and fix $\Sigma=(0,T)\times\partial\Omega$, $Q=(0,T)\times\Omega$ with $0<T<\infty$. We consider the wave equation

$$\partial_t^2 u-\Delta_x u+q(t,x)u=0,\quad (t,x)\in Q, \tag{1.1}$$

where the potential q is assumed to be an unbounded real valued coefficient. In this paper we seek unique determination of q from observations of solutions of (1.1) on ∂Q.

1.2. **Obstruction to uniqueness and set of full data for our problem.** Let ν be the outward unit normal vector to $\partial\Omega$, $\partial_\nu=\nu\cdot\nabla_x$ the normal derivative and from now on let $\Box$ be the differential operators $\Box:=\partial_t^2-\Delta_x$. It has been proved by [40], that, for $T>\mathrm{Diam}(\Omega)$, the data

$$\mathcal{A}_q=\{(u_{|\Sigma},\partial_\nu u_{|\Sigma}):\ u\in L^2(Q),\ \Box u+qu=0,\ u_{|t=0}=\partial_t u_{|t=0}=0\} \tag{1.2}$$

determines uniquely a time-independent potential q. On the other hand, due to domain of dependence arguments, there is no hope to recover even smooth time-dependent coefficients restricted to the set

$$D=\{(t,x)\in Q:\ t\in(0,\mathrm{Diam}(\Omega)/2)\cup(T-\mathrm{Diam}(\Omega)/2,T),\ \mathrm{dist}(x,\partial\Omega)>\min(t,T-t)\}$$

from the data $\mathcal{A}_q$ (see [32, Subsection 1.1]). Therefore, even when T is large, for the global recovery of general time-dependent coefficients the information on the bottom $t=0$ and the top $t=T$ of Q are unavoidable. Thus, for our problem the extra information on $\{t=0\}$ and $\{t=T\}$, of solutions u of (1.1), can not be completely removed. In this context, we introduce the set of data

$$C_q=\{(u_{|\Sigma},u_{t=0},\partial_t u_{|t=0},\partial_\nu u_{|\Sigma},u_{|t=T},\partial_t u_{|t=T}):\ u\in L^2(Q),\ \Box u+qu=0\}.$$

and we recall that [25] proved that, for $q\in L^\infty(Q)$, the data C_q determines uniquely q. From now on we will refer to C_q as the set of full data for our problem and we mention that [31, 32, 33] proved recovery of bounded time-dependent coefficients q from partial data corresponding to partial knowledge of the set C_q. The goal of the present paper is to prove recovery of singular time-dependent coefficients q from full and partial data.

1.3. **Physical and mathematical motivations.** Physically speaking, our inverse problem consists of determining unstable properties such as some rough time evolving density of an inhomogeneous medium from disturbances generated on the boundary and at initial time, and measurements of the response. The goal is to determine the function q which describes the property of the medium. Moreover, singular time-dependent coefficients can be associated to some unstable time-evolving phenomenon that can not be modeled by bounded time-dependent coefficients or time independent coefficients.

Let us also observe that, according to [11, 27], for parabolic equations the recovery of nonlinear terms, appearing in some suitable nonlinear equations, can be reduced to the determination of time-dependent coefficients. In this context, the information that allows to recover the nonlinear term is transferred, throw a linearization process, to a time-dependent coefficient depending explicitly on some solutions of the nonlinear problem. In contrast to parabolic equations, due to the weak regularity of solutions, it is not clear that this process allows to transfer the recovery of nonlinear terms, appearing in a nonlinear wave equation, to a bounded time-dependent coefficient. Thus, in order to expect an application of the strategy set by [11, 27] to the recovery of nonlinear terms for nonlinear wave equations, it seems important to consider recovery of singular time-dependent coefficients.

1.4. **known results.** The problem of determining coefficients appearing in hyperbolic equations has attracted many attention over the last decades. This problem has been stated in terms of recovery of a time-independent potential q from the set $\mathcal{A}_q$. For instance, [40] proved that $\mathcal{A}_q$ determines uniquely a time-independent potential q, while [16] proved that partial boundary observations are sufficient for this problem. We recall also that [4, 5, 30, 44] studied the stability issue for this problem.

Several authors considered also the problem of determining time-dependent coefficients appearing in wave equations. In [43], the authors shown that the knowledge of scattering data determines uniquely a smooth time-dependent potential. In [41], the authors studied the recovery of a time-dependent potential q from data on the boundary $\partial\Omega$ for all time given by $(u_{|\mathbb{R}\times\partial\Omega}, \partial_\nu u_{|\mathbb{R}\times\partial\Omega})$ of forward solutions of (1.1) on the infinite time-space cylindrical domain $\mathbb{R}_t \times \Omega$ instead of Q. As for [39], the authors considered this problem at finite time on Q and they proved the recovery of q restricted to some strict subset of Q from $\mathcal{A}_q$. Isakov established in [25, Theorem 4.2] unique global determination of general time-dependent potentials on the whole domain Q from the important set of full data C_q. By applying a result of unique continuation for wave equation, which is valid only for coefficients analytic with respect to the time variable (see for instance the counterexample of [1]), [17] proved unique recovery of time-dependent coefficients from partial knowledge of the data $\mathcal{A}_q$. In [42], the author extended the result of [41]. Moreover, [46] established the stable recovery of X-ray transforms of time-dependent potentials and [2, 6] proved log-type stability in the determination of time-dependent coefficients with data similar to [25] and [39]. In [31, 32, 33], the author proved uniqueness and stability in the recovery of several time-dependent coefficients from partial knowledge of the full set of data C_q. It seems that the results of [31, 32, 33] are stated with the weakest conditions so far that allows to recover general bounded time-dependent coefficients. More recently, [34] proved unique determination of such coefficients on Riemannian manifolds. We mention also the work of [45] who determined some information about time-dependent coefficients from the Dirichlet-to-Neumann map on a cylinder-like Lorentzian manifold related to the wave equation. We refer to the work [10, 12, 20, 21, 35] for determination of time-dependent coefficients for fractional diffusion, parabolic and Schrödinger equations have been considered.

In all the above mentioned results, the authors considered time-dependent coefficients that are at least bounded. There have been several works dealing with recovery of non-smooth coefficients appearing in elliptic equations such as [9, 15, 19, 23]. Nevertheless, to our best knowledge, except the present paper, there is no work in the mathematical literature dealing with the recovery of singular time-dependent coefficients q even from the important set of full data C_q.

1.5. **Main results.** The main purpose of this paper is to prove the unique global determination of time-dependent and unbounded coefficient q from partial knowledge of the observation of solutions on $\partial Q = (\{0\} \times \overline{\Omega}) \cup \Sigma \cup (\{T\} \times \overline{\Omega})$. More precisely, we would like to prove unique recovery of unbounded coefficient

$q \in L^{p_1}(0,T;L^{p_2}(\Omega))$, $p_1 \geqslant 2$, $p_2 \geqslant n$, from partial knowledge of the full set of data C_q. We start by considering the recovery of some general unbounded coefficient q from restriction of C_q on the bottom $t=0$ and top $t=T$ of the time-space cylindrical domain Q. More precisely, for $q \in L^{p_1}(0,T;L^{p_2}(\Omega))$, $p_1 \geqslant 2$, $p_2 \geqslant n$, we consider the recovery of q from the set of data

$$C_q(0) = \{(u_{|\Sigma}, \partial_t u_{|t=0}, \partial_\nu u, u_{|t=T}, \partial_t u_{|t=T}) :\ u \in \mathcal{K}(Q),\ \Box u + qu = 0,\ u_{|t=0} = 0\},$$

or the set of data

$$C_q(T) = \{(u_{|\Sigma}, u_{t=0}, \partial_t u_{|t=0}, \partial_\nu u, u_{|t=T}) :\ u \in \mathcal{K}(Q),\ \Box u + qu = 0\},$$

where $\mathcal{K}(Q) = \mathcal{C}([0,T];H^1(\Omega)) \cap \mathcal{C}^1([0,T];L^2(\Omega))$. In addition, assuming that $T > \mathrm{Diam}(\Omega)$, we prove the recovery of q from the set of data

$$C_q(0,T) = \{(u_{|\Sigma}, \partial_t u_{|t=0}, \partial_\nu u, u_{|t=T}) :\ u \in \mathcal{K}(Q),\ \Box u + qu = 0,\ u_{|t=0} = 0\}.$$

Our first main result can be stated as follows

Theorem 1.1. *Let $p_1 \in (2,+\infty)$, $p_2 \in (n,+\infty)$ and let $q_1,\ q_2 \in L^{p_1}(0,T;L^{p_2}(\Omega))$. Then, either of the following conditions:*

$$C_{q_1}(0) = C_{q_2}(0), \tag{1.3}$$

$$C_{q_1}(T) = C_{q_2}(T), \tag{1.4}$$

implies that $q_1 = q_2$. Moreover, assuming that $T > Diam(\Omega)$, the condition

$$C_{q_1}(0,T) = C_{q_2}(0,T) \tag{1.5}$$

implies that $q_1 = q_2$.

We consider also the recovery of a time-dependent and unbounded coefficient q from restriction of the data C_q on the lateral boundary Σ. Namely, for all $\omega \in \mathbb{S}^{n-1} = \{x \in \mathbb{R}^n :\ |x| = 1\}$ we introduce the ω-shadowed and ω-illuminated faces

$$\partial\Omega_{+,\omega} = \{x \in \partial\Omega :\ \nu(x)\cdot\omega > 0\}, \quad \partial\Omega_{-,\omega} = \{x \in \partial\Omega :\ \nu(x)\cdot\omega \leqslant 0\}$$

of $\partial\Omega$. Here, for all $k \in \mathbb{N}^*$, $\cdot$ denotes the scalar product in $\mathbb{R}^k$ given by

$$x\cdot y = x_1y_1 + \ldots + x_ky_k, \quad x = (x_1,\ldots,x_k) \in \mathbb{R}^k,\ y = (y_1,\ldots,y_k) \in \mathbb{R}^k.$$

We define also the parts of the lateral boundary Σ taking the form $\Sigma_{\pm,\omega} = (0,T)\times\partial\Omega_{\pm,\omega}$. We fix $\omega_0 \in \mathbb{S}^{n-1}$ and we consider $V = (0,T)\times V'$ with V' a closed neighborhood of $\partial\Omega_{-,\omega_0}$ in $\partial\Omega$. Then, we study the recovery of $q \in L^p(Q)$, $p > n+1$, from the data

$$C_q(T,V) = \{(u_{|\Sigma}, u_{|t=0}, \partial_t u_{|t=0}, \partial_\nu u_{|V}, u_{|t=T}) :\ u \in H^1(Q),\ \Box u + qu = 0\}$$

and the determination of a time-dependent coefficient $q \in L^\infty(0,T;L^p(\Omega))$, $p > n$, from the data

$$C_q(0,T,V) = \{(u_{|\Sigma}, \partial_t u_{|t=0}, \partial_\nu u_{|V}, u_{|t=T}) :\ u \in L^2(0,T;H^1(\Omega)),\ \Box u + qu = 0,\ u_{|t=0} = 0\}.$$

We refer to Section 2 for the definition of this set. Our main result can be stated as follows.

Theorem 1.2. *Let $p \in (n+1,+\infty)$ and let $q_1,\ q_2 \in L^p(Q)$. Then, the condition*

$$C_{q_1}(T,V) = C_{q_2}(T,V) \tag{1.6}$$

implies that $q_1 = q_2$.

Theorem 1.3. *Let $p \in (n,+\infty)$ and let $q_1,\ q_2 \in L^\infty(0,T;L^p(\Omega))$. Then, the condition*

$$C_{q_1}(0,T,V) = C_{q_2}(0,T,V) \tag{1.7}$$

implies that $q_1 = q_2$.

To our best knowledge the result of Theorem 1.1, 1.2 and 1.3 are the first results claiming unique determination of unbounded time-dependent coefficients for the wave equation. In Theorem 1.1, we prove recovery of coefficients q, that can admit some singularities, by making restriction on the set of full data C_q on the bottom $t = 0$ and the top $t = T$ of Q. While, in Theorem 1.2 and 1.3, we consider less singular time-dependent coefficients, in order to restrict the data on the lateral boundary $\Sigma = (0, T) \times \partial\Omega$.

We mention also that the uniqueness result of Thorem 1.3 is stated with data close to the one considered by [31, 32], where determination of bounded time-dependent potentials is proved with conditions that seems to be one of the weakest so far. More precisely, the only difference between [31, 32] and Theorem 1.3 comes from the restriction on the Dirichlet boundary condition ([31, 32] consider Dirichlet boundary condition supported on a neighborhood of the ω_0-shadowed face, while in Theorem 1.3 we do not restrict the support of the Dirichlet boundary).

In the present paper we consider two different approaches which depend mainly on the restriction that we make on the set of full data C_q. For Theorem 1.1, we use geometric optics solutions corresponding to oscillating solutions of the form

$$u(t, x) = \sum_{j=1}^{N} a_j(t, x)e^{i\lambda\psi_j(t,x)} + R_\lambda(t, x), \quad (t, x) \in Q, \tag{1.8}$$

with $\lambda > 1$ a large parameter, R_λ a remainder term that admits a decay with respect to the parameter λ and ψ_j, $j = 1, .., N$, real valued. For $N = 1$, these solutions correspond to a classical tool for proving determination of time independent or time-dependent coefficients (e. g. [2, 3, 4, 6, 39, 41, 40]). In a similar way to [34], we consider in Theorem 1.1 solutions of the form (1.8) with $N = 2$ in order to be able to restrict the data at $t = 0$ and $t = T$ while avoiding a "reflection". It seems that in the approach set so far for the construction of solutions of the form (1.8), the decay of the remainder term R_λ relies in an important way to the fact that the coefficient q is bounded (or time independent). In this paper, we prove how this construction can be extended to unbounded time-dependent coefficients.

The approach used for Theorem 1.1 allows in a quite straightforward way to restrict the data on the bottom $t = 0$ and on the top $t = T$ of Q. Nevertheless, it is not clear how one can extend this approach to restriction on the lateral boundary Σ without requiring additional smoothness or geometrical assumptions. For this reason, in order to consider restriction on Σ, we use a different approach where the oscillating solutions (1.8) are replaced by exponentially growing and exponentially decaying solutions of the form

$$u(t, x) = e^{\pm\lambda(t+x\cdot\omega)}(a(t, x) + w_\lambda(t, x)), \quad (t, x) \in Q, \tag{1.9}$$

where $\omega \in \mathbb{S}^{n-1}$ and w_λ admits a decay with respect to the parameter λ. The idea of this approach, which is inspired by [5, 31, 32, 33] (see also [8, 29] for elliptic equations), consists of combining results of density of products of solutions with Carleman estimates with linear weight in order to be able to restrict at the same time the data on the bottom $t = 0$, on the top $t = T$ and on the lateral boundary Σ of Q. For the construction of these solutions, we use Carleman estimates in negative order Sobolev space. To our best knowledge this is the first extension of this approach to singular time-dependent coefficients.

1.6. **Outline.** This paper is organized as follows. In Section 2, we start with some preliminary results and we define the set of data $C_q(0)$, $C_q(T)$, $C_q(0, T)$, $C_q(T, V)$ and $C_q(0, T, V)$. In Section 3, we prove Theorem 1.1 by mean of geometric optics solutions of the form (1.8). Then, Section 4 and Section 5 are respectively devoted to the proof of Theorem 1.2 and Theorem 1.3.

2. Preliminary results

In the present section we define the set of data $C_q(T, V)$, $C_q(0, T, V)$ and we recall some properties of the solutions of (1.1) for any $q \in L^{p_1}(Q)$, with $p_1 > n + 1$, or, for $q \in L^\infty(0, T; L^{p_2}(\Omega))$, with $p_2 > n$. For this purpose, in a similar way to [32], we will introduce some preliminary tools. We define the space

$$H_\Box(Q) = \{u \in H^1(Q) : \ \Box u = (\partial_t^2 - \Delta_x)u \in L^2(Q)\},$$

$$H_{\Box,*}(Q)=\{u\in L^2(0,T;H^1(\Omega)):\ \Box u=(\partial_t^2-\Delta_x)u\in L^2(Q)\},$$

with the norm

$$\|u\|^2_{H_\Box(Q)}=\|u\|^2_{H^1(Q)}+\left\|(\partial_t^2-\Delta_x)u\right\|^2_{L^2(Q)},$$

$$\|u\|^2_{H_{\Box,*}(Q)}=\|u\|^2_{L^2(0,T;H^1(\Omega))}+\left\|(\partial_t^2-\Delta_x)u\right\|^2_{L^2(Q)}.$$

We consider also the space

$$S=\{u\in H^1(Q):\ (\partial_t^2-\Delta_x)u=0\}\ (\text{resp } S_*=\{u\in L^2(0,T;H^1(\Omega)):\ (\partial_t^2-\Delta_x)u=0\})$$

and topologize it as a closed subset of $H^1(Q)$ (resp $L^2(0,T;H^1(\Omega))$). In view of [32, Proposition 4], the maps

$$\tau_0w=(w_{|\Sigma},w_{|t=0},\partial_tw_{|t=0}),\quad \tau_1w=(\partial_\nu w_{|\Sigma},w_{|t=T},\partial_tw_{|t=T}),\quad w\in\mathcal{C}^\infty(\overline{Q}),$$

can be extended continuously to $\tau_0:H_{\Box,*}(Q)\to H^{-3}(0,T;H^{-\frac12}(\partial\Omega))\times H^{-2}(\Omega)\times H^{-4}(\Omega)$, $\tau_1:H_{\Box,*}(Q)\to H^{-3}(0,T;H^{-\frac32}(\partial\Omega))\times H^{-2}(\Omega)\times H^{-4}(\Omega)$. Here for all $w\in\mathcal{C}^\infty(\overline{Q})$ we set

$$\tau_0w=(\tau_{0,1}w,\tau_{0,2}w,\tau_{0,3}w),\quad \tau_1w=(\tau_{1,1}w,\tau_{1,2}w,\tau_{1,3}w),$$

where

$$\tau_{0,1}w=w_{|\Sigma},\ \tau_{0,2}w=w_{|t=0},\ \tau_{0,3}w=\partial_tw_{|t=0},\ \tau_{1,1}w=\partial_\nu w_{|\Sigma},\ \tau_{1,2}w=w_{|t=T},\ \tau_{1,3}w=\partial_tw_{|t=T}.$$

Therefore, we can introduce

$$\mathcal{H}=\{\tau_0u:\ u\in H_\Box(Q)\}\subset H^{-3}(0,T;H^{-\frac12}(\partial\Omega))\times H^{-2}(\Omega)\times H^{-4}(\Omega),$$

$$\mathcal{H}_*=\{(\tau_{0,1}u,\tau_{0,3}u):\ u\in H_{\Box,*}(Q),\ \tau_{0,2}u=0\}\subset H^{-3}(0,T;H^{-\frac12}(\partial\Omega))\times H^{-4}(\Omega).$$

By repeating the arguments used in [32, Proposition 1], one can check that the restriction of τ_0 to S (resp S_*) is one to one and onto. Thus, we can use $(\tau_{0|S})^{-1}$ (resp $(\tau_{0|S_*})^{-1}$) to define the norm of $\mathcal{H}$ (resp $\mathcal{H}_*$) by

$$\|(f,v_0,v_1)\|_{\mathcal{H}}=\left\|(\tau_{0|S})^{-1}(f,v_0,v_1)\right\|_{H^1(Q)},\quad (f,v_0,v_1)\in\mathcal{H},$$

$$(\text{resp } \|(f,v_1)\|_{\mathcal{H}_*}=\left\|(\tau_{0|S_*})^{-1}(f,0,v_1)\right\|_{L^2(0,T;H^1(\Omega))},\quad (f,v_1)\in\mathcal{H}_*).$$

Let us consider the initial boundary value problem (IBVP in short)

$$\begin{cases}\partial_t^2v-\Delta_xv+qv=F(t,x), & (t,x)\in Q,\\ v(0,x)=v_0(x),\quad \partial_tv(0,x)=v_1(x), & x\in\Omega\\ v(t,x)=0, & (t,x)\in\Sigma.\end{cases}\tag{2.1}$$

We have the following well-posedness result for this IBVP when q is unbounded.

Proposition 2.1. *Let $p_1\in(1,+\infty)$ and $p_2\in(n,+\infty)$. For $q\in L^{p_1}(0,T;L^{p_2}(\Omega))$, $v_0\in H^1_0(\Omega)$, $v_1\in L^2(\Omega)$ and $F\in L^2(Q)$, problem (2.1) admits a unique solution $v\in\mathcal{C}([0,T];H^1_0(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))$ satisfying*

$$\|v\|_{\mathcal{C}([0,T];H^1_0(\Omega))}+\|v\|_{\mathcal{C}^1([0,T];L^2(\Omega))}\leqslant C(\|v_0\|_{H^1(\Omega)}+\|v_1\|_{L^2(\Omega)}+\|F\|_{L^2(Q)}),\tag{2.2}$$

with C depending only on p_1, p_2, n, T, Ω and any $M\geqslant\|q\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}$.

Proof. According to the second part of the proof of [37, Theorem 8.1, Chapter 3], [37, Remark 8.2, Chapter 3] and [37, Theorem 8.3, Chapter 3], the proof of this proposition will be completed if we show that for any $v\in W^{2,\infty}(0,T;H^1_0(\Omega))$ solving (2.1) the a priori estimate (2.2) holds true. Without lost of generality we assume that v is real valued. From now on we consider this estimate. We define the enery $E(t)$ at time $t\in[0,T]$ by

$$E(t):=\int_\Omega\left(|\partial_tv(t,x)|^2+|\nabla_xv(t,x)|^2\right)dx.$$

Multiplying (2.1) by ∂_tv and integrating by parts we get

$$E(t)-E(0)=-2\int_0^t\int_\Omega q(s,x)v(s,x)\partial_tv(s,x)dxds+2\int_0^t\int_\Omega F(s,x)\partial_tv(s,x)dxds.\tag{2.3}$$

On the other hand, we have

$$\left|\int_0^t\int_\Omega q(s,x)v(s,x)\partial_t v(s,x)dxds\right|\leqslant\int_0^t\|qv(s,\cdot)\|_{L^2(\Omega)}\|\partial_t v(s,\cdot)\|_{L^2(\Omega)}\,ds. \tag{2.4}$$

Applying the Sobolev embedding theorem and the Hölder inequality, for all $s\in(0,T)$ we get

$$\begin{aligned}\|qv(s,\cdot)\|_{L^2(\Omega)}&\leqslant\|q(s,\cdot)\|_{L^{p_2}(\Omega)}\|v(s,\cdot)\|_{L^{\frac{2p_2}{p_2-2}}(\Omega)}\\&\leqslant C\|q(s,\cdot)\|_{L^{p_2}(\Omega)}\|v(s,\cdot)\|_{H^{\frac{n}{p_2}}(\Omega)}\\&\leqslant C\|q(s,\cdot)\|_{L^{p_2}(\Omega)}\|v(s,\cdot)\|_{H^1(\Omega)}\end{aligned}$$

with C depending only on Ω. Then, the Poincarré inequality implies

$$\|qv(s,\cdot)\|_{L^2(\Omega)}\leqslant C\|q(s,\cdot)\|_{L^{p_2}(\Omega)}\|\nabla_x v(s,\cdot)\|_{L^2(\Omega)}\leqslant C\|q(s,\cdot)\|_{L^{p_2}(\Omega)}E(s)^{\frac12},$$

where C depends only on Ω. Thus, from (2.4), we get

$$\left|\int_0^t\int_\Omega q(s,x)v(s,x)\partial_t v(s,x)dxds\right|\leqslant C\int_0^t\|q(s,\cdot)\|_{L^{p_2}(\Omega)}E(s)ds\leqslant\|q\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}\left(\int_0^t E(s)^{\frac{p_1}{p_1-1}}ds\right)^{\frac{p_1-1}{p_1}}. \tag{2.5}$$

In the same way, an application of the Hölder inequality yields

$$\begin{aligned}\left|\int_0^t\int_\Omega F(s,x)\partial_t v(s,x)dxds\right|&\leqslant T^{\frac{1}{2p_1}}\|F\|_{L^2(Q)}\left(\int_0^t E(s)^{\frac{p_1}{p_1-1}}ds\right)^{\frac{p_1-1}{2p_1}}\\&\leqslant T^{\frac{1}{p_1}}\|F\|_{L^2(Q)}^2+\left(\int_0^t E(s)^{\frac{p_1}{p_1-1}}\right)^{\frac{p_1-1}{p_1}}.\end{aligned}$$

Combining this estimate with (2.3)-(2.5), we deduce that

$$E(t)\leqslant E(0)+C\|F\|_{L^2(Q)}^2+C\left(\int_0^t E(s)^{\frac{p_1}{p_1-1}}\right)^{\frac{p_1-1}{p_1}},$$

where C depends only on T, Ω and any $M\geqslant\|q\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}$. By taking the power $\frac{p_1}{p_1-1}$ on both side of this inequality, we get

$$E(t)^{\frac{p_1}{p_1-1}}\leqslant C\left(\|v_0\|_{H^1(\Omega)}+\|v_1\|_{L^2(\Omega)}+\|F\|_{L^2(Q)}\right)^{\frac{2p_1}{p_1-1}}+C\int_0^t E(s)^{\frac{p_1}{p_1-1}}ds.$$

Then, the Gronwall inequality implies

$$\begin{aligned}E(t)^{\frac{p_1}{p_1-1}}&\leqslant C\left(\|v_0\|_{H^1(\Omega)}+\|v_1\|_{L^2(\Omega)}+\|F\|_{L^2(Q)}\right)^{\frac{2p_1}{p_1-1}}e^{Ct}\\&\leqslant C\left(\|v_0\|_{H^1(\Omega)}+\|v_1\|_{L^2(\Omega)}+\|F\|_{L^2(Q)}\right)^{\frac{2p_1}{p_1-1}}e^{CT}.\end{aligned}$$

From this last estimate one can easily deduce (2.2).

□

Let us introduce the IBVP

$$\begin{cases}\partial_t^2u-\Delta_x u+q(t,x)u=0, & \text{in } Q,\\ u(0,\cdot)=v_0,\quad \partial_t u(0,\cdot)=v_1, & \text{in } \Omega,\\ u=g, & \text{on } \Sigma.\end{cases} \tag{2.6}$$

We are now in position to state existence and uniqueness of solutions of this IBVP for $(g,v_0,v_1)\in\mathcal{H}$ and $q\in L^p(Q)$, $p>n+1$.

Proposition 2.2. *Let* $(g, v_0, v_1) \in \mathcal{H}$, $q \in L^p(Q)$, $p > n+1$. *Then, the IBVP* (2.6) *admits a unique weak solution* $u \in H^1(Q)$ *satisfying*

$$\|u\|_{H^1(Q)} \leqslant C \|(g, v_0, v_1)\|_{\mathcal{H}} \tag{2.7}$$

and the boundary operator $B_q : (g, v_0, v_1) \mapsto (\tau_{1,1} u_{|V}, \tau_{1,2} u)$ *is a bounded operator from* $\mathcal{H}$ *to* $H^{-3}(0,T; H^{-\frac{3}{2}}(V')) \times H^{-2}(\Omega)$.

Proof. We split u into two terms $u = v + \tau_0^{-1}(g, v_0, v_1)$ where v solves

$$\left\{ \begin{array}{rcl} \partial_t^2 v - \Delta_x v + qv &=& q\tau_0^{-1}(g, v_0, v_1), \quad (t,x) \in Q, \\ v_{|t=0} = \partial_t v_{|t=0} &=& 0, \\ v_{|\Sigma} &=& 0. \end{array} \right. \tag{2.8}$$

Since $q \in L^p(Q)$ and $\tau_0^{-1}(g, v_0, v_1) \in H^1(Q)$, by the Sobolev embedding theorem we have $q\tau_0^{-1}(g, v_0, v_1) \in L^2(Q)$. Thus, according to Proposition 2.1 one can check that the IBVP (2.8) has a unique solution $v \in \mathcal{C}^1([0,T]; L^2(\Omega)) \cap \mathcal{C}([0,T]; H_0^1(\Omega))$ satisfying

$$\begin{aligned} \|v\|_{\mathcal{C}^1([0,T];L^2(\Omega))} + \|v\|_{\mathcal{C}([0,T];H_0^1(\Omega))} &\leqslant C \left\|q\tau_0^{-1}(g, v_0, v_1)\right\|_{L^2(Q)} \\ &\leqslant C \|q\|_{L^p(Q)} \left\|\tau_0^{-1}(g, v_0, v_1)\right\|_{H^1(Q)}. \end{aligned} \tag{2.9}$$

Thus, $u = v + \tau_0^{-1}(g, v_0, v_1)$ is the unique solution of (2.6) and estimate (2.9) implies (2.7). Now let us consider the last part of the proposition. For this purpose, let $(g, v_0, v_1) \in \mathcal{H}$ and let $u \in H^1(Q)$ be the solution of (2.6). Note first that $(\partial_t^2 - \Delta_x)u = -qu \in L^2(Q)$. Therefore, $u \in H_\Box(Q)$ and $\tau_{1,1} u \in H^{-3}(0,T; H^{-\frac{3}{2}}(\partial\Omega))$, $\tau_{1,2} u \in H^{-2}(\Omega)$ with

$$\begin{aligned} \|\tau_{1,1} u\|^2 + \|\tau_{1,2} u\|^2 \leqslant C^2 \|u\|_{H_\Box(Q)}^2 &= C^2(\|u\|_{H^1(Q))}^2 + \|qu\|_{L^2(Q)}^2) \\ &\leqslant C^2(1 + \|q\|_{L^p(Q)}^2) \|u\|_{H^1(Q)}^2. \end{aligned}$$

Combining this with (2.7), we find that B_q is a bounded operator from $\mathcal{H}$ to $H^{-3}(0,T; H^{-\frac{3}{2}}(V')) \times H^{-2}(\Omega)$. □

From now on, we define the set $C_q(T, V)$ by

$$C_q(T, V) = \{(g, v_0, v_1, B_q(g, v_0, v_1)) : \ (g, v_0, v_1) \in \mathcal{H}\}.$$

In the same way, for $q \in L^{p_1}(0,T; L^{p_2}(\Omega))$, $p_1 \geqslant 2, p_2 > n$, we consider the set $C_q(T)$, $C_q(0)$, $C_q(0,T)$ introduced before Theorem 1.1. Using similar arguments to Proposition 2.2 we can prove the following.

Proposition 2.3. *Let* $(g, v_1) \in \mathcal{H}_*$ *with* $v_0 = 0$ *and let* $q \in L^\infty(0,T; L^p(\Omega))$, $p > n$. *Then, the IBVP* (2.6) *admits a unique weak solution* $u \in L^2(0,T; H^1(\Omega))$ *satisfying*

$$\|u\|_{L^2(0,T;H^1(\Omega))} \leqslant C \|(g, v_1)\|_{\mathcal{H}_*} \tag{2.10}$$

and the boundary operator $B_{q,*} : (g, v_1) \mapsto (\tau_{1,1} u_{|V}, \tau_{1,2} u)$ *is a bounded operator from* $\mathcal{H}_*$ *to* $H^{-3}(0,T; H^{-\frac{3}{2}}(V')) \times H^{-2}(\Omega)$.

We define the set $C_q(0, T, V)$ by

$$C_q(0, T, V) = \{(g, v_1, B_{q,*}(g, v_1)) : \ (g, v_1) \in \mathcal{H}_*\}.$$

3. Proof of Theorem 1.1

The goal of this section is to prove Theorem 1.1. For this purpose, we consider special solutions u_j of the equation

$$\partial_t^2 u_j - \Delta_x u_j + q_j u_j = 0 \tag{3.1}$$

taking the form

$$u_j = a_{j,1} e^{i\lambda\psi_1(t,x)} + a_{j,2} e^{i\lambda\psi_2(t,x)} + R_{j,\lambda} \tag{3.2}$$

with a large parameter $\lambda > 0$ and a remainder term $R_{j,\lambda}$ that admits some decay with respect to λ. The use of such a solutions, also called oscillating geometric optics solutions, goes back to [40] who have proved unique recovery of time-independent coefficients. Since then, such approach has been used by various authors in different context including recovery of a bounded time-dependent coefficient by [34]. In this section we will prove how one can extend this approach, that has been specifically designed for the recovery of time-independent coefficients or bounded time-dependent coefficients, to the recovery of singular time-dependent coefficients.

3.1. **Oscillating geometric optics solutions.** Fixing $\omega \in \mathbb{S}^{n-1}$, $\lambda > 1$ and $a_{j,k} \in \mathcal{C}^\infty(\overline{Q})$, $j = 1, 2$, $k = 1, 2$, we consider solutions of (3.1) taking the form

$$u_1(t,x) = a_{1,1}(t,x) e^{-i\lambda(t+x\cdot\omega)} + a_{1,2}(t,x) e^{-i\lambda((2T-t)+x\cdot\omega)} + R_{1,\lambda}(t,x), \quad (t,x) \in Q, \tag{3.3}$$

$$u_2(t,x) = a_{2,1}(t,x) e^{i\lambda(t+x\cdot\omega)} + a_{2,2}(t,x) e^{i\lambda(-t+x\cdot\omega)} + R_{2,\lambda}(t,x), \quad (t,x) \in Q. \tag{3.4}$$

Here, the expression $a_{j,k}$, $j,k = 1,2$, are independent of λ and they are respectively solutions of the transport equation

$$\partial_t a_{j,k} + (-1)^k \omega \cdot \nabla_x a_{j,k} = 0, \quad (t,x) \in Q, \tag{3.5}$$

and the expression $R_{j,\lambda}$, $j = 1,2$, solves respectively the IBVP

$$\begin{cases} \partial_t^2 R_{1,\lambda} - \Delta_x R_{1,\lambda} + q_1 R_{1,\lambda} &= F_{1,\lambda}, \quad (t,x) \in Q, \\ R_{1,\lambda}(T,x) = 0, \quad \partial_t R_{1,\lambda}(T,x) &= 0, \quad\ \ x \in \Omega \\ R_{1,\lambda}(t,x) = 0, & \qquad\quad (t,x) \in \Sigma, \end{cases} \tag{3.6}$$

$$\begin{cases} \partial_t^2 R_{2,\lambda} - \Delta_x R_{2,\lambda} + q_2 R_{2,\lambda} &= F_{2,\lambda}, \quad (t,x) \in Q, \\ R_{2,\lambda}(0,x) = 0, \quad \partial_t R_{2,\lambda}(0,x) &= 0, \quad\ \ x \in \Omega \\ R_{2,\lambda}(t,x) = 0, & \qquad\quad (t,x) \in \Sigma, \end{cases} \tag{3.7}$$

with $F_{j,\lambda} = -[(\Box + q_j)(u_j - R_{j,\lambda})]$. The main point in the construction of such solutions, also called oscillating geometric optics (GO in short) solutions, consists of proving the decay of the expression $R_{j,\lambda}$ with respect to $\lambda \to +\infty$. Actually, we would like to prove the following,

$$\lim_{\lambda \to +\infty} \|R_{j,\lambda}\|_{L^\infty(0,T;L^2(\Omega))} = 0. \tag{3.8}$$

For $q \in L^\infty(Q)$, the construction of GO solutions of the form (3.3)-(3.4), with $a_{j,k}$ satisfying (3.5) and $R_{j,\sigma}$ satisfying (3.6)-(3.8), has been proved in [34, Lemma 2.2]. The fact that q is bounded plays an important role in the arguments of [34, Lemma 2.2]. For this reason we can not apply the result of [34] and we need to consider the following.

Lemma 3.1. *Let $q_j \in L^{p_1}(0,T;L^{p_2}(\Omega))$, $j = 1,2$, $p_1 > 2$, $p_2 > n$. Then, we can find $u_j \in \mathcal{K}(Q)$ solving (3.1), of the form (3.3)-(3.4), with $R_{j,\lambda}$, $j = 1,2$, satisfying (3.8) and the following estimate*

$$\sup_{\lambda > 1} \max_{j=1,2} \|R_{j,\lambda}\|_{L^\infty(0,T;H^1(\Omega))} < \infty. \tag{3.9}$$

Proof. We will consider this result only for $j=2$, the proof for $j=1$ being similar by symmetry. Note first that, (3.5) implies that

$$\begin{aligned} F_{2,\lambda}(t,x) &= -e^{i\lambda(t+x\cdot\omega)}(\Box+q_2)a_{2,1}(t,x) - e^{i\lambda(-t+x\cdot\omega)}(\Box+q_2)a_{2,2}(t,x) \\ &= H_\lambda(t,x) \\ &= e^{i\lambda t}H_{1,\lambda}(t,x) + e^{-i\lambda t}H_{2,\lambda}(t,x), \end{aligned}$$

with

$$\|H_\lambda\|_{L^2(Q)} \leqslant \|(\Box+q_2)a_{2,1}\|_{L^2(Q)} + \|(\Box+q_2)a_{2,2}\|_{L^2(Q)}. \tag{3.10}$$

Thus, in light Proposition 2.1, we have $R_{2,\lambda}\in\mathcal{K}(Q)$ with

$$\|R_{2,\lambda}\|_{\mathcal{C}^1([0,T];L^2(\Omega))} + \|R_{2,\lambda}\|_{\mathcal{C}([0,T];H^1(\Omega))} \leqslant C(1+\|q_2\|_{L^{p_1}(0,T;L^{p_2}(\Omega))})(\|a_{2,1}\|_{W^{2,\infty}(Q)} + \|a_{2,2}\|_{W^{2,\infty}(Q)}). \tag{3.11}$$

In particular, this proves (3.9). The only point that we need to check is the decay with respect to λ given by (3.8). For this purpose, we consider $v(t,x) := \int_0^t R_{2,\lambda}(s,x)ds$ and we easily check that v solves

$$\begin{cases} \partial_t^2 v - \Delta_x v = G_\lambda, & (t,x)\in Q, \\ v(0,x)=0, \quad \partial_t v(0,x) = 0, & x\in\Omega \\ v(t,x)=0, & (t,x)\in\Sigma, \end{cases} \tag{3.12}$$

with

$$G_\lambda(t,x) = -\int_0^t q_2(s,x)R_{2,\lambda}(s,x)ds + \int_0^t H_\lambda(s,x)ds, \quad (t,x)\in Q.$$

In view of [38, Theorem 2.1, Chapter 5], since $G_\lambda\in H^1(0,T;L^2(\Omega))$ we have $v\in H^2(Q)$. We define the energy $E(t)$ at time t associated with v and given by

$$E(t) := \int_\Omega \left(|\partial_t v|^2(t,x) + |\nabla_x v|^2(t,x)\right)dx \geqslant \int_\Omega |R_{2,\lambda}(t,x)|^2 dx.$$

Multiplying (3.12) by $\overline{\partial_t v}$ and taking the real part, we find

$$E(t) = -2\Re\left(\int_0^t\int_\Omega\left(\int_0^s q_2(\tau,x)R_{2,\lambda}(\tau,x)d\tau\right)\overline{\partial_t v(s,x)}dxds\right) + 2\Re\left(\int_0^t\int_\Omega\left(\int_0^s H_\lambda(\tau,x)d\tau\right)\overline{\partial_t v(s,x)}dxds\right).$$

Applying Fubini's theorem, we obtain

$$\begin{aligned} E(t) &= -2\Re\left(\int_0^t\int_\Omega q_2(\tau,x)R_{2,\lambda}(\tau,x)\left(\overline{\int_\tau^t \partial_t v(s,x)ds}\right)dxd\tau\right) + 2\Re\left(\int_0^t\int_\Omega\left(\int_0^s H_\lambda(\tau,x)d\tau\right)\overline{\partial_t v(s,x)}dxds\right) \\ &= -2\Re\left(\int_0^t\int_\Omega q_2(\tau,x)R_{2,\lambda}(\tau,x)(\overline{v(t,x)} - \overline{v(\tau,x)})dxd\tau\right) + 2\Re\left(\int_0^t\int_\Omega\left(\int_0^s H_\lambda(\tau,x)d\tau\right)\overline{\partial_t v(s,x)}dxds\right). \end{aligned} \tag{3.13}$$

On the other hand, applying the Hölder inequality, we get

$$\begin{aligned} \left|\int_0^t\int_\Omega q_2(\tau,x)R_{2,\lambda}(\tau,x)\overline{v(t,x)}dxd\tau\right| &\leqslant \int_0^t \|\partial_t v(\tau,\cdot)\|_{L^2(\Omega)}\|q_2(\tau,\cdot)v(t,\cdot)\|_{L^2(\Omega)}d\tau \\ &\leqslant \left(\int_0^t \|\partial_t v(\tau,\cdot)\|_{L^2(\Omega)}\|q_2(\tau,\cdot)\|_{L^{p_2}(\Omega)}d\tau\right)\|v(t,\cdot)\|_{L^{\frac{p_2-2}{2p_2}}(\Omega)}. \end{aligned}$$

Then, combining the Sobolev embedding theorem with the Poincarré inequality, we deduce that

$$\begin{aligned}\left|\int_0^t\int_\Omega q_2(\tau,x)R_{2,\lambda}(\tau,x)\overline{v(t,x)}dxd\tau\right| &\leqslant C\left(\int_0^t \|\partial_t v(\tau,\cdot)\|_{L^2(\Omega)}\|q_2(\tau,\cdot)\|_{L^{p_2}(\Omega)}d\tau\right)\|v(t,\cdot)\|_{H^1(\Omega)}\\ &\leqslant C\left(\int_0^t E(\tau)^{1/2}\|q_2(\tau,\cdot)\|_{L^{p_2}(\Omega)}d\tau\right)E(t)^{1/2}\\ &\leqslant C^2\left(\int_0^t E(\tau)^{1/2}\|q_2(\tau,\cdot)\|_{L^{p_2}(\Omega)}d\tau\right)^2+\frac{E(t)}{4}\\ &\leqslant C^2T\left(\int_0^t E(\tau)\|q_2(\tau,\cdot)\|^2_{L^{p_2}(\Omega)}d\tau\right)+\frac{E(t)}{4},\end{aligned}$$

with C depending only on Ω. Applying again the Hölder inequality, we get

$$\left|\int_0^t\int_\Omega q_2(\tau,x)R_{2,\lambda}(\tau,x)\overline{v(t,x)}dxd\tau\right| \leqslant C\left(\int_0^t E(\tau)^{\frac{p_1}{p_1-2}}d\tau\right)^{\frac{p_1-2}{p_1}}\|q_2\|^2_{L^{p_1}(0,T;L^{p_2}(\Omega))}+\frac{E(t)}{4}. \tag{3.14}$$

In the same way, we obtain

$$\begin{aligned}\left|\int_0^t\int_\Omega q_2(\tau,x)R_{2,\lambda}(\tau,x)\overline{v(\tau,x)}dxd\tau\right| &\leqslant C\int_0^t E(\tau)\|q_2(\tau,\cdot)\|_{L^{p_2}(\Omega)}d\tau\\ &\leqslant C\left(\int_0^t E(\tau)^{\frac{p_1}{p_1-2}}d\tau\right)^{\frac{p_1-2}{p_1}}\|q_2\|_{L^{\frac{p_1}{2}}(0,T;L^{p_2}(\Omega))}.\end{aligned} \tag{3.15}$$

Finally, fixing

$$\beta_\lambda(t,x):=\int_0^t H_\lambda(\tau,x)d\tau,$$

we find

$$\begin{aligned}\left|\int_0^t\int_\Omega\left(\int_0^s H_\lambda(\tau,x)d\tau\right)\overline{\partial_t v(s,x)}dxds\right| &\leqslant \|\beta_\lambda\|_{L^{\frac{p_1}{2}}(0,T;L^2(\Omega))}\left(\int_0^t E(\tau)^{\frac{p_1}{2(p_1-2)}}d\tau\right)^{\frac{p_1-2}{p_1}}\\ &\leqslant \|\beta_\lambda\|^2_{L^{\frac{p_1}{2}}(0,T;L^2(\Omega))}+\left(\int_0^t E(\tau)^{\frac{p_1}{2(p_1-2)}}d\tau\right)^{\frac{2(p_1-2)}{p_1}}\\ &\leqslant \|\beta_\lambda\|^2_{L^{\frac{p_1}{2}}(0,T;L^2(\Omega))}+T^{\frac{p_1-2}{p_1}}\left(\int_0^t E(\tau)^{\frac{p_1}{p_1-2}}d\tau\right)^{\frac{p_1-2}{p_1}}.\end{aligned} \tag{3.16}$$

Combining (3.13)-(3.16), we deduce that

$$E(t)\leqslant\frac{E(t)}{4}+C(\|q_2\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}+1)^2\left(\int_0^t E(\tau)^{\frac{p_1}{p_1-2}}d\tau\right)^{\frac{p_1-2}{p_1}}+\|\beta_\lambda\|^2_{L^{\frac{p_1}{2}}(0,T;L^2(\Omega))}$$

and we get

$$E(t)\leqslant C(\|q_2\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}+1)^2\left(\int_0^t E(\tau)^{\frac{p_1}{p_1-2}}d\tau\right)^{\frac{p_1-2}{p_1}}+\frac{4\|\beta_\lambda\|^2_{L^{\frac{p_1}{2}}(0,T;L^2(\Omega))}}{3},$$

with C depending only on Ω and T. Now taking the power $\frac{p_1}{p_1-2}$ on both side of this inequality, we get

$$E(t)^{\frac{p_1}{p_1-2}}\leqslant 2^{\frac{p_1}{p_1-2}}C^{\frac{p_1}{p_1-2}}(\|q_2\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}+1)^{\frac{2p_1}{p_1-2}}\int_0^t E(\tau)^{\frac{p_1}{p_1-2}}d\tau+2^{\frac{p_1}{p_1-2}}\left(\frac{4\|\beta_\lambda\|^2_{L^{\frac{p_1}{2}}(0,T;L^2(\Omega))}}{3}\right)^{\frac{p_1}{p_1-2}}$$

and applying the Gronwall inequality, we obtain

$$E(t)^{\frac{p_1}{p_1-2}} \leqslant C_1 \|\beta_\lambda\|_{L^{\frac{p_1}{2}}(0,T;L^2(\Omega))}^{\frac{2p_1}{p_1-2}} e^{C_2 t} \leqslant C_1 \|\beta_\lambda\|_{L^{\frac{p_1}{2}}(0,T;L^2(\Omega))}^{\frac{2p_1}{p_1-2}} e^{C_2 T},$$

where C_1 depends only on p_1 and C_2 on $\|q_2\|_{L^{p_1}(0,T;L^{p_2}(\Omega))}$, Ω and T. According to this estimate, the proof of the lemma will be completed if we prove that

$$\lim_{\lambda\to+\infty} \|\beta_\lambda\|_{L^\infty(0,T;L^2(\Omega))} = 0. \tag{3.17}$$

This follows from some arguments similar to the end of the proof of [34, Lemma 2.2] that we recall for sake of completeness. Applying the Riemann-Lebesgue lemma, for all $t \in [0,T]$ and almost every $x \in \Omega$, we have

$$\lim_{\lambda\to+\infty} \int_0^t e^{i\lambda\tau} H_{1,\lambda}(\tau,x)d\tau = \lim_{\lambda\to+\infty} \int_0^t e^{-i\lambda\tau} H_{2,\lambda}(\tau,x)d\tau = 0.$$

Therefore, for all $t \in [0,T]$ and almost every $x \in \Omega$, we obtain

$$\lim_{\lambda\to+\infty} \beta_\lambda(t,x) = \lim_{\lambda\to+\infty} \int_0^t H_\lambda(\tau,x)d\tau = 0.$$

Moreover, from the definition of H_λ, we get

$$\left|\int_0^t H_\lambda(\tau,x)d\tau\right| \leqslant \int_0^t (|(\Box+q_2)a_{2,1}| + |(\Box+q_2)a_{2,2}|)ds, \quad t\in[0,T], \quad x\in\Omega.$$

Thus, we deduce from Lebesgue's dominated convergence theorem that

$$\lim_{\lambda\to+\infty} \left\|\int_0^t H_\lambda(\tau,\cdot)d\tau\right\|_{L^2(\Omega)} = 0, \quad t\in[0,T].$$

Combining this with the estimate

$$\begin{aligned}&\left\|\int_0^{t_2} H_\lambda(\tau,\cdot)d\tau - \int_0^{t_1} H_\lambda(\tau,\cdot)d\tau\right\|_{L^2(\Omega)}\\ &\quad\leqslant (t_2-t_1)^{\frac12}[\|(\Box+q_2)a_{2,1}\|_{L^2(Q)} + \|(\Box+q_2)a_{2,2}\|_{L^2(Q))}], \quad 0\leqslant t_1<t_2\leqslant T,\end{aligned}$$

we deduce (3.17). This completes the proof the lemma. □

3.2. **Proof of Theorem 1.1 with restriction at** $t=0$ **or** $t=T$**.** In this section we will prove that (1.3) or (1.4) implies that $q_1=q_2$. We start by assuming that (1.3) is fulfilled and we fix $q=q_2-q_1$ on Q extended by 0 on $\mathbb{R}^{1+n}\setminus Q$. We fix $\lambda>1$, $\omega\in\mathbb{S}^{n-1}$ and we fix $\xi\in\mathbb{R}^{1+n}$ satisfying $(1,-\omega)\cdot\xi=0$. Then, in view of Lemma 3.1, we can consider $u_j\in\mathcal{K}(Q)$, $j=1,2$, solving (3.1), of the form (3.3)-(3.4), with $a_{1,1}(t,x)=(2\pi)^{\frac{n+1}{2}}e^{-i(t,x)\cdot\xi}$, $a_{1,2}=0$, $a_{2,1}=1$, $a_{2,2}=-1$ and with condition (3.8)-(3.9) fulfilled, that is,

$$\begin{aligned}u_1(t,x) &= (2\pi)^{\frac{n+1}{2}}e^{-i(t,x)\cdot\xi}e^{-i\lambda(t+x\cdot w)} + R_{1,\lambda}(t,x),\\ u_2(t,x) &= e^{i\lambda(t+x\cdot w)} - e^{i\lambda(-t+x\cdot w)} + R_{2,\lambda}(t,x).\end{aligned} \tag{3.18}$$

Obviously, we have $u_2(0,x)=0$, since $R_{2,\lambda}(0,x)=0$ by (3.7). In view of Proposition 2.1, there exists a unique weak solution $v\in\mathcal{K}(Q)$ to the IBVP:

$$\begin{aligned}&\partial_t^2 v - \Delta v + q_1 v = 0 \quad \text{in} \quad Q,\\ &v|_{t=0} = u_2|_{t=0} = 0,\ \partial_t v|_{t=0} = \partial_t u_2|_{t=0}, \quad v|_\Sigma = u_2|_\Sigma.\end{aligned}$$

Setting $u := v - u_2$, we see

$$\begin{aligned}&\partial_t^2 u - \Delta u + q_1 u = (q_2-q_1)u_2 \quad \text{in} \quad Q,\\ &u|_{t=0} = 0,\ \partial_t u|_{t=0} = 0, \quad u|_\Sigma = 0.\end{aligned} \tag{3.19}$$

Noting that the inhomogeneous term $(q_2-q_1)u_2\in L^2(Q)$, due to the fact that $q_2-q_1\in L^2(0,T;L^{p_2}(\Omega))$ and $u_2\in L^\infty(0,T;H^1(\Omega))$. Hence, again using Proposition 2.1 gives that $u\in\mathcal{C}([0,T];H_0^1(\Omega))\cap\mathcal{C}^1([0,T];L^2(\Omega))$. Therefore, we have $u\in\mathcal{K}(Q)\cap H_\Box(Q)$. Combining this with $u_1\in\mathcal{K}(Q)\cap H_\Box(Q)$, we deduce that

$$(\partial_t u,-\nabla_x u),(\partial_t u_1,-\nabla_x u_1)\in H_{\rm div}(Q)=\{F\in L^2(Q;\mathbb{C}^{n+1}):\ {\rm div}_{(t,x)}F\in L^2(Q)\}.$$

Now, in view of [28, Lemma 2.2] we can multiply u_1 to the equation in (3.19) and apply Green formula to get

$$\begin{aligned}
&\int_Q (q_2-q_1)u_2u_1\,dxdt\\
&=\int_Q(\partial_t^2u-\Delta u+q_1u)u_1\,dxdt\\
&=\int_Q(\Box u+q_1u)u_1-(\Box u_1+q_1u_1)u\,dxdt\\
&=\int_Q\Box u u_1-\Box u_1 u\,dxdt\\
&=\langle(\partial_t u,-\nabla_x u)\cdot\mathbf{n},u_1\rangle_{H^{-\frac12}(\partial Q),H^{\frac12}(\partial Q)}-\langle(\partial_t u_1,-\nabla_x u_1)\cdot\mathbf{n},u\rangle_{H^{-\frac12}(\partial Q),H^{\frac12}(\partial Q)}
\end{aligned}\tag{3.20}$$

with $\mathbf{n}$ the outward unite normal vector to ∂Q. Since $C_{q_1}(0)=C_{q_2}(0)$ and $v|_{t=0}=u_2|_{t=0}=0$, we see $\partial_\nu u|_\Sigma=u|_{t=T}=\partial_t u|_{t=T}=0$, in addition to the boundary conditions of u in (3.19). Consequently, it follows from (3.20) that

$$\int_Q(q_2-q_1)u_2u_1\,dxdt=0.$$

Inserting the expressions of u_j $(j=1,2)$ given by (3.18) to the previous identity gives the relation

$$\begin{aligned}
0&=(2\pi)^{(n+1)/2}\int_Q q(t,x)e^{-i(t,x)\cdot\xi}\,dxdt+\tilde R_\lambda,\\
\tilde R_\lambda&:=(2\pi)^{(n+1)/2}\int_Q q(t,x)e^{-i(t,x)\cdot\xi}\left(-e^{-2i\lambda t}+e^{-i\lambda(t+x\cdot w)}\,R_{2,\lambda}(t,x)\right)\,dxdt\\
&\quad+\int_Q q(t,x)R_{1,\lambda}(t,x)\left(e^{i\lambda(t+x\cdot w)}-e^{i\lambda(-t+x\cdot w)}+R_{2,\lambda}(t,x)\right)\,dxdt
\end{aligned}$$

for all $\lambda>1$. Using the fact that $q\in L^2(Q)$ and applying the Riemann-Lebesgue lemma and (3.8), we deduce that

$$\begin{aligned}
&\left|\int_Q q(t,x)e^{-i(t,x)\cdot\xi}\left(-e^{-2i\lambda t}+e^{-i\lambda(t+x\cdot w)}\,R_{2,\lambda}(t,x)\right)\,dxdt\right|\to0,\\
&\left|\int_Q q(t,x)R_{1,\lambda}(t,x)\left(e^{i\lambda(t+x\cdot w)}-e^{i\lambda(-t+x\cdot w)}\right)\,dxdt\right|\to0
\end{aligned}$$

as $\lambda\to\infty$. On the other hand, by Cauchy-Schwarz inequality it holds that

$$\begin{aligned}
\left|\int_Q q(t,x)R_{1,\lambda}(t,x)R_{2,\lambda}(t,x)\,dxdt\right|&\leqslant||q\,R_{1,\lambda}||_{L^2(Q)}\,||R_{2,\lambda}||_{L^2(Q)}\\
&\leqslant C\,||q||_{L^{p_1}(0,T;L^{p_2}(\Omega))}\,||R_{1,\lambda}||_{L^\infty(0,T;H^1(\Omega))}\,||R_{2,\lambda}||_{L^\infty(0,T;L^2(\Omega))},
\end{aligned}$$

which tends to zero as $\lambda\to\infty$ due to the decaying behavior of $R_{j,\lambda}$ (see (3.8)) and estimate (3.9). Therefore, $|\tilde R_\lambda|\to0$ as $\lambda\to\infty$. It then follows that

$$\mathcal{F}q(\xi)=(2\pi)^{\frac{n+1}{2}}\int_{\mathbb{R}^{1+n}}q(t,x)e^{-i(t,x)\cdot\xi}dxdt=0.\tag{3.21}$$

Since $\omega\in\mathbb{S}^{n-1}$ is arbitrary chosen, we deduce that for any $\omega\in\mathbb{S}^{n-1}$ and any ξ lying in the hyperplane $\{\zeta\in\mathbb{R}^{1+n}:\ \zeta\cdot(1,-\omega)=0\}$ of $\mathbb{R}^{1+n}$, the Fourier transform $\mathcal{F}q$ is null at ξ. On the other hand, since

$q \in L^1(\mathbb{R}^{1+n})$ is compactly supported in $\overline{Q}$, we know that $\mathcal{F}q$ is a complex valued real-analytic function and it follows that $\mathcal{F}q = 0$. By inverse Fourier transform this yields the vanishing of q, which implies that $q_1 = q_2$ in Q.

To prove that the relation (1.4) implies $q_1 = q_2$, we shall consider $u_j \in \mathcal{K}(Q)$, $j = 1, 2$, solving (3.1), of the form (3.3)-(3.4), with $a_{1,1} = 1$, $a_{1,2} = -1$, $a_{2,1} = (2\pi)^{\frac{n+1}{2}} e^{-i(t,x)\cdot\xi}$, $a_{2,2} = 0$ and with condition (3.8)-(3.9) fulfilled. Then, by using the fact that $u_1(T, x) = 0$, $x \in \Omega$, and by repeating the above arguments, we deduce that $q_1 = q_2$. For brevity we omit the details.

We have proved so far that either of the conditions (1.3) and (1.4) implies $q_1 = q_2$. It remains to prove that for $T > \mathrm{Diam}(\Omega)$, the condition (1.5) implies $q_1 = q_2$.

3.3. **Proof of Theorem 1.1 with restriction at $t = 0$ and $t = T$.** In this section, we assume that $T > \mathrm{Diam}(\Omega)$ is fulfilled and we will show that (1.5) implies $q_1 = q_2$. For this purpose, we fix $\lambda > 1$, $\omega \in \mathbb{S}^{n-1}$ and $\varepsilon = \frac{T-\mathrm{Diam}(\Omega)}{4}$. We set also $\chi \in \mathcal{C}_0^\infty(-\varepsilon, T + \mathrm{Diam}(\Omega) + \varepsilon)$ satisfying $\chi = 1$ on $[0, T + \mathrm{Diam}(\Omega)]$ and $x_0 \in \overline{\Omega}$ such that

$$x_0 \cdot \omega = \inf_{x \in \overline{\Omega}} x \cdot \omega.$$

We introduce the solutions $u_j \in \mathcal{K}(Q)$, $j = 1, 2$, of (3.1), of the form (3.3)-(3.4), with

$$a_{1,1}(t,x) = (2\pi)^{\frac{n+1}{2}} \chi(t + (x - x_0)\cdot\omega) e^{-i(t,x)\cdot\xi}, \quad a_{1,2}(t,x) = -(2\pi)^{\frac{n+1}{2}} \chi((2T - t) + (x - x_0)\cdot\omega) e^{-i(2T-t,x)\cdot\xi},$$

$$a_{2,1}(t,x) = \chi(t + (x - x_0)\cdot\omega), \quad a_{2,2}(t,x) = -\chi(-t + (x - x_0)\cdot\omega)$$

and with condition (3.8)-(3.9) fulfilled. Then, one can check that $u_1(T, x) = u_2(0, x) = 0$, $x \in \Omega$, and repeating the arguments of the previous subsection we deduce that condition (1.5) implies the orthogonality identity

$$\int_Q q(t,x) u_2(t,x) u_1(t,x) dx dt = 0. \tag{3.22}$$

It remains to proves that this implies $q = 0$. Note that

$$\begin{aligned}\int_Q q(t,x)u_2(t,x)u_1(t,x)dxdt =&(2\pi)^{\frac{n+1}{2}} \int_{\mathbb{R}^{1+n}} q(t,x)\chi^2(t + (x - x_0)\cdot\omega)e^{-i(t,x)\cdot\xi} dxdt + \int_Q e^{-2i\lambda t} a_{1,1}a_{2,2} dxdt \\ &+ \int_Q e^{-2i\lambda(T-t)} a_{1,2}a_{2,1} dxdt + e^{-2i\lambda T} \int_Q a_{1,2}a_{2,2} dxdt + \int_Q Z_\lambda(t,x) dxdt,\end{aligned} \tag{3.23}$$

with

$$Z_\lambda = q(u_1 - R_{1,\lambda})R_{2,\lambda} + q(u_2 - R_{2,\lambda})R_{1,\lambda} + qR_{2,\lambda}R_{1,\lambda}.$$

In a similar way to the previous subsection, one can check that (3.8)-(3.9) imply that

$$\lim_{\lambda\to+\infty} \int_Q Z_\lambda dxdt = 0.$$

Moreover, the Riemann-Lebesgue lemma implies

$$\lim_{\lambda\to+\infty} \left(\int_Q e^{-2i\lambda t} a_{1,1}a_{2,2} dxdt + \int_Q e^{-2i\lambda(T-t)} a_{1,2}a_{2,1} dxdt \right) = 0.$$

In addition, using the fact that for $(t, x) \in Q$ we have

$$0 \leqslant t + (x - x_0)\cdot\omega \leqslant T + |x - x_0| \leqslant T + \mathrm{Diam}(\Omega),$$

we deduce that

$$q(t,x)\chi^2(t + (x - x_0)\cdot\omega) = q(t,x), \quad (t,x) \in \mathbb{R}^{1+n}$$

and that

$$(2\pi)^{\frac{n+1}{2}} \int_{\mathbb{R}^{1+n}} q(t,x)\chi^2(t + (x - x_0)\cdot\omega)e^{-i(t,x)\cdot\xi} dxdt = \mathcal{F}q(\xi).$$

Thus, repeating the arguments of the previous subsection we can deduce that $q_1 = q_2$ provided that

$$\int_Q a_{1,2}a_{2,2}dxdt = 0. \tag{3.24}$$

Since $a_{2,2}(t,x) = -\chi(-t+(x-x_0)\cdot\omega)$ and $a_{1,2} = -(2\pi)^{\frac{n+1}{2}}\chi((2T-t)+(x-x_0)\cdot\omega)e^{-i(t,x)\cdot\xi}$, we deduce that

$$\mathrm{supp}(a_{2,2}) \subset \{(t,x)\in\mathbb{R}^{1+n}:\ (x-x_0)\cdot\omega \geqslant t-\varepsilon\},$$

$$\mathrm{supp}(a_{1,2}) \subset \{(t,x)\in\mathbb{R}^{1+n}:\ 2T-t+(x-x_0)\cdot\omega \leqslant T+\mathrm{Diam}(\Omega)+\varepsilon\}.$$

But, for any $(t,x)\in\{(t,x)\in\mathbb{R}^{1+n}:\ (x-x_0)\cdot\omega \geqslant t-\varepsilon\}$, one can check that

$$2T-t+(x-x_0)\cdot\omega \geqslant 2T-\varepsilon = T+\mathrm{Diam}(\Omega)+3\varepsilon > T+\mathrm{Diam}(\Omega)+\varepsilon.$$

Therefore, we have

$$\{(t,x)\in\mathbb{R}^{1+n}:\ (x-x_0)\cdot\omega \geqslant t-\varepsilon\}\cap\{(t,x)\in\mathbb{R}^{1+n}:\ 2T-t+(x-x_0)\cdot\omega \leqslant T+\mathrm{Diam}(\Omega)+\varepsilon\}=\emptyset$$

and by the same way that $\mathrm{supp}(a_{2,2})\cap\mathrm{supp}(a_{1,2})=\emptyset$. This implies (3.24) and by the same way that $q_1 = q_2$. Thus, the proof of Theorem 1.1 is completed.

4. Proof of Theorem 1.2

In the previous section we have seen that the oscillating geometric optics solutions (3.2) can be used for the recovery of some general singular time-dependent potential. We have even proved that, by adding a second term, we can restrict the data on the bottom $t = 0$ and top $t = T$ of Q while avoiding a "reflection". Nevertheless, as mentioned in the introduction, it is not clear how one can adapt this approach to restrict data on the lateral boundary Σ without requiring additional smoothness or geometrical assumptions. In this section, we use a different strategy for restricting the data at Σ. Namely, we replace the oscillating GO solutions (3.2) by exponentially growing and decaying solutions, of the form (1.9), in order to restrict the data on Σ by mean of a Carleman estimate. In this section, we assume that $q_1, q_2 \in L^p(Q)$, with $p > n+1$, and we will prove that (1.6) implies $q_1 = q_2$. For this purpose, we will start with the construction of solutions of (1.1) taking the form (1.9). Then we will show Carleman estimates for unbounded potentials and we will complete the proof of Theorem 1.2.

4.1. **Geometric optics solutions for Theorem 1.2.** Let $\omega \in \mathbb{S}^{n-1}$ and let $\xi \in \mathbb{R}^{1+n}$ be such that $\xi\cdot(1,-\omega) = 0$. In this section we consider exponentially decaying solutions $u_1 \in H^1(Q)$ of the equation $(\partial_t^2 - \Delta_x + q_1)u_1 = 0$ in Q taking the form

$$u_1(t,x) = e^{-\lambda(t+x\cdot\omega)}(e^{-i(t,x)\cdot\xi} + w_1(t,x)), \tag{4.1}$$

and exponentially growing solution $u_2 \in H^1(Q)$ of the equation $(\partial_t^2 - \Delta_x + q_2)u_2 = 0$ in Q taking the form

$$u_2(t,x) = e^{\lambda(t+x\cdot\omega)}(1 + w_2(t,x)) \tag{4.2}$$

where $\lambda > 1$ and the term $w_j \in H^1(Q)$, $j = 1,2$, satisfies

$$\|w_j\|_{H^1(Q)} + \lambda\|w_j\|_{L^2(Q)} \leqslant C, \tag{4.3}$$

with C independent of λ. We summarize these results in the following way.

Proposition 4.1. *There exists $\lambda_1 > 1$ such that for $\lambda > \lambda_1$ we can find a solution $u_1 \in H^1(Q)$ of $\Box u_1 + q_1u_1 = 0$ in Q taking the form (4.1) with $w_1 \in H^1(Q)$ satisfying (4.3) for $j = 1$.*

Proposition 4.2. *There exists $\lambda_2 > \lambda_1$ such that for $\lambda > \lambda_2$ we can find a solution $u_2 \in H^1(Q)$ of $\Box u_2 + q_2u_2 = 0$ in Q taking the form (4.2) with $w_2 \in H^1(Q)$ satisfying (4.3) for $j = 2$.*

We start by considering Proposition 4.1. To build solutions $u_1 \in H^1(Q)$ of the form (4.1), we first recall some preliminary tools and a suitable Carleman estimate in Sobolev space of negative order borrowed from [33]. For all $m \in \mathbb{R}$, we introduce the space $H^m_\lambda(\mathbb{R}^{1+n})$ defined by

$$H^m_\lambda(\mathbb{R}^{1+n}) = \{u \in \mathcal{S}'(\mathbb{R}^{1+n}) : \ (|(\tau,\xi)|^2 + \lambda^2)^{\frac{m}{2}}\,\hat{u} \in L^2(\mathbb{R}^{1+n})\},$$

with the norm

$$\|u\|^2_{H^m_\lambda(\mathbb{R}^{1+n})} = \int_{\mathbb{R}}\int_{\mathbb{R}^n} (|(\tau,\xi)|^2 + \lambda^2)^m |\hat{u}(\tau,\xi)|^2 d\xi d\tau.$$

Note that here we consider these spaces with $\lambda > 1$ and, for $\lambda = 1$, one can check that $H^m_\lambda(\mathbb{R}^{1+n}) = H^m(\mathbb{R}^{1+n})$. Here for all tempered distribution $u \in \mathcal{S}'(\mathbb{R}^{1+n})$, we denote by $\hat{u}$ the Fourier transform of u. We fix the weighted operator

$$P_{\omega,\pm\lambda} := e^{\mp\lambda(t+x\cdot\omega)}\Box e^{\pm\lambda(t+x\cdot\omega)} = \Box \pm 2\lambda(\partial_t - \omega\cdot\nabla_x)$$

and we recall the following Carleman estimate

Lemma 4.1. (Lemma 5.1, [33]) *There exists $\lambda_1' > 1$ such that*

$$\|v\|_{L^2(\mathbb{R}^{1+n})} \leqslant C\,\|P_{\omega,\lambda}v\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})}, \quad v \in \mathcal{C}^\infty_0(Q), \quad \lambda > \lambda_1', \tag{4.4}$$

with $C > 0$ independent of v and λ.

From this result we can deduce the Carleman estimate

Lemma 4.2. *Let $p_1 \in (n+1,+\infty)$, $p_2 \in (n,+\infty)$ and $q \in L^{p_1}(Q) \cup L^\infty(0,T;L^{p_2}(\Omega))$. Then, there exists $\lambda_1'' > \lambda_1'$ such that*

$$\|v\|_{L^2(\mathbb{R}^{1+n})} \leqslant C\,\|P_{\omega,\lambda}v + qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})}, \quad v \in \mathcal{C}^\infty_0(Q), \quad \lambda > \lambda_1'', \tag{4.5}$$

with $C > 0$ independent of v and λ.

Proof. We start by considering the case $q \in L^{p_1}(Q)$. Note first that

$$\|P_{\omega,\lambda}v + qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})} \geqslant \|P_{\omega,\lambda}v\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})} - \|qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})}. \tag{4.6}$$

On the other hand, fixing

$$r = \frac{n+1}{p_1}, \quad \frac{1}{p_3} = \frac{1}{2} - \frac{1}{p_1} = \frac{n+1-2r}{2(n+1)},$$

by the Sobolev embedding theorem we deduce that

$$\begin{aligned}\|qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})} &\leqslant \lambda^{r-1}\,\|qv\|_{H^{-r}_\lambda(\mathbb{R}^{n+1})}\\ &\leqslant \lambda^{r-1}\,\|qv\|_{H^{-r}(\mathbb{R}^{n+1})}\\ &\leqslant C\lambda^{r-1}\,\|qv\|_{L^{\frac{p_3}{p_3-1}}(Q)}\end{aligned}$$

Combining this with the fact that

$$\frac{p_3-1}{p_3} = 1 - \frac{1}{p_3} = \frac{1}{2} + \frac{1}{p_1},$$

we deduce from the Hölder inequality that

$$\|qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})} \leqslant C\lambda^{r-1}\,\|q\|_{L^{p_1}(Q)}\,\|v\|_{L^2(Q)}.$$

Thus, applying (4.4) and (4.6), we deduce (4.5) for $\lambda > 1$ sufficiently large. Now let us consider the case $q \in L^\infty(0,T;L^{p_2}(\Omega))$. Note first that

$$\|qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})} \leqslant \|qv\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))}.$$

Therefore, by repeating the above arguments, we obtain

$$\|qv\|_{H^{-1}_\lambda(\mathbb{R}^{1+n})} \leqslant C\lambda^{\frac{n}{p_2}-1}\,\|q\|_{L^\infty(0,T;L^{p_2}(\Omega))}\,\|v\|_{L^2(Q)}$$

which implies (4.5) for $\lambda > 1$ sufficiently large. Combining these two results, one can find $\lambda_1'' > \lambda_1'$ such that (4.5) is fulfilled. $\square$

Using this new carleman estimate we are now in position to complete the proof of Proposition 4.1.

Proof of Proposition 4.1. Note first that

$$\begin{aligned}\Box(e^{-\lambda(t+x\cdot\omega)}e^{-i\xi\cdot(t,x)}) &= [2i\lambda(1,-\omega)\cdot\xi e^{-i\xi\cdot(t,x)} + \Box e^{-i\xi\cdot(t,x)}]e^{-\lambda(t+x\cdot\omega)}\\ &= [\Box e^{-i\xi\cdot(t,x)}]e^{-\lambda(t+x\cdot\omega)},\end{aligned}$$

$$\Box(e^{-\lambda(t+x\cdot\omega)}w_1) = e^{-\lambda(t+x\cdot\omega)}P_{\omega,-\lambda}w_1.$$

Therefore, we need to consider $w_1 \in H^1(Q)$ a solution of

$$P_{\omega,-\lambda}w_1 + q_1w_1 = -e^{\lambda(t+x\cdot\omega)}(\Box + q_1)(e^{-\lambda(t+x\cdot\omega)}e^{-i\xi\cdot(t,x)}) = -(\Box + q_1)e^{-i\xi\cdot(t,x)} = F \tag{4.7}$$

and satisfying (4.3) for $j = 1$. For this purpose, we will use estimate (4.5). From now on, we fix $\lambda_1 = \lambda_1''$. Applying the Carleman estimate (4.5), we define the linear form $\mathcal{L}$ on $\{P_{\omega,\lambda}z + q_1z :\ z \in \mathcal{C}_0^\infty(Q)\}$, considered as a subspace of $H_\lambda^{-1}(\mathbb{R}^{1+n})$ by

$$\mathcal{L}(P_{\omega,\lambda}z + q_1z) = \int_Q zF\,dxdt, \quad z \in \mathcal{C}_0^\infty(Q).$$

Then, (4.5) implies

$$|\mathcal{L}(P_{\omega,\lambda}z + q_1z)| \leqslant C\,\|F\|_{L^2(Q)}\,\|P_{\omega,\lambda}z + q_1z\|_{H_\lambda^{-1}(\mathbb{R}^{1+n})}, \quad z \in \mathcal{C}_0^\infty(Q).$$

Thus, by the Hahn Banach theorem we can extend $\mathcal{L}$ to a continuous linear form on $H_\lambda^{-1}(\mathbb{R}^{1+n})$ still denoted by $\mathcal{L}$ and satisfying $\|\mathcal{L}\| \leqslant C\,\|F\|_{L^2(Q)}$. Therefore, there exists $w_1 \in H_\lambda^1(\mathbb{R}^{1+n})$ such that

$$\langle h, w_1\rangle_{H_\lambda^{-1}(\mathbb{R}^{1+n}),H_\lambda^1(\mathbb{R}^{1+n})} = \mathcal{L}(h), \quad h \in H_\lambda^{-1}(\mathbb{R}^{1+n}).$$

Choosing $h = P_{\omega,\lambda}z + q_1z$ with $z \in \mathcal{C}_0^\infty(Q)$ proves that w_1 satisfies $P_{\omega,-\lambda}w_1 + q_1w_1 = F$ in Q. Moreover, we have $\|w_1\|_{H_\lambda^1(\mathbb{R}^{1+n})} \leqslant \|\mathcal{L}\| \leqslant C\,\|F\|_{L^2(Q)}$. This proves that w_1 fufills (4.3) which completes the proof of the proposition. $\square$

Now let us consider the construction of the exponentially growing solutions given by Proposition 4.2. Combining [33, Lemma 5.4] with the arguments used in Lemma 4.2 we obtain the Carleman estimate.

Lemma 4.3. *There exists $\lambda_2' > 0$ such that for $\lambda > \lambda_2'$, we have*

$$\|v\|_{L^2(\mathbb{R}^{1+n})} \leqslant C\,\|P_{\omega,-\lambda}v + q_2v\|_{H_\lambda^{-1}(\mathbb{R}^{1+n})}, \quad v \in \mathcal{C}_0^\infty(Q), \quad \lambda > \lambda_3', \tag{4.8}$$

with $C > 0$ independent of v and λ.

In a similar way to Proposition 4.1, we can complete the proof of Proposition 4.2 by applying estimate (4.8).

4.2. **Carleman estimates for unbounded potential.** This subsection is devoted to the proof of a Carleman estimate similar to [33, Theorem 3.1]. More precisely, we consider the following estimate.

Theorem 4.1. *Let $p_1 \in (n+1, +\infty)$, $p_2 \in (n, +\infty)$ and assume that $q \in L^{p_1}(Q)$ (resp $q \in L^\infty(0,T;L^{p_2}(\Omega))$) and $u \in \mathcal{C}^2(\overline{Q})$. If u satisfies the condition*

$$u_{|\Sigma} = 0, \quad u_{|t=0} = \partial_t u_{|t=0} = 0, \tag{4.9}$$

then there exists $\lambda_3>\lambda_2$ depending only on Ω, T and $M\geqslant\|q\|_{L^{p_1}(Q)}$ (resp $M\geqslant\|q\|_{L^\infty(0,T;L^{p_2}(\Omega))}$) such that the estimate

$$\begin{aligned}&\lambda\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|\partial_t u(T,x)\right|^2dx+\lambda\int_{\Sigma_{+,\omega}}e^{-2\lambda(t+\omega\cdot x)}\left|\partial_\nu u\right|^2\left|\omega\cdot\nu(x)\right|d\sigma(x)dt\\&+\int_Q e^{-2\lambda(t+\omega\cdot x)}[\lambda^2\left|u\right|^2+\left|\partial_t u\right|^2+\left|\nabla_x u\right|^2]dxdt\\&\leqslant C\left(\int_Q e^{-2\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x+q)u\right|^2dxdt+\lambda^3\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|u(T,x)\right|^2dx\right)\\&\quad+C\left(\lambda\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|\nabla_x u(T,x)\right|^2dx+\lambda\int_{\Sigma_{-,\omega}}e^{-2\lambda(t+\omega\cdot x)}\left|\partial_\nu u\right|^2\left|\omega\cdot\nu(x)\right|d\sigma(x)dt\right)\end{aligned}\tag{4.10}$$

holds true for $\lambda\geqslant\lambda_3$ with C depending only on Ω, T and M.

Proof. Since the proof of this result is similar for $q\in L^{p_1}(Q)$ or $q\in L^\infty(0,T;L^{p_2}(\Omega))$, we assume without lost of generality that $q\in L^{p_1}(Q)$. Note first that for $q=0$, (4.9) follows from [33, Theorem 3.1]. On the other hand, we have

$$\left\|e^{-\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x+q)u\right|\right\|_{L^2(Q)}\geqslant\left\|e^{-\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x)u\right|\right\|_{L^2(Q)}-\left\|e^{-\lambda(t+\omega\cdot x)}qu\right\|_{L^2(Q)}$$

and by the Hölder inequality we deduce that

$$\left\|e^{-\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x+q)u\right|\right\|_{L^2(Q)}\geqslant\left\|e^{-2\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x)u\right|\right\|_{L^2(Q)}-\|q\|_{L^{p_1}(Q)}\left\|e^{-\lambda(t+\omega\cdot x)}u\right\|_{L^{p_3}(Q)}$$

with $p_3=\frac{2p_1}{p_1-2}$. Now fix $s:=\frac{n+1}{p_1}\in(0,1)$ and notice that

$$\frac{1}{p_3}=\frac{n+1-2s}{2(n+1)}.$$

Thus, by the Sobolev embedding theorem, we have

$$\left\|e^{-\lambda(t+\omega\cdot x)}u\right\|_{L^{p_3}(Q)}\leqslant C\left\|e^{-\lambda(t+\omega\cdot x)}u\right\|_{H^s(Q)}$$

and by interpolation we deduce that

$$\begin{aligned}\left\|e^{-\lambda(t+\omega\cdot x)}u\right\|_{L^{p_3}(Q)}&\leqslant C\left(\int_Q e^{-2\lambda(t+\omega\cdot x)}(\lambda^2|u|^2+|\partial_t u|^2+|\nabla_x u|^2dxdt\right)^{\frac{s}{2}}\left\|e^{-\lambda(t+\omega\cdot x)}u\right\|_{L^2(Q)}^{1-s}\\&\leqslant C\lambda^{s-1}\left(\int_Q e^{-2\lambda(t+\omega\cdot x)}(\lambda^2|u|^2+|\partial_t u|^2+|\nabla_x u|^2dxdt\right)^{\frac12}.\end{aligned}$$

On the other hand, in view of [33, Theorem 3.1], there exists $\lambda_3'>1$ such that, for $\lambda>\lambda_3'$, we have

$$\begin{aligned}&\int_Q e^{-2\lambda(t+\omega\cdot x)}(\lambda^2|u|^2+|\partial_t u|^2+|\nabla_x u|^2)dxdt\\&\leqslant C\left(\int_Q e^{-2\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x)u\right|^2dxdt+\lambda^3\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|u(T,x)\right|^2dx\right)\\&\quad+C\left(\lambda\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|\nabla_x u(T,x)\right|^2dx+\lambda\int_{\Sigma_{-,\omega}}e^{-2\lambda(t+\omega\cdot x)}\left|\partial_\nu u\right|^2\left|\omega\cdot\nu(x)\right|d\sigma(x)dt\right).\end{aligned}$$

Thus, we get

$$\begin{aligned}&\int_Q e^{-2\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x+q)u\right|^2dxdt+\lambda^3\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|u(T,x)\right|^2dx\\&\quad+\lambda\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|\nabla_x u(T,x)\right|^2dx+\lambda\int_{\Sigma_{-,\omega}}e^{-2\lambda(t+\omega\cdot x)}\left|\partial_\nu u\right|^2\left|\omega\cdot\nu(x)\right|d\sigma(x)dt\\&\geqslant\frac12\int_Q e^{-2\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x)u\right|^2dxdt+\lambda^3\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|u(T,x)\right|^2dx\\&\quad+\lambda\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|\nabla_x u(T,x)\right|^2dx+\lambda\int_{\Sigma_{-,\omega}}e^{-2\lambda(t+\omega\cdot x)}\left|\partial_\nu u\right|^2\left|\omega\cdot\nu(x)\right|d\sigma(x)dt\\&\quad-C\left\|q\right\|_{L^{p_1}(Q)}^2\lambda^{2(s-1)}\left(\int_Q e^{-2\lambda(t+\omega\cdot x)}\left|(\partial_t^2-\Delta_x)u\right|^2dxdt+\lambda^3\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|u(T,x)\right|^2dx\right)\\&\quad-C\left\|q\right\|_{L^{p_1}(Q)}^2\lambda^{2(s-1)}\left(\lambda\int_\Omega e^{-2\lambda(T+\omega\cdot x)}\left|\nabla_x u(T,x)\right|^2dx+\lambda\int_{\Sigma_{-,\omega}}e^{-2\lambda(t+\omega\cdot x)}\left|\partial_\nu u\right|^2\left|\omega\cdot\nu(x)\right|d\sigma(x)dt\right).\end{aligned}$$

Therefore, fixing λ sufficiently large and applying [33, Theorem 3.1] with $a=q=0$ we deduce (4.10).

□

Remark 4.1. *Note that, by density,* (4.10) *remains true for* $u \in \mathcal{C}^1([0,T];L^2(\Omega)) \cap \mathcal{C}([0,T];H^1(\Omega))$ *satisfying* (4.9), $(\partial_t^2 - \Delta_x)u \in L^2(Q)$ *and* $\partial_\nu u \in L^2(\Sigma)$.

4.3. **Completion of the proof of Theorem 1.2.** This subsection is devoted to the proof of Theorem 1.2. From now on, we set $q = q_2 - q_1$ on Q and we assume that $q = 0$ on $\mathbb{R}^{1+n} \setminus Q$. For all $\theta \in \mathbb{S}^{n-1}$ and all $r > 0$, we set

$$\partial\Omega_{+,r,\theta} = \{x \in \partial\Omega : \ \nu(x)\cdot\theta > r\}, \quad \partial\Omega_{-,r,\theta} = \{x \in \partial\Omega : \ \nu(x)\cdot\theta \leqslant r\}$$

and $\Sigma_{\pm,r,\theta} = (0,T)\times\partial\Omega_{\pm,r,\theta}$. Here and in the remaining of this text we always assume, without mentioning it, that θ and r are chosen in such way that $\partial\Omega_{\pm,r,\pm\theta}$ contain a non-empty relatively open subset of $\partial\Omega$. Without lost of generality we assume that there exists $\varepsilon > 0$ such that for all $\omega \in \{\theta \in \mathbb{S}^{n-1} : |\theta - \omega_0| \leqslant \varepsilon\}$ we have $\partial\Omega_{-,\varepsilon,\omega} \subset V'$. In order to prove Theorem 1.2, we will use the Carleman estimate stated in Theorem 4.1. Let $\lambda > \lambda_3$ and fix $\omega \in \{\theta \in \mathbb{S}^{n-1} : |\theta - \omega_0| \leqslant \varepsilon\}$. According to Proposition 4.1, we can introduce

$$u_1(t,x) = e^{-\lambda(t+x\cdot\omega)}(e^{-i(t,x)\cdot\xi} + w_1(t,x)), \ (t,x) \in Q,$$

where $u_1 \in H^1(Q)$ satisfies $\partial_t^2 u_1 - \Delta_x u_1 + q_1 u_1 = 0$, $\xi\cdot(1,-\omega) = 0$ and w_1 satisfies (4.3) for $j = 1$. Moreover, in view of Proposition 4.2, we consider $u_2 \in H^1(Q)$ a solution of $\partial_t^2 u_2 - \Delta_x u_2 + q_2 u_2 = 0$, of the form

$$u_2(t,x) = e^{\lambda(t+x\cdot\omega)}(1 + w_2(t,x)), \quad (t,x) \in Q,$$

where w_2 satisfies (4.3) for $j = 2$. In view of Proposition 2.2, there exists a unique weak solution $z_1 \in H_\Box(Q)$ of

$$\begin{cases} \partial_t^2 z_1 - \Delta_x z_1 + q_1 z_1 = 0 & \text{in } Q, \\ \tau_0 z_1 = \tau_0 u_2. \end{cases} \tag{4.11}$$

Then, $u = z_1 - u_2$ solves

$$\begin{cases} \partial_t^2 u - \Delta_x u + q_1 u = (q_2 - q_1)u_2 & \text{in } Q, \\ u(0,x) = \partial_t u(0,x) = 0 & \text{on } \Omega, \\ u = 0 & \text{on } \Sigma. \end{cases} \tag{4.12}$$

Since $u_2 \in H^1(Q)$, by the Sobolev embedding theorem we have $(q_2 - q_1)u_2 \in L^2(Q)$. Thus, repeating the arguments of Theorem 1.1, we derive the formula (3.20). On the other hand, we have $u_{|t=0} = \partial_t u_{|t=0} = u_{|\Sigma} = 0$ and condition (1.6) implies that $u_{|t=T} = \partial_\nu u_{|V} = 0$. In addition, in view of [36, Theorem 2.1], we have $\partial_\nu u \in L^2(\Sigma)$. Combining this with the fact that $u \in \mathcal{C}^1([0,T];L^2(\Omega))$ and $u_1 \in H^1(Q) \subset H^1(0,T;L^2(\Omega))$, we obtain

$$\int_Q q u_2 u_1 dx dt = -\int_{\Sigma\setminus V} \partial_\nu u u_1 d\sigma(x)dt + \int_\Omega \partial_t u(T,x) u_1(T,x) dx. \tag{4.13}$$

Applying the Cauchy-Schwarz inequality to the first expression on the right hand side of this formula and using the fact that $(\Sigma \setminus V) \subset \Sigma_{+,\varepsilon,\omega}$, we get

$$\begin{aligned}\left|\int_{\Sigma\setminus V} \partial_\nu u u_1 d\sigma(x)dt\right| &\leqslant \int_{\Sigma_{+,\varepsilon,\omega}} \left|\partial_\nu u e^{-\lambda(t+\omega\cdot x)}(e^{-i(t,x)\cdot\xi} + w_1)\right| d\sigma(x)dt \\ &\leqslant C(1 + \|w_1\|_{L^2(\Sigma)}) \left(\int_{\Sigma_{+,\varepsilon,\omega}} \left|e^{-\lambda(t+\omega\cdot x)}\partial_\nu u\right|^2 d\sigma(x)dt\right)^{\frac{1}{2}},\end{aligned}$$

for some C independent of λ. On the other hand, one can check that

$$\|w_1\|_{L^2(\Sigma))} \leqslant C\, \|w_1\|_{H^1(Q)}.$$

Combining this with (4.3), we obtain

$$\left|\int_{\Sigma\setminus V} \partial_\nu u u_1 d\sigma(x)dt\right| \leqslant C \left(\int_{\Sigma_{+,\varepsilon,\omega}} \left|e^{-\lambda(t+\omega\cdot x)}\partial_\nu u\right|^2 d\sigma(x)dt\right)^{\frac{1}{2}}.$$

In the same way, we have

$$\begin{aligned}\left|\int_\Omega \partial_t u(T,x)u_1(T,x)dx\right| &\leqslant \int_\Omega \left|\partial_t u(T,x)e^{-\lambda(T+\omega\cdot x)}\left(e^{-i(T,x)\cdot\xi}+w_1(T,x)\right)\right|dx\\ &\leqslant C\left(\int_\Omega \left|e^{-\lambda(T+\omega\cdot x)}\partial_t u(T,x)\right|^2 dx\right)^{\frac12}.\end{aligned}$$

Combining these estimates with the Carleman estimate (4.10) and applying the fact that $u_{|t=T}=\partial_\nu u_{|\Sigma_{-,\omega}}=0$, $\partial\Omega_{+,\varepsilon,\omega}\subset\partial\Omega_{+,\omega}$, we find

$$\begin{aligned}&\left|\int_Q qu_2u_1dxdt\right|^2\\ &\leqslant C\left(\int_{\Sigma_{+,\varepsilon,\omega}}\left|e^{-\lambda(t+\omega\cdot x)}\partial_\nu u\right|^2 d\sigma(x)dt+\int_\Omega\left|e^{-\lambda(T+\omega\cdot x)}\partial_t u(T,x)\right|^2dx\right)\\ &\leqslant \varepsilon^{-1}C\lambda^{-1}\left(\lambda\int_{\Sigma_{+,\omega}}\left|e^{-\lambda(t+\omega\cdot x)}\partial_\nu u\right|^2\omega\cdot\nu(x)d\sigma(x)dt+\lambda\int_\Omega\left|e^{-\lambda(T+\omega\cdot x)}\partial_t u(T,x)\right|^2dx\right)\\ &\leqslant \varepsilon^{-1}C\lambda^{-1}\left(\int_Q\left|e^{-\lambda(t+\omega\cdot x)}(\partial_t^2-\Delta_x+q_1)u\right|^2dxdt\right)\\ &\leqslant \varepsilon^{-1}C\lambda^{-1}\left(\int_Q\left|e^{-\lambda(t+\omega\cdot x)}qu_2\right|^2dxdt\right)\\ &\leqslant \varepsilon^{-1}C\lambda^{-1}\left(\int_Q|q(1+w_2)|^2dxdt\right).\end{aligned}\tag{4.14}$$

Here $C>0$ stands for some generic constant independent of λ. On the other hand, in a similar way to Lemma 4.2, combining the Hölder inequality and the Sobolev embedding theorem we get

$$\int_Q |q(1+w_2)|^2dxdt\leqslant C\left\|q\right\|^2_{L^p(Q)}\left\|(1+w_2)\right\|^2_{H^{\frac{n+1}{p}}(Q)}\leqslant C\left\|q\right\|^2_{L^p(Q)}(1+\left\|w_2\right\|_{H^1(Q)})^2.$$

Combining this with (4.3) and (4.14), we obtain

$$\left|\int_Q qu_2u_1dxdt\right|\leqslant C\lambda^{-1/2}.$$

It follows

$$\lim_{\lambda\to+\infty}\int_Q qu_2u_1dxdt=0.\tag{4.15}$$

Moreover, (4.1)-(4.3) imply

$$\int_Q qu_2u_1dxdt=\int_{\mathbb{R}^{1+n}}q(t,x)e^{-i\xi\cdot(t,x)}dxdt+\int_Q W_\lambda(t,x)dxdt,$$

with

$$\int_Q |W_\lambda(t,x)|dxdt\leqslant C\lambda^{-1}.$$

Combining this with (4.15), for all $\omega\in\{y\in\mathbb{S}^{n-1}:|y-\omega_0|\leqslant\varepsilon\}$ and all $\xi\in(1,-\omega)^\perp:=\{\zeta\in\mathbb{R}^{1+n}:\zeta\cdot(1,-\omega)=0\}$, the Fourier transform $\mathcal{F}(q)$ of q satisfies $\mathcal{F}(q)(\xi)=0$. On the other hand, since $q\in L^1(\mathbb{R}^{1+n})$ is supported on $\overline{Q}$ which is compact, $\mathcal{F}(q)$ is a complex valued real-analytic function and it follows that $q=0$ and $q_1=q_2$. This completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

Let us first remark that, in contrast to Theorem 1.1, in Theorem 1.2 we do not restrict the data to solutions of (1.1) satisfying $u_{|t=0} = 0$. In this section we will show Theorem 1.3 by combining the restriction on the bottom $t = 0$, the top $t = T$ of Q stated in Theorem 1.1 with the restriction on the lateral boundary Σ stated in Theorem 1.2. From now on, we fix $q_1, q_2 \in L^\infty(0,T; L^p(\Omega))$, $p > n$, and we will show that condition (1.7) implies $q_1 = q_2$. For this purpose we still consider exponentially growing and decaying GO solutions close to those of the previous subsection, but this time we need to take into account the constraint $u_2(0,x) = 0$ required in Theorem 1.3. For this purpose, we will consider a different construction comparing to the one of the previous section which will follow from a Carleman estimate in negative order Sobolev space only with respect to the space variable.

5.1. **Carleman estimate in negative Sobolev space for Theorem 1.3.** In this subsection we will derive a Carleman estimate in negative order Sobolev space which will be one of the main tool for the construction of exponentially growing solutions u_2 of (3.1) taking the form

$$u_2(t,x) = e^{\lambda(t+x\cdot\omega)}(1 + w_2(t,x)) - e^{\lambda(-t+x\cdot\omega)}, \tag{5.1}$$

with the restriction $\tau_{0,2}u_2 = 0$ (recall that for $v \in \mathcal{C}^\infty(\overline{Q})$, $\tau_{0,2}v = v_{|t=0}$). In a similar way to the previous section, for all $m \in \mathbb{R}$, we introduce the space $H^m_\lambda(\mathbb{R}^n)$ defined by

$$H^m_\lambda(\mathbb{R}^n) = \{u \in \mathcal{S}'(\mathbb{R}^n) : \ (|\xi|^2 + \lambda^2)^{\frac{m}{2}}\hat{u} \in L^2(\mathbb{R}^n)\},$$

with the norm

$$\|u\|^2_{H^m_\lambda(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (|\xi|^2 + \lambda^2)^m |\hat{u}(\xi)|^2 d\xi.$$

In order to construct solutions u_2 of the form (4.2) and satisfying $\tau_{0,2}u_2 = 0$, instead of the Carleman estimate (4.8), we consider the following.

Theorem 5.1. *There exists $\lambda_2' > 0$ such that for $\lambda > \lambda_2'$ and for all $v \in \mathcal{C}^2([0,T]; \mathcal{C}^\infty_0(\Omega))$ satisfying*

$$v(T,x) = \partial_t v(T,x) = v(0,x) = 0, \quad x \in \mathbb{R}^n, \tag{5.2}$$

we have

$$\|v\|_{L^2((0,T)\times\mathbb{R}^n)} \leqslant C \left\| P_{\omega,-\lambda}v + q_2 v \right\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))}, \tag{5.3}$$

with $C > 0$ independent of v and λ.

In order to prove this theorem, we start by recalling the following intermediate tools. From now on, for $m \in \mathbb{R}$ and $\xi \in \mathbb{R}^n$, we set

$$\langle \xi, \lambda \rangle = (|\xi|^2 + \lambda^2)^{\frac{1}{2}}$$

and $\langle D_x, \lambda \rangle^m u$ defined by

$$\langle D_x, \lambda \rangle^m u = \mathcal{F}^{-1}(\langle \xi, \lambda \rangle^m \mathcal{F}u).$$

For $m \in \mathbb{R}$ we define also the class of symbols

$$S^m_\lambda = \{c_\lambda \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n) : \ |\partial_x^\alpha \partial_\xi^\beta c_\lambda(x,\xi)| \leqslant C_{\alpha,\beta} \langle \xi, \lambda \rangle^{m-|\beta|}, \ \alpha, \beta \in \mathbb{N}^n\}.$$

Following [24, Theorem 18.1.6], for any $m \in \mathbb{R}$ and $c_\lambda \in S^m_\lambda$, we define $c_\lambda(x, D_x)$, with $D_x = -i\nabla_x$, by

$$c_\lambda(x, D_x)u(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} c_\lambda(x,\xi)\hat{u}(\xi)e^{ix\cdot\xi}d\xi.$$

For all $m \in \mathbb{R}$, we define $OpS^m_\lambda := \{c_\lambda(x, D_x) : \ c_\lambda \in S^m_\lambda\}$ and for $m = -\infty$ we set

$$OpS^{-\infty}_\lambda := \bigcap_{m\in\mathbb{R}} OpS^m_\lambda.$$

Now let us consider the following intermediate result.

Lemma 5.1. *There exists $\lambda_2''>0$ such that for $\lambda>\lambda_2''$ and for all $v\in\mathcal{C}^2([0,T];\mathcal{C}_0^\infty(\Omega))$ satisfying (5.2), we have*

$$\|v\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))}\leqslant C\,\|P_{\omega,-\lambda}v\|_{L^2((0,T)\times\mathbb{R}^n))}\,, \tag{5.4}$$

with $C>0$ independent of v and λ.

Proof. Consider $w(t,x)=v(T-t,x)$ and note that according to (5.2), we have $w\in\mathcal{C}^2([0,T];\mathcal{C}_0^\infty(\Omega))$ and

$$w(0,x)=\partial_t w(0,x)=w(T,x)=0,\quad x\in\mathbb{R}^n.$$

Therefore, in a similar way to the proof of [32, Lemma 4.1], one can check that

$$\int_Q |P_{-\omega,\lambda}w|^2dxdt\geqslant\int_Q|\Box w|^2dxdt+c\lambda^2\int_Q|w|^2dxdt, \tag{5.5}$$

with $c>0$ independent of w and λ. Now, recalling that w solves

$$\left\{\begin{array}{rl}\partial_t^2w-\Delta_x w=\Box w, & (t,x)\in Q,\\ w(0,x)=0,\quad \partial_t w(0,x)=0, & x\in\Omega\\ w(t,x)=0 & (t,x)\in\Sigma,\end{array}\right.$$

we deduce that

$$\int_Q|\nabla_x w|^2dxdt\leqslant C\int_Q|\Box w|^2dxdt,$$

where C depends only on T and Ω. Combining this with (5.5), we obtain

$$\|w\|^2_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))}\leqslant C\int_Q|P_{-\omega,\lambda}w|^2dxdt.$$

Using the fact that $P_{-\omega,\lambda}w(t,x)=P_{\omega,-\lambda}v(T-t,x)$, we deduce (5.4). □

Armed with this Carleman estimate, we are now in position of completing the proof of Theorem 5.1.
Proof of Theorem 5.1. Let $v\in\mathcal{C}^2([0,T];\mathcal{C}_0^\infty(\Omega))$ satisfy (5.2), consider Ω_j, $j=1,2$, two bounded open smooth domains of $\mathbb{R}^n$ such that $\overline{\Omega}\subset\Omega_1$, $\overline{\Omega_1}\subset\Omega_2$ and let $\psi\in\mathcal{C}_0^\infty(\Omega_2)$ be such that $\psi=1$ on $\overline{\Omega_1}$. We consider $w\in\mathcal{C}^2([0,T];\mathcal{C}_0^\infty(\overline{\Omega}))$ given by

$$w(t,\cdot)=\psi\left\langle D_x,\lambda\right\rangle^{-1}v(t,\cdot)$$

and we remark that w satisfies

$$w(T,x)=\partial_t w(T,x)=w(0,x)=0,\quad x\in\mathbb{R}^n. \tag{5.6}$$

Now let us consider the quantity $\left\langle D_x,\lambda\right\rangle^{-1}P_{\omega,-\lambda}\left\langle D_x,\lambda\right\rangle w$. Note first that

$$\|P_{\omega,-\lambda}\left\langle D_x,\lambda\right\rangle w\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))}=\left\|\left\langle D_x,\lambda\right\rangle^{-1}P_{\omega,-\lambda}\left\langle D_x,\lambda\right\rangle w\right\|_{L^2((0,T)\times\mathbb{R}^n)}.$$

Moreover, it is clear that

$$\left\langle D_x,\lambda\right\rangle^{-1}P_{\omega,-\lambda}\left\langle D_x,\lambda\right\rangle=P_{\omega,-\lambda}.$$

Therefore, we have

$$\|P_{\omega,-\lambda}\left\langle D_x,\lambda\right\rangle w\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))}=\|P_{\omega,-\lambda}w\|_{L^2((0,T)\times\mathbb{R}^n)}$$

and, since w satisfies (5.6), combining this with (5.4) we deduce that

$$\|w\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))}\leqslant C\,\|P_{\omega,-\lambda}\left\langle D_x,\lambda\right\rangle w\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))}\,. \tag{5.7}$$

On the other hand, fixing $\psi_1\in\mathcal{C}_0^\infty(\Omega_1)$ satisfying $\psi_1=1$ on $\overline{\Omega}$, we get

$$w(t,\cdot)=\left\langle D_x,\lambda\right\rangle^{-1}v(t,\cdot)+(\psi-1)\left\langle D_x,\lambda\right\rangle^{-1}\psi_1 v(t,\cdot)$$

and, combining this with (5.7), we deduce that

$$\begin{aligned}
&\|v\|_{L^2((0,T)\times\mathbb{R}^n)}\\
&=\left\|\langle D_x,\lambda\rangle^{-1} v\right\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))}\\
&\leqslant \|w\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))}+\left\|(\psi-1)\langle D_x,\lambda\rangle^{-1}\psi_1 v\right\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))}\\
&\leqslant C\left\|P_{\omega,-\lambda}\langle D_x,\lambda\rangle w\right\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))}+\left\|(\psi-1)\langle D_x,\lambda\rangle^{-1}\psi_1 v\right\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))}\\
&\leqslant C\left\|P_{\omega,-\lambda} v\right\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))}+C\left\|P_{\omega,-\lambda}\langle D_x,\lambda\rangle(\psi-1)\langle D_x,\lambda\rangle^{-1}\psi_1 v\right\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))}\\
&\quad+\left\|(\psi-1)\langle D_x,\lambda\rangle^{-1}\psi_1 v\right\|_{L^2(0,T;H^1_\lambda(\mathbb{R}^n))}.
\end{aligned} \tag{5.8}$$

Moreover, since $(\psi-1)=0$ on neighborhood of $\mathrm{supp}(\psi_1)$, in view of [24, Theorem 18.1.8], we have $(\psi-1)\langle D_x,\lambda\rangle^{-1}\psi_1\in OpS^{-\infty}_\lambda$. In the same way, [24, Theorem 18.1.8] implies that

$$P_{\omega,-\lambda}\langle D_x,\lambda\rangle(\psi-1)\langle D_x,\lambda\rangle^{-1}\psi_1\in OpS^{-\infty}_\lambda$$

and we deduce that

$$\begin{aligned}
&C\left\|P_{\omega,-\lambda}\langle D_x,\lambda\rangle(\psi-1)\langle D_x,\lambda\rangle^{-1}\psi_1 v\right\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))}+\left\|(\psi-1)\langle D_x,\lambda\rangle^{-1}\psi_1 v\right\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))}\\
&\leqslant \frac{\|v\|_{L^2((0,T)\times\mathbb{R}^n)}}{\lambda^2}.
\end{aligned}$$

Combining this with (5.8) and choosing λ sufficiently large, we deduce (5.3) for $q_2=0$. Then, we deduce (5.3) for $q_2\neq 0$ by applying arguments similar to Lemma 4.2. ☐

Applying the Carleman estimate (5.3), we can now build solutions u_2 of the form (5.1) and satisfying $\tau_{0,2}u_2=0$ and complete the proof of Theorem 1.3.

5.2. **Completion of the proof of Theorem 1.3.** We start by proving existence of a solution $u_2\in L^2(0,T;H^1(\Omega))$ of the form (5.1) with the term $w_2\in L^2(0,T;H^1(\Omega))\cap e^{-\lambda(t+x\cdot\omega)}H_{\Box,*}(Q)$, satisfying

$$\|w_2\|_{L^2(0,T;H^1(\Omega))}+\lambda\|w_2\|_{L^2(Q)}\leqslant C, \tag{5.9}$$

$$\tau_{0,2}w_2=0. \tag{5.10}$$

This result is summarized in the following way.

Proposition 5.1. *There exists $\lambda_2>\lambda_1$ such that for $\lambda>\lambda_2$ we can find a solution $u_2\in L^2(0,T;H^1(\Omega))$ of $\Box u_2+q_2u_2=0$ in Q taking the form* (5.1) *with $w_2\in L^2(0,T;H^1(\Omega))\cap e^{-\lambda(t+x\cdot\omega)}H_{\Box,*}(Q)$ satisfying* (5.9)*-*(5.10).

Proof. We need to consider $w_2\in L^2(0,T;H^1(\Omega))$ a solution of

$$P_{\omega,\lambda}w_2+q_2w_2=-e^{-\lambda(t+x\cdot\omega)}(\Box+q_2)(e^{\lambda(t+x\cdot\omega)}-e^{\lambda(-t+x\cdot\omega)})=-q_2(1-e^{-2\lambda t}), \tag{5.11}$$

satisfying (5.9)-(5.10). Note that here, we use (5.11) and the fact that $P_{\omega,\lambda}w_2=e^{-\lambda(t+x\cdot\omega)}\Box e^{\lambda(t+x\cdot\omega)}w_2$ in order to prove that $w_2\in e^{-\lambda(t+x\cdot\omega)}H_{\Box,*}(Q)$ and we define $\tau_{0,2}w_2$ by $\tau_{0,2}w_2=e^{-\lambda x\cdot\omega}\tau_{0,2}e^{\lambda(t+x\cdot\omega)}w_2$. We will construct such a function w_2 by applying estimate (5.3). From now on, we fix $\lambda_2=\lambda_2''$. Applying the Carleman estimate (5.3), we define the linear form $\mathcal{M}$ on

$$\mathcal{I}=\{P_{\omega,-\lambda}v+q_2v:\ v\in\mathcal{C}^2([0,T];\mathcal{C}^\infty_0(\Omega))\ \text{ satisfying (5.2)}\},$$

considered as a subspace of $L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))$, by

$$\mathcal{M}(P_{\omega,-\lambda}v+q_2v)=-\int_Q vq_2(1-e^{-2\lambda t})dxdt,\quad v\in\mathcal{I}.$$

Then, (5.3) implies

$$|\mathcal{M}(P_{\omega,-\lambda}v+q_1v)|\leqslant C\|q_2\|_{L^2(Q)}\|P_{\omega,-\lambda}v+q_1v\|_{L^2(0,T;H^{-1}_\lambda(\mathbb{R}^n))},\quad v\in\mathcal{I},$$

with $C>0$ independent of λ and v. Thus, by the Hahn Banach theorem we can extend $\mathcal{M}$ to a continuous linear form on $L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n))$ still denoted by $\mathcal{M}$ and satisfying $\|\mathcal{M}\| \leqslant C\|q_2\|_{L^2(Q)}$. Therefore, there exists $w_2 \in L^2(0,T;H_\lambda^1(\mathbb{R}^n))$ such that

$$\langle g, w_2\rangle_{L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n)),L^2(0,T;H_\lambda^1(\mathbb{R}^n))} = \mathcal{M}(g), \quad g\in L^2(0,T;H_\lambda^{-1}(\mathbb{R}^n)).$$

Choosing $g = P_{\omega,-\lambda}v + q_2 v$ with $v\in\mathcal{C}_0^\infty(Q)$ proves that w_2 satisfies $P_{\omega,\lambda}w_2 + q_2 w_1 = -q_2(1-e^{-2\lambda t})$ in Q. Moreover, choosing $g = P_{\omega,-\lambda}v + q_1 v$, with $v\in\mathcal{I}$ and $\partial_t v_{|t=0}$ arbitrary, proves that (5.10) is fulfilled. Finally, using the fact that $\|w_2\|_{L^2(0,T;H_\lambda^1(\mathbb{R}^n))} \leqslant \|\mathcal{M}\| \leqslant C\|q\|_{L^2(Q)}$ proves that w_2 fulfills (5.9) which completes the proof of the proposition. □

Using this proposition, we are now in position to complete the proof of Theorem 1.3.

Proof of Theorem 1.3. Let us remark that since Lemma 4.2 and Theorem 4.1 are valid when $q\in L^\infty(0,T;L^p(\Omega))$ one can easily extend Proposition 4.1 to the case $q_1\in L^\infty(0,T;L^p(\Omega))$. Therefore, in the context of this section, Proposition 4.1 holds true. Combining Proposition 4.1 with Proposition 5.1, we deduce existence of a solution $u_1\in H^1(Q)$ of $\Box u_1 + q_1 u_1 = 0$ in Q taking the form (4.1), with $w_1\in H^1(Q)$ satisfying (4.3) for $j=1$, as well as the existence of a solution $u_2\in L^2(0,T;H^1(\Omega))$ of $\Box u_2 + q_2 u_2 = 0$ in Q, $\tau_{0,2}u_2 = 0$, taking the form (5.1) with the term $w_2\in L^2(0,T;H^1(\Omega))$ satisfying (5.9). Repeating the arguments of the end of the proof of Theorem 1.2 (see Subsection 4.4), we can deduce the following orthogonality identity

$$\lim_{\lambda\to+\infty}\int_Q q u_1 u_2 dxdt = 0. \tag{5.12}$$

Moreover, one can check that

$$\int_Q q u_1 u_2 dxdt = \int_{\mathbb{R}^{1+n}} q(t,x)e^{-i\xi\cdot(t,x)}dxdt + \int_Q Y_\lambda(t,x)dxdt,$$

with

$$Y_\lambda(t,x) = q[e^{-2\lambda t}e^{-i(t,x)\cdot\xi} + e^{-i(t,x)\cdot\xi}w_2 + w_1 + w_1 w_2].$$

Combining (4.3), (5.9) with the fact that

$$\int_Q |q(t,x)|\left|e^{-2\lambda t}e^{-i(t,x)\cdot\xi}\right| dxdt \leqslant \|q\|_{L^2(Q)}\left(\int_0^{+\infty} e^{-2\lambda t}dt\right)^{\frac12} \leqslant \|q\|_{L^2(Q)}\lambda^{-\frac12},$$

we deduce that

$$\lim_{\lambda\to+\infty}\int_Q Y_\lambda(t,x)dxdt = 0.$$

Combining this asymptotic property with (5.12), we can conclude in a similar way to Theorem 1.2 that $q_1 = q_2$. □

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References

[1] S. Alinhac and M.S. Baouendi, *A non uniqueness result for operators of principal type*, Math. Z., **220** (1995), 561-568.

[2] M. Bellassoued and I. Ben Aicha, *Stable determination outside a cloaking region of two time-dependent coefficients in an hyperbolic equation from Dirichlet to Neumann map*, Journal of Mathematical Analysis and Applications, **449** (2017), 46-76.

[3] M. Bellassoued, M. Choulli, M. Yamamoto, *Stability estimate for an inverse wave equation and a multidimensional Borg-Levinson theorem*, J. Diff. Equat., **247** (2) (2009), 465-494.

[4] M. Bellassoued and D. Dos Santos Ferreira, *Stability estimates for the anisotropic wave equation from the Dirichlet-to-Neumann map*, Inverse Probl. Imaging, **5** (4) (2011), 745-773.

[5] M. Bellassoued, D. Jellali, M. Yamamoto, *Lipschitz stability for a hyperbolic inverse problem by finite local boundary data*, Appl. Anal., **85** (2006), 1219-1243.

[6] I. Ben Aicha, *Stability estimate for hyperbolic inverse problem with time-dependent coefficient*, Inverse Problems, **31** (2015), 125010.

[7] A. Bukhgeim and M. Klibanov, *Global uniqueness of a class of multidimensional inverse problem*, Sov. Math.-Dokl., **24** (1981), 244-247.

[8] A. L. Bukhgeim and G. Uhlmann, *Recovering a potential from partial Cauchy data*, Commun. Partial Diff. Eqns., **27** (2002), no 3-4, 653-668.

[9] P. Caro and K. M. Rogers, *Global Uniqueness for The Calderón Problem with Lipschitz Conductivities*, Forum of Mathematics, Pi, **4** (2016), p. e2. Cambridge University Press.

[10] M. Choulli and Y. Kian, *Stability of the determination of a time-dependent coefficient in parabolic equations*, MCRF, **3** (2) (2013), 143-160.

[11] M. Choulli and Y. Kian, *Logarithmic stability in determining the time-dependent zero order coefficient in a parabolic equation from a partial Dirichlet-to-Neumann map. Application to the determination of a nonlinear term*, preprint, arXiv:1605.08672.

[12] M. Choulli, Y. Kian, E. Soccorsi, *Determining the time dependent external potential from the DN map in a periodic quantum waveguide*, SIAM J.Math. Anal., **47** (6) (2015), 4536-4558.

[13] M. Choulli, Y. Kian, E. Soccorsi, *Stability result for elliptic inverse periodic coefficient problem by partial Dirichlet-to-Neumann map*, to appear Journal of Spectral Theory, arXiv:1601.05355.

[14] M. Choulli, Y. Kian, E. Soccorsi, *On the Calderón problem in periodic cylindrical domain with partial Dirichlet and Neumann data*, to appear in Mathematical Methods in the Applied Sciences, arXiv:1601.05358.

[15] F. Chung, L. Tzou, *The L^p Carleman estimate and a partial data inverse problem*, preprint, arXiv:1610.01715.

[16] G. Eskin, *A new approach to hyperbolic inverse problems*, Inverse Problems, **22** no. 3 (2006), 815-831.

[17] G. Eskin, *Inverse hyperbolic problems with time-dependent coefficients*, Commun. Partial Diff. Eqns., **32** (11) (2007), 1737-1758.

[18] G. Eskin, *Inverse problems for general second order hyperbolic equations with time-dependent coefficients*, to appear in Bull. Math. Sci., arXiv:1503.00825.

[19] D.D. Feirrera, C. Kenig, M. Salo, *Determining an unbounded potential from Cauchy data in admissible geometries*, Comm. Partial Differential Equations, **38** (2013), 50-68.

[20] K. Fujishiro and Y. Kian, *Determination of time dependent factors of coefficients in fractional diffusion equations*, MCRF, **6** (2016), 251-269.

[21] P. Gaitan and Y. Kian, *A stability result for a time-dependent potential in a cylindrical domain*, Inverse Problems, **29** (6) (2013), 065006.

[22] P. Grisvard, *Elliptic problems in nonsmooth domains*, Pitman, London, 1985.

[23] B. Haberman and D. Tataru, *Uniqueness in Calderón's problem with Lipschitz conductivities*, Duke Math. Journal, **162** (3) (2013), 497-516.

[24] L. Hörmander, *The Analysis of linear partial differential operators*, Vol III, Springer-Verlag, Berlin, Heidelberg, 1983.

[25] V. Isakov, *Completness of products of solutions and some inverse problems for PDE*, J. Diff. Equat., **92** (1991), 305-316.

[26] V. Isakov, *An inverse hyperbolic problem with many boundary measurements*, Commun. Partial Diff. Eqns., **16** (1991), 1183-1195.

[27] V. Isakov, *On uniqueness in inverse problems for semilinear parabolic equations*, Arch. Rat. Mech. Anal., **124** (1993), 1-12.

[28] O. Kavian, *Four Lectures on Parameter Identification*, Three Courses on Partial Differential Equations, pp. 125-162, IRMA Lect. Math. Theor. Phys., 4, de Gruyter, Berlin, 2003.

[29] C.E. Kenig, J. Sjöstrand, G. Uhlmann, *The Calderon problem with partial data*, Ann. of Math., **165** (2007), 567-591.

[30] Y. Kian, *Stability of the determination of a coefficient for wave equations in an infinite waveguide*, Inverse Probl. Imaging, **8** (3) (2014), 713-732.

[31] Y. Kian, *Stability in the determination of a time-dependent coefficient for wave equations from partial data*, Journal of Mathematical Analysis and Applications, **436** (2016), 408-428.

[32] Y. Kian, *Unique determination of a time-dependent potential for wave equations from partial data*, Annales de l'IHP (C) nonlinear analysis, **34** (2017), 973-990.
[33] Y. Kian, *Recovery of time-dependent damping coefficients and potentials appearing in wave equations from partial data*, SIAM J. Math. Anal., **48** (6) (2016), 4021-4046.
[34] Y. Kian and L. Oksanen, *Recovery of time-dependent coefficient on Riemanian manifold for hyperbolic equations*, preprint, arXiv:1606.07243.
[35] Y. Kian and E. Soccorsi, *Hölder stably determining the time-dependent electromagnetic potential of the Schrödinger equation*, preprint, arXiv:1705.01322.
[36] I. Lasiecka, J-L. Lions, R. Triggiani, *Non homogeneous boundary value problems for second order hyperbolic operators* J. Math. Pures Appl., **65** (1986), 149-192.
[37] J-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. I, Dunod, Paris, 1968.
[38] J-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Vol. II, Dunod, Paris, 1968.
[39] Rakesh and A. G. Ramm, *Property C and an Inverse Problem for a Hyperbolic Equation*, J. Math. Anal. Appl., **156** (1991), 209-219.
[40] Rakesh and W. Symes, *Uniqueness for an inverse problem for the wave equation*, Commun. Partial Diff. Eqns., **13** (1) (1988), 87-96.
[41] A. G. Ramm and J. Sjöstrand, *An inverse problem of the wave equation*, Math. Z., **206** (1991), 119-130.
[42] R. Salazar, *Determination of time-dependent coefficients for a hyperbolic inverse problem*, Inverse Problems, **29** (9) (2013), 095015.
[43] P. Stefanov, *Uniqueness of the multi-dimensional inverse scattering problem for time dependent potentials*, Math. Z., **201** (4) (1989), 541-559.
[44] P. Stefanov and G. Uhlmann, *Stable determination of the hyperbolic Dirichlet-to-Neumann map for generic simple metrics*, International Math Research Notices (IMRN), **17** (2005), 1047-1061.
[45] P. Stefanov and Y. Yang, *The inverse problem for the Dirichlet-to-Neumann map on Lorentzian manifolds*, preprint, arXiv:1607.08690.
[46] A. Waters, *Stable determination of X-ray transforms of time dependent potentials from partial boundary data*, Commun. Partial Diff. Eqns., **39** (2014), 2169-2197.

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