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Van Duong Dinh

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# GLOBAL EXISTENCE FOR THE DEFOCUSING MASS-CRITICAL NONLINEAR FOURTH-ORDER SCHRÖDINGER EQUATION BELOW THE ENERGY SPACE

VAN DUONG DINH

ABSTRACT. In this paper, we consider the defocusing mass-critical nonlinear fourth-order Schrödinger equation. Using the  $I$ -method combined with the interaction Morawetz estimate, we prove that the problem is globally well-posed in  $H^\gamma(\mathbb{R}^d)$ ,  $5 \leq d \leq 7$  with  $\gamma(d) < \gamma < 2$ , where  $\gamma(5) = \frac{8}{5}$ ,  $\gamma(6) = \frac{5}{3}$  and  $\gamma(7) = \frac{13}{7}$ .

## 1. INTRODUCTION

Consider the defocusing mass-critical nonlinear fourth-order Schrödinger equation, namely

$$\begin{cases} i\partial_t u(t, x) + \Delta^2 u(t, x) &= -(|u|^{\frac{8}{d}} u)(t, x), \quad t \geq 0, x \in \mathbb{R}^d, \\ u(0, x) &= u_0(x) \in H^\gamma(\mathbb{R}^d), \end{cases} \quad (\text{NL4S})$$

where  $u(t, x)$  is a complex valued function in  $\mathbb{R}^+ \times \mathbb{R}^d$ .

The fourth-order Schrödinger equation was introduced by Karpman [Kar96] and Karpman-Shagalov [KS00] taking into account the role of small fourth-order dispersion terms in the propagation of intense laser beams in a bulk medium with Kerr nonlinearity. The study of nonlinear fourth-order Schrödinger equation has attracted a lot of interest in the past several years (see [Pau1], [Pau2], [HHW06], [HHW07], [HJ05], [MXZ09], [MXZ11], [MWZ15] and references therein).

It is known (see [Din1] or [Din2]) that (NL4S) is locally well-posed in  $H^\gamma(\mathbb{R}^d)$  for  $\gamma > 0$  satisfying for  $d \neq 1, 2, 4$ ,

$$[\gamma] \leq 1 + \frac{8}{d}. \quad (1.1)$$

Here  $[\gamma]$  is the smallest integer greater than or equal to  $\gamma$ . This condition ensures the nonlinearity to have enough regularity. The time of existence depends only on the  $H^\gamma$ -norm of initial data. Moreover, the local solution enjoys mass conservation, i.e.

$$M(u(t)) := \|u(t)\|_{L^2(\mathbb{R}^d)}^2 = \|u_0\|_{L^2(\mathbb{R}^d)}^2,$$

and  $H^2$ -solution has conserved energy, i.e.

$$E(u(t)) := \int_{\mathbb{R}^d} \frac{1}{2} |\Delta u(t, x)|^2 + \frac{d}{2d+8} |u(t, x)|^{\frac{2d+8}{d}} dx = E(u_0).$$

The conservations of mass and energy together with the persistence of regularity (see [Din2]) yield the global well-posedness for (NL4S) in  $H^\gamma(\mathbb{R}^d)$  with  $\gamma \geq 2$  satisfying for  $d \neq 1, 2, 4$ , (1.1). We also have (see [Din1] or [Din2]) the local well-posedness for (NL4S) with initial data  $u_0 \in L^2(\mathbb{R}^d)$  but the time of existence depends on the profile of  $u_0$  instead of its  $H^\gamma$ -norm. The global existence holds for small  $L^2$ -norm initial data. For large  $L^2$ -norm initial data, the conservation of mass

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does not immediately give the global well-posedness in  $L^2(\mathbb{R}^d)$ . For the global well-posedness with large  $L^2$ -norm initial data, we refer the reader to [PS10] where the authors established the global well-posedness and scattering for (NL4S) in  $L^2(\mathbb{R}^d)$ ,  $d \geq 5$ .

The main goal of this paper is to prove the global well-posedness for (NL4S) in a low regularity space  $H^\gamma(\mathbb{R}^d)$ ,  $d \geq 5$  with  $\gamma < 2$ . Since we are working with low regularity data, the conservation of energy does not hold. In order to overcome this problem, we make use of the  $I$ -method and the interaction Morawetz inequality. Due to the high-order term  $\Delta^2 u$ , we requires the nonlinearity to have at least two orders of derivatives in order to successfully establish the almost conservation law. We thus restrict ourself in spatial space of dimensions  $d = 5, 6, 7$ .

Let us recall some known results about the global existence below the energy space for the nonlinear fourth-order Schrödinger equation. To our knowledge, the first result to address this problem belongs to Guo in [Guo10], where the author considered a more general fourth-order Schrödinger equation, namely

$$i\partial_t u + \lambda \Delta u + \mu \Delta^2 u + \nu |u|^{2m} u = 0,$$

and established the global existence in  $H^\gamma(\mathbb{R}^d)$  for  $1 + \frac{md-9+\sqrt{(4m-md+7)^2+16}}{4m} < \gamma < 2$  where  $m$  is an integer satisfying  $4 < md < 4m+2$ . The proof is based on the  $I$ -method which is a modification of the one invented by  $I$ -Team [CKSTT02] in the context of nonlinear Schrödinger equation. Later, Miao-Wu-Zhang studied the defocusing cubic fourth-order Schrödinger equation, namely

$$i\partial_t u + \Delta^2 u + |u|^2 u = 0,$$

and proved the global well-posedness and scattering in  $H^\gamma(\mathbb{R}^d)$  with  $\gamma(d) < \gamma < 2$  where  $\gamma(5) = \frac{16}{11}$ ,  $\gamma(6) = \frac{16}{9}$  and  $\gamma(7) = \frac{45}{23}$ . The proof relies on the combination of  $I$ -method and a new interaction Morawetz inequality. Recently, the author in [Din3] showed that the defocusing cubic fourth-order Schrödinger equation is globally well-posed in  $H^\gamma(\mathbb{R}^4)$  with  $\frac{60}{53} < \gamma < 2$ . The analysis is carried out in Bourgain spaces  $X^{\gamma,b}$  which is similar to those in [CKSTT02]. Note that in the above considerations, the nonlinearity is algebraic. This allows to write explicitly the commutator between the  $I$ -operator and the nonlinearity by means of the Fourier transform, and then control it by multi-linear analysis. When one considers the mass-critical nonlinear fourth-order Schrödinger equation in dimensions  $d \geq 5$ , this method does not work. We thus rely purely on Strichartz and interaction Morawetz estimates. The main result of this paper is the following:

**Theorem 1.1.** *Let  $d = 5, 6, 7$ . The initial value problem (NL4S) is globally well-posed in  $H^\gamma(\mathbb{R}^d)$ , for any  $\gamma(d) < \gamma < 2$ , where  $\gamma(5) = \frac{8}{5}$ ,  $\gamma(6) = \frac{5}{3}$  and  $\gamma(7) = \frac{13}{7}$ .*

The proof of the above theorem is based on the combination of the  $I$ -method and the interaction Morawetz inequality which is similar to those given in [DPST07]. The  $I$ -method was first introduced by  $I$ -Team in [CKSTT02] in order to treat the nonlinear Schrödinger equation at low regularity. The idea is to replace the non-conserved energy  $E(u)$  when  $\gamma < 2$  by an “almost conserved” variance  $E(Iu)$  with  $I$  a smoothing operator which is the identity at low frequency, and behaves like a fractional integral operator of order  $2 - \gamma$  at high frequency. Since  $Iu$  is not a solution of (NL4S), we may expect an energy increment. The key is to show that the modified energy  $E(Iu)$  is an “almost conserved” quantity in the sense that the time derivative of  $E(Iu)$  decays with respect to a large parameter  $N$  (see Section 2 for the definition of  $I$  and  $N$ ). To do so, we need delicate estimates on the commutator between the  $I$ -operator and the nonlinearity. Note that in our setting, the nonlinearity is not algebraic. Thus we can not apply the Fourier transform technique. Fortunately, thanks to a special Strichartz estimate (2.4), we are able to apply the technique given in [VZ09] to control the commutator. The interaction Morawetz inequality for the nonlinear fourth-order Schrödinger equation was first introduced in [Pau2] for  $d \geq 7$ , and was

extended for  $d \geq 5$  in [MWZ15]. With this estimate, the interpolation argument and Sobolev embedding give for any compact interval  $J$ ,

$$\|u\|_{M(J)} := \|u\|_{L_t^{\frac{8(d-3)}{d}} L_x^{\frac{2(d-3)}{d-4}}} \lesssim |J|^{\frac{d-4}{8(d-3)}} \|u_0\|_{L_x^2}^{\frac{1}{d-3}} \|u\|_{L_t^\infty \dot{H}_x^{\frac{1}{2}}}^{\frac{d-4}{d-3}}. \quad (1.2)$$

As a byproduct of the Strichartz estimates and  $I$ -method, we show the almost conservation law for the modified energy of (NL4S), that is if  $u \in L^\infty(J, \mathcal{S}'(\mathbb{R}^d))$  is a solution to (NL4S) on a time interval  $J = [0, T]$ , and satisfies  $\|Iu_0\|_{H_x^2} \leq 1$  and if  $u$  satisfies in addition the a priori bound  $\|u\|_{M(J)} \leq \mu$  for some small constant  $\mu > 0$ , then

$$\sup_{t \in [0, T]} |E(Iu(t)) - E(Iu_0)| \lesssim N^{-(2-\gamma+\delta)}.$$

for  $\max\{3 - \frac{8}{d}, \frac{8}{d}\} < \gamma < 2$  and  $0 < \delta < \gamma + \frac{8}{d} - 3$ .

We now briefly outline the idea of the proof. Let  $u$  be a global in time solution to (NL4S). Observe that for any  $\lambda > 0$ ,

$$u_\lambda(t, x) := \lambda^{-\frac{d}{2}} u(\lambda^{-4}t, \lambda^{-1}x) \quad (1.3)$$

is also a solution to (NL4S). By choosing

$$\lambda \sim N^{\frac{2-\gamma}{\gamma}}, \quad (1.4)$$

and using some harmonic analysis, we can make  $E(Iu_\lambda(0)) \leq \frac{1}{4}$  by taking  $\lambda$  sufficiently large depending on  $\|u_0\|_{H_x^\gamma}$  and  $N$ . Fix an arbitrary large time  $T$ . The main goal is to show

$$E(Iu_\lambda(\lambda^4 T)) \leq 1. \quad (1.5)$$

With this bound, we can easily obtain the growth of  $\|u(T)\|_{H_x^\gamma}$ , and the global well-posedness in  $H^\gamma(\mathbb{R}^d)$  follows immediately. In order to get (1.5), we claim that

$$\|u_\lambda\|_{M([0, t])} \leq K t^{\frac{d-4}{8(d-3)}}, \quad \forall t \in [0, \lambda^4 T],$$

for some constant  $K$ . If it is not so, then there exists  $T_0 \in [0, \lambda^4 T]$  such that

$$\|u_\lambda\|_{M([0, T_0])} > K T_0^{\frac{d-4}{8(d-3)}}, \quad (1.6)$$

$$\|u_\lambda\|_{M([0, T_0])} \leq 2K T_0^{\frac{d-4}{8(d-3)}}. \quad (1.7)$$

Using (1.7), we can split  $[0, T_0]$  into  $L$  subintervals  $J_k, k = 1, \dots, L$  so that

$$\|u_\lambda\|_{M(J_k)} \leq \mu.$$

The number  $L$  must satisfy

$$L \sim T_0^{\frac{d-4}{d}}. \quad (1.8)$$

Thus we can apply the almost conservation law to get

$$\sup_{[0, T_0]} E(Iu_\lambda(t)) \leq E(Iu_\lambda(0)) + N^{-(2-\gamma+\delta)} L.$$

Since  $E(Iu_\lambda(0)) \leq \frac{1}{4}$ , in order to have  $E(Iu_\lambda(t)) \leq 1$  for all  $t \in [0, T_0]$ , we need

$$N^{-(2-\gamma+\delta)} L \ll \frac{1}{4}. \quad (1.9)$$

Combining (1.4), (1.8) and (1.9), we obtain the condition on  $\gamma$ . Next, using (1.2) together with some harmonic analysis, we estimate

$$\|u_\lambda\|_{M([0, T_0])} \lesssim T_0^{\frac{d-4}{8(d-3)}} \|u_0\|_{L_x^2}^{\frac{1}{d-3}} \sup_{[0, T_0]} \left( \|u_0\|^{\frac{3}{4}} \|Iu_\lambda(t)\|_{\dot{H}_x^2}^{\frac{1}{4}} + N^{-\frac{3}{4}} \|Iu_\lambda(t)\|_{\dot{H}_x^2} \right)^{\frac{d-4}{d-3}}.$$

Since  $\|Iu_\lambda(t)\|_{\dot{H}_x^2} \lesssim E(Iu_\lambda(t)) \leq 1$  for all  $t \in [0, T_0]$ , we get

$$\|u_\lambda\|_{M([0, T_0])} \leq CT_0^{\frac{d-4}{8(d-3)}},$$

for some constant  $C > 0$ . This leads to a contradiction to (1.6) for an appropriate choice of  $K$ . Thus we have the claim and also

$$E(Iu_\lambda(t)) \leq 1, \quad \forall t \in [0, \lambda^4 T].$$

For more details, we refer the reader to Section 4.

This paper is organized as follows. In Section 2, we introduce some notations and recall some results related to our problem. In Section 3, we show the almost conservation law for the modified energy. Finally, the proof of our main result is given in Section 4.

## 2. PRELIMINARIES

In the sequel, the notation  $A \lesssim B$  denotes an estimate of the form  $A \leq CB$  for some constant  $C > 0$ . The notation  $A \sim B$  means that  $A \lesssim B$  and  $B \lesssim A$ . We write  $A \ll B$  if  $A \leq cB$  for some small constant  $c > 0$ . We also use  $\langle a \rangle := 1 + |a|$ .

**2.1. Nonlinearity.** Let  $F(z) := |z|^{\frac{8}{d}} z$ ,  $d = 5, 6, 7$  be the function that defines the nonlinearity in (NL4S). The derivative  $F'(z)$  is defined as a real-linear operator acting on  $w \in \mathbb{C}$  by

$$F'(z) \cdot w := w \partial_z F(z) + \bar{w} \partial_{\bar{z}} F(z),$$

where

$$\partial_z F(z) = \frac{2d+8}{2d} |z|^{\frac{8}{d}}, \quad \partial_{\bar{z}} F(z) = \frac{4}{d} |z|^{\frac{8}{d}} \frac{z}{\bar{z}}.$$

We shall identify  $F'(z)$  with the pair  $(\partial_z F(z), \partial_{\bar{z}} F(z))$ , and define its norm by

$$|F'(z)| := |\partial_z F(z)| + |\partial_{\bar{z}} F(z)|.$$

It is clear that  $|F'(z)| = O(|z|^{\frac{8}{d}})$ . We also have the following chain rule

$$\partial_k F(u) = F'(u) \partial_k u,$$

for  $k \in \{1, \dots, d\}$ . In particular, we have

$$\nabla F(u) = F'(u) \nabla u.$$

We next recall the fractional chain rule to estimate the nonlinearity.

**Lemma 2.1.** *Suppose that  $G \in C^1(\mathbb{C}, \mathbb{C})$ , and  $\alpha \in (0, 1)$ . Then for  $1 < q \leq q_2 < \infty$  and  $1 < q_1 \leq \infty$  satisfying  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ ,*

$$\| |\nabla|^\alpha G(u) \|_{L_x^q} \lesssim \|G'(u)\|_{L_x^{q_1}} \| |\nabla|^\alpha u \|_{L_x^{q_2}}.$$

We refer the reader to [CW91, Proposition 3.1] for the proof of the above estimate when  $1 < q_1 < \infty$ , and to [KPV93, Theorem A.6] for the proof when  $q_1 = \infty$ .

When  $G$  is no longer  $C^1$ , but Hölder continuous, we have the following fractional chain rule.

**Lemma 2.2.** *Suppose that  $G \in C^{0,\beta}(\mathbb{C}, \mathbb{C})$ ,  $\beta \in (0, 1)$ . Then for every  $0 < \alpha < \beta$ ,  $1 < q < \infty$ , and  $\frac{\alpha}{\beta} < \rho < 1$ ,*

$$\| |\nabla|^\alpha G(u) \|_{L_x^q} \lesssim \| |u|^{\beta-\frac{\alpha}{\rho}} \|_{L_x^{q_1}} \| |\nabla|^\rho u \|_{L_x^{\frac{\alpha}{\rho} q_2}},$$

provided  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$  and  $(1 - \frac{\alpha}{\beta\rho}) q_1 > 1$ .

The reader can find the proof of this result in [Vis06, Proposition A.1].

**2.2. Strichartz estimates.** Let  $I \subset \mathbb{R}$  and  $p, q \in [1, \infty]$ . We define the mixed norm

$$\|u\|_{L_t^p(I, L_x^q)} := \left( \int_I \left( \int_{\mathbb{R}^d} |u(t, x)|^q dx \right)^{\frac{1}{q}} dt \right)^{\frac{1}{p}}$$

with a usual modification when either  $p$  or  $q$  are infinity. When there is no risk of confusion, we may write  $L_t^p L_x^q$  instead of  $L_t^p(I, L_x^q)$ . We also use  $L_{t,x}^p$  when  $p = q$ .

**Definition 2.3.** A pair  $(p, q)$  is said to be **Schrödinger admissible**, for short  $(p, q) \in S$ , if

$$(p, q) \in [2, \infty]^2, \quad (p, q, d) \neq (2, \infty, 2), \quad \frac{2}{p} + \frac{d}{q} \leq \frac{d}{2}.$$

We also denote for  $(p, q) \in [1, \infty]^2$ ,

$$\gamma_{p,q} = \frac{d}{2} - \frac{d}{q} - \frac{4}{p}. \quad (2.1)$$

**Definition 2.4.** A pair  $(p, q)$  is called **biharmonic admissible**, for short  $(p, q) \in B$ , if

$$(p, q) \in S, \quad \gamma_{p,q} = 0.$$

**Proposition 2.5** (Strichartz estimate for fourth-order Schrödinger equation [Din1]). *Let  $\gamma \in \mathbb{R}$  and  $u$  be a (weak) solution to the linear fourth-order Schrödinger equation namely*

$$u(t) = e^{it\Delta^2} u_0 + \int_0^t e^{i(t-s)\Delta^2} F(s) ds,$$

for some data  $u_0, F$ . Then for all  $(p, q)$  and  $(a, b)$  Schrödinger admissible with  $q < \infty$  and  $b < \infty$ ,

$$\| |\nabla|^\gamma u \|_{L_t^p(\mathbb{R}, L_x^q)} \lesssim \| |\nabla|^{\gamma+\gamma_{p,q}} u_0 \|_{L_x^2} + \| |\nabla|^{\gamma+\gamma_{p,q}-\gamma_{a',b'}-4} F \|_{L_t^{a'}(\mathbb{R}, L_x^{b'})}. \quad (2.2)$$

Here  $(a, a')$  and  $(b, b')$  are conjugate pairs, and  $\gamma_{p,q}, \gamma_{a',b'}$  are defined as in (2.1).

Note that the estimate (2.2) is exactly the one given in [MZ07], [Pau1] or [Pau2] where the author considered  $(p, q)$  and  $(a, b)$  are either sharp Schrödinger admissible, i.e.

$$p, q \in [2, \infty]^2, \quad (p, q, d) \neq (2, \infty, 2), \quad \frac{2}{p} + \frac{d}{q} = \frac{d}{2},$$

or biharmonic admissible. We refer the reader to [Din1, Proposition 2.1] for the proof of Proposition 2.5. The proof is based on the scaling technique instead of using a dedicate dispersive estimate of [BKS00] for the fundamental solution of the homogeneous fourth-order Schrödinger equation.

The following result is a direct consequence of (2.2).

**Corollary 2.6.** *Let  $u$  be a (weak) solution to the linear fourth-order Schrödinger equation for some data  $u_0, F$ . Then for all  $(p, q)$  and  $(a, b)$  biharmonic admissible satisfying  $q < \infty$  and  $b < \infty$ ,*

$$\|u\|_{L_t^p(\mathbb{R}, L_x^q)} \lesssim \|u_0\|_{L_x^2} + \|F\|_{L_t^{a'}(\mathbb{R}, L_x^{b'})}, \quad (2.3)$$

and

$$\|\Delta u\|_{L_t^p(\mathbb{R}, L_x^q)} \lesssim \|\Delta u_0\|_{L_x^2} + \|\nabla F\|_{L_t^2(\mathbb{R}, L_x^{\frac{2d}{d+2})}}. \quad (2.4)$$

**2.3. Littlewood-Paley decomposition.** Let  $\varphi$  be a radial smooth bump function supported in the ball  $|\xi| \leq 2$  and equal to 1 on the ball  $|\xi| \leq 1$ . For  $M = 2^k, k \in \mathbb{Z}$ , we define the Littlewood-Paley operators:

$$\begin{aligned}\widehat{P_{\leq M} f}(\xi) &:= \varphi(M^{-1}\xi)\hat{f}(\xi), \\ \widehat{P_{> M} f}(\xi) &:= (1 - \varphi(M^{-1}\xi))\hat{f}(\xi), \\ \widehat{P_M f}(\xi) &:= (\varphi(M^{-1}\xi) - \varphi(2M^{-1}\xi))\hat{f}(\xi),\end{aligned}$$

where  $\hat{\cdot}$  is the spatial Fourier transform. Similarly, we can define

$$P_{< M} := P_{\leq M} - P_M, \quad P_{\geq M} := P_{> M} + P_M,$$

and for  $M_1 \leq M_2$ ,

$$P_{M_1 < \cdot \leq M_2} := P_{\leq M_2} - P_{\leq M_1} = \sum_{M_1 < M \leq M_2} P_M.$$

We recall the following standard Bernstein inequalities (see e.g. [BCD11, Chapter 2] or [Tao06, Appendix]):

**Lemma 2.7** (Bernstein inequalities). *Let  $\gamma \geq 0$  and  $1 \leq p \leq q \leq \infty$ . We have*

$$\begin{aligned}\|P_{\geq M} f\|_{L_x^p} &\lesssim M^{-\gamma} \|\nabla^\gamma P_{\geq M} f\|_{L_x^p}, \\ \|P_{\leq M} |\nabla|^\gamma f\|_{L_x^p} &\lesssim M^\gamma \|P_{\leq M} f\|_{L_x^p}, \\ \|P_M |\nabla|^{\pm\gamma} f\|_{L_x^p} &\sim M^{\pm\gamma} \|P_M f\|_{L_x^p}, \\ \|P_{\leq M} f\|_{L_x^q} &\lesssim M^{\frac{d}{p} - \frac{d}{q}} \|P_{\leq M} f\|_{L_x^p}, \\ \|P_M f\|_{L_x^q} &\lesssim M^{\frac{d}{p} - \frac{d}{q}} \|P_M f\|_{L_x^p}.\end{aligned}$$

**2.4.  $I$ -operator.** Let  $0 \leq \gamma < 2$  and  $N \gg 1$ . We define the Fourier multiplier  $I_N$  by

$$\widehat{I_N f}(\xi) := m_N(\xi)\hat{f}(\xi),$$

where  $m_N$  is a smooth, radially symmetric, non-increasing function such that

$$m_N(\xi) := \begin{cases} 1 & \text{if } |\xi| \leq N, \\ (N^{-1}|\xi|)^{\gamma-2} & \text{if } |\xi| \geq 2N. \end{cases}$$

We shall drop the  $N$  from the notation and write  $I$  and  $m$  instead of  $I_N$  and  $m_N$ . We collect some basic properties of the  $I$ -operator in the following lemma.

**Lemma 2.8.** *Let  $0 \leq \sigma \leq \gamma < 2$  and  $1 < q < \infty$ . Then*

$$\|If\|_{L_x^q} \lesssim \|f\|_{L_x^q}, \tag{2.5}$$

$$\|\nabla^\sigma P_{> N} f\|_{L_x^q} \lesssim N^{\sigma-2} \|\Delta If\|_{L_x^q}, \tag{2.6}$$

$$\|\langle \nabla \rangle^\sigma f\|_{L_x^q} \lesssim \|\langle \Delta \rangle If\|_{L_x^q}, \tag{2.7}$$

$$\|f\|_{H_x^\gamma} \lesssim \|If\|_{H_x^2} \lesssim N^{2-\gamma} \|f\|_{H_x^\gamma}, \tag{2.8}$$

$$\|If\|_{\dot{H}_x^2} \lesssim N^{2-\gamma} \|f\|_{\dot{H}_x^\gamma}. \tag{2.9}$$

*Proof.* The estimate (2.5) is a direct consequence of the Hörmander-Mikhlin multiplier theorem. To prove (2.6), we write

$$\|\nabla^\sigma P_{> N} f\|_{L_x^q} = \|\nabla^\sigma P_{> N} (\Delta I)^{-1} \Delta If\|_{L_x^q}.$$

The desired estimate (2.6) follows again from the Hörmander-Mikhlin multiplier theorem. In order to get (2.7), we estimate

$$\|\langle \nabla \rangle^\sigma f\|_{L_x^q} \leq \|P_{\leq N} \langle \nabla \rangle^\sigma f\|_{L_x^q} + \|P_{> N} f\|_{L_x^q} + \|P_{> N} |\nabla|^\sigma f\|_{L_x^q}.$$

Thanks to the fact that the  $I$ -operator is the identity at low frequency  $|\xi| \leq N$ , the multiplier theorem and (2.6) imply

$$\|\langle \nabla \rangle^\sigma f\|_{L_x^q} \lesssim \|\langle \Delta \rangle I f\|_{L_x^q} + \|\Delta I f\|_{L_x^q}.$$

This proves (2.7). Finally, by the definition of the  $I$ -operator and (2.6), we have

$$\begin{aligned} \|f\|_{H_x^\gamma} &\lesssim \|P_{\leq N} f\|_{H_x^\gamma} + \|P_{> N} f\|_{L_x^2} + \| |\nabla|^\gamma P_{> N} f\|_{L_x^2} \\ &\lesssim \|P_{\leq N} I f\|_{H_x^\gamma} + N^{-2} \|\Delta I f\|_{L_x^2} + N^{\gamma-2} \|\Delta I f\|_{L_x^2} \lesssim \|I f\|_{H_x^2}. \end{aligned}$$

This shows the first inequality in (2.8). For the second inequality in (2.8), we estimate

$$\|I f\|_{H_x^2} \lesssim \|P_{\leq N} \langle \nabla \rangle^2 I f\|_{L_x^2} + \|P_{> N} \langle \nabla \rangle^2 I f\|_{L_x^2} \lesssim N^{2-\gamma} \|f\|_{H_x^\gamma}.$$

Here we use the definition of  $I$ -operator to get

$$\|P_{\leq N} I \langle \nabla \rangle^{2-\gamma}\|_{L_x^2 \rightarrow L_x^2}, \quad \|P_{> N} I \langle \nabla \rangle^{2-\gamma}\|_{L_x^2 \rightarrow L_x^2} \lesssim N^{2-\gamma}.$$

The estimate (2.9) is proved as for the second estimate in (2.8). The proof is complete.  $\square$

When the nonlinearity  $F(u)$  is algebraic, one can use the Fourier transform to write the commutator like  $F(Iu) - IF(u)$  as a product of Fourier transforms of  $u$  and  $Iu$ , and then measure the frequency interactions. However, in our setting, the nonlinearity is no longer algebraic, we thus need the following rougher estimate which is a modified version of the Schrödinger context (see [VZ09]).

**Lemma 2.9.** *Let  $1 < \gamma < 2$ ,  $0 < \delta < \gamma - 1$  and  $1 < q, q_1, q_2 < \infty$  be such that  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ . Then*

$$\|I(fg) - (If)g\|_{L_x^q} \lesssim N^{-(2-\gamma+\delta)} \|If\|_{L_x^{q_1}} \|\langle \nabla \rangle^{2-\gamma+\delta} g\|_{L_x^{q_2}}. \quad (2.10)$$

The proof is a slight modification of the one given in Lemma 2.5 of [VZ09]. We thus only give a sketch of the proof.

*Sketch of the proof.* By the Littlewood-Paley decomposition, we write

$$\begin{aligned} I(fg) - (If)g &= I(fP_{\leq 1}g) - (If)P_{\leq 1}g + \sum_{M>1} [I(P_{\lesssim M}fP_Mg) - (IP_{\lesssim M}f)P_Mg] \\ &\quad + \sum_{M>1} [I(P_{\gg M}fP_Mg) - (IP_{\gg M}f)P_Mg] \\ &= I(P_{\gtrsim N}fP_{\leq 1}g) - (IP_{\gtrsim N}f)P_{\leq 1}g + \sum_{M \gtrsim N} [I(P_{\lesssim M}fP_Mg) - (IP_{\lesssim M}f)P_Mg] \\ &\quad + \sum_{M>1} [I(P_{\gg M}fP_Mg) - (IP_{\gg M}f)P_Mg] \\ &= \text{Term}_1 + \text{Term}_2 + \text{Term}_3. \end{aligned}$$

Here we use the definition of the  $I$ -operator to get

$$I(P_{\ll N}fP_{\leq 1}g) = (IP_{\ll N}f)P_{\leq 1}g, \quad I(P_{\lesssim M}fP_Mg) = (IP_{\lesssim M}f)P_Mg,$$

for all  $M \ll N$ .

For the second term, using Lemma 2.7 and Lemma 2.8, we estimate

$$\begin{aligned} \|I(P_{\lesssim M} f P_M g) - (IP_{\lesssim M} f) P_M g\|_{L_x^q} &\lesssim \|P_{\lesssim M} f\|_{L_x^{q_1}} \|P_M g\|_{L_x^{q_2}}, \quad M \gtrsim N \\ &\lesssim \left(\frac{M}{N}\right)^{2-\gamma} \|If\|_{L_x^{q_1}} \|P_M g\|_{L_x^{q_2}} \\ &\lesssim M^{-\delta} N^{-(2-\gamma)} \|If\|_{L_x^{q_1}} \|\nabla\|^{2-\gamma+\delta} g\|_{L_x^{q_2}}. \end{aligned}$$

Summing over all  $N \lesssim M \in 2^{\mathbb{Z}}$ , we get

$$\|\text{Term}_2\|_{L_x^q} \lesssim N^{-(2-\gamma+\delta)} \|If\|_{L_x^{q_1}} \|\nabla\|^{2-\gamma+\delta} g\|_{L_x^{q_2}}.$$

For the third term, we write

$$\begin{aligned} I(P_{\gg M} f P_M g) - (IP_{\gg M} f) P_M g &= \sum_{1 \ll k \in \mathbb{N}} [I(P_{2^k M} f P_M g) - (IP_{2^k M} f) P_M g] \\ &= \sum_{\substack{1 \ll k \in \mathbb{N} \\ N \lesssim 2^k M}} I(P_{2^k M} f P_M g) - (IP_{2^k M} f) P_M g. \end{aligned}$$

We note that

$$[I(P_{2^k M} f P_M g) - (IP_{2^k M} f) P_M g]^\wedge(\xi) = \int_{\xi=\xi_1+\xi_2} (m_N(\xi_1 + \xi_2) - m_N(\xi_1)) \widehat{P_{2^k M} f}(\xi_1) \widehat{P_M g}(\xi_2).$$

For  $|\xi_1| \sim 2^k M \gtrsim N$  and  $|\xi_2| \sim M$ , the mean value theorem implies

$$|m_N(\xi_1 + \xi_2) - m_N(\xi_1)| \lesssim |\nabla m_N(\xi_1)| |\xi_2| \lesssim 2^{-k} \left(\frac{2^k M}{N}\right)^{\gamma-2}.$$

The Coifman-Meyer multiplier theorem (see e.g. [CM75, CM91]) then yields

$$\begin{aligned} \|I(P_{2^k M} f P_M g) - (IP_{2^k M} f) P_M g\|_{L_x^q} &\lesssim 2^{-k} \left(\frac{2^k M}{N}\right)^{\gamma-2} \|P_{2^k M} f\|_{L_x^{q_1}} \|P_M g\|_{L_x^{q_2}} \\ &\lesssim 2^{-k} M^{-(2-\gamma+\delta)} \|If\|_{L_x^{q_1}} \|\nabla\|^{2-\gamma+\delta} g\|_{L_x^{q_2}}. \end{aligned}$$

By rewrite  $2^{-k} M^{-(2-\gamma+\delta)} = 2^{-k(\gamma-1-\delta)} (2^k M)^{-(2-\gamma+\delta)}$ , we sum over all  $k \gg 1$  with  $\gamma-1 > \delta$  and  $N \lesssim 2^k M$  to get

$$\|\text{Term}_3\|_{L_x^q} \lesssim N^{-(2-\gamma+\delta)} \|If\|_{L_x^{q_1}} \|\nabla\|^{2-\gamma+\delta} g\|_{L_x^{q_2}}.$$

Finally, we consider the first term. It is proved by the same argument as for the third term. We estimate

$$\begin{aligned} \|\text{Term}_1\|_{L_x^q} &\lesssim \sum_{k \in \mathbb{N}, 2^k \gtrsim N} \|I(P_{2^k} f P_{\leq 1} g) - (IP_{2^k} f) P_{\leq 1} g\|_{L_x^q} \\ &\lesssim \sum_{k \in \mathbb{N}, 2^k \gtrsim N} 2^{-k} \|If\|_{L_x^{q_1}} \|g\|_{L_x^{q_2}} \\ &\lesssim N^{-1} \|If\|_{L_x^{q_1}} \|g\|_{L_x^{q_2}}. \end{aligned}$$

Note that the condition  $\gamma-1 > \delta$  ensures that  $N^{-1} \lesssim N^{-(2-\gamma+\delta)}$ . This completes the proof.  $\square$

As a direct consequence of Lemma 2.9 with the fact that

$$\nabla F(u) = \nabla u F'(u),$$

we have the following corollary. Note that the  $I$ -operator commutes with  $\nabla$ .

**Corollary 2.10.** *Let  $1 < \gamma < 2, 0 < \delta < \gamma - 1$  and  $1 < q, q_1, q_2 < \infty$  be such that  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$ . Then*

$$\|\nabla IF(u) - (I\nabla u)F'(u)\|_{L_x^q} \lesssim N^{-(2-\gamma+\delta)} \|\nabla Iu\|_{L_x^{q_1}} \|\langle \nabla \rangle^{2-\gamma+\delta} F'(u)\|_{L_x^{q_2}}. \quad (2.11)$$

**2.5. Interaction Morawetz inequality.** We end this section by recalling the interaction Morawetz inequality for the nonlinear fourth-order Schrödinger equation. This estimate was first established by Pausader in [Pau2] for  $d \geq 7$ . Later, Miao-Wu-Zhang in [MWZ15] extended this interaction Morawetz estimate to  $d \geq 5$ .

**Proposition 2.11** (Interaction Morawetz inequality [Pau2], [MWZ15]). *Let  $d \geq 5, J$  be a compact time interval and  $u$  a solution to (NL4S) on the spacetime slab  $J \times \mathbb{R}^d$ . Then we have the following a priori estimate:*

$$\|\langle \nabla \rangle^{-\frac{d-5}{4}} u\|_{L_t^4(J, L_x^4)} \lesssim \|u_0\|_{L_x^2}^{\frac{1}{2}} \|u\|_{L_t^\infty(J, \dot{H}_x^{\frac{1}{2}})}^{\frac{1}{2}}. \quad (2.12)$$

By interpolating (2.12) and the trivial estimate

$$\|u\|_{L_t^\infty(J, \dot{H}_x^{\frac{1}{2}})} \leq \|u\|_{L_t^\infty(J, \dot{H}_x^{\frac{1}{2}})},$$

we obtain

$$\|u\|_{L_t^{2(d-3)}(J, L_x^{\frac{2(d-3)}{d-4}})} \lesssim \left( \|u_0\|_{L_x^2} \|u\|_{L_t^\infty(J, \dot{H}_x^{\frac{1}{2}})} \right)^{\frac{1}{d-3}} \|u\|_{L_t^\infty(J, \dot{H}_x^{\frac{1}{2}})}^{\frac{d-5}{d-3}} = \|u_0\|_{L_x^2}^{\frac{1}{d-3}} \|u\|_{L_t^\infty(J, \dot{H}_x^{\frac{1}{2}})}^{\frac{d-4}{d-3}}.$$

Using Sobolev embedding in time, we get

$$\|u\|_{M(J)} := \|u\|_{L_t^{\frac{8(d-3)}{d}}(J, L_x^{\frac{2(d-3)}{d-4}})} \lesssim |J|^{\frac{d-4}{8(d-3)}} \|u_0\|_{L_x^2}^{\frac{1}{d-3}} \|u\|_{L_t^\infty(J, \dot{H}_x^{\frac{1}{2}})}^{\frac{d-4}{d-3}}. \quad (2.13)$$

Here  $(\frac{8(d-3)}{d}, \frac{2(d-3)}{d-4})$  is a biharmonic admissible pair.

### 3. ALMOST CONSERVATION LAW

For any spacetime slab  $J \times \mathbb{R}^d$ , we define

$$Z_I(J) := \sup_{(p,q) \in B} \|\langle \Delta \rangle Iu\|_{L_t^p(J, L_x^q)}.$$

Note that in our consideration  $5 \leq d \leq 7$ , the biharmonic admissible condition  $(p, q) \in B$  ensures  $q < \infty$ . Let us start with the following commutator estimates.

**Lemma 3.1.** *Let  $5 \leq d \leq 7, 1 < \gamma < 2$  and  $0 < \delta < \gamma - 1$ . Then*

$$\|\nabla IF(u) - (I\nabla u)F'(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+2}})} \lesssim N^{-(2-\gamma+\delta)} (Z_I(J))^{1+\frac{8}{d}}, \quad (3.1)$$

$$\|\nabla IF(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+2}})} \lesssim \|u\|_{M(J)}^{\frac{8}{d}} Z_I(J) + N^{-(2-\gamma+\delta)} (Z_I(J))^{1+\frac{8}{d}}, \quad (3.2)$$

where  $\|u\|_{M(J)}$  is given in (2.13). In particular,

$$\|\nabla IF(u)\|_{L_t^2(J, L_x^{\frac{2d}{d+2}})} \lesssim (Z_I(J))^{1+\frac{8}{d}}. \quad (3.3)$$

*Proof.* For simplifying the notation, we shall drop the dependence on the time interval  $J$ . We apply (2.11) with  $q = \frac{2d}{d+2}, q_1 = \frac{2d(d-3)}{d^2-9d+22}$  and  $q_2 = \frac{d(d-3)}{2(2d-7)}$  to get

$$\|\nabla IF(u) - (I\nabla u)F'(u)\|_{L_x^{\frac{2d}{d+2}}} \lesssim N^{-(2-\gamma+\delta)} \|\nabla Iu\|_{L_x^{\frac{2d(d-3)}{d^2-9d+22}}} \|\langle \nabla \rangle^{2-\gamma+\delta} F'(u)\|_{L_x^{\frac{d(d-3)}{2(2d-7)}}}.$$

We then apply Hölder's inequality to have

$$\|\nabla IF(u) - (I\nabla u)F'(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim N^{-\alpha} \|\nabla Iu\|_{L_t^{\frac{2(d-3)}{d-4}} L_x^{\frac{2d(d-3)}{d^2-9d+22}}} \|\langle \nabla \rangle^\alpha F'(u)\|_{L_t^{2(d-3)} L_x^{\frac{d(d-3)}{2(2d-7)}}},$$

where  $\alpha = 2 - \gamma + \delta \in (0, 1)$  by our assumptions. For the first factor in the right hand side, we use the Sobolev embedding to obtain

$$\|\nabla Iu\|_{L_t^{\frac{2(d-3)}{d-4}} L_x^{\frac{2d(d-3)}{d^2-9d+22}}} \lesssim \|\Delta Iu\|_{L_t^{\frac{2(d-3)}{d-4}} L_x^{\frac{2d(d-3)}{d^2-7d+16}}} \lesssim Z_I, \quad (3.4)$$

where  $\left(\frac{2(d-3)}{d-4}, \frac{2d(d-3)}{d^2-7d+16}\right)$  is a biharmonic admissible pair. For the second factor, we estimate

$$\|\langle \nabla \rangle^\alpha F'(u)\|_{L_t^{2(d-3)} L_x^{\frac{d(d-3)}{2(2d-7)}}} \lesssim \|F'(u)\|_{L_t^{2(d-3)} L_x^{\frac{d(d-3)}{2(2d-7)}}} + \|\langle \nabla \rangle^\alpha F'(u)\|_{L_t^{2(d-3)} L_x^{\frac{d(d-3)}{2(2d-7)}}}. \quad (3.5)$$

Since  $F'(u) = O(|u|^{\frac{8}{d}})$ , we use (2.7) to have

$$\|F'(u)\|_{L_t^{2(d-3)} L_x^{\frac{d(d-3)}{2(2d-7)}}} \lesssim \|u\|_{L_t^{\frac{8}{d}} L_x^{\frac{16(d-3)}{d} L_x^{\frac{4(d-3)}{2d-7}}}}^{\frac{8}{d}} \lesssim Z_I^{\frac{8}{d}}, \quad (3.6)$$

where  $\left(\frac{16(d-3)}{d}, \frac{4(d-3)}{2d-7}\right)$  is biharmonic admissible. In order to treat the second term in (3.5), we apply Lemma 2.1 with  $q = \frac{d(d-3)}{2(2d-7)}$ ,  $q_1 = \frac{2d(d-3)}{-d^2+11d-26}$  and  $q_2 = \frac{2d(d-3)}{d^2-3d-2}$  to get

$$\|\langle \nabla \rangle^\alpha F'(u)\|_{L_t^{2(d-3)} L_x^{\frac{d(d-3)}{2(2d-7)}}} \lesssim \|F''(u)\|_{L_t^{4(d-3)} L_x^{\frac{2d(d-3)}{-d^2+11d-26}}} \|\langle \nabla \rangle^\alpha u\|_{L_t^{4(d-3)} L_x^{\frac{2d(d-3)}{d^2-3d-2}}}.$$

As  $F''(u) = O(|u|^{\frac{8}{d}-1})$ , we have

$$\|F''(u)\|_{L_t^{4(d-3)} L_x^{\frac{2d(d-3)}{-d^2+11d-26}}} \lesssim \|u\|_{L_t^{\frac{8}{d}-1} L_x^{\frac{4(8-d)(d-3)}{d} L_x^{\frac{2(8-d)(d-3)}{-d^2+11d-26}}}}^{\frac{8}{d}-1} \lesssim Z_I^{\frac{8}{d}-1}. \quad (3.7)$$

Here  $\left(\frac{4(8-d)(d-3)}{d}, \frac{2(8-d)(d-3)}{-d^2+11d-26}\right)$  is biharmonic admissible. Since  $\left(4(d-3), \frac{2d(d-3)}{d^2-3d-2}\right)$  is also a biharmonic admissible, we have from (2.7) that

$$\|\langle \nabla \rangle^\alpha u\|_{L_t^{4(d-3)} L_x^{\frac{2d(d-3)}{d^2-3d-2}}} \lesssim Z_I. \quad (3.8)$$

Note that  $\alpha < 1 < \gamma$ . Collecting (3.4), (3.6), (3.7) and (3.8), we prove (3.1).

We now prove (3.2). We have from (3.1) and the triangle inequality that

$$\|\nabla IF(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \|(\nabla Iu)F'(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} + N^{-(2-\gamma+\delta)} Z_I^{1+\frac{8}{d}}. \quad (3.9)$$

The Hölder inequality gives

$$\|(\nabla Iu)F'(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \|\nabla Iu\|_{L_t^{\frac{2(d-3)}{d-5}} L_x^{\frac{2d(d-3)}{d^2-9d+26}}} \|F'(u)\|_{L_t^{d-3} L_x^{\frac{d(d-3)}{4(d-4)}}}.$$

We use the Sobolev embedding to estimate

$$\|\nabla Iu\|_{L_t^{\frac{2(d-3)}{d-5}} L_x^{\frac{2d(d-3)}{d^2-9d+26}}} \lesssim \|\Delta Iu\|_{L_t^{\frac{2(d-3)}{d-5}} L_x^{\frac{2d(d-3)}{d^2-7d+20}}} \lesssim Z_I. \quad (3.10)$$

Here  $\left(\frac{2(d-3)}{d-5}, \frac{2d(d-3)}{d^2-7d+20}\right)$  is biharmonic admissible. Since  $F'(u) = O(|u|^{\frac{8}{d}})$ , we have

$$\|F'(u)\|_{L_t^{d-3} L_x^{\frac{d(d-3)}{4(d-4)}}} \lesssim \|u\|_{L_t^{\frac{8}{d}} L_x^{\frac{8(d-3)}{d} L_x^{\frac{2(d-3)}{d-4}}}}^{\frac{8}{d}} = \|u\|_M^{\frac{8}{d}}. \quad (3.11)$$

Combining (3.9), (3.10) and (3.11), we obtain (3.2). The estimate (3.3) follows directly from (3.2) and (2.7). Note that  $\left(\frac{8(d-3)}{d}, \frac{2(d-3)}{d-4}\right)$  is biharmonic admissible. The proof is complete.  $\square$

We are now able to prove the almost conservation law for the modified energy functional  $E(Iu)$ , where

$$E(Iu(t)) = \frac{1}{2} \|Iu(t)\|_{H_x^2}^2 + \frac{d}{2d+8} \|Iu(t)\|_{L_x^{\frac{2d+8}{d}}}^{\frac{2d+8}{d}}.$$

**Proposition 3.2.** *Let  $5 \leq d \leq 7$ ,  $\max\{3 - \frac{8}{d}, \frac{8}{d}\} < \gamma < 2$  and  $0 < \delta < \gamma + \frac{8}{d} - 3$ . Assume that  $u \in L^\infty([0, T], \mathcal{S}(\mathbb{R}^d))$  is a solution to (NL4S) on a time interval  $J = [0, T]$ , and satisfies  $\|Iu_0\|_{H_x^2} \leq 1$ . Assume in addition that  $u$  satisfies the a priori bound*

$$\|u\|_{M(J)} \leq \mu,$$

for some small constant  $\mu > 0$ . Then, for  $N$  sufficiently large,

$$\sup_{t \in [0, T]} |E(Iu(t)) - E(Iu_0)| \lesssim N^{-(2-\gamma+\delta)}. \quad (3.12)$$

Here the implicit constant depends only on the size of  $E(Iu_0)$ .

*Proof.* We again drop the notation  $J$  for simplicity. Our first step is to control the size of  $Z_I$ . Applying  $I$ ,  $\Delta I$  to (NL4S), and then using Strichartz estimates (2.3), (2.4), we have

$$Z_I \lesssim \|Iu_0\|_{H_x^2} + \|IF(u)\|_{L_t^2 L_x^{\frac{2d}{d+4}}} + \|\nabla IF(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}}. \quad (3.13)$$

Using (3.2), we have

$$\|\nabla IF(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \|u\|_{M}^{\frac{8}{d}} Z_I + N^{-(2-\gamma+\delta)} Z_I^{1+\frac{8}{d}} \lesssim \mu^{\frac{8}{d}} Z_I + N^{-(2-\gamma+\delta)} Z_I^{1+\frac{8}{d}}. \quad (3.14)$$

We next drop the  $I$ -operator and use Hölder's inequality to estimate

$$\begin{aligned} \|IF(u)\|_{L_t^2 L_x^{\frac{2d}{d+4}}} &\lesssim \| |u|^{\frac{8}{d}} \|_{L_t^{d-3} L_x^{\frac{4(d-3)}{d(d-4)}}} \|u\|_{L_t^{\frac{2(d-3)}{d-5}} L_x^{\frac{2d(d-3)}{d^2-7d+20}}} \\ &\lesssim \|u\|_{L_t^{\frac{8}{d}} L_x^{\frac{8(d-3)}{d}}}^{\frac{8}{d}} \|u\|_{L_t^{\frac{2(d-3)}{d-5}} L_x^{\frac{2d(d-3)}{d^2-7d+20}}} \\ &\lesssim \|u\|_{M}^{\frac{8}{d}} Z_I \lesssim \mu^{\frac{8}{d}} Z_I. \end{aligned} \quad (3.15)$$

The last inequality follows from (2.7) and the fact  $\left(\frac{2(d-3)}{d-5}, \frac{2d(d-3)}{d^2-7d+20}\right)$  is biharmonic admissible. Collecting from (3.13) to (3.15), we obtain

$$Z_I \lesssim \|Iu_0\|_{H_x^2} + \mu^{\frac{8}{d}} Z_I + N^{-(2-\gamma+\delta)} Z_I^{1+\frac{8}{d}}.$$

By taking  $\mu$  sufficiently small and  $N$  sufficiently large, the continuity argument gives

$$Z_I \lesssim \|Iu_0\|_{H_x^2} \leq 1. \quad (3.16)$$

Next, we have from a direct computation that

$$\partial_t E(Iu(t)) = \operatorname{Re} \int \overline{I\partial_t u} (\Delta^2 Iu + F(Iu)) dx.$$

By the Fundamental Theorem of Calculus,

$$E(Iu(t)) - E(Iu_0) = \int_0^t \partial_s E(Iu(s)) ds = \operatorname{Re} \int_0^t \int \overline{I\partial_s u} (\Delta^2 Iu + F(Iu)) dx ds.$$

Using  $I\partial_t u = i\Delta^2 Iu + iIF(u)$ , we see that

$$\begin{aligned} E(Iu(t)) - E(Iu_0) &= \operatorname{Re} \int_0^t \int \overline{I\partial_s u} (F(Iu) - IF(u)) dx ds \\ &= \operatorname{Im} \int_0^t \int \overline{\Delta^2 Iu + IF(u)} (F(Iu) - IF(u)) dx ds \\ &= \operatorname{Im} \int_0^t \int \overline{\Delta Iu} \Delta (F(Iu) - IF(u)) dx ds \\ &\quad + \operatorname{Im} \int_0^t \int \overline{IF(u)} (F(Iu) - IF(u)) dx ds. \end{aligned}$$

We next write

$$\begin{aligned} \Delta(F(Iu) - IF(u)) &= (\Delta Iu)F'(Iu) + |\nabla Iu|^2 F''(Iu) - I(\Delta F'(u)) - I(|\nabla u|^2 F''(u)) \\ &= (\Delta Iu)(F'(Iu) - F'(u)) + |\nabla Iu|^2 (F''(Iu) - F''(u)) + \nabla Iu \cdot (\nabla Iu - \nabla u) F''(u) \\ &\quad + (\Delta Iu)F'(u) - I(\Delta u F'(u)) + (I\nabla u) \cdot \nabla u F''(u) - I(\nabla u \cdot \nabla u F''(u)). \end{aligned}$$

Therefore,

$$E(Iu(t)) - E(Iu_0) = \operatorname{Im} \int_0^t \int \overline{\Delta Iu} \Delta Iu (F'(Iu) - F'(u)) dx ds \quad (3.17)$$

$$+ \operatorname{Im} \int_0^t \int \overline{\Delta Iu} |\nabla Iu|^2 (F''(Iu) - F''(u)) dx ds \quad (3.18)$$

$$+ \operatorname{Im} \int_0^t \int \overline{\Delta Iu} \nabla Iu \cdot (\nabla Iu - \nabla u) F''(u) dx ds \quad (3.19)$$

$$+ \operatorname{Im} \int_0^t \int \overline{\Delta Iu} [(\Delta Iu)F'(u) - I(\Delta u F'(u))] dx ds \quad (3.20)$$

$$+ \operatorname{Im} \int_0^t \int \overline{\Delta Iu} [(I\nabla u) \cdot \nabla u F''(u) - I(\nabla u \cdot \nabla u F''(u))] dx ds \quad (3.21)$$

$$+ \operatorname{Im} \int_0^t \int \overline{IF(u)} (F(Iu) - IF(u)) dx ds. \quad (3.22)$$

Let us consider (3.17). By Hölder's inequality, we estimate

$$\begin{aligned} |(3.17)| &\lesssim \|\Delta Iu\|_{L_t^4 L_x^{\frac{2d}{d-2}}}^2 \|F'(Iu) - F'(u)\|_{L_t^2 L_x^{\frac{d}{2}}} \\ &\lesssim Z_I^2 \| |Iu - u| (|Iu| + |u|)^{\frac{8}{d}-1} \|_{L_t^2 L_x^{\frac{d}{2}}} \\ &\lesssim Z_I^2 \|P_{>N} u\|_{L_t^{\frac{16}{d}} L_x^4} \|u\|_{L_t^{\frac{16}{d}} L_x^4}^{\frac{8}{d}-1}. \end{aligned} \quad (3.23)$$

By (2.6), we bound

$$\|P_{>N} u\|_{L_t^{\frac{16}{d}} L_x^4} \lesssim N^{-2} \|\Delta Iu\|_{L_t^{\frac{16}{d}} L_x^4} \lesssim N^{-2} Z_I, \quad (3.24)$$

where  $(\frac{16}{d}, 4)$  is biharmonic admissible. Similarly, we have from (2.7) that

$$\|u\|_{L_t^{\frac{16}{d}} L_x^4} \lesssim Z_I. \quad (3.25)$$

Combining (3.23) – (3.25), we get

$$|(3.17)| \lesssim N^{-2} Z_I^{2+\frac{8}{d}}. \quad (3.26)$$

We next bound

$$\begin{aligned} |(3.18)| &\lesssim \|\Delta Iu\|_{L_t^4 L_x^{\frac{2d}{d-2}}} \|\nabla Iu\|_{L_t^{\frac{16}{11}} L_x^{\frac{4d}{4d-11}}}^2 \|F''(Iu) - F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}} \\ &\lesssim \|\Delta Iu\|_{L_t^4 L_x^{\frac{2d}{d-2}}} \|\nabla Iu\|_{L_t^{\frac{32}{11}} L_x^{\frac{8d}{4d-11}}}^2 \|F''(Iu) - F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}} \\ &\lesssim Z_I^3 \|Iu - u\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}}^{\frac{8}{d}-1} \\ &\lesssim Z_I^3 \|P_{>N}u\|_{L_t^{\frac{16(8-d)}{d}} L_x^{\frac{4(8-d)}{15-2d}}}^{\frac{8}{d}-1} \\ &\lesssim N^{-2(\frac{8}{d}-1)} Z_I^{2+\frac{8}{d}}. \end{aligned} \quad (3.27)$$

Here we drop the  $I$ -operator and apply (2.7) with the fact  $\gamma > 1$  to get the third line. We also use the fact that for  $5 \leq d \leq 7$ ,

$$|F''(z) - F''(\zeta)| \lesssim |z - \zeta|^{\frac{8}{d}-1}, \quad \forall z, \zeta \in \mathbb{C}.$$

The last estimate uses (3.24). Note that  $\left(\frac{32}{11}, \frac{8d}{4d-11}\right)$  and  $\left(\frac{16(8-d)}{d}, \frac{4(8-d)}{15-2d}\right)$  are biharmonic admissible. Similarly, we estimate

$$\begin{aligned} |(3.19)| &\lesssim \|\Delta Iu\|_{L_t^4 L_x^{\frac{2d}{d-2}}} \|\nabla Iu\|_{L_t^{\frac{32}{11}} L_x^{\frac{8d}{4d-11}}} \|\nabla Iu - \nabla u\|_{L_t^{\frac{32}{11}} L_x^{\frac{8d}{4d-11}}} \|F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}} \\ &\lesssim Z_I^2 \|\nabla P_{>N}u\|_{L_t^{\frac{32}{11}} L_x^{\frac{8d}{4d-11}}} \|F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}}. \end{aligned}$$

We next use (2.6) to have

$$\|\nabla P_{>N}u\|_{L_t^{\frac{32}{11}} L_x^{\frac{8d}{4d-11}}} \lesssim N^{-1} \|\Delta Iu\|_{L_t^{\frac{32}{11}} L_x^{\frac{8d}{4d-11}}} \lesssim N^{-1} Z_I.$$

As  $F''(u) = O(|u|^{\frac{8}{d}-1})$ , we use (2.7) to get

$$\|F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}} \lesssim \|u\|_{L_t^{\frac{16(8-d)}{d}} L_x^{\frac{4(8-d)}{15-2d}}}^{\frac{8}{d}-1} \lesssim Z_I^{\frac{8}{d}-1}. \quad (3.28)$$

We thus obtain

$$|(3.19)| \lesssim N^{-1} Z_I^{2+\frac{8}{d}}. \quad (3.29)$$

By Hölder's inequality,

$$|(3.20)| \lesssim \|\Delta Iu\|_{L_t^2 L_x^{\frac{2d}{d-4}}} \|(\Delta Iu)F'(u) - I(\Delta uF'(u))\|_{L_t^2 L_x^{\frac{2d}{d+4}}}.$$

We then apply Lemma 2.9 with  $q = \frac{2d}{d+4}$ ,  $q_1 = \frac{2d(d-3)}{d^2-7d+16}$  and  $q_2 = \frac{d(d-3)}{2(2d-7)}$  to get

$$\|(\Delta Iu)F'(u) - I(\Delta uF'(u))\|_{L_x^{\frac{2d}{d+4}}} \lesssim N^{-\alpha} \|\Delta Iu\|_{L_x^{\frac{2d(d-3)}{d^2-7d+16}}} \|\langle \nabla \rangle^\alpha F'(u)\|_{L_x^{\frac{d(d-3)}{2(2d-7)}}},$$

where  $\alpha = 2 - \gamma + \delta$ . The Hölder inequality then implies

$$\|(\Delta Iu)F'(u) - I(\Delta uF'(u))\|_{L_t^2 L_x^{\frac{2d}{d+4}}} \lesssim N^{-\alpha} \|\Delta Iu\|_{L_t^{\frac{2(d-3)}{d-4}} L_x^{\frac{2d(d-3)}{d^2-7d+16}}} \|\langle \nabla \rangle^\alpha F'(u)\|_{L_t^{2(d-3)} L_x^{\frac{d(d-3)}{2(2d-7)}}}.$$

We have from (3.5), (3.6), (3.7) and (3.8) that

$$\|\langle \nabla \rangle^\alpha F'(u)\|_{L_t^{2(d-3)} L_x^{\frac{d(d-3)}{2(2d-7)}}} \lesssim Z_I^{\frac{8}{d}}.$$

Thus

$$|(3.20)| \lesssim N^{-(2-\gamma+\delta)} Z_I^{2+\frac{8}{d}}. \quad (3.30)$$

Similarly, we bound

$$|(3.21)| \lesssim \|\Delta I u\|_{L_t^4 L_x^{\frac{2d}{d-2}}} \|(I \nabla u) \cdot \nabla u F''(u) - I(\nabla u \cdot \nabla u F''(u))\|_{L_t^{\frac{4}{3}} L_x^{\frac{2d}{d+2}}}. \quad (3.31)$$

Applying Lemma 2.9 with  $q = \frac{2d}{d+2}$ ,  $q_1 = \frac{8d}{4d-11}$  and  $q_2 = \frac{8d}{19}$  and using Hölder inequality, we have

$$\|(I \nabla u) \cdot \nabla u F''(u) - I(\nabla u \cdot \nabla u F''(u))\|_{L_t^{\frac{4}{3}} L_x^{\frac{2d}{d+2}}} \lesssim N^{-\alpha} \|I \nabla u\|_{L_t^{\frac{32}{11}} L_x^{\frac{8d}{4d-11}}} \|\langle \nabla \rangle^\alpha (\nabla u F''(u))\|_{L_t^{\frac{8}{5}} L_x^{\frac{8d}{19}}}. \quad (3.32)$$

The fractional chain rule implies

$$\begin{aligned} \|\langle \nabla \rangle^\alpha (\nabla u F''(u))\|_{L_t^{\frac{8}{5}} L_x^{\frac{8d}{19}}} &\lesssim \|\langle \nabla \rangle^{\alpha+1} u\|_{L_t^{\frac{32}{11}} L_x^{\frac{8d}{4d-11}}} \|F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}} \\ &\quad + \|\nabla u\|_{L_t^{\frac{32}{11}} L_x^{\frac{8d}{4d-11}}} \|\langle \nabla \rangle^\alpha F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}}. \end{aligned} \quad (3.33)$$

By our assumptions on  $\gamma$  and  $\delta$ , we see that  $\alpha + 1 < \gamma$ . By (2.7) (and dropping the  $I$ -operator if necessary) and (3.28),

$$\|I \nabla u\|_{L_t^{\frac{32}{11}} L_x^{\frac{8d}{4d-11}}}, \|\nabla u\|_{L_t^{\frac{32}{11}} L_x^{\frac{8d}{4d-11}}}, \|\langle \nabla \rangle^{\alpha+1} u\|_{L_t^{\frac{32}{11}} L_x^{\frac{8d}{4d-11}}} \lesssim Z_I, \quad \|F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}} \lesssim Z_I^{\frac{8}{d}-1}. \quad (3.34)$$

Here  $(\frac{32}{11}, \frac{8d}{4d-11})$  is biharmonic admissible. It remains to bound  $\|\langle \nabla \rangle^\alpha F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}}$ . To do so, we use

$$\|\langle \nabla \rangle^\alpha F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}} \lesssim \|F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}} + \|\langle \nabla \rangle^\alpha F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}}. \quad (3.35)$$

The first term in the right hand side is treated in (3.28). For the second term in the right hand side, we make use of the fractional chain rule given in Lemma 2.2 with  $\beta = \frac{8}{d} - 1$ ,  $\alpha = 2 - \gamma + \delta$ ,  $q = \frac{4d}{15-2d}$  and  $q_1, q_2$  satisfying

$$\left(\frac{8}{d} - 1 - \frac{\alpha}{\rho}\right) q_1 = \frac{\alpha}{\rho} q_2 = \frac{4(8-d)}{15-2d},$$

and  $\frac{\alpha}{\frac{8}{d}-1} < \rho < 1$ . Note that the choice of  $\rho$  is possible since  $\alpha < \frac{8}{d} - 1$  by our assumptions. With these choices, we have

$$\left(1 - \frac{\alpha}{\beta \rho}\right) q_1 = \frac{4d}{15-2d} > 1,$$

for  $5 \leq d \leq 7$ . Then,

$$\|\langle \nabla \rangle^\alpha F''(u)\|_{L_x^{\frac{4d}{15-2d}}} \lesssim \| |u|^{\frac{8}{d}-1-\frac{\alpha}{\rho}} \|_{L_x^{q_1}} \|\langle \nabla \rangle^\alpha \rho u\|_{L_x^{\frac{\alpha}{\rho} q_2}} \lesssim \| |u|^{\frac{8}{d}-1-\frac{\alpha}{\rho}} \|_{L_x^{(\frac{8}{d}-1-\frac{\alpha}{\rho}) q_1}} \|\langle \nabla \rangle^\alpha \rho u\|_{L_x^{\frac{\alpha}{\rho} q_2}}.$$

By Hölder's inequality,

$$\begin{aligned} \|\langle \nabla \rangle^\alpha F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}} &\lesssim \| |u|^{\frac{8}{d}-1-\frac{\alpha}{\rho}} \|_{L_t^{(\frac{8}{d}-1-\frac{\alpha}{\rho}) p_1} L_x^{(\frac{8}{d}-1-\frac{\alpha}{\rho}) q_1}} \|\langle \nabla \rangle^\alpha \rho u\|_{L_t^{\frac{\alpha}{\rho} p_2} L_x^{\frac{\alpha}{\rho} q_2}} \\ &= \| |u|^{\frac{8}{d}-1-\frac{\alpha}{\rho}} \|_{L_t^{\frac{16(8-d)}{d}} L_x^{\frac{4(8-d)}{15-2d}}} \|\langle \nabla \rangle^\alpha \rho u\|_{L_t^{\frac{16(8-d)}{d}} L_x^{\frac{4(8-d)}{15-2d}}}, \end{aligned}$$

provided

$$\left(\frac{8}{d} - 1 - \frac{\alpha}{\rho}\right)p_1 = \frac{\alpha}{\rho}p_2 = \frac{16(8-d)}{d}.$$

Since  $\left(\frac{16(8-d)}{d}, \frac{4(8-d)}{15-2d}\right)$  is biharmonic admissible, we have from (2.7) with the fact  $0 < \rho < 1 < \gamma$  that

$$\|\lvert \nabla \rvert^\alpha F''(u)\|_{L_t^{16} L_x^{\frac{4d}{15-2d}}} \lesssim Z_I^{\frac{8}{d}-1}. \quad (3.36)$$

Collecting from (3.31) to (3.36), we get

$$|(3.21)| \lesssim N^{-(2-\gamma+\delta)} Z_I^{2+\frac{8}{d}}. \quad (3.37)$$

Finally, we consider (3.22). We bound

$$\begin{aligned} |(3.22)| &\lesssim \|\lvert \nabla \rvert^{-1} IF(u)\|_{L_t^2 L_x^{\frac{2d}{d-2}}} \|\nabla(F(Iu) - IF(u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \\ &\lesssim \|\nabla IF(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \|\nabla(F(Iu) - IF(u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}}. \end{aligned} \quad (3.38)$$

By (3.3),

$$\|\nabla IF(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim Z_I^{1+\frac{8}{d}}.$$

By the triangle inequality, we estimate

$$\|\nabla(F(Iu) - IF(u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim \|(\nabla Iu)(F'(Iu) - F'(u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}} + \|(\nabla Iu)F'(u) - \nabla IF(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}}.$$

We firstly use Hölder's inequality and estimate as in (3.23) to get

$$\begin{aligned} \|(\nabla Iu)(F'(Iu) - F'(u))\|_{L_t^2 L_x^{\frac{2d}{d+2}}} &\lesssim \|\nabla Iu\|_{L_t^\infty L_x^{\frac{2d}{d-2}}} \|F'(Iu) - F'(u)\|_{L_t^2 L_x^{\frac{d}{2}}} \\ &\lesssim \|\Delta Iu\|_{L_t^\infty L_x^2} \|P_{>N} u\|_{L_t^{\frac{16}{d}} L_x^4} \|u\|_{L_t^{\frac{16}{d}} L_x^4}^{\frac{8}{d}-1} \\ &\lesssim N^{-2} Z_I^{1+\frac{8}{d}}. \end{aligned} \quad (3.39)$$

By (3.1),

$$\|(\nabla Iu)F'(u) - \nabla IF(u)\|_{L_t^2 L_x^{\frac{2d}{d+2}}} \lesssim N^{-(2-\gamma+\delta)} Z_I^{1+\frac{8}{d}}. \quad (3.40)$$

Combining (3.38) – (3.40), we get

$$|(3.22)| \lesssim Z_I^{1+\frac{8}{d}} (N^{-2} Z_I^{1+\frac{8}{d}} + N^{-(2-\gamma+\delta)} Z_I^{1+\frac{8}{d}}). \quad (3.41)$$

The desired estimate (3.12) follows from (3.26), (3.27), (3.29), (3.30), (3.41) and (3.16). The proof is complete.  $\square$

#### 4. GLOBAL WELL-POSEDNESS

Let us now show the global existence given in Theorem 1.1. By density argument, we assume that  $u_0 \in C_0^\infty(\mathbb{R}^d)$ . Let  $u$  be a global solution to (NL4S) with initial data  $u_0$ . In order to apply the almost conservation law, we need the modified energy of initial data to be small. Since  $E(Iu_0)$  is not necessarily small, we will use the scaling (1.3) to make  $E(Iu_\lambda(0))$  small. We have

$$E(Iu_\lambda(0)) = \frac{1}{2} \|Iu_\lambda(0)\|_{\dot{H}_x^2}^2 + \frac{d}{2d+8} \|Iu_\lambda(0)\|_{L_x^{\frac{2d+8}{d}}}^{\frac{2d+8}{d}}. \quad (4.1)$$

We use (2.9) to estimate

$$\|Iu_\lambda(0)\|_{\dot{H}_x^2} \lesssim N^{2-\gamma} \|u_\lambda(0)\|_{\dot{H}_x^\gamma} = N^{2-\gamma} \lambda^{-\gamma} \|u_0\|_{\dot{H}_x^\gamma}. \quad (4.2)$$

In order to make  $\|Iu_\lambda(0)\|_{\dot{H}_x^2} \leq \frac{1}{8}$ , we choose

$$\lambda \approx N^{\frac{2-\gamma}{\gamma}}. \quad (4.3)$$

We next bound  $\|Iu_\lambda(0)\|_{L_x^{\frac{2d+8}{d}}}$ . Note that we can easily estimate this norm by the Sobolev embedding

$$\|Iu_\lambda(0)\|_{L_x^{\frac{2d+8}{d}}} \lesssim \|u_\lambda(0)\|_{L_x^{\frac{2d+8}{d}}} = \lambda^{-\frac{2d}{d+4}} \|u_0\|_{L_x^{\frac{2d+8}{d}}} \lesssim \lambda^{-\frac{2d}{d+4}} \|u_0\|_{H_x^\gamma},$$

but it requires  $\gamma \geq \frac{2d}{d+4}$ . In order to remove this requirement, we use the technique of [CKSTT04] (see also [MWZ15]). We firstly separate the frequency space into the domains

$$\Omega_1 := \left\{ \xi \in \mathbb{R}^d, |\xi| \lesssim \frac{1}{\lambda} \right\}, \quad \Omega_2 := \left\{ \xi \in \mathbb{R}^d, \frac{1}{\lambda} \lesssim |\xi| \lesssim N \right\}, \quad \Omega_3 := \left\{ \xi \in \mathbb{R}^d, |\xi| \gtrsim N \right\},$$

and then write

$$[u_\lambda(0)]^\sim(\xi) = (\chi_1(\xi) + \chi_2(\xi) + \chi_3(\xi))[u_\lambda(0)]^\sim(\xi),$$

for non-negative smooth functions  $\chi_j$  supported in  $\Omega_j, j = 1, 2, 3$  respectively and satisfying  $\sum \chi_j(\xi) = 1$ . Thus

$$Iu_\lambda(0) = \chi_1(D)Iu_\lambda(0) + \chi_2(D)Iu_\lambda(0) + \chi_3(D)Iu_\lambda(0).$$

We now use the Sobolev embedding to have

$$\|\chi_1(D)Iu_\lambda(0)\|_{L_x^{\frac{2d+8}{d}}} \lesssim \|\nabla|^{\frac{2d}{d+4}} \chi_1(D)Iu_\lambda(0)\|_{L_x^2} \lesssim \|\nabla|^{\frac{2d}{d+4}} \chi_1(D)I\|_{L_x^2 \rightarrow L_x^2} \|u_\lambda(0)\|_{L_x^2}.$$

Thanks to the support of  $\chi_1$ , the functional calculus gives

$$\|\nabla|^{\frac{2d}{d+4}} \chi_1(D)I\|_{L_x^2 \rightarrow L_x^2} \lesssim \|\xi\|^{\frac{2d}{d+4}-\alpha} |\xi|^\alpha \chi_1(\xi)\|_{L_\xi^\infty} \lesssim \lambda^{\alpha-\frac{2d}{d+4}}, \quad (4.4)$$

provided  $0 < \alpha < \frac{2d}{d+4}$ . Similarly,

$$\|\chi_3(D)Iu_\lambda(0)\|_{L_x^{\frac{2d+8}{d}}} \lesssim \|\nabla|^{\frac{2d}{d+4}} \chi_3(D)Iu_\lambda(0)\|_{L_x^2} \lesssim \|\nabla|^{\frac{2d}{d+4}-\gamma} \chi_3(D)I\|_{L_x^2 \rightarrow L_x^2} \|u_\lambda(0)\|_{\dot{H}_x^\gamma}.$$

A direct computation shows

$$\|u_\lambda(0)\|_{\dot{H}_x^\gamma} = \lambda^{-\gamma} \|u_0\|_{\dot{H}_x^\gamma}. \quad (4.5)$$

Using the support of  $\chi_3$ , the functional calculus again gives

$$\|\nabla|^{\frac{2d}{d+4}-\gamma} \chi_3(D)I\|_{L_x^2 \rightarrow L_x^2} \lesssim \|\xi\|^{\frac{2d}{d+4}-\gamma} \chi_3(\xi) (N|\xi|^{-1})^{2-\gamma}\|_{L_\xi^\infty} \lesssim N^{\frac{2d}{d+4}-\gamma}. \quad (4.6)$$

To obtain this bound, we split into two cases.

When  $\frac{2d}{d+4} \geq \gamma$ , we simply bound

$$\|\xi\|^{\frac{2d}{d+4}-\gamma} \chi_3(\xi) (N|\xi|^{-1})^{2-\gamma}\|_{L_\xi^\infty} \lesssim 1 \lesssim N^{\frac{2d}{d+4}-\gamma}.$$

When  $\gamma > \frac{2d}{d+4}$ , we write

$$\|\xi\|^{\frac{2d}{d+4}-\gamma} \chi_3(\xi) (N|\xi|^{-1})^{2-\gamma}\|_{L_\xi^\infty} = N^{\frac{2d}{d+4}-\gamma} \|(N|\xi|^{-1})^{\gamma-\frac{2d}{d+4}} \chi_3(\xi) (N|\xi|^{-1})^{2-\gamma}\|_{L_\xi^\infty} \lesssim N^{\frac{2d}{d+4}-\gamma}.$$

Combining (4.5) and (4.6), we get

$$\|\chi_3(D)Iu_\lambda(0)\|_{L_x^{\frac{2d+8}{d}}} \lesssim N^{\frac{2d}{d+4}-\gamma} \lambda^{-\gamma} \|u_0\|_{\dot{H}_x^\gamma}. \quad (4.7)$$

We treat the intermediate case as

$$\|\chi_2(D)Iu_\lambda(0)\|_{L_x^{\frac{2d+8}{d}}} \lesssim \|\nabla|^{2d-d+4-\gamma}\chi_2(D)I\|_{L_x^2 \rightarrow L_x^2} \|u_\lambda(0)\|_{\dot{H}_x^\gamma}.$$

We have

$$\|\nabla|^{2d-d+4-\gamma}\chi_2(D)I\|_{L_x^2 \rightarrow L_x^2} \lesssim \|\xi|^{2d-d+4-\gamma}\chi_2(\xi)\|_{L_\xi^\infty}.$$

When  $\frac{2d}{d+4} \geq \gamma$ , we bound

$$\|\xi|^{2d-d+4-\gamma}\chi_2(\xi)\|_{L_\xi^\infty} \lesssim N^{\frac{2d}{d+4}-\gamma}.$$

When  $\gamma > \frac{2d}{d+4}$ , we write

$$\|\xi|^{2d-d+4-\gamma}\chi_2(\xi)\|_{L_\xi^\infty} = \|\xi|^{2d-d+4-\gamma-\beta}|\xi|^\beta\chi_2(\xi)\|_{L_\xi^\infty} \lesssim \lambda^{\beta+\gamma-\frac{2d}{d+4}},$$

provided  $\frac{2d}{d+4} - \gamma < \beta < \frac{2d}{d+4}$ . These estimates together with (4.5) yield

$$\|\chi_2(D)Iu_\lambda(0)\|_{L_x^{\frac{2d+8}{d}}} \lesssim \begin{cases} N^{\frac{2d}{d+4}-\gamma}\lambda^{-\gamma}\|u_0\|_{\dot{H}_x^\gamma} & \text{if } \frac{2d}{d+4} \geq \gamma, \\ \lambda^{\beta-\frac{2d}{d+4}}\|u_0\|_{\dot{H}_x^\gamma} & \text{if } \gamma > \frac{2d}{d+4}. \end{cases} \quad (4.8)$$

Collecting (4.4), (4.7), (4.8) and use (4.3), we obtain

$$\|Iu_\lambda(0)\|_{L_x^{\frac{2d+8}{d}}} \lesssim (\lambda^{\alpha-\frac{2d}{d+4}} + \lambda^{\beta-\frac{2d}{d+4}} + \lambda^{-\frac{8\gamma}{(d+4)(2-\gamma)}})\|u_0\|_{H_x^\gamma}, \quad (4.9)$$

for some  $0 < \alpha < \frac{2d}{d+4}$  and  $\frac{2d}{d+4} - \gamma < \beta < \frac{2d}{d+4}$ . Therefore, it follows from (4.1), (4.2), (4.3) and (4.9) by taking  $\lambda$  sufficiently large depending on  $\|u_0\|_{H_x^\gamma}$  and  $N$  (which will be chosen later and depends only on  $\|u_0\|_{H_x^\gamma}$ ) that

$$E(Iu_\lambda(0)) \leq \frac{1}{4}.$$

Now let  $T$  be arbitrarily large. We define

$$X := \{0 \leq t \leq \lambda^4 T \mid \|u_\lambda\|_{M([0,t])} \leq Kt^{\frac{d-4}{8(d-3)}}\},$$

with  $K$  a constant to be chosen later. Here  $M(J)$  is given in (2.13). We claim that  $X = [0, \lambda^4 T]$ . Assume by contradiction that it is not so. Since  $\|u_\lambda\|_{M([0,t])}$  is a continuous function of time, there exists  $T_0 \in [0, \lambda^4 T]$  such that

$$\|u_\lambda\|_{M([0,T_0])} > KT_0^{\frac{d-4}{8(d-3)}}, \quad (4.10)$$

$$\|u_\lambda\|_{M([0,T_0])} \leq 2KT_0^{\frac{d-4}{8(d-3)}}. \quad (4.11)$$

Using (4.11), we are able to split  $[0, T_0]$  into subintervals  $J_k, k = 1, \dots, L$  in such a way that

$$\|u_\lambda\|_{M(J_k)} \leq \mu,$$

where  $\mu$  is as in Proposition 3.2. The number  $L$  of possible subinterval must satisfy

$$L \sim \left( \frac{2KT_0^{\frac{d-4}{8(d-3)}}}{\mu} \right)^{\frac{8(d-3)}{d}} \sim T_0^{\frac{d-4}{d}}. \quad (4.12)$$

Next, thanks to Proposition 3.2, we see that for  $1 < \gamma < 2$  and any  $0 < \delta < \gamma - 1$ ,

$$\sup_{[0, T_0]} E(Iu_\lambda(t)) \lesssim E(Iu_\lambda(0)) + N^{-(2-\gamma+\delta)}L,$$

for  $\max\{3 - \frac{8}{d}, \frac{8}{d}\} < \gamma < 2$  and  $0 < \delta < \gamma + \frac{8}{d} - 3$ . Since  $E(Iu_\lambda(0)) \leq \frac{1}{4}$ , we need

$$N^{-(2-\gamma+\delta)}L \ll \frac{1}{4} \quad (4.13)$$

in order to guarantee

$$E(Iu_\lambda(t)) \leq 1, \quad (4.14)$$

for all  $t \in [0, T_0]$ . As  $T_0 \leq \lambda^4 T$ , we have from (4.12) and (4.13) and the choice of  $\lambda$  given in (4.3) that

$$N^{-(2-\gamma+\delta)} N^{\frac{4(2-\gamma)(d-4)}{\gamma d}} T^{\frac{d-4}{d}} \ll \frac{1}{4}, \quad (4.15)$$

or

$$\frac{4(2-\gamma)(d-4)}{\gamma d} < 2 - \gamma + \delta, \quad (4.16)$$

for  $\max\{3 - \frac{8}{d}, \frac{8}{d}\} < \gamma < 2$  and  $0 < \delta < \gamma + \frac{8}{d} - 3$ . Since  $2 - \gamma + \delta < \frac{8}{d} - 1$ , the condition (4.16) is possible if we have

$$\frac{4(2-\gamma)(d-4)}{\gamma d} < \frac{8}{d} - 1.$$

This implies  $\gamma > \frac{8(d-4)}{3d-8}$ . Thus

$$\gamma > \max\left\{3 - \frac{8}{d}, \frac{8}{d}, \frac{8(d-4)}{3d-8}\right\}.$$

Next, by (2.13),

$$\|u_\lambda\|_{M([0, T_0])} \lesssim T_0^{\frac{d-4}{8(d-3)}} \|u_0\|_{L_x^2}^{\frac{1}{d-3}} \|u_\lambda\|_{L_t^\infty([0, T_0], \dot{H}_x^{\frac{1}{2}})}^{\frac{d-4}{d-3}}.$$

We use (2.6) and the definition of the  $I$ -operator to estimate

$$\begin{aligned} \|u_\lambda(t)\|_{\dot{H}_x^{\frac{1}{2}}} &\leq \|P_{\leq N} u_\lambda(t)\|_{\dot{H}_x^{\frac{1}{2}}} + \|P_{> N} u_\lambda(t)\|_{\dot{H}_x^{\frac{1}{2}}} \\ &\lesssim \|P_{\leq N} u_\lambda(t)\|_{L_x^2}^{\frac{3}{4}} \|P_{\leq N} u_\lambda(t)\|_{\dot{H}_x^2}^{\frac{1}{4}} + N^{-\frac{3}{2}} \|Iu_\lambda(t)\|_{\dot{H}_x^2} \\ &\lesssim \|u_0\|_{L_x^2}^{\frac{3}{4}} \|Iu_\lambda(t)\|_{\dot{H}_x^2}^{\frac{1}{4}} + N^{-\frac{3}{2}} \|Iu_\lambda(t)\|_{\dot{H}_x^2}. \end{aligned}$$

Thus,

$$\|u_\lambda\|_{M([0, T_0])} \lesssim T_0^{\frac{d-4}{8(d-3)}} \|u_0\|_{L_x^2}^{\frac{1}{d-3}} \sup_{[0, T_0]} \left( \|u_0\|_{L_x^2}^{\frac{3}{4}} \|Iu_\lambda(t)\|_{\dot{H}_x^2}^{\frac{1}{4}} + N^{-\frac{3}{2}} \|Iu_\lambda(t)\|_{\dot{H}_x^2} \right)^{\frac{d-4}{d-3}}. \quad (4.17)$$

Since  $\|Iu_\lambda(t)\|_{\dot{H}_x^2} \lesssim \sqrt{E(Iu_\lambda(t))}$ , we obtain from (4.14) and (4.17),

$$\|u_\lambda\|_{M([0, T_0])} \leq C T_0^{\frac{d-4}{8(d-3)}},$$

for some constant  $C > 0$ . This contradicts with (4.10) for an appropriate choice of  $K$ . We get  $X = [0, \lambda^4 T]$  with  $T$  arbitrarily large and

$$E(Iu_\lambda(\lambda^4 T)) \leq 1. \quad (4.18)$$

Note that under the condition of  $\gamma$ , we see from (4.15) that the choice of  $N$  makes sense for arbitrarily large  $T$ . Now, by the conservation of mass and (4.18), we bound

$$\begin{aligned} \|u(T)\|_{H_x^\gamma} &\lesssim \|u(T)\|_{L_x^2} + \|u(T)\|_{\dot{H}_x^\gamma} \lesssim \|u_0\|_{L_x^2} + \lambda^\gamma \|u_\lambda(\lambda^4 T)\|_{\dot{H}_x^\gamma} \\ &\lesssim \|u_0\|_{L_x^2} + \lambda^\gamma \|Iu_\lambda(\lambda^4 T)\|_{H_x^2} \\ &\lesssim \lambda^\gamma \lesssim N^{2-\gamma} \lesssim T^{\alpha(\gamma, d)}, \end{aligned}$$

where  $\alpha(\gamma, d)$  is a positive number that depends on  $\gamma$  and  $d$ . This a priori bound gives the global existence in  $H^\gamma$ . The proof is now complete.

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(V. D. Dinh) INSTITUT DE MATHÉMATIQUES DE TOULOUSE, UNIVERSITÉ TOULOUSE III PAUL SABATIER, 31062 TOULOUSE CEDEX 9, FRANCE

*E-mail address:* [dinhvan.duong@math.univ-toulouse.fr](mailto:dinhvan.duong@math.univ-toulouse.fr)