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Uniqueness and Continuous Dependence on Data for One-Dimensional Impact Problems

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Abstract—One dimensional dynamics with impact are described by the data of a closed interval K , a function f of time, position, and velocity, and a restitution coefficient $e \in [0, 1]$. When u is in the interior of K , it satisfies the ordinary differential equation $\dot{u} = f(t, u, \dot{u})$. When it hits the end points of K , the velocity is reversed and multiplied by e . If f is analytic with respect to its three arguments, it is proved that uniqueness holds for the forward Cauchy problem. © 1998 Elsevier Science Ltd. All rights reserved.

Keywords—Vibro-impact system, Uniqueness of the Cauchy problem, Continuous dependence.

1. INTRODUCTION

In this article, we study the uniqueness and continuous dependence on data of solutions to a differential system with impact. We use the mathematical formulation we gave in [1] and we recall the description and the main results given there, specialized to the one-dimensional case: a continuous function f from $[T, T'] \times \mathbb{R} \times \mathbb{R}$ is given; it is assumed to be Lipschitz continuous with respect to its last two arguments; a convex closed set K is also given that is not a singleton. We let $C^0([T, T']; K)$ be the set of continuous functions from $[T, T']$ to \mathbb{R} which take their values in K . We seek a function $u \in C^0([T, T']; \mathbb{R})$ whose derivative is of bounded variation, and a real measure μ on $[T, T']$ which satisfy the following set of relations:

$$u \in C^0([T, T']; K), \tag{1.1}$$

$$\ddot{u} = f(\cdot, u, \dot{u}) + \mu, \text{ in the sense of distributions,} \tag{1.2}$$

$$\langle \mu, v - u \rangle \leq 0, \quad \forall v \in C^0([T, T']; K). \tag{1.3}$$

Observe that \dot{u} must have discontinuities at the boundary ∂K of K : when $u(t) = 0$ with strictly negative velocity $\dot{u}(t - 0)$, the velocity after impact has to be nonnegative, and the second derivative of u must have a Dirac mass at t_0 . Condition (1.3) is equivalent to

$$\text{supp}(\mu) \subset \{t : u(t) \in \partial K\}, \quad \mu \geq 0 \text{ on } \{t : u(t) = \min K\}, \quad \mu \leq 0 \text{ on } \{t : u(t) = \max K\}.$$

In the language of subdifferentials as described by Rockafellar in [2,3], (1.1)–(1.3) is equivalent to

$$\ddot{u} + \partial\psi_K(u) \ni f(\cdot, u, \dot{u}),$$

where ψ_K is the indicator function of K .

Let us describe now the admissible Cauchy data: the tangent cone at a to K is given by

$$T_K(a) = \mathbb{R} \text{ if } a \in \text{int}(K), \quad T_K(a) = \mathbb{R}^- \text{ if } a = \max K, \quad T_K(a) = \mathbb{R}^+ \text{ if } a = \min K.$$

The set of admissible Cauchy data for (1.1)–(1.3) is the subset C_K of \mathbb{R}^2 given by

$$C_K = \{(a, b) : a \in K, b \in T_K(a)\}.$$

We are given Cauchy data $(u_0, u_1) \in C_K$ and an initial time $t_0 \in [T, T']$. The solution has to satisfy the initial conditions

$$u(t_0) = u_0, \quad \dot{u}(t_0) = u_1. \quad (1.4)$$

We will say that we solve a forward (respectively, backward) Cauchy problem if we seek a solution of (1.1)–(1.4) on $[t_0, T']$ (respectively, $[T, t_0]$).

Conditions (1.1)–(1.4) do not suffice to define interesting solutions: in the very simple case $N = 1$, $f = 0$, and $K = \mathbb{R}^+$, with initial data $u(0) = u_0 > 0$ and $\dot{u}(0) = u_1 < 0$, it is easy to see that the solution is defined uniquely up to time $t_1 = -u_0/u_1$ by $u(t) = u_0 - u_1 t$. Conditions (1.1) to (1.3) imply that $\dot{u}(t_1 + 0) \geq 0$. It can be checked that all solutions of (1.1)–(1.3) with these initial data are given by

$$u(t) = \begin{cases} u_0 - u_1 t, & \text{for } 0 \leq t \leq t_1, \\ 0, & \text{for } t_1 \leq t \leq t_2, \\ u'_1(t - t_2), & \text{for } t \geq t_2, \end{cases}$$

where u'_1 is any positive number and t_2 is any number larger than t_1 ; we can even take t_2 infinite. In this case, the third line in the above formula has to be suppressed.

Therefore, the variational formulation (1.1)–(1.3) lacks a constitutive law for the impacts; we choose to use the classical model of Newton: given a restitution coefficient $e \in [0, 1]$, we require that at an impact the normal component of the velocity be reversed and multiplied by e . Since we consider a one-dimensional problem, we must have

$$u(t) \in \partial K \implies \dot{u}(t + 0) = -e\dot{u}(t - 0). \quad (1.5)$$

It has been proved in [1] that under the above assumptions, the problem (1.1)–(1.5) possesses a solution.

However, the solution of (1.1)–(1.5) is not unique in general.

Let us check that $e > 1$ precludes forward uniqueness, even in the simplest case. Let $f = -1$, $u_0 = u_1 = 0$. Choose $t_0 > 0$ and $v_0 > 0$ such that

$$t_0 > \frac{2v_0}{1 - e}.$$

We define the sequences $\{v_n\}$ and $\{t_n\}$, for all $n \in \mathbb{Z}$, by

$$v_n = e^{-n}v_0, \quad t_{n-1} - t_n = 2e^{-n}v_0.$$

As n tends to $+\infty$, t_n tends to $t_\infty = t_0 - 2v_0/(1 - e)$. On (t_n, t_{n-1}) , we let

$$u(t) = v_n(t - t_n) - \frac{(t - t_n)^2}{2}.$$

This defines u on $(t_\infty, +\infty)$; we extend u by 0 for $t \leq t_0$. Therefore, we obtained a solution of (1.1)–(1.5).

An example of forward nonuniqueness has been given in [4] when $e = 1$, $K = \mathbb{R}^+$, and f is a C^∞ function depending only on t . Such an example can be constructed only if the function f has a flat point, i.e., a point t_0 at which f vanishes as well as all its derivatives; moreover, f should not vanish identically in any right neighborhood $[t_0, t_1]$ of t_0 .

Percivale gave some sufficient conditions for uniqueness in [5]. His setting is Hamiltonian: u takes its values in a Riemannian manifold, bounces on a smooth submanifold of this manifold

with restitution coefficient $e = 1$, and the Hamiltonian is the sum of kinetic energy and potential energy which may depend on time. In particular, in dimension $N = 1$ when f is an analytic function of time only, uniqueness holds.

In this article, the following theorem is proved.

THEOREM 4.1. *Assume that f is a real analytic function of (t, a, b) on a domain \mathcal{O} , satisfying the geometric property (3.16). If (t_0, u_0, u_1) belongs to $\mathcal{O} \cap (\mathbb{R} \times C_K)$, there exists at most one forward local solution of (1.1)–(1.5).*

In fact, I conjecture that if K is a closed subset of \mathbb{R}^N with analytic boundary, $e \in [0, 1]$, and f is a real analytic function of (t, a, b) on a domain \mathcal{O} of $\mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$, the solution of the forward Cauchy problem is unique for the N -dimensional generalization of (1.1)–(1.5).

This theorem assumes that we have a reasonable notion of a local solution of (1.1)–(1.5). In order to build such a notion, we define a phase space in Section 2 by a gluing process. If $e > 0$, the gluing process produces a C^∞ manifold, provided that we remove the points $(a, 0)$ and $(b, 0)$. If $e = 0$, this gluing process produces a badly behaved object, which is to be expected. However, even in the case $e > 0$, there is no smooth metric which would serve the analysis of the dynamical system of interest; therefore, we use a nonsmooth metric which is obtained explicitly. The phase space equipped with this metric is denoted V_e . To apply Ascoli-Arzelá's theorem, we show that V_e is complete, and we characterize the compact subsets of V_e .

Let $U(t)$ be the projection in the phase space V_e of the solution $(u(t), \dot{u}(t \pm 0))$; in Section 3, we give some local estimates on U ; in particular, we show that U is Lipschitz continuous, and we estimate its Lipschitz constant. Moreover, we can define a local solution when f is defined only on an open subset of \mathbb{R}^3 and is only locally Lipschitz continuous with respect to its last two arguments. Then we estimate the minimal time of existence.

In Section 4, we prove the uniqueness of the local solution.

Finally, we prove in Section 5 that the local solutions depend continuously on the data, provided that we measure distances in the metric of V_e ; more precisely, if the sequence f_n converges uniformly on compact sets to an analytic function f , and if the initial data $(t_{0,n}, u_{0,n}, u_{1,n})$ converge to (t_0, u_0, u_1) , then the inferior limit of the existence time of the solution u_n is greater than or equal to the existence time of the solution u ; moreover, U_n converges to U uniformly on compact sets.

2. THE PHASE SPACE

It is convenient to work with $K = \mathbb{R}^+$; the general case is easy to construct from this particular case.

A first and naïve definition of the phase space starts from the reflection condition: let

$$\Delta_- = \{0\} \times \mathbb{R}^-, \quad V = \mathbb{R}^+ \times \mathbb{R},$$

and define a relation R_e on V by

$$\xi R_e \eta \quad \text{iff} \quad \begin{cases} \xi = \eta, \\ \text{or } \xi \in \Delta_-, \eta = -e\xi, \\ \text{or } \eta \in \Delta_-, \xi = -e\eta. \end{cases}$$

This is clearly an equivalence relation; the class of $\xi \in V$ is denoted ξ^* , and the projection $\xi \mapsto \xi^*$ is denoted π . Unfortunately, R_e is not an open equivalence relation, and the quotient topology is not interesting: if $e > 0$ and \mathcal{O} is an open set of $(0, \infty) \times \mathbb{R}$ intersecting Δ_- , then $\pi(\mathcal{O} \cap V)$ does not contain any point $\pi(\xi)$ such that $\xi_1 > 0$ and $\xi_2 > 0$, though it contains the points of $\pi(-e\mathcal{O} \cap \Delta_+)$. A correct concept of gluing should preclude this situation.

There are two ways to correct this situation. The first one consists in adding flaps to V , i.e., we let, when $e > 0$,

$$F_+ = \left\{ (u, v) : v > 0, |u| < \frac{v}{e} \right\}, \quad F_- = \{(u, v) : v < 0, |u| < -v\}.$$

Consider

$$\mathbf{V} = V \cup F_+ \cup F_- \setminus \{(0, 0)\},$$

and define a linear transformation H by its matrix

$$H = \begin{pmatrix} -1 & 0 \\ 0 & -e \end{pmatrix}.$$

We extend the relation R_e to all of \mathbf{V} by

$$\xi R_e \eta \quad \text{iff} \quad \begin{cases} \xi = \eta, \\ \text{or } \xi \in F_-, \eta = H\xi, \\ \text{or } \eta \in F_-, \xi = H\eta. \end{cases}$$

Then it is clear that R_e is a regular equivalence relation, so that \mathbf{V}/R_e is endowed with a structure of C^∞ manifold. If R_e is extended to $\bar{\mathbf{V}}$ simply by taking its closure \bar{R}_e in $\mathbf{V} \times \mathbf{V}$, we still obtain an open and closed equivalence relation: however, the quotient space cannot be smooth in the neighborhood of $(0,0)$. The manifold $\bar{\mathbf{V}}/\bar{R}_e$ is homeomorphic to a plane.

However, a distance on \mathbf{V}/R_e cannot be simultaneously smooth and convenient for the study of the dynamical properties; in particular, whenever $\pi^{-1}(x) = \{\xi\}$ and $\pi^{-1}(y) = \{\eta\}$, ξ and η are close enough, we would like this distance to be the norm of $\xi - \eta$, so as to work as in ordinary space; this is clearly impossible if $e < 1$. Moreover, the status of the point $(0,0)$ is unpleasant and the case $e = 0$ is left out altogether.

The second way to remedy the situation is more simple-minded and explicit: it consists of constructing explicitly a distance function d_e on V/R_e which defines a coarser topology than the quotient topology. We proceed as follows: let $\xi = (\xi_1, \xi_2)$ be an element of \mathbb{R}^2 , and let its norm be

$$|\xi| = |\xi_1| + |\xi_2|.$$

For all x and y in V/R_e , we let

$$\delta(x, y) = \min\{|\xi - \eta| : \xi \in x, \eta \in y\}. \quad (2.1)$$

The function δ is symmetric, nonnegative, and it vanishes if and only if $x = y$. However, δ is not a distance if $e \in [0, 1)$: it suffices to take $a > 0$, $x = (a, -2)^\bullet$, $y = (0, -2)^\bullet$, $z = (0, -1)^\bullet$. Then

$$\delta(x, z) = a + 1, \quad \delta(x, y) = a, \quad \delta(y, z) = e.$$

Thus,

$$\delta(x, y) > \delta(x, y) + \delta(y, z),$$

whenever $e < 1$. We define

$$d_e(x, y) = \inf \left\{ \sum_{j=0}^{N-1} \delta(x^j, x^{j+1}) : N \in \mathbb{N}, x_0 = x, x_N = y \right\}. \quad (2.2)$$

LEMMA 2.1. *The function d_e is a distance over V/R_e . The metric topology defined by d_e on V/R_e is coarser than the quotient topology. For all ξ and η in V ,*

$$d_e(\xi^\bullet, \eta^\bullet) \leq |\xi - \eta|. \quad (2.3)$$

The distance d_e is given explicitly by

$$d_e(\xi^\bullet, \eta^\bullet) = \min[|\xi_1 - \eta_1| + |\xi_2 - \eta_2|, \xi_1 + \eta_1 + |\psi_e(\xi_2) - \psi_e(\eta_2)|], \quad (2.4)$$

where

$$\psi_e(t) = \begin{cases} -et, & \text{if } t < 0, \\ t, & \text{if } t \geq 0. \end{cases} \quad (2.5)$$

PROOF. It is immediate that (2.2) defines a distance on V/R_e . This distance is maximal among all the distances on V_e which are estimated by δ , indeed, let d' be any distance on V/R_e which satisfies for all x and y

$$d'(x, y) \leq \delta(x, y).$$

For all sequence $(x^j)_{0 \leq j \leq N}$ such that $x^0 = x$ and $x^N = y$, we have

$$d'(x, y) \leq \sum_{j=0}^{N-1} d'(x^j, x^{j+1}) \leq \sum_{j=0}^{N-1} \delta(x^j, x^{j+1}).$$

Therefore,

$$d'(x, y) \leq d_e(x, y).$$

Therefore, we will have proved (2.4) if we can establish that its right-hand side is a distance on V/R_e which is situated between d_e and δ . For $x = (\xi_1, \xi_2)^\bullet \in V/R_e$, we define $\tilde{x} = (0, \xi_2)^\bullet$, and similarly for $y = (\eta_1, \eta_2)^\bullet$, $\tilde{y} = (0, \eta_2)^\bullet$, and we let

$$d'(x, y) = \min(\delta(x, y), \delta(x, \tilde{x}) + \delta(\tilde{x}, \tilde{y}) + \delta(\tilde{y}, y)).$$

According to (2.2), and the definition of d' ,

$$d_e(x, y) \leq d'(x, y) \leq \delta(x, y).$$

We will show now that d' is a distance on V/R_e ; first we obtain an explicit expression for d' : let $x = \{\xi\}$; if $e > 0$, this means that $\xi_1 > 0$; if $e = 0$, this means that $\xi_1 > 0$ or $\xi_2 > 0$, and similarly for $y = \{\eta\}$. A straightforward calculation gives

$$\delta(\tilde{x}, \tilde{y}) = |\psi_e(\xi_2) - \psi_e(\eta_2)|. \quad (2.6)$$

Therefore,

$$d'(x, y) = \min(|\xi - \eta|, \xi_1 + \eta_1 + |\psi_e(\xi_2) - \psi_e(\eta_2)|). \quad (2.7)$$

The function δ is continuous on $(V/R_e) \times (V/R_e)$, since $(\xi, \eta) \mapsto \delta(\xi^\bullet, \eta^\bullet)$ is continuous on $V \times V$, as can be seen by a simple observation. In the same way, the mapping $x \mapsto \tilde{x}$ is continuous from V/R_e to itself: it suffices to check that $\xi \mapsto \tilde{\xi}^\bullet$ is continuous, but this is so since this mapping can be decomposed as

$$(\xi_1, \xi_2) \mapsto (\xi_1, 0) \mapsto (\xi_1, 0)^\bullet,$$

which is obviously continuous. Therefore, by continuity, (2.7) holds for all x and y in V/R_e .

By continuity, it is enough to prove the triangle inequality

$$d'(x, y) \leq d'(x, z) + d'(z, y),$$

when $x \neq \tilde{x}$, $y \neq \tilde{y}$, and $z \neq \tilde{z}$, because the set of x such that $x \neq \tilde{x}$ is dense in V/R_e for the quotient topology. We have to consider several cases. Assume first that

$$d'(x, y) = \delta(x, y),$$

i.e.,

$$\delta(x, y) \leq \delta(x, \tilde{x}) + \delta(\tilde{x}, \tilde{y}) + \delta(\tilde{y}, y).$$

On the one hand, we have

$$\delta(x, y) \leq \delta(x, z) + \delta(z, y).$$

If $d'(x, z) = \delta(x, z)$ and $d'(z, y) = \delta(z, y)$, we are done. On the other hand,

$$\delta(\tilde{x}, \tilde{y}) \leq \delta(\tilde{x}, \tilde{z}) + \delta(\tilde{z}, \tilde{y}),$$

thanks to (2.6). Therefore, if $d'(x, z) < \delta(x, z)$, and $d'(z, y) < \delta(z, y)$, we have

$$d'(x, y) \leq \delta(x, \tilde{x}) + \delta(\tilde{x}, \tilde{y}) + \delta(\tilde{y}, y) \leq \delta(x, \tilde{x}) + \delta(\tilde{x}, \tilde{z}) + \delta(\tilde{z}, \tilde{y}) + \delta(\tilde{y}, y),$$

and the inequality holds. If $d'(x, z) = \delta(x, z)$ and $d'(z, y) < \delta(z, y)$, our inequality will hold if we show that

$$\delta(x, \tilde{x}) + \delta(\tilde{x}, \tilde{y}) + \delta(\tilde{y}, y) \leq \delta(x, z) + \delta(z, \tilde{z}) + \delta(\tilde{z}, \tilde{y}) + \delta(\tilde{y}, y). \quad (2.8)$$

More explicitly, if $y = \{\eta\}$ and $z = \{\zeta\}$, (2.8) can be rewritten as

$$\xi_1 + \eta_1 + |\psi_e(\xi_2) - \psi_e(\eta_2)| \leq |\xi_1 - \zeta_1| + |\xi_2 - \zeta_2| + \zeta_1 + |\psi_e(\zeta_2) - \psi_e(\eta_2)| + \eta_2.$$

This relation holds because ψ_e is a contraction.

The case $d'(x, y) < \delta(x, y)$ is treated analogously, and the details are left to the reader.

Therefore, we have proved that d' given by (2.7) is identical to d_e , i.e., we have proved (2.4).

The fact that the topology of V/R_e , equipped with the distance function d_e , is coarser than the quotient topology follows from the observation that the mapping $(\xi, \eta) \mapsto d_e(\xi^\bullet, \eta^\bullet)$ is continuous from $V \times V$ to \mathbb{R}^+ . \blacksquare

From now on, we will denote by V_e the quotient space V/R_e equipped with the distance d_e . The mapping ϕ is the composition of π and the injection $V/R_e \hookrightarrow V_e$. It is continuous since the topology of V_e is coarser than the topology of V/R_e .

The spaces \bar{V}/\bar{R}_e can be identified to V_e if $e > 0$. This fact is not needed in the sequel, and its proof is left to the reader.

The closed ball of center $U_0 \in V_e$ and radius ρ will be denoted $B_e(U_0, \rho)$:

$$B_e(U_0, \rho) = \{x \in V_e : d_e(x, U_0) \leq \rho\}.$$

If $B(\xi, \rho)$ is the closed ball of \mathbb{R}^2 of center ξ and radius ρ , we have that

$$B(\xi, \rho) \cap V \subset \phi^{-1}B_e(\phi(\xi), \rho).$$

If $e > 0$, the inverse images $\phi^{-1}B_e(x, \rho)$ are compact; if $e = 0$, some of these inverse images are not compact; if $\phi(\Delta_-)$ is included in $B_e(x, \rho)$, then $\phi^{-1}B_e(x, \rho)$ will contain the unbounded equivalence class Δ_- .

Figures 1–4 depict some pictures of balls in V_e .

The essential topological properties of V_e are summarized in the next lemma.

LEMMA 2.2. *The metric space V_e is complete, and all the bounded closed sets of V_e are compact if $e > 0$; if $e = 0$, the compact sets are the bounded closed sets C of V_e for which there exists $a \geq 0$ such that C is included in $\phi(\mathbb{R}^+ \times [-a, +\infty))$.*

PROOF. Let $(x_n)_n$ be a Cauchy sequence of elements of V_e . Let $\xi_n = (\xi_{1,n}, \xi_{2,n}) \in V$ be an element of x_n . Assume first that there is a subsequence $(\eta_p)_p$ of $(\xi_n)_n$ such that

$$\liminf \eta_{1,p} > 0. \quad (2.9)$$

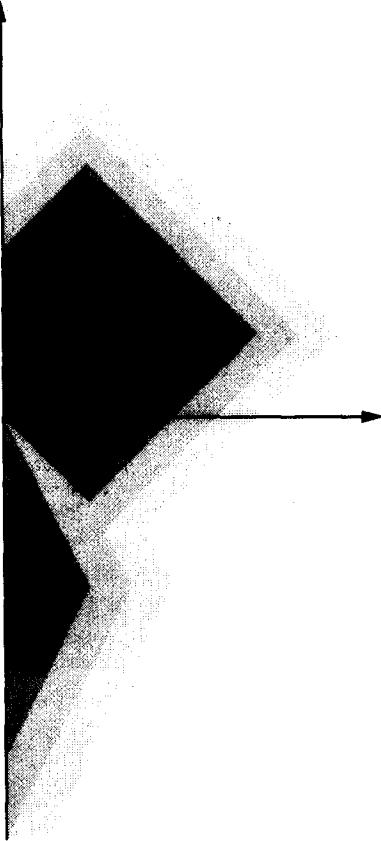


Figure 1. Balls in V_e for $e = 1/2$.

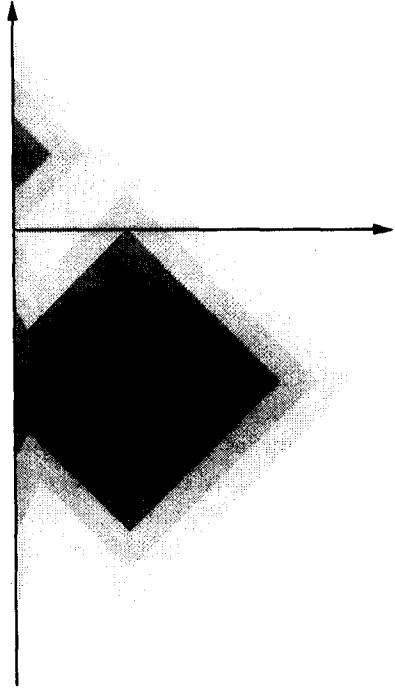


Figure 2. Balls in V_e for $e = 1/2$.

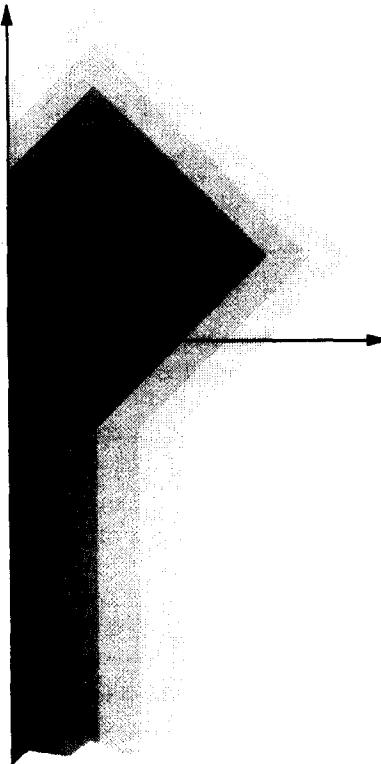


Figure 3. Balls in V_e for $e = 0$.

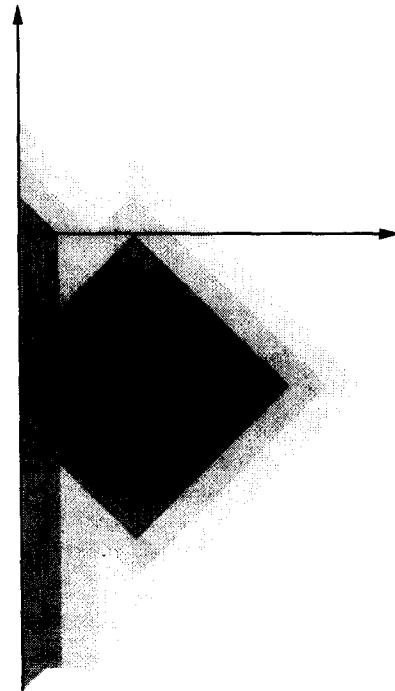


Figure 4. Balls in V_e for $e = 0$.

Then, relation (2.4) implies that $(\eta_p)_p$ is a Cauchy sequence in the closed subset V of \mathbb{R}^2 , whose limit will be denoted η_∞ . We infer from (2.3) that $(\eta_p^\bullet)_p$ converges to η_∞^\bullet .

Assume now that the subsequence (η_p) satisfies

$$\lim_{p \rightarrow \infty} \eta_{1,p} = 0, \quad (2.10)$$

and

$$\eta_{2,p} \geq 0. \quad (2.11)$$

Then, (2.3) implies that $(\eta_{2,p})_p$ is a Cauchy sequence whose limit will be denoted $\eta_{2,\infty}$, and if $\eta_\infty = (0, \eta_{2,\infty})$, then $(\eta_p^\bullet)_p$ converges to η_∞^\bullet . If instead of (2.11), we have

$$\eta_{2,p} \leq 0, \quad (2.12)$$

we have to consider separately the case $e > 0$ and the case $e = 0$. In the case $e > 0$, we conclude as above that $\eta_{2,p}$ is a Cauchy sequence whose limit is $\eta_{2,\infty}$ and $(\eta_p^\bullet)_p$ converges to η_∞^\bullet . If e vanishes, $(\eta_{2,p})_p$ is not a Cauchy sequence, but $(\eta_p^\bullet)_p$ converges to $(0, 0)^\bullet$ as we can see immediately.

It remains to join together all these results. Observe first that if we have a subsequence $(\eta_p)_p$ of $(\xi_n)_n$ which satisfies (2.9) and a subsequence $(\zeta_p)_p$ of (ξ_n) which satisfies (2.10), we have a contradiction if we apply (2.10) to the distance between η_p and ζ_p . Observe now that if $(\eta_p)_p$ satisfies (2.10) and (2.11) and $(\zeta_p)_p$ satisfies (2.10) and (2.12), then, in the case $e > 0$, (2.4) implies that $\eta_{2,\infty} = -e\zeta_{2,\infty}$; in the case $e = 0$, (2.4) implies that $\eta_{2p} = 0$.

Therefore, in all the possible cases, we have established that $(\xi_n)_n$ has a limit in V_e .

We characterize the compact sets of V_e in a very straightforward manner: if $e > 0$, and $(x_n)_n$ is a bounded sequence in V_e , we can find a bounded sequence $(\xi_n)_n$ in V such that $x_n = \xi_n^\bullet$, thanks to (2.4), and the remainder of the argument is clear. If $e = 0$, and $(x_n)_n$ is a bounded sequence, we find $\xi_n \in x_n$, and we can make sure that $(\xi_n)_n$ is bounded, only if we know that $\xi_{2,n}$ cannot tend to $-\infty$; this is ensured by the extra condition of the statement of our lemma in the case $e = 0$. ■

3. LOCAL ESTIMATES AND LOCAL EXISTENCE

Let us assume first that f is a continuous function on \mathbb{R}^3 which is uniformly Lipschitz continuous with respect to its last two arguments. For all $(\bar{t}_0, \bar{u}_0, \bar{u}_1)$ in $\mathbb{R} \times C_K$ and for all $\rho > 0$, we define the compact subsets $D(\bar{t}_0, \bar{u}_0, \bar{u}_1, \rho)$ of \mathbb{R}^3 by

$$D(\bar{t}_0, \bar{u}_0, \bar{u}_1, \rho) = \begin{cases} [-\rho, \rho] \times \phi^{-1}B_e((\bar{u}_0, \bar{u}_1)^\bullet, \rho), & \text{if } e \in (0, 1], \\ [-\rho, \rho] \times (\phi^{-1}B_0((\bar{u}_0, \bar{u}_1)^\bullet, \rho) \\ \cap \{(a, b) : a + b \geq \bar{u}_0 + \bar{u}_1 - 2\rho\}), & \text{if } e = 0. \end{cases} \quad (3.1)$$

According to [1], for all initial data (t_0, u_0, u_1) in \mathbb{R}^3 , and for all $T > t_0$, there exists a solution of (1.1)–(1.5) on $[t_0, T)$. This solution is such that at t_0 , u and \dot{u} are continuous from the right. If we take (t_0, u_0, u_1) in $D(\bar{t}_0, \bar{u}_0, \bar{u}_1, \rho/2)$, there exists $\tau > t_0$ such that for all $t \in [t_0, \tau)$, $(t, u(t), \dot{u}(t \pm 0))$ belongs to $D(\bar{t}_0, \bar{u}_0, \bar{u}_1, \rho)$. Our purpose in the beginning of this section is to give an estimate from below on $\tau - t_0$. To this end, we let

$$M = \max\{|f(t, a, b)| : (t, a, b) \in D(\bar{t}_0, \bar{u}_0, \bar{u}_1, \rho)\}, \quad (3.2)$$

we denote $U(t) = (u(t), \dot{u}(t \pm 0))^\bullet$, and we prove an estimate on $d_e(U(t), U(t_0))$.

LEMMA 3.1. *For all $t \in [t_0, \tau)$ and all $t' \in [t, \tau]$, the following estimate holds:*

$$d_e(U(t), U(t')) \leq M(t' - t) + \frac{M}{2}(t' - t)^2 + |\dot{u}(t + 0)|(t' - t). \quad (3.3)$$

PROOF. Define

$$\mathcal{U} = \{t \in [t_0, \tau] : u(t) > 0\} \quad \text{and} \quad \mathcal{F} = \{t \in [t_0, \tau] : u(t) = 0\}.$$

When (t, t') is included in \mathcal{U} , u satisfies the ordinary differential equation $\ddot{u} = f(\cdot, u, \dot{u})$; hence, we infer the estimates

$$|\dot{u}(t') - \dot{u}(t)| \leq M(t' - t), \quad (3.4)$$

$$|u(t') - u(t)| \leq (t' - t)|\dot{u}(t + 0)| + M \frac{(t' - t)^2}{2}, \quad (3.5)$$

which yield (3.3) immediately.

Assume now that (t, t') contains exactly one point t_1 of \mathcal{F} , then the triangle inequality and the previous case imply that

$$\begin{aligned} d_e(U(t), U(t')) &\leq M(t_1 - t) + \frac{M}{2}(t_1 - t)^2 + |\dot{u}(t + 0)|(t_1 - t) + M(t' - t_1) \\ &\quad + \frac{M}{2}(t' - t_1)^2 + |\dot{u}(t_1 + 0)|(t' - t_1). \end{aligned} \quad (3.6)$$

But

$$|\dot{u}(t_1 + 0)| \leq |\dot{u}(t_1 - 0)| \leq |\dot{u}(t + 0)| + M(t_1 - t).$$

If we insert this inequality into (3.6), we obtain (3.3).

Assume that (t, t') contains $p \geq 2$ points of \mathcal{F} , $t_1 < t_2 < \dots < t_p$; the sign conditions $\dot{u}(t_{j-1} + 0) \geq 0$, $\dot{u}(t_j - 0) \leq 0$, and (3.4) imply

$$|\dot{u}(t_j - 0) - \dot{u}(t_{j-1} + 0)| = |\dot{u}(t_j - 0)| + |\dot{u}(t_{j-1} + 0)| \leq M(t_j - t_{j-1}). \quad (3.7)$$

Thus,

$$d_e(U(t_p), U(t_1)) \leq \sum_{j=2}^p d_e(U(t_j), U(t_{j-1})) \leq M(t_p - t_1).$$

Moreover, if we apply (3.3) to the interval (t_p, t') and if we observe that (3.7) gives $|\dot{u}(t_p + 0)| \leq M(t_p - t_{p-1})$, we can see that

$$d_e(U(t'), U(t_p)) \leq M(t' - t_p) + \frac{M}{2}(t' - t_p)^2 + M(t_p - t_{p-1})(t' - t_p).$$

Therefore,

$$\begin{aligned} d(U(t), U(t')) &\leq M(t_1 - t) + \frac{M}{2}(t_1 - t)^2 + |\dot{u}(t + 0)|(t_1 - t) \\ &\quad + M(t_p - t_1) + M(t' - t_p) + \frac{M}{2}(t' - t_p)^2 + M(t_p - t_{p-1})(t' - t_p). \end{aligned} \quad (3.8)$$

We observe that

$$(t_1 - t)^2 + (t' - t_p)^2 + 2(t_p - t_{p-1})(t' - t_p) \leq (t' - t)^2,$$

and gathering the terms on the right-hand side of (3.8), we infer that (3.3) holds.

Let us assume now that (t, t') contains an infinite number of points of \mathcal{F} . Let \mathcal{F}^a denote the set of nonisolated points of \mathcal{F} and let

$$s = \inf(\mathcal{F}^a \cap (t, t')) \quad \text{and} \quad s' = \sup(\mathcal{F}^a \cap (t, t')).$$

At s , $\dot{u}(s+0)$ or $\dot{u}(s-0)$ vanish; if $e > 0$, both vanish; if $e = 0$, at least $\dot{u}(s+0)$ vanishes; the same observations hold at s' . If $s > t$, for all $\varepsilon > 0$, the interval $(t, s - \varepsilon)$ contains at most a finite number of points of \mathcal{F} ; therefore, for all $\sigma \in (t, s)$,

$$d_e(U(t), U(\sigma)) \leq M(\sigma - t) + \frac{M}{2}(\sigma - t)^2 + |\dot{u}(t+0)|(\sigma - t). \quad (3.9)$$

We pass to the limit in (3.9) as σ tends to s , which is permissible, thanks to (2.3) and the continuity on the right of $u(\sigma)$ and $\dot{u}(\sigma - 0)$. Therefore,

$$d_e(U(t), U(s)) \leq M(s - t) + \frac{M}{2}(s - t)^2 + |\dot{u}(t+0)|(s - t). \quad (3.10)$$

The same argument proves that if $s' < t'$, then

$$d_e(U(s'), U(t')) \leq M(t' - s') + \frac{M}{2}(t' - s')^2 + |\dot{u}(s'+0)|(t' - s').$$

But we have seen that $\dot{u}(s'+0)$ vanishes, thus

$$d_e(U(s'), U(t')) \leq M(t' - s') + \frac{M}{2}(t' - s')^2. \quad (3.11)$$

Of course, (3.10) and (3.11) hold also if $t = s$ or $t' = s'$; as $d_e(U(s), U(s'))$ vanishes, we obtain (3.3) with the help of the triangle inequality. \blacksquare

In the case $e = 0$, the definition of $D(t_0, u_0, u_1, \rho)$ is more complicated. The following lemma enables us to deal with the supplementary condition.

LEMMA 3.2. *If $e = 0$, then for all $t \in [t_0, \tau]$, the following inequality holds:*

$$u(t) + |\dot{u}(t \pm 0)| \geq u_0 + u_1 - \left[M(t - t_0) + \frac{M}{2}(t - t_0)^2 + |u_1|(t - t_0) \right]. \quad (3.12)$$

PROOF. If \mathcal{U} contains an interval (t_0, t_1) , (3.4) and (3.5) imply that

$$|u(t) - u_0| + |\dot{u}(t) - u_1| \leq M(t - t_0) + \frac{M}{2}(t - t_0)^2 + |u_1|(t - t_0), \quad (3.13)$$

and (3.12) is obvious. If $[t_0, t] \cap \mathcal{F}$ is not empty, let $s = \inf \mathcal{F}$. On (t_0, s) , (3.13) implies

$$|\dot{u}(s-0) - u_1| + |u_0| \leq M(s - t_0) + \frac{M}{2}(s - t_0)^2 + |u_1|(s - t_0).$$

Thanks to the sign conditions $\dot{u}(s-0) \leq 0$, this relation implies

$$u_1 + u_0 \leq M(s - t_0) + \frac{M}{2}(s - t_0)^2 + |u_1|(s - t_0). \quad (3.14)$$

Let $s' = \sup(\mathcal{F} \cap [t_0, t])$. If $s' < t$, (3.3) gives

$$|u(t)| + |\dot{u}(t-0)| \leq M(t - s') + \frac{M}{2}(t - s')^2,$$

which implies

$$u(t) + \dot{u}(t+0) \geq u(t) + \dot{u}(t-0) \geq - \left[M(t - s') + \frac{M}{2}(t - s')^2 \right]. \quad (3.15)$$

If we subtract (3.14) from (3.15), we obtain

$$\begin{aligned} u(t) + \dot{u}(t \pm 0) &\geq u_0 + u_1 - \left[M(t - s') + \frac{M}{2}(t - s')^2 + M(s - t_0) + \frac{M}{2}(s - t_0)^2 + |u_1|(s - t_0) \right] \\ &\geq u_0 + u_1 - \left[M(t - t_0) + \frac{M}{2}(t - t_0)^2 + |u_1|(t - t_0) \right], \end{aligned}$$

and our conclusion holds. \blacksquare

We are able now to estimate from below the length of the largest time interval $[t_0, \tau]$ during which $(t, u(t), \dot{u}(t \pm 0))$ remains in $D(t_0, u_0, u_1, \rho)$, provided that (t_0, u_0, u_1) belongs to $D(t_0, u_0, u_1, \rho/2)$.

THEOREM 3.3. *Let ρ be a strictly positive number, let $(\bar{t}_0, \bar{u}_0, \bar{u}_1)$ be an element of $\mathbb{R} \times C_K$, let M be as in (3.2), and let (t_0, u_0, u_1) belong to $D(t_0, u_0, u_1, \rho/2)$. Let σ be the strictly positive root of the equation*

$$M\sigma + \frac{M}{2}\sigma^2 + \left(|u_1| + \frac{\rho}{2}\right)\sigma - \frac{\rho}{2} = 0,$$

and let $\hat{\sigma} = \min(\sigma, \rho/2)$. Then, $(t, u(t), \dot{u}(t \pm 0))$ belongs to $D(t_0, u_0, u_1, \rho)$ for all $t \in [t_0, t_0 + \hat{\sigma}]$.

PROOF. Let $\tau > t_0$ be the largest time such that $(t, u(t), \dot{u}(t \pm 0))$ belongs to $D(t_0, u_0, u_1, \rho)$ for $t_0 \leq t \leq \tau$. If $e > 0$, (3.3) holds on $[t_0, \tau]$; if $\tau - t_0 < \hat{\sigma}$, (3.3) implies

$$d_e(U(t_0 + \hat{\sigma}), \bar{U}_0) \leq \frac{\rho}{2} + M(\tau - t_0) + \frac{M}{2}(\tau - t_0)^2 + |\dot{u}_1|(\tau - t_0) < \rho.$$

Therefore, there exists a nonempty time interval $[\tau, \tau + \varepsilon]$ on which $(t, u(t), \dot{u}(t \pm 0))$ still belongs to $D(t_0, u_0, u_1, \rho)$, which contradicts the definition of τ .

If $e = 0$, we use also (3.12), and then the same argument leads to the same conclusion. \blacksquare

We are able now to prove the existence of local solutions of (1.1)–(1.5): let \mathcal{O} be an open bounded subset of \mathbb{R}^3 which has the following property:

$$\begin{aligned} \mathcal{O} \cap (\mathbb{R} \times C_K) &\text{ is not empty;} \\ \forall (t, a, b) \in \mathcal{O} \cap (\mathbb{R} \times V), \exists \rho > 0 &\text{ such that } D(t, a, b, \rho) \text{ is included in } \mathcal{O}. \end{aligned} \tag{3.16}$$

Condition (3.16) means that if e is strictly positive, $(t, 0, b)$ belongs to \mathcal{O} if and only if $(t, 0, -\min(b, 0)e - \max(b, 0)/e)$ belongs to \mathcal{O} ; if e vanishes, it means that $(t, 0, b)$ belongs to \mathcal{O} if and only if $\{t\} \times \{0\} \times [b, \max(b, 0)]$ belongs to \mathcal{O} .

Assume that the continuous function f is locally Lipschitz continuous with respect to its last two arguments on \mathcal{O} . Choose an arbitrary triplet (t_0, u_0, u_1) in \mathcal{O} , and $\rho > 0$ such that $D(t_0, u_0, u_1, \rho)$ is included in \mathcal{O} . It is possible to find a continuous function g on \mathbb{R}^3 which is uniformly Lipschitz continuous with respect to its last two arguments, and which coincides with f on $D(t_0, u_0, u_1, \rho)$. Thus, we have a solution of (1.1)–(1.5) on $[t_0, +\infty)$, with f replaced by g . Theorem 3.3 implies that there exists $\hat{\sigma}$ such that $(t, u(t), \dot{u}(t \pm 0))$ remains in $D(t_0, u_0, u_1, \rho)$ for all $t \in [t_0, t_0 + \hat{\sigma}]$. Therefore, on $[t_0, t_0 + \hat{\sigma}]$, we have a solution of (1.1)–(1.5). As in the classical theory of ordinary differential equations, a local solution of (1.1)–(1.5) will be defined by giving an interval I whose left end is t_0 and a function $u \in C^0(I; \mathbb{R})$ whose derivative is of locally bounded variation, with u satisfying (1.1)–(1.5) on every compact subinterval of I . It is then possible to speak as in the classical theory of a maximal solution.

4. UNIQUENESS

We assume in this section that \mathcal{O} is a domain of \mathbb{R}^3 which possesses the property (3.16) and that f is a real analytic function on \mathcal{O} . Therefore, as stated in Section 3, for all data $(t_0, u_0, u_1) \in \mathcal{O}$, there exists a local solution of (1.1)–(1.5) on some time interval starting at t_0 .

THEOREM 4.1. *Under the above assumptions, if (t_0, u_0, u_1) belongs to $\mathcal{O} \cap (\mathbb{R} \times C_K)$, there exists at most one solution of (1.1)–(1.5) on any interval $[t_0, t'_0]$.*

PROOF. In this proof, we assume $K = \mathbb{R}^+$. When K is not of this form, local arguments and elementary geometric transformations enable us to reach the same conclusion.

Assume that there exist two distinct solutions u and v of (1.1)–(1.5) on $[t_0, t'_0]$, with initial data u_0 and u_1 . Let

$$\tau = \inf\{t > t_0 : u(t) \neq v(t)\}.$$

We can see that $u = v$ on $[t_0, \tau]$; the nonuniqueness assumption implies that $\tau < t'_0$. By definition of the solutions, $u(\tau) = v(\tau) \geq 0$. Moreover, if $u(\tau) = v(\tau) = 0$, then $\dot{u}(\tau - 0) = \dot{v}(\tau - 0) \leq 0$. Condition (1.5) implies then that

$$\dot{u}(\tau + 0) = \dot{v}(\tau + 0) = -e\dot{u}(\tau - 0).$$

This means that the Cauchy conditions for u and v coincide at τ . Therefore, without loss of generality, we may assume that

$$t_0 = \inf\{t > t_0 : u(t) \neq v(t)\}. \quad (4.1)$$

Let us first observe that if there exists a nonempty interval (t_0, τ) , on which u and v are strictly positive, we have a contradiction: on (t_0, τ) , u and v satisfy the classical ordinary differential equation

$$\ddot{w} = f(\cdot, w, \dot{w}),$$

with the same initial data at t_0 ; therefore, $u = v$ on $[t_0, \tau]$ which implies that $\inf\{t > t_0 : u(t) \neq v(t)\} \geq \tau > t_0$, which contradicts (4.1).

In particular, these remarks preclude the situations where $u_0 > 0$, or where $u_0 = 0$ and $u_1 > 0$: in this case, we could find by continuity a nonempty interval (t_0, τ) on which u and v would be strictly positive.

Hence, from now on, we may assume that

$$u_0 = u_1 = 0. \quad (4.2)$$

Moreover, we must have $f(t_0, 0, 0) \leq 0$. If this condition is not satisfied, then by continuity of u and by continuity on the right of \dot{u} at 0, there exists a time interval (t_0, τ) on which $f(t, u(t), \dot{u}(t)) > 0$. Relation (1.3) implies that μ is a nonnegative measure; we infer from (1.1) and (1.2) that $u > 0$ on (t_0, t_1) . The same argument applies to v , and thus we have a contradiction.

From now on, we assume that

$$f(t_0, 0, 0) \leq 0. \quad (4.3)$$

The previous analysis shows that one of the two functions u or v has to vanish at times arbitrarily close to t_0 . Let us assume, for instance, that u vanishes for times arbitrarily close to t_0 . We will show that there must exist an interval $[t_0, t_2]$, with $t_2 > t_0$ such that u vanishes identically on $[t_0, t_2]$.

If such an interval does not exist, it means that the set

$$\mathcal{U} = \{t \in (t_0, t'_0) : u(t) > 0\} \quad (4.4)$$

has t_0 in its closure. Therefore, there exists an infinite sequence of nonempty intervals (a_n, b_n) which are connected components of \mathcal{U} such that for all $n \geq 1$,

$$b_{n+1} \leq a_n < b_n, \quad (4.5)$$

and a_n tends to t_0 as n tends to infinity.

We recall some classical results in the theory of complex ordinary differential equations: if \mathcal{P} is an open connected region of $\mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^p$, and F is an analytic mapping from \mathcal{P} to \mathbb{C}^n , the ordinary differential system

$$\mathbf{w}' = F(z, \mathbf{w}, \lambda),$$

with initial condition

$$\mathbf{w}(z_0) = \mathbf{w}_0,$$

possesses a local unique solution for each (z_0, w_0, λ) . If the value of this solution at z is denoted by $\mathbf{W}(z, z_0, \mathbf{w}_0, \lambda)$, it is locally an analytic function of all its arguments. More precisely, one can find for all $(\tilde{z}_0, \tilde{\mathbf{w}}_0, \tilde{\lambda}_0) \in \mathcal{P}$ a number $\rho > 0$ such that \mathbf{W} is well defined and analytic on a set containing the multicylinder of \mathbb{C}^{2+n+p} ,

$$\{|z - \tilde{z}_0| \leq \rho\} \times \{|z_0 - \tilde{z}_0| \leq \rho\} \times \left\{ \max_{1 \leq j \leq n} |\mathbf{w}_j - \tilde{\mathbf{w}}_{0,j}| \leq \rho \right\} \times \left\{ \max_{1 \leq k \leq p} |\lambda_k - \tilde{\lambda}_{0,k}| \leq \rho \right\}.$$

We apply this result first to the problem

$$\dot{w} = f(t, w, \dot{w}), \quad w(\tau) = \alpha, \quad \dot{w}(\tau) = \beta.$$

There exists $\rho > 0$ such that its solution $w = W(t, \tau, \alpha, \beta)$ is defined and analytic on an open set containing $[t_0 - \rho, t_0 + \rho]^2 \times [-\rho, \rho]^2$, and $(t, W(t, \tau, \alpha, \beta), \partial_1 W(t, \tau, \alpha, \beta))$ belongs to \mathcal{O} .

Let us define

$$f_1(t) = f(t, 0, 0), \quad f_2(t, \alpha) = \frac{f(t, \alpha, 0) - f(t, 0, 0)}{\alpha}, \quad f_3(t, \alpha, \beta) = \frac{f(t, \alpha, \beta) - f(t, \alpha, 0)}{\beta},$$

and

$$\lambda = (\tau, \alpha, \beta), \quad m_2 = (t, \lambda) = f_2(t, W(t, \tau, \alpha, \beta)), \\ m_3(t, \lambda) = f_3(t, W(t, \tau, \alpha, \beta), \partial_1 W(t, \tau, \alpha, \beta)).$$

Let E be the solution of

$$\partial_1^2 E(t, s, \lambda) = m_3(t, \lambda) \partial_1 E(t, s) + m_2(t, \lambda) E(t, s), \quad (4.6)$$

with initial conditions

$$E(s, s, \lambda) = 0, \quad \partial_1 E(s, s, \lambda) = 1.$$

There exists $\sigma \in (0, \rho)$ such that E is well defined and analytic on an open set containing

$$A_\sigma = [t_0 - \sigma, t_0 + \sigma]^3 \times [-\sigma, \sigma]^4.$$

In fact, since (4.6) is linear, it is even possible to take $\sigma = \rho$.

In particular, there exists a constant C such that

$$|\partial_1^2 E(t, s, \lambda)| \leq C, \quad \forall (t, s, \lambda) \in A_\sigma.$$

This implies that

$$\forall (t, s, \lambda) \in A_\sigma, \quad |E(t, s, \lambda) - t + s| \leq \frac{C(t-s)^2}{2}, \quad \text{and} \quad (4.7)$$

$$\forall (t, s, \lambda) \in A_\sigma, \quad |\partial_1 E(t, s, \lambda) - 1| \leq C(t-s). \quad (4.8)$$

If we choose σ small enough, we may conclude from (4.7) that

$$E(t, s, \lambda)(t - s) > 0 \quad (4.9)$$

and from (4.8) that

$$\partial_1 E(t, s, \lambda) > 0, \quad (4.10)$$

for all $(t, s, \lambda) \in A_\sigma$.

On (a_n, b_n) , we may write

$$\ddot{u}(t) = f_1(t) + f_2(t, u(t))u(t) + f_3(t, u(t), \dot{u}(t))\dot{u}(t).$$

Using the previously introduced notations and

$$\lambda_n = (a_n, 0, \dot{u}(t_n + 0)),$$

we may write, with the help of a Duhamel's formula, that

$$u(t) = E(t, a_n, \lambda_n)\dot{u}(a_n + 0) + \int_{a_n}^t E(t, s, \lambda_n)f_1(s) ds. \quad (4.11)$$

As f_1 is analytic at t_0 , there exists t_1 such that either $f_1(t) < 0$ on (t_0, t_1) or $f_1(t) \geq 0$ on $(0, t_1)$. Let us assume first that

$$f_1(t) \geq 0, \quad \forall t \in (t_0, t_1).$$

By construction, $\dot{u}(a_n + 0) \geq 0$; if $b_n \leq t_1$ is small enough, (4.9) implies that $E(t, a_n)$ is strictly positive on (a_n, b_n) , and $u(t) \geq 0$ on (a_n, b_n) ; we can also see that $u(b_n)$ vanishes only if $\dot{u}(a_n + 0)$ and the restriction of f_1 to (a_n, b_n) vanish, then u vanishes identically on (a_n, b_n) which contradicts the definition of (a_n, b_n) . Therefore, $f_1 < 0$ on (t_0, t_1) .

If $\dot{u}(a_n + 0)$ vanishes, then (4.11) implies that $u(t) < 0$ on (a_n, b_n) . This also contradicts the definition of (a_n, b_n) . Therefore, $\dot{u}(a_n + 0) > 0$.

Thus, if $e = 0$, we have proved that t_0 cannot be in the closure of \mathcal{U} .

From now on, in the proof, we assume that $e \in (0, 1]$. The previous considerations show that we may find a strictly decreasing sequence $(\tau_k)_k$ such that (τ_k, τ_{k-1}) is a connected component of \mathcal{U} . The limit of this sequence as k tends to infinity is $\tau_\infty \geq t_0$. We may choose sequence $(\tau_k)_k$, so that τ_∞ is arbitrarily close to 0. We denote

$$w_k = \dot{u}(\tau_k + 0).$$

Let us first assume that

$$f(t_0, 0, 0) < 0. \quad (4.12)$$

By continuity, there exists τ such that for $t \in [t_0, \tau]$,

$$f(t, u(t), \dot{u}(t)) \leq \frac{1}{2}f(t_0, 0, 0).$$

Let

$$\gamma_k = \inf_{t \in (\tau_k, \tau_{k-1})} [-f(t, u(t), \dot{u}(t))], \quad \text{and} \quad \gamma^k = \sup_{t \in (\tau_k, \tau_{k-1})} [-f(t, u(t), \dot{u}(t))].$$

On (τ_k, τ_{k-1}) , we have the two inequalities

$$w_k(t - \tau_k) - \gamma^k \frac{(t - \tau_k)^2}{2} \leq u(t) \leq w_k(t - \tau_k) - \gamma_k \frac{(t - \tau_k)^2}{4}. \quad (4.13)$$

Hence,

$$\tau_{k-1} - \tau_k \in \left[\frac{2w_k}{\gamma^k}, \frac{2w_k}{\gamma_k} \right]. \quad (4.14)$$

An inequality similar to (4.13) yields

$$\dot{u}(\tau_{k-1} - 0) \in \left[w_k \left(1 - \frac{2\gamma^k}{\gamma_k} \right), w_k \left(1 - \frac{2\gamma^k}{\gamma^k} \right) \right].$$

This implies that

$$w_k \geq \frac{w_{k-1}}{e(2\gamma^k/\gamma_k - 1)}. \quad (4.15)$$

Both sequences $(\gamma_k)_k$ and $(\gamma^k)_k$ tend to $-f(t_0, 0, 0)$ as k tends to infinity; thus, if $e \in (0, 1)$, we can take k so large that

$$e \left(\frac{2\gamma^k}{\gamma_k} - 1 \right) < 1.$$

This implies that for k large, w_k increases faster than a diverging geometric sequence, a contradiction.

Let us take now the case $e = 1$. We need another estimate to reach our conclusion: on the relatively compact set $\{(t, u(t), \dot{u}(t+0)) : t \in [0, \tau]\}$, f is Lipschitz continuous with respect to its arguments; therefore, there exists a constant C such that

$$\gamma^k - \gamma_k \leq C \left[\max_{(\tau_k, \tau_{k-1})} u(t) + \max(w_k, w_{k-1}) + \tau_{k-1} - \tau_k \right].$$

We infer from (4.13) that for $t \in (\tau_k, \tau_{k-1})$,

$$u(t) \leq \frac{w_k^2}{(2\gamma^k)}.$$

From this relation, (4.15), and (4.14), we infer that there exist C' such that

$$\gamma^k - \gamma_k \leq C' v_k, \quad (4.16)$$

for k large enough. Assume that (4.16) holds for $k \geq m$; relation (4.15) implies that

$$\frac{w_k}{w_m} = \frac{w_k}{w_{k-1}} \frac{w_{k-1}}{w_{k-2}} \dots \frac{w_{m+1}}{w_m} \geq \prod_{j=m+1}^k \frac{1}{2\gamma^j/\gamma_j - 1}.$$

We know that w_k tends to zero as k tends to infinity, then the theory of infinite products implies that

$$\sum_{j=m}^{\infty} \left(\frac{\gamma^j}{\gamma_j} - 1 \right) = +\infty.$$

Now (4.16) and (4.12) imply that

$$\sum_{j=m}^{\infty} v_j = 0,$$

and therefore, thanks to (4.14),

$$\sum_{j=m}^{\infty} (\tau_{k-1} - \tau_k) = +\infty,$$

a contradiction. Therefore, we have proved that t_0 cannot belong to the closure of \mathcal{U} .

Assume now that

$$f(t_0, 0, 0) = 0. \quad (4.17)$$

We also let $\varepsilon_k = \tau_{k-1} - \tau_k$ and

$$\lambda_k^+ = (\tau_k, 0, w_k).$$

Relation (4.11) implies that

$$w_k = -\varepsilon_k \int_0^1 \frac{E(\tau_k + \varepsilon_k, \tau_k + \varepsilon_k s, \lambda_k^+)}{E(\tau_k + \varepsilon_k, \tau_k, \lambda_k^+)} f_1(\tau_k + \varepsilon_k s) ds. \quad (4.18)$$

If we define

$$\lambda_k^- = \left(\tau_k, 0, -\frac{w_k}{e} \right),$$

we can write the analogous relation

$$\frac{w_k}{e} = -\varepsilon_{k+1} \int_0^1 \frac{E(\tau_k - \varepsilon_{k+1} s, \tau_k, \lambda_k^-)}{E(\tau_k - \varepsilon_{k+1}, \tau_k, \lambda_k^-)} f_1(\tau_k - \varepsilon_{k+1} s) ds. \quad (4.19)$$

Define a function

$$\Phi(t, \varepsilon, s, \lambda) = \frac{E(t + \varepsilon, t + \varepsilon s, \lambda)}{E(t + \varepsilon, t)}.$$

The function Φ can be rewritten using Taylor's formula as

$$\Phi(t, \varepsilon, s, \lambda) = (1-s) \int_0^1 \partial_1 E(s + \varepsilon s + t' \varepsilon (1-s), t + \varepsilon s, \lambda) dt' \left(\int_0^1 \partial_1 E(s + \varepsilon t', t, \lambda) dt' \right)^{-1}$$

Then, relation (4.10) implies that Φ is well defined and analytical on an open set containing the set

$$B_\sigma = \{(t, \varepsilon, s, \lambda) : (t + \varepsilon, t, \lambda) \in A_\sigma, s \in [0, 1]\}.$$

Observe that

$$\Phi(t, \varepsilon, 0, \lambda) = 1, \quad \Phi(t, \varepsilon, 1, \lambda) = 1.$$

Let

$$\Phi_1(t, \varepsilon, s, \lambda) = \frac{\Phi(t, \varepsilon, s, \lambda)}{1-s}.$$

By Taylor's expansion, it is clear that Φ_1 remains strictly positive on A_σ . It is immediate that

$$\Phi_1(t, \varepsilon, 0, \lambda) = 1,$$

so that

$$\partial_2 \Phi_1(t, \varepsilon, 0, \lambda) = 0.$$

Therefore, there exists an analytic function Φ_2 on an open set containing B_σ such that

$$\partial_2 \Phi(t, \varepsilon, s, \lambda) = s(1-s) \Phi_2(t, \varepsilon, s, \lambda).$$

Let us define now

$$\psi(t, \varepsilon, \lambda) = - \int_0^1 \Phi(t, \varepsilon, s, \lambda) f_1(t + \varepsilon s) ds.$$

We show that ψ is strictly increasing with respect to ε if $\varepsilon > -t$, $t > 0$, and λ , ε , and t are small enough: we differentiate ψ with respect to ε and obtain

$$\partial_2 \psi(t, \varepsilon, \lambda) = - \int_0^1 s(1-s) f_1'(t + \varepsilon s) \Phi_1(t, \varepsilon, s, \lambda) \left[1 + \frac{f_1(t + \varepsilon s) \Phi_2(t, \varepsilon, s, \lambda)}{f_1'(t + \varepsilon s) \Phi_1(t, \varepsilon, s, \lambda)} \right] ds.$$

The elementary properties of the analytic function f_1' show that $\partial_2\psi$ is strictly positive if $t > 0$ and $\varepsilon > -t$ and ε , t , and λ are sufficiently small.

Let us show now that $\varepsilon_{k+1} \geq \varepsilon_k$ for all k large enough. Suppose to the contrary that $\varepsilon_{k+1} < \varepsilon_k$; the monotonicity properties of ψ , relations (4.18) and (4.19), and the fact that

$$\psi(\tau_k, 0, \lambda_k^+) = \psi(\tau_k, 0, \lambda_k^-) = -f_1(\tau_k),$$

imply that

$$w_k = \varepsilon_k \psi^k(\varepsilon_k) \geq \varepsilon_{k+1} \psi^k(\varepsilon_{k+1}) > \psi_k(\varepsilon_k) = w_{k/\varepsilon},$$

which is a contradiction. Therefore, for k large enough, $\varepsilon_k \geq \varepsilon_{k+1}$ which precludes the convergence of the series of general term ε_k . Hence, in the case (4.17) too, t_0 cannot be in the closure of \mathcal{U} .

Next, if we assume that the two solutions u and v vanish on (t_0, t_1) , we obtain a contradiction to (4.1). Finally assume that $v > 0$ and $u \equiv 0$ on (t_0, t_1) . According to (1.1), $f(t, 0, 0) \leq 0$ on $[t_0, t_1]$; therefore, if $\lambda_0 = (t_0, 0, 0)$, we have, according to (4.11),

$$v(t) = \int_{t_0}^t E(t, s, \lambda_0) f_1(s) ds.$$

This implies immediately that v is strictly negative on $(t_0, t_0 + \sigma)$; therefore, we have a contradiction. Our proof of uniqueness is complete. \blacksquare

5. CONTINUOUS DEPENDENCE OF SOLUTIONS

The continuous dependence of the solution of (1.1)–(1.5) on the data under these assumptions is given here for the sake of completeness. We make the following assumptions: \mathcal{O} is a domain of \mathbb{R}^3 which has property (3.16), f and the f_n , $n \in \mathbb{N}$ are real analytic functions on \mathcal{O} , (t_0, u_0, u_1) and (t_{0n}, u_{0n}, u_{1n}) , $n \in \mathbb{N}$ are given in $(\mathbb{R} \times C_K) \cap \mathcal{O}$, and u_n is the solution of (1.1)–(1.5) with t_{0n}, u_{0n}, u_{1n} , and f_n instead of (t_0, u_0, u_1, f) . We assume that u and the u_n , $n \in \mathbb{N}$ are maximal solutions, with respective intervals of existence $[t_0, \tau]$ and $[t_{0n}, \tau_n]$.

THEOREM 5.1. *Assume that $(f_n)_{n \in \mathbb{N}}$ converges uniformly to f on the compact subsets of \mathcal{O} , and that the sequence $(t_{0n}, u_{0n}, u_{1n})_{n \in \mathbb{N}}$ converges to (t_0, u_0, u_1) in C_K . Then, the sequence of maximal solutions $([t_{0n}, \tau_n], u_n)$ converges to the maximal solution $([t_0, \tau], u)$ in the following sense:*

$$\liminf_{n \rightarrow \infty} \tau_n \geq \tau, \tag{5.1}$$

$$u_n \text{ converges uniformly to } u \text{ on compact subsets of } [t_0, \tau], \tag{5.2}$$

$$\dot{u}_n \text{ converges to } \dot{u} \text{ almost everywhere and in } L_{\text{loc}}^p \text{ for all finite } p, \tag{5.3}$$

$$\mu_n \text{ converges weak}^* \text{ to } \mu, \tag{5.4}$$

$$U_n \text{ converges uniformly to } U \text{ on compact subsets of } [t_0, \tau]. \tag{5.5}$$

PROOF. For all $T \in (t_0, \tau)$, the set $S = \{(t, u(t), \dot{u}(t-0)); t \in (t_0, T)\} \cup \{(t, u(t), \dot{u}(t+0)); t \in [t_0, T]\}$ is compact; it is covered by the open sets $\text{int } D(t, u(t), \dot{u}(t \pm 0), \rho)$ which are included in \mathcal{O} . We cover S by a finite union of these open sets, i.e., we can find $p \geq 1$, $\rho_j > 0$, $t_j \in [t_0, \sigma]$, $1 \leq j \leq p$ such that

$$S \subset \bigcup_{j=1}^p \text{int } D(t_j, u(t_j), \dot{u}(t_j \pm 0), \rho_j).$$

Without loss of generality, we may assume that the first of the t_j 's is t_0 . Let

$$L = \bigcup_{j=1}^p D(t_j, u(t_j), \dot{u}(t_j \pm 0), \rho_j),$$

and

$$M = \sup \left\{ \sup \left(|f(t, a, b)|, \sup_n |f_n(t, a, b)| \right) : (t, a, b) \in L \right\}.$$

Theorem 3.3 implies that if $(t_{0n}, u_{0n}, u_{1n}) \in D(t_0, u_0, u_1, \rho_0/2)$, then the maximal interval $[t_{0n}, s_n]$ on which u_n belongs to L contains $[t_{0n}, t_{0n} + \hat{\sigma}]$, where $\hat{\sigma}$ is a strictly positive number which depends only on ρ_0 and M . Therefore

$$\liminf_{n \rightarrow \infty} \tau_n \geq \liminf_{n \rightarrow \infty} s_n \geq t_0 + \hat{\sigma}.$$

Let us assume that

$$\liminf s_n = s_\infty < \sigma.$$

Let α be some strictly positive number such that $\alpha < \hat{\sigma}$. The functions u_n and $(\dot{u}_n(t \pm 0))$ are bounded on $I(n) = [t_{0n}, s_\infty - \alpha]$ uniformly in n for n large enough. Moreover, if μ_n is the measure associated to u_n , we have

$$\int_{I(n)} d\mu_n \leq |\dot{u}(s_\infty - \alpha)| + |u_{1n}| + M(s_\infty - \alpha - t_{0n}),$$

i.e., the measures μ_n are uniformly bounded. This suffices to extract a subsequence which has the convergence properties described at (5.1), thanks to the uniqueness theorem. Of course, this means that all the sequence converges to u . Let $D(s_\infty, u(s_\infty), \dot{u}(s_\infty + 0), \rho')$ be included in \mathcal{O} ; let $\hat{\sigma}'$ be determined as in Theorem 3.3; choose $\alpha \geq s_\infty - \hat{\sigma}'/2$, and n sufficiently large for $(s_\infty - \alpha, u_n(s_\infty - \alpha), \dot{u}_n(s_\infty - \alpha + 0))$ to belong to $D(s_\infty, u(s_\infty), \dot{u}(s_\infty + 0), \rho'/2)$. Then, Theorem 3.3 implies that $(s_\infty - \alpha, u_n(s_\infty - \alpha), \dot{u}_n(s_\infty - \alpha + 0))$ belongs to $D(s_\infty, u(s_\infty), \dot{u}(s_\infty + 0), \rho'/2)$ for $t \in [s_\infty - \alpha, s_\infty - \alpha + \hat{\sigma}']$. We infer that in fact

$$\liminf s_n \geq \sigma.$$

The last statement follows from the observation that the sequence $(U_n)_n$ is equicontinuous from $[t_0, \sigma]$ to V_e , thanks to Lemmas 3.1 and 3.2, and each of the sets $\{U_n(x) : n \in \mathbb{N}\}$ is relatively compact in V_e . Therefore, we may apply Ascoli-Arzelá's theorem, and our conclusion holds. ■

REMARK 5.2. The analyticity of f_n has not been used in the proof of the theorem. When the f_n 's are only locally Lipschitz continuous on \mathcal{O} and have the convergence property described above, the conclusion still holds.

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