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# A stable Langevin model with diffusive-reflective boundary conditions

J.-F. Jabir\* and C. Profeta<sup>†‡</sup>

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## Abstract

In this note, we consider the construction of a one-dimensional stable Langevin type process confined in the upper half-plane and submitted to reflective-diffusive boundary conditions whenever the particle position hits 0. We show that two main different regimes appear according to the values of the chosen parameters. We then use this study to construct the law of a (free) stable Langevin process conditioned to stay positive, thus extending earlier works on integrated Brownian motion. This construction further allows to obtain the exact asymptotics of the persistence probability of the integrated stable Lévy process. In addition, the paper is concluded by solving the associated trace problem in the symmetric case.

## 1 Introduction

Let  $(L_t, t \geq 0)$  be a strictly  $\alpha$ -stable Lévy process, defined on some filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t, t \geq 0), \mathbb{P})$ , with scaling parameter  $\alpha \in (0, 2]$  and positivity parameter  $\rho$ . Its characteristic exponent is chosen as

$$\Psi(\lambda) = \log(\mathbb{E}[e^{i\lambda L_1}]) = -(i\lambda)^\alpha e^{-i\pi\alpha\rho \operatorname{sgn}(\lambda)}, \quad \lambda \in \mathbb{R},$$

where the positivity parameter  $\rho$  is given via the usual asymmetric parameter  $\beta \in [-1, 1]$  by

$$\rho = \mathbb{P}(L_1 \geq 0) = \frac{1}{2} + \frac{1}{\pi\alpha} \arctan(\beta \tan(\pi\alpha/2)).$$

We assume that  $|L|$  is not a subordinator, i.e.  $\rho \in (0, 1)$ . For  $\theta, c$  two (strictly) positive constants, we consider the SDE :

$$\begin{cases} X_t = X_0 + \int_0^t U_s ds, \\ U_t = U_0 + L_t + \sum_{n \geq 1} \left( (1 - \beta_n)(\theta^n M_n - U_{\tau_n^-}) - \beta_n(1 + c)U_{\tau_n^-} \right) \mathbb{I}_{\{\tau_n \leq t\}}, \end{cases} \quad (1)$$

where  $(\tau_n, n \in \mathbb{N})$  are the successive hitting times of  $(X_t, t \geq 0)$  at the boundary  $x = 0$ ; namely

$$\tau_n = \inf\{t > \tau_{n-1}; X_t = 0\}, \quad \tau_0 = 0,$$

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with

$$\mathbb{I}_{\{\tau_n \leq t\}} = \begin{cases} 1 & \text{if } \tau_n \leq t, \\ 0 & \text{otherwise,} \end{cases}$$

and where the sequences  $(\beta_n, n \geq 1)$  and  $(M_n, n \geq 1)$  are independent random variables, also independent from  $(X_0, U_0)$  and  $(L_t, t \geq 0)$ , such that

1. the random variables  $(\beta_n, n \geq 1)$  are i.i.d. Bernoulli r.v.'s with parameter  $p := \mathbb{P}(\beta_1 = 1)$ ,
2. the random variables  $(M_n, n \geq 1)$  are i.i.d., non-negative and such that  $\mathbb{P}(M_1 = 0) = 0$ . We further assume that they admit moment of order at least  $\alpha$ .

The model (1) describes a type of Langevin model where, at each time  $t$ , the components  $X_t$  and  $U_t$  represent respectively the position and the velocity of some fluid particle, interacting with a physical wall located at the axis  $x = 0$ . The particle positions are confined within the upper-half space  $[0, \infty)$ , the velocity paths are càdlàg, governed by the Lévy process  $(L_t, t \geq 0)$  when  $X_t$  is in the interior  $(0, \infty)$  and, whenever the particle hits the frontier  $x = 0$ , the velocity component is submitted to either a partially absorbing boundary condition or a diffusive boundary condition, respectively quantified by  $c$  and  $\theta^n M_n$ . More precisely, for all  $n$ , the left-hand limit  $U_{\tau_n^-} = \lim_{t \rightarrow \tau_n, t < \tau_n} U_t$  describes the outgoing velocity of the particle and the reflective-diffusive interaction between the particle and the confinement frontier implies that the velocity  $U_{\tau_n} = \lim_{t \rightarrow \tau_n, t > \tau_n} U_t$  immediately after the impact is given by :

$$U_{\tau_n} = \Delta U_{\tau_n} + U_{\tau_n^-} = \begin{cases} -cU_{\tau_n^-} & \text{if } \beta_n = 1, \\ \theta^n M_n & \text{if } \beta_n = 0. \end{cases} \quad (2)$$

As  $U_{\tau_n^-}$  is necessarily non-positive, the particle either re-emerges in  $(0, \infty)$  or remains stuck to the axis  $x = 0$ , and, in any cases, remains confined within  $[0, \infty)$ .

The (partially) absorbing case ( $p = 1$ , i.e.  $\beta_n = 1$  a.s.) was previously introduced and intensively investigated in Bertoin [2, 3] (in the case of a totally absorbing wall,  $c = 0$ ) and Jacob [14, 15] when  $L = B$  is a one dimensional Brownian motion. Jacob [14] exhibited the critical level  $c_{\text{crit}} = \exp(-\pi/\sqrt{3})$  separating sticky and non-sticky situations; whenever  $c \geq c_{\text{crit}}$ ,  $\lim_n \tau_n = \infty$  a.s., whereas if  $c < c_{\text{crit}}$ ,  $\lim_n \tau_n < \infty$  a.s. In the situation where  $L = B$  is a standard Brownian motion, the corresponding zeroes of the integrated Brownian motion  $(x + tu + \int_0^t B_s ds, t \geq 0)$  has been the subject of a long history of studies starting from the early work of McKean [19] and the numerous works of Aimé Lachal (see [18] among others). In the case of a stable Lévy process, few results on the distribution of the zeroes of an integrated Lévy process are known. Nevertheless the necessary and sufficient conditions ensuring the non-accumulation of  $(\tau_n, n \geq 1)$  in finite time will be exhibited in Section 2.

The totally diffusive situation ( $p = 0$ , i.e.  $\beta_n = 0$  a.s.) models the particular case of Maxwell boundary conditions introduced in the kinetic theory of gases (see e.g. Chapter 8 in Cercignani, Illner and Pulvirenti [7]). The particular situation where  $\theta = 1$  and  $(M_n, n \geq 1)$  is distributed according to a Maxwellian distribution of the form:

$$\frac{v}{\Theta} \exp\left\{-\frac{|v|^2}{2\Theta}\right\} \mathbb{I}_{\{v \geq 0\}}, \quad \text{with } \Theta > 0,$$

corresponds to the situation where a (gas) particle interacts with a surface in thermodynamical equilibrium at temperature  $\Theta$ , and where the particle re-emerges from the wall with a velocity  $M_n$  after each

impact. (1) allow slightly more general diffusive phenomena where the magnitude of the re-emerging velocity  $M_n$  is further controlled by the parameter  $\theta^n$ , allowing situations where the diffusive velocity progressively vanishes at  $n \rightarrow \infty$ . Let us also point out that the connection between transport equation endowing diffusive-reflective boundary conditions and Langevin models driven by a Poisson process has been studied in Costantini and Kurtz [9] and Costantini [8]. The link between (1) and kinetic equations will be further discussed in Section 4.

Our aim in this paper is to study the limit behaviour of the sequence  $(\tau_n, n \geq 1)$ , according to the parameters  $\theta$  and  $c$ . Such results then allow to discuss some related problems for the construction of a conditioned stable Langevin process, as well as some relations to kinetic equations.

- a) In the next Section 2, we describe the asymptotic behaviour of the sequence  $(\tau_n, n \geq 1)$  in the case where  $(X_0, U_0)$  belongs to the semi-finite line  $\{0\} \times (0, \infty)$ , showing necessary and sufficient conditions on the parameters  $c$  and  $\theta$  to characterize sticky situations (see Theorem 2.1.1) in the case of an absorbing wall ( $p = 1$ ), of a diffusive wall ( $p = 0$ ) and mixed boundary condition ( $0 < p < 1$ ). When the lifetime of the process is infinite, we further give some a.s. asymptotics for the behavior of  $(\tau_n, n \geq 1)$  (see Theorem 2.2.1).
- b) Section 3 is dedicated to the construction of an integrated stable Lévy process conditioned to never hit 0, thus extending the previous results obtained in Jacob [15] and in Groeneboom, Jongbloed and Wellner [12] for Brownian motion. As a by product, we deduce the asymptotic behavior of the upper tail distribution of  $\tau_1$  (Corollary 3.2.2), improving the result in Profeta and Simon [22].
- c) Section 4 is dedicated to the link between (1) and trace problems for kinetic equations. Such link was previously studied in Bossy and Jabir [5] for Langevin model driven by a Brownian motion and singular nonlinear (in the sense of McKean-Vlasov) drift component. We show the existence of trace functions under appropriate assumptions on  $c, \theta$  and the distribution of  $(X_0, U_0)$ .

## 2 Estimation on the asymptotic behavior of $(\tau_n, n \in \mathbb{N})$ starting from $\{0\} \times (0, +\infty)$

We start by decomposing the paths of  $X$  into a sum of excursions. To simplify the expressions, we shall assume that a.s.  $X_0 = 0$  and  $U_0 > 0$ .

For  $n \geq 1$ , we define the restarting velocity after the  $n^{\text{th}}$  passage time at the boundary :

$$V_n = U_{\tau_n} = (1 - \beta_n)\theta^n M_n + \beta_n c |U_{\tau_n^-}| \quad \text{and} \quad V_0 = U_0. \quad (3)$$

Observe that by the scaling property of  $L$ , for  $n \geq 1$  :

$$(\tau_n, U_{\tau_n^-}) \stackrel{(d)}{=} (\tau_{n-1} + V_{n-1}^\alpha \xi_1, V_{n-1} \ell_1) \quad (4)$$

where the pair  $(\xi_1, \ell_1)$  is independent from  $V_{n-1}$  and is distributed as  $(\tau_1, U_{\tau_1^-})$  when the process  $(X, U)$  is started from  $(0, 1)$ . Although the law of the pair  $(\xi_1, \ell_1)$  is not explicitly known, we have the following estimates from [22]: for  $\lambda \geq 0$ ,

$$\mathbb{E}[\xi_1^\lambda] < +\infty \iff \mathbb{E}[|\ell_1|^{\alpha\lambda}] < +\infty \iff \lambda < \frac{1 - \rho}{1 + \alpha\rho}, \quad (5)$$

as well as the Mellin's transform,

$$\mathbb{E}[|\ell_1|^{\nu-1}] = \mathbb{E}_{(0,1)}[|U_{\tau_1^-}|^{\nu-1}] = \frac{\sin(\pi\gamma\nu)}{\sin(\pi\nu(1-\gamma))} \quad (6)$$

where  $\gamma$  is given by

$$\gamma = \frac{\alpha(1-\rho)}{1+\alpha}. \quad (7)$$

By defining

$$g_n = \sup\{k \leq n-1; \beta_k = 0\} \quad \text{and} \quad M_0 = U_0,$$

we deduce by iteration from (3) and (4) that

$$(V_n, n \geq 1) \stackrel{(d)}{=} \left( (1-\beta_n)\theta^n M_n + \beta_n \theta^{g_n} M_{g_n} \prod_{i=g_n+1}^n c|\ell_i|, n \geq 1 \right) \quad (8)$$

where  $(\ell_n, n \geq 1)$  are i.i.d. random variables with the same law that of  $U_{\tau_1^-}$  when  $(X, U)$  starts from  $(0, 1)$ . Note that the r.v.'s  $(V_n, n \geq 1)$  are of course not independent, except if  $p = 0$ .

The same scaling and the independent increments properties further allow to decompose the passage time  $\tau_n$  as :

$$\tau_n = \tau_1 + \sum_{k=2}^n \frac{\tau_k - \tau_{k-1}}{V_{k-1}^\alpha} V_{k-1}^\alpha \stackrel{(d)}{=} \xi_1 U_0^\alpha + \sum_{k=2}^n \xi_k V_{k-1}^\alpha \quad (9)$$

where  $(\xi_k, k \geq 1)$  are i.i.d. r.v.'s with the same law as  $\tau_1$  when  $(X, U)$  is started from  $(0, 1)$ . Furthermore, for every fixed  $n \geq 1$ ,  $\xi_k$  is independent from  $V_{k-1}$ .

In the following, we shall be interested in the study of

$$\tau_\infty = \lim_{n \rightarrow +\infty} \tau_n = \inf\{t > 0; (X_t, U_t) = (0, 0)\}.$$

## 2.1 Absorption in finite time

We start by looking at the conditions under which the particle is absorbed at the boundary in finite time, that is  $\tau_\infty < \infty$  a.s.

**Theorem 2.1.1.** *Assume that  $X_0 = 0$  and  $U_0 > 0$  with  $U_0^\alpha$  integrable. Then we have the following situations:*

1. If  $p = 1$ , then

$$\tau_\infty < \infty \quad \mathbb{P} - a.s. \quad \iff \quad c < c_{crit}$$

where

$$c_{crit} = \exp(-\pi \cot(\pi\gamma)).$$

In particular, for  $\lambda > 0$ ,

$$\mathbb{E}[\tau_\infty^\lambda] < +\infty \quad \iff \quad \left\{ c < c_{crit} \text{ and } c^{\alpha\lambda} \mathbb{E}[|\ell_1|^{\alpha\lambda}] < 1 \right\}.$$

2. If  $p = 0$ , then

$$\tau_\infty < \infty \quad \mathbb{P} - a.s. \quad \iff \quad \theta < 1.$$

In particular, for  $\lambda > 0$ ,

$$\mathbb{E}[\tau_\infty^\lambda] < +\infty \quad \iff \quad \left\{ \theta < 1 \text{ and } \lambda < \frac{1-\rho}{1+\alpha\rho} \right\}$$

3. If  $0 < p < 1$ , then

$$\tau_\infty < \infty \quad \mathbb{P} - a.s. \quad \iff \quad \theta < 1.$$

In particular, for  $\lambda > 0$ ,

$$\mathbb{E}[\tau_\infty^\lambda] < +\infty \quad \iff \quad \{\theta < 1 \text{ and } c^{\alpha\lambda} \mathbb{E}[|\ell_1|^{\alpha\lambda}] p < 1\}.$$

The proof of this theorem will rely on the following lemma, which is a special case of Kolmogorov's three-series theorem (see e.g. Durrett [10, p.64]).

**Lemma 2.1.2.** *Let  $(X_n, n \geq 0)$  be i.i.d. and positive random variables. We assume that there exists  $\lambda \in (0, 1)$  such that  $0 < \mathbb{E}[X_1^\lambda] < +\infty$ . Then, for  $a \geq 0$  :*

$$\sum_{n=1}^{+\infty} X_n a^n < +\infty \quad a.s. \quad \iff \quad a < 1.$$

*Proof.* If  $a < 1$ ,

$$\mathbb{E} \left[ \left( \sum_{n=1}^{+\infty} X_n a^n \right)^\lambda \right] \leq \sum_{n=1}^{+\infty} \mathbb{E} [X_n^\lambda] a^{\lambda n} = \frac{a^\lambda \mathbb{E} [X_1^\lambda]}{1 - a^\lambda} < +\infty$$

which implies that the series converges a.s. If  $a \geq 1$ , we deduce from the monotone convergence theorem, for all  $\lambda > 0$ ,

$$\mathbb{E} \left[ \exp \left( -\lambda \sum_{n=1}^N X_n a^n \right) \right] \leq \mathbb{E} [\exp(-\lambda X_1)]^N \xrightarrow{N \rightarrow +\infty} 0$$

which concludes the proof.  $\square$

**Remark 2.1.3.** *Note that we cannot totally remove the assumption on the moments in the previous lemma. Indeed, consider for instance a sequence of positive i.i.d. random variables with distribution :*

$$\mathbb{P}(X_1 \in dx) = \frac{\ln(2)}{x \ln^2(x)} \mathbb{I}_{\{x \geq 2\}} dx.$$

Then, integrating by parts,

$$\mathbb{E} [\exp(-\lambda X_1 a^n)] = \int_2^{+\infty} e^{-\lambda a^n x} \frac{\ln(2)}{x \ln^2(x)} dx = e^{-2\lambda a^n} - \lambda a^n \int_2^{+\infty} e^{-\lambda a^n x} \frac{\ln(2)}{\ln(x)} dx.$$

Taking  $a < 1$ , we deduce by the Tauberian theorem that :

$$1 - \mathbb{E} [\exp(-\lambda X_1 a^n)] \underset{n \rightarrow +\infty}{\sim} -\frac{\ln(2)}{n \ln(a)}$$

hence

$$\mathbb{E} \left[ \exp \left( -\lambda \sum_{n=1}^N X_n a^n \right) \right] = \prod_{n=1}^N \mathbb{E} [\exp(-\lambda X_1 a^n)] \xrightarrow{N \rightarrow +\infty} 0$$

which proves that  $\sum_{n=1}^{+\infty} X_n a^n = +\infty$  a.s. for any  $a > 0$  in this case.

*Proof.* We shall now prove Theorem 2.1.1 and deal with the three cases separately.

1. Assume first that  $p = 1$ . We follow the ideas of Jacob [14, Corollary 1]. According to (8), we have the decomposition

$$V_n \stackrel{(d)}{=} U_0 \prod_{i=1}^n c|\ell_i|.$$

Therefore, from (9),

$$\tau_\infty \stackrel{(d)}{=} \xi_1 U_0^\alpha + U_0^\alpha \sum_{n=2}^{+\infty} \xi_n c^{\alpha(n-1)} \prod_{i=1}^{n-1} |\ell_i|^\alpha.$$

Next, from Profeta and Simon [23], the law of large numbers implies that :

$$\frac{1}{n} \sum_{i=1}^{n-1} \ln(|\ell_i|) \xrightarrow[n \rightarrow +\infty]{\text{a.s.}} \mathbb{E}[\ln(|\ell_1|)] = \pi \cot(\pi\gamma) = -\ln(c_{\text{crit}})$$

hence, for any fixed  $\varepsilon > 0$ , we have for  $n$  large enough

$$c_{\text{crit}}^{-n+n\varepsilon} \leq \prod_{i=1}^{n-1} |\ell_i| \leq c_{\text{crit}}^{-n-n\varepsilon}. \quad (10)$$

and the finiteness of  $\tau_\infty$  follows from (5) and Lemma 2.1.2 when  $c < c_{\text{crit}}$ . When  $c = c_{\text{crit}}$ , observe that the random walk  $Z_n = \sum_{i=1}^n \ln(|\ell_i|) + \ln(c_{\text{crit}})$  is recurrent, hence the restarting velocity  $V_n = U_0 \exp(Z_n)$  does not converge to 0, which implies that  $\tau_\infty = \infty$  a.s.

To get the condition on the moments, we then notice that  $\tau_\infty$  is solution of a renewal equation :

$$\tau_\infty \stackrel{(d)}{=} \xi_1 U_0^\alpha (1 - c^\alpha |\ell_1|^\alpha) + U_0^\alpha \xi_2 c^\alpha |\ell_1|^\alpha + c^\alpha |\ell_1|^\alpha \tau_\infty.$$

From Goldie [11, Theorem 4.1], we deduce that there exists a constant  $\kappa > 0$  such that :

$$\mathbb{P}(\tau_\infty > t) \underset{t \rightarrow +\infty}{\sim} \frac{\kappa}{t^\eta(c)}$$

where  $\eta(c) > 0$  is such that  $c^{\alpha\eta(c)} \mathbb{E}[|\ell_1|^{\alpha\eta(c)}] = 1$ . Point 1. then follows from the fact that for  $c < c_{\text{crit}}$ , the function  $\lambda \rightarrow c^{\alpha\lambda} \mathbb{E}[|\ell_1|^{\alpha\lambda}]$  is convex with a negative derivative at  $0^+$  given by  $\alpha(\ln(c) - \ln(c_{\text{crit}})) < 0$ .

2. Assume now that  $p = 0$ . Then, the particle will always restart afresh when hitting the zero axis, namely  $V_n = \theta^n M_n$  for  $n \geq 1$ . Hence

$$\tau_\infty \stackrel{(d)}{=} \xi_1 U_0^\alpha + \sum_{n=2}^{+\infty} \xi_n \theta^{\alpha(n-1)} M_{n-1}^\alpha$$

and from Lemma 2.1.2, this series converges if and only if  $\theta < 1$ , in which case the moments of  $\tau_\infty$  are finite if and only if those of  $\xi_1$  are (since  $U_0^\alpha$  and  $M_1^\alpha$  are assumed to be integrable).

3. Assume finally that  $0 < p < 1$ . Observe first that since the r.v.'s  $(\beta_k, k \geq 1)$  only take the values 0 or 1 and all the terms are positive,  $\tau_n$  may be decomposed, after a change of indices, as :

$$\tau_n \stackrel{(d)}{=} \xi_1 U_0^\alpha + \sum_{k=1}^{n-1} \xi_{k+1} (1 - \beta_k) (\theta^k M_k)^\alpha + \sum_{k=1}^{n-1} \xi_{k+1} \beta_k \left( \theta^{g_k} M_{g_k} \prod_{i=g_k+1}^k c|\ell_i| \right)^\alpha$$

with the convention that  $M_0 = U_0$ . Since in this case the r.v.'s  $(\beta_k, k \geq 1)$  take infinitely many often the value 0, we first observe by Lemma 2.1.2 that  $\tau_\infty = \infty$  as soon as  $\theta \geq 1$ . Taking  $0 < \lambda < 1$  small enough, we then have :

$$\begin{aligned} \mathbb{E}[\tau_n^\lambda] &\leq \mathbb{E}[\xi_1^\lambda U_0^{\lambda\alpha}] + \sum_{k=1}^{n-1} \mathbb{E} \left[ (1 - \beta_k) (\xi_{k+1} \theta^{\alpha k} M_k^\alpha)^\lambda \right] + \sum_{k=1}^{n-1} \mathbb{E} \left[ \beta_k \left( \xi_{k+1} \theta^{\alpha g_k} M_{g_k}^\alpha \prod_{i=g_k+1}^k c^\alpha |\ell_i|^\alpha \right)^\lambda \right] \\ &\leq \mathbb{E}[\xi_1^\lambda] \left( \mathbb{E}[U_0^{\lambda\alpha}] + (1-p)\mathbb{E}[M_1^{\alpha\lambda}] \frac{\theta^{\alpha\lambda} - \theta^{\alpha\lambda n}}{1 - \theta^{\alpha\lambda}} \right) + p\mathbb{E}[\xi_1^\lambda] \sum_{k=1}^{n-1} \mathbb{E} \left[ \theta^{\alpha g_k \lambda} M_{g_k}^{\alpha\lambda} \prod_{i=g_k+1}^k c^{\alpha\lambda} |\ell_i|^{\alpha\lambda} \right]. \end{aligned}$$

We shall compute the remaining sum in the following Lemma.

**Lemma 2.1.4.** *For  $n \geq 2$ ,  $\theta \neq 1$  and  $0 < \lambda < \frac{1-p}{1+\alpha p}$ :*

$$\begin{aligned} &\sum_{k=1}^n \mathbb{E} \left[ \theta^{\alpha g_k \lambda} M_{g_k}^{\alpha\lambda} \prod_{i=g_k+1}^k c^{\alpha\lambda} |\ell_i|^{\alpha\lambda} \right] \\ &= \mathbb{E}[U_0^{\alpha\lambda}] c^{\alpha\lambda n} \mathbb{E} [|\ell_1|^{\alpha\lambda}]^n p^{n-1} + \sum_{i=1}^{n-1} c^{\alpha\lambda i} \mathbb{E} [|\ell_1|^{\alpha\lambda}]^i p^{i-1} \left( \mathbb{E}[U_0^{\alpha\lambda}] + (1-p)\theta^{\alpha\lambda} \mathbb{E}[M_1^{\alpha\lambda}] \frac{\theta^{\alpha\lambda(n-i)} - 1}{\theta^{\alpha\lambda} - 1} \right). \end{aligned}$$

In particular, for  $\lambda$  small enough such that  $c^{\alpha\lambda} \mathbb{E} [|\ell_1|^{\alpha\lambda}] p < 1$ , there exist two constants  $A_\lambda, B_\lambda$  independent of  $n$  such that

$$\mathbb{E} [\tau_n^\lambda] \leq A_\lambda \theta^{\alpha\lambda n} + B_\lambda.$$

*Proof.* The law of  $g_k$  is given by :

$$\mathbb{P}(g_k = l) = (1-p)p^{k-l-1} \text{ for } l \in \{1, \dots, k-1\} \quad \text{and} \quad \mathbb{P}(g_k = 0) = p^{k-1}. \quad (11)$$

We decompose the expectation according to  $g_k$ . When  $g_k = 0$  :

$$\mathbb{E} \left[ \theta^{\alpha g_k \lambda} M_{g_k}^{\alpha\lambda} \prod_{i=g_k+1}^k c^{\alpha\lambda} |\ell_i|^{\alpha\lambda} \mathbb{I}_{\{g_k=0\}} \right] = \mathbb{E}[U_0^{\alpha\lambda}] c^{\alpha\lambda k} \mathbb{E} [|\ell_1|^{\alpha\lambda}]^k p^{k-1}. \quad (12)$$

When  $g_k > 0$ , similar computations yield :

$$\begin{aligned} \mathbb{E} \left[ \theta^{\alpha g_k \lambda} M_{g_k}^{\alpha\lambda} \prod_{i=g_k+1}^k c^{\alpha\lambda} |\ell_i|^{\alpha\lambda} \mathbb{I}_{\{g_k>0\}} \right] &= \sum_{l=1}^{k-1} \mathbb{E} \left[ \theta^{\alpha l \lambda} M_l^{\alpha\lambda} \prod_{i=l+1}^k c^{\alpha\lambda} |\ell_i|^{\alpha\lambda} \right] (1-p)p^{k-l-1} \\ &= (1-p)\mathbb{E}[M_1^{\alpha\lambda}] \sum_{i=1}^{k-1} \theta^{\alpha\lambda(k-i)} c^{\alpha\lambda i} \mathbb{E} [|\ell_1|^{\alpha\lambda}]^i p^{i-1}. \end{aligned} \quad (13)$$

Applying Fubini-Tonelli's theorem, we deduce that

$$\begin{aligned} \sum_{k=2}^n \sum_{i=1}^{k-1} \theta^{\alpha\lambda(k-i)} c^{\alpha\lambda i} \mathbb{E} [|\ell_1|^{\alpha\lambda}]^i p^{i-1} &= \sum_{i=1}^{n-1} \theta^{-\alpha\lambda i} c^{\alpha\lambda i} \mathbb{E} [|\ell_1|^{\alpha\lambda}]^i p^{i-1} \sum_{k=i+1}^n \theta^{k\alpha\lambda} \\ &= \theta^{\alpha\lambda} \sum_{i=1}^{n-1} c^{\alpha\lambda i} \mathbb{E} [|\ell_1|^{\alpha\lambda}]^i p^{i-1} \frac{1 - \theta^{\alpha\lambda(n-i)}}{1 - \theta^{\alpha\lambda}} \end{aligned} \quad (14)$$

which concludes the proof of Lemma 2.1.4.  $\square$



Now, letting  $n \rightarrow +\infty$  in Lemma 2.1.4 and using the fact that  $\theta < 1$ , we deduce on the one hand that  $\mathbb{E}[\tau_\infty^\lambda] < +\infty$  as soon as  $\lambda$  is small enough so that  $c^{\alpha\lambda}\mathbb{E}[|\ell_1|^{\alpha\lambda}] p < 1$ . In particular,  $\tau_\infty < +\infty$  a.s. for any  $c > 0$ . On the other hand, we have :

$$\mathbb{E}[\tau_\infty^\lambda] \geq \mathbb{E} \left[ \left( U_0^\alpha \sum_{k=1}^{+\infty} \xi_{k+1} \beta_k \left( \prod_{i=1}^k c |\ell_i| \right)^\alpha \mathbb{I}_{\{g_k=0\}} \right)^\lambda \right].$$

Observe now that the random variable appearing on the right-hand side of the previous equation, say  $\chi_\infty$ , is a solution of the following renewal equation :

$$\chi_\infty \stackrel{(d)}{=} \xi_2 \beta_1 U_0^\alpha c^\alpha |\ell_1|^\alpha \mathbb{I}_{\{g_1=0\}} + c^\alpha |\ell_1|^\alpha \mathbb{I}_{\{\beta_1=1\}} \chi_\infty.$$

The proof is then concluded using the asymptotics given by Goldie [11, Theorem 4.1] as in Point 1., since

$$\mathbb{E} \left[ c^{\alpha\lambda} |\ell_1|^{\alpha\lambda} 1_{\{\beta_1=1\}}^\lambda \right] = c^{\alpha\lambda} \mathbb{E} [ |\ell_1|^{\alpha\lambda} ] p.$$

□

## 2.2 Asymptotics of $(\tau_n, n \geq 1)$

We now study the rate of divergence of  $\tau_n$  when  $\tau_n \xrightarrow[n \rightarrow +\infty]{} +\infty$ .

**Theorem 2.2.1.** *Assume now that  $\tau_\infty = +\infty$  a.s. We have the following asymptotics :*

1. When  $p = 1$  :

(a) If  $c > c_{crit}$  :

$$\frac{\ln(\tau_n)}{n} \xrightarrow[n \rightarrow +\infty]{a.s.} \alpha (\pi \cot(\pi\gamma) + \ln(c)).$$

(b) If  $c = c_{crit}$  : for any  $0 < \lambda < 2$ ,

$$\frac{\ln(\tau_n)}{n^{1/\lambda}} \xrightarrow[n \rightarrow +\infty]{a.s.} 0.$$

2. When  $p = 0$  :

(a) If  $\theta > 1$  :

$$\frac{\ln(\tau_n)}{n} \xrightarrow[n \rightarrow +\infty]{a.s.} \alpha \ln(\theta).$$

(b) If  $\theta = 1$  : for any  $0 < \lambda < \frac{1-\rho}{1+\alpha\rho}$ ,

$$\frac{\tau_n}{n^{1/\lambda}} \xrightarrow[n \rightarrow +\infty]{a.s.} 0.$$

3. When  $0 < p < 1$  :

(a) If  $\theta > 1$  :

$$\frac{\ln(\tau_n)}{n} \xrightarrow[n \rightarrow +\infty]{a.s.} \alpha \ln(\theta).$$

(b) If  $\theta = 1$  : for any  $\lambda > 0$  such that  $c^{\alpha\lambda}\mathbb{E}[|\ell_1|^{\alpha\lambda}] p < 1$  :

$$\frac{\tau_n}{n^{1/\lambda}} \xrightarrow[n \rightarrow +\infty]{a.s.} 0.$$

Before proving Theorem 2.2.1, we state a short lemma regarding the negative moments of  $\xi_1$ .

**Lemma 2.2.2.**

1. If  $0 < \alpha < 2$ , then for  $\lambda \geq 0$  :

$$\mathbb{E}[\xi_1^{-\lambda}] < +\infty \iff \lambda \in [0, 1).$$

2. If  $\alpha = 2$ , then for any  $\lambda \geq 0$ , we have  $\mathbb{E}[\xi_1^{-\lambda}] < +\infty$

*Proof.* We denote by  $(A_t = \int_0^t L_u du, t \geq 0)$  the (free) integrated stable Lévy process, and by  $\mathbb{P}_{(x,y)}$  the law of  $(A, L)$  when started from  $(x, y)$ . Applying the Markov property, we have for  $\mu > 0$  :

$$\int_0^{+\infty} e^{-\mu t} \mathbb{P}_{(0,1)}(A_t \leq 0) dt = \mathbb{E} \left[ e^{-\mu \xi_1} \int_0^{+\infty} e^{-\mu t} \mathbb{P}_{(0,\ell_1)}(A_t \leq 0) dt \right].$$

Recall then that  $A_t \stackrel{(d)}{=} \frac{t^{1+1/\alpha}}{1+\alpha} L_1$  under  $\mathbb{P}_{(0,0)}$ . Therefore, since  $\ell_1 < 0$  a.s., we deduce that

$$1 \geq \mathbb{P}_{(0,\ell_1)}(A_t \leq 0) = \mathbb{P}(L_1 \leq -(1+\alpha)t^{-1/\alpha}\ell_1) \geq 1 - \rho,$$

hence

$$\mu \int_0^{+\infty} e^{-\mu t} \mathbb{P}_{(0,1)}(A_t \leq 0) dt \leq \mathbb{E} [e^{-\mu \xi_1}] \leq \frac{\mu}{1-\rho} \int_0^{+\infty} e^{-\mu t} \mathbb{P}_{(0,1)}(A_t \leq 0) dt.$$

Integrating against  $\mu^{\lambda-1}$  on  $(0, +\infty)$  with  $\lambda > 0$ , we obtain :

$$\lambda \int_0^{+\infty} t^{-\lambda-1} \mathbb{P}_{(0,1)}(A_t \leq 0) dt \leq \mathbb{E} [\xi_1^{-\lambda}] \leq \frac{\lambda}{1-\rho} \int_0^{+\infty} t^{-\lambda-1} \mathbb{P}_{(0,1)}(A_t \leq 0) dt.$$

The result now follows from the asymptotics of the stable laws, i.e. for  $0 < \alpha < 2$ ,

$$\mathbb{P}(L_1 \leq -t^{-1/\alpha}) \underset{t \rightarrow 0}{\sim} \kappa t$$

for some  $\kappa > 0$ . In the Brownian case  $\alpha = 2$ , this asymptotics is well-known to be exponential. □

*Proof.* We now come back to the proof of Theorem 2.2.1.

1.(a) The proof of Point 1.(a) is a direct adaptation of Profeta-Simon [23, Theorem A], using the decomposition :

$$\tau_{2p} - \tau_{2p-1} \stackrel{(d)}{=} L_{\tau_1} \times \tau_1 \times \left( \prod_{i=1}^{p-1} c|\ell_i| \right)^{2\alpha}.$$

Note that due to the reflection, there is no need to use the dual process  $-L$  here.

1.(b) Let  $\varepsilon > 0$ . Using the Markov property :

$$\begin{aligned} \mathbb{P} \left( \frac{\ln(\tau_n)}{n^{1/\lambda}} \geq \varepsilon \right) &\leq e^{-\varepsilon n^{1/\lambda-1/2}} \mathbb{E}[\tau_n^{1/\sqrt{n}}] \\ &\leq e^{-\varepsilon n^{1/\lambda-1/2}} \left( \mathbb{E} [(\xi_1 U_0^\alpha)^{1/\sqrt{n}}] + \sum_{k=2}^n \mathbb{E} [(\xi_k U_0^\alpha)^{1/\sqrt{n}}] \mathbb{E} [(c_{\text{crit}}|\ell_1|)^{\alpha/\sqrt{n}}]^{k-1} \right) \\ &\leq e^{-\varepsilon n^{1/\lambda-1/2}} \mathbb{E} [(\xi_1 U_0^\alpha)^{1/\sqrt{n}}] \left( 1 + (n-2) \mathbb{E} [(c_{\text{crit}}|\ell_1|)^{\alpha/\sqrt{n}}]^n \right) \end{aligned}$$

since, from Jensen's inequality,

$$\mathbb{E} \left[ (c_{\text{crit}} |\ell_1|)^{\alpha/\sqrt{n}} \right] \geq \exp \left( \frac{\alpha}{\sqrt{n}} \mathbb{E}[\ln(|\ell_1|) + \ln(c_{\text{crit}})] \right) = 1.$$

Next, using the explicit Mellin transform (6) and Taylor expansions, we may compute the limit :

$$\mathbb{E} \left[ (c_{\text{crit}} |\ell_1|)^{\alpha/\sqrt{n}} \right]^n = e^{-\sqrt{n}\alpha\pi \cot(\pi\gamma)} \left( \frac{\sin(\pi\gamma(1 + \frac{\alpha}{\sqrt{n}}))}{\sin(\pi(1-\gamma)(1 + \frac{\alpha}{\sqrt{n}}))} \right)^n \xrightarrow{n \rightarrow +\infty} \exp \left( \frac{\pi^2\alpha^2}{2}(1-2\gamma) \right).$$

The a.s. convergence then follows from the usual application of the Borel-Cantelli's lemma.

2.(a) and 3.(a) Both cases may be dealt with in the same way, by taking  $p \in [0, 1[$ . We start with the Markov's inequality :

$$\mathbb{P} \left( \left| \frac{\ln(\tau_n)}{n} - \alpha \ln(\theta) \right| > \varepsilon \right) \leq e^{-\lambda \varepsilon n} (\mathbb{E} [\tau_n^\lambda \theta^{-\alpha \lambda n}] + \mathbb{E} [\tau_n^{-\lambda} \theta^{\alpha \lambda n}])$$

Using Lemma 2.1.4 with  $\lambda$  small enough, the first term may be bounded by

$$\mathbb{E} [\tau_n^\lambda \theta^{-\alpha \lambda n}] \leq A_\lambda + B_\lambda \theta^{-\alpha \lambda n} \leq A_\lambda + B_\lambda < +\infty$$

since  $\theta > 1$ . Similarly, since  $\beta_n$  only takes the values 0 or 1, the second term may be bounded by :

$$\begin{aligned} \mathbb{E} [\tau_n^{-\lambda} \theta^{\alpha \lambda n}] &\leq \theta^{\alpha \lambda n} \mathbb{E} \left[ \left( \xi_{n+1} (1 - \beta_n) \theta^{\alpha n} M_n^\alpha + \xi_{n+1} \beta_n \theta^{\alpha g_n} M_{g_n}^\alpha \prod_{i=g_n+1}^n c^\alpha |\ell_i|^\alpha \right)^{-\lambda} \right] \\ &\leq \theta^{\alpha \lambda n} \left( (1-p) \theta^{-n\alpha\lambda} \mathbb{E} [\xi_1^{-\lambda} M_1^{-\alpha\lambda}] + p \mathbb{E} [\xi_1^{-\lambda}] \mathbb{E} \left[ \left( \theta^{g_n} M_{g_n} \prod_{i=g_n+1}^n c |\ell_i| \right)^{-\alpha\lambda} \right] \right). \end{aligned}$$

Observe next that, decomposing the expectation with respect to the law of  $g_n$  (see (11)), we obtain

$$\begin{aligned} \theta^{\alpha \lambda n} \mathbb{E} \left[ \left( \theta^{g_n} M_{g_n} \prod_{i=g_n+1}^n c |\ell_i| \right)^{-\alpha\lambda} \right] \\ = \theta^{\alpha \lambda n} \mathbb{E}[U_0^{-\alpha\lambda}] \mathbb{E}[|c\ell_1|^{-\alpha\lambda}]^n p^{n-1} + (1-p) \mathbb{E}[M_1^{-\alpha\lambda}] \sum_{k=1}^{n-1} \theta^{\alpha \lambda k} \mathbb{E}[|c\ell_1|^{-\alpha\lambda}]^k p^{k-1}. \end{aligned}$$

This term may be bounded by a constant independent of  $n$  as soon as  $\lambda$  is small enough so that

$$\theta^{\alpha \lambda} \mathbb{E}[|c\ell_1|^{-\alpha\lambda}] p < 1.$$

The result then follows again from the Borel-Cantelli's lemma.

2.(b) and 3.(b) are consequences of the following result by Petrov [21], which we adapt here to our set-up. Assume that  $(X_k, k \geq 1)$  are positive r.v.'s such that  $\mathbb{E}[X_k^\nu]$  for some positive  $\nu \leq 1$  and all  $k \geq 1$ . If  $A_n = \sum_{k=1}^n \mathbb{E}[X_k^\nu] \xrightarrow{n \rightarrow +\infty} +\infty$ , then for any  $0 < \lambda < \nu$ ,  $\sum_{k=1}^n X_k = o(A_n^{1/\lambda})$  a.s.

We therefore apply the aforementioned result with  $X_k = \tau_k - \tau_{k-1}$ . When  $p = 0$  and  $\theta = 1$ , we choose  $\nu < \frac{1-\rho}{1+\alpha\rho}$ . This yields :

$$\sum_{k=2}^n \mathbb{E}[(\tau_k - \tau_{k-1})^\nu] = \sum_{k=1}^n \mathbb{E}[(\xi_k M_{k-1}^\alpha)^\nu] \underset{n \rightarrow +\infty}{\sim} \mathbb{E}[(\xi_1 M_1^\alpha)^\nu] n.$$

When  $p \in (0, 1)$  and  $\theta = 1$ , we choose  $\nu > 0$  such that  $c^{\alpha\nu} \mathbb{E}[|\ell_1|^{\alpha\nu}] p < 1$ . This yields :

$$\sum_{k=2}^n \mathbb{E}[(\tau_k - \tau_{k-1})^\nu] = (1-p) \mathbb{E}[(\xi_1 M_1^\alpha)^\nu] (n-2) + p \mathbb{E}[\xi_1^\nu] \sum_{k=2}^n \mathbb{E} \left[ \left( M_{g_{k-1}} \prod_{i=g_{k-1}+1}^{k-1} c|\ell_i| \right)^{\alpha\nu} \right].$$

Passing to the limit in Lemma 2.1.4 as  $\theta \rightarrow 1$ , we further deduce that

$$\sum_{k=2}^n \mathbb{E} \left[ \left( M_{g_{k-1}} \prod_{i=g_{k-1}+1}^{k-1} c|\ell_i| \right)^{\alpha\nu} \right] \underset{n \rightarrow +\infty}{\sim} \kappa n$$

for some constant  $\kappa > 0$ . Points 2.(b) and 3.(b) thus follow directly from Petrov's result.  $\square$

### 3 Langevin processes conditioned of not hitting $(0, 0)$

We shall construct in this section the law of an integrated  $\alpha$ -stable Lévy process conditioned to stay positive, thus extending some earlier results by Groeneboom, Jongbloed and Wellner [12] on integrated Brownian motion. Note that a direct construction seems difficult as we do not have the exact asymptotic of  $\mathbb{P}_{(x,y)}(\tau_1 > t)$  but only lower and upper bounds, see [22, Theorem A]. We assume in this section that  $p = 1$ , and  $c < c_{\text{crit}}$  so that  $\tau_\infty < +\infty$  a.s. We now denote by  $\mathbb{P}^{(c)}$  the law of the solution of (1), i.e. of the integrated  $\alpha$ -stable Langevin process reflected on a partially elastic boundary, and, to simplify  $\mathbb{P} = \mathbb{P}^{(0)}$ . The general idea of this section is to first condition the process  $(X, U)$  under  $\mathbb{P}^{(c)}$  to not hit the boundary  $(0, 0)$ , which is done using a renewal result, and then to let  $c \rightarrow 0$ .

We start by recalling that the law of  $U_{\tau_1^-}$  (which is the same under  $\mathbb{P}^{(c)}$  for any  $c \geq 0$ ) is given, for  $\nu \in (0, 1)$  via the Mellin transform (see [23, Formula (2.1)]):

$$\mathbb{E}_{(x,u)} \left[ |U_{\tau_1^-}|^{\nu-1} \right] = (1+\alpha)^{1-\frac{\nu}{1+\alpha}} \frac{\pi \int_0^\infty \int_0^\infty \lambda^{-\nu} q \left( - (1+x\lambda^{1+\alpha} + ut\lambda) t^{-1-\frac{1}{\alpha}} \right) d\lambda t^{-1-1/\alpha} dt}{\Gamma^2(\nu/(1+\alpha)) \Gamma(1-\nu) \sin(\pi\nu(1-\gamma))}$$

where  $q$  is the stable density whose Fourier transform is given by :

$$\int_{\mathbb{R}} e^{i\lambda z} q(z) dz = \mathbb{E} \left[ \exp \left( i\lambda \int_0^1 L_s ds \right) \right] = \exp \left( - \frac{1}{\alpha+1} (i\lambda)^\alpha e^{-i\pi\alpha\rho \text{sgn}(\lambda)} \right).$$

Integrating by parts in  $\lambda$  to remove the term  $\Gamma(1-\nu)$ , we obtain

$$\begin{aligned} & \mathbb{E}_{(x,u)} \left[ |U_{\tau_1^-}|^{\nu-1} \right] \\ &= (1+\alpha)^{1-\frac{\nu}{1+\alpha}} \frac{\pi \int_0^\infty \lambda^{1-\nu} \int_0^\infty \frac{(1+\alpha)x\lambda^\alpha + ut}{t^{2+2/\alpha}} q' \left( - (1+x\lambda^{1+\alpha} + ut\lambda) t^{-1-\frac{1}{\alpha}} \right) dt d\lambda}{\Gamma^2(\nu/(1+\alpha)) \Gamma(2-\nu) \sin(\pi\nu(1-\gamma))} \end{aligned} \quad (15)$$

which is valid for  $\nu \in \left(0, \frac{1}{1-\gamma}\right)$ .

**Remark 3.0.1.** *In the following, to avoid complicated notations in conditional expectations, we shall systematically remove the superscript  $(c)$  when taking the expectation of  $\mathcal{F}_{\tau_1}$ -measurable random variables.*

### 3.1 The case $0 < c < c_{\text{crit}}$

We follow Jacob [15, Section 3]. Let  $\eta(c) > 0$  be the unique solution of the equation

$$c^{\alpha\eta(c)} \mathbb{E}_{(0,1)} \left[ |U_{\tau_1^-}|^{\alpha\eta(c)} \right] = 1 \quad \iff \quad c^{\alpha\eta(c)} \frac{\sin(\pi\gamma(\alpha\eta(c) + 1))}{\sin(\pi(1-\gamma)(\alpha\eta(c) + 1))} = 1. \quad (16)$$

Looking at Formula (6), we see that  $\eta(c)$  is a decreasing function of  $c$ , and is such that

$$\lim_{c \rightarrow 0} \eta(c) = \frac{1-\rho}{1+\alpha\rho} = \eta.$$

We define the harmonic function  $h^c$  for  $\{x > 0 \text{ and } u \in \mathbb{R}\}$  or  $\{x = 0 \text{ and } u > 0\}$  by

$$h^c(x, u) = c^{\alpha\eta(c)} \mathbb{E}_{(x,u)} [|U_{\tau_1^-}|^{\alpha\eta(c)}] \quad (17)$$

Note that  $h^c$  enjoys the following scaling property :

$$h^c(x, u) = x^{\frac{\alpha\eta(c)}{1+\alpha}} h^c(1, ux^{-1/(\alpha+1)}). \quad (18)$$

In particular, for  $x > 0$  and  $u > 0$ , we have

$$h^c(0, u) = u^{\alpha\eta(c)} \quad \text{and} \quad h^c(x, 0) = x^{\frac{\alpha\eta(c)}{1+\alpha}} h^c(1, 0).$$

To simplify the expressions, we set  $a_- = \max(-a, 0)$  and

$$\kappa_{\alpha,\rho}^c = \frac{\pi(1+\alpha)^{\frac{(1-\eta(c))\alpha}{1+\alpha}}}{\Gamma^2\left(\frac{1+\eta(c)\alpha}{1+\alpha}\right)\Gamma(1-\eta(c)\alpha)\sin(\pi\gamma(1+\eta(c)\alpha))}.$$

Integrating by parts (15) with  $\nu = 1 + \alpha\eta(c)$ , and using several changes of variables, we obtain :

$$\begin{aligned} h^c(x, u) &= \alpha\eta(c)\kappa_{\alpha,\rho}^c \iint_0^{+\infty} \lambda^{-\alpha\eta(c)-1} \left( q\left(-\frac{1}{t^{1+1/\alpha}}\right) - q\left(-\frac{1+x\lambda^{1+\alpha}+ut\lambda}{t^{1+1/\alpha}}\right) \right) t^{-1-1/\alpha} d\lambda dt \\ &= \frac{\alpha\eta(c)}{1+\alpha} \kappa_{\alpha,\rho}^c \int_0^{+\infty} \mathbb{E} \left[ \left( \int_0^t L_s ds \right)_-^{\frac{\alpha(\eta(c)-1)}{1+\alpha}} \right] - \mathbb{E} \left[ \left( x + ut + \int_0^t L_s ds \right)_-^{\frac{\alpha(\eta(c)-1)}{1+\alpha}} \right] dt \end{aligned}$$

which proves that  $h^c$  is increasing in both variables  $x$  and  $u$ . Note that  $h^c$  also enjoys the following scaling property :

$$h^c(x, u) = x^{\frac{\alpha\eta(c)}{1+\alpha}} h^c(1, ux^{-1/(\alpha+1)}). \quad (19)$$

**Proposition 3.1.1.** *For  $0 < c < c_{\text{crit}}$ , there exists a probability  $\mathbb{P}_{(x,u)}^{(c)\uparrow}$  on  $(\Omega, \mathcal{F}_\infty)$  such that*

$$\forall \Lambda_s \in \mathcal{F}_s, \quad \lim_{t \rightarrow +\infty} \mathbb{P}_{(x,y)}^{(c)}(\Lambda_s | \tau_\infty > t) = \mathbb{P}_{(x,u)}^{(c)\uparrow}(\Lambda_s).$$

$\mathbb{P}_{(x,u)}^{(c)\uparrow}$  may be described by an  $h$ -transform with respect to  $\mathbb{P}_{(x,u)}^{(c)}$  as follows :

$$\forall \Lambda_s \in \mathcal{F}_s, \quad \mathbb{P}_{(x,u)}^{(c)\uparrow}(\Lambda_s) = \frac{1}{h^c(x, u)} \mathbb{E}_{(x,u)}^{(c)} \left[ \mathbb{1}_{\Lambda_s} h^c(X_s, U_s) \mathbb{1}_{\{s < \tau_\infty\}} \right].$$

*Proof.* From Goldie [11, Theorem 4.1], there exists  $\kappa > 0$  such that:

$$\mathbb{P}_{(0,u)}^{(c)}(\tau_\infty > t) \underset{t \rightarrow +\infty}{\sim} \kappa \frac{u^{\alpha\eta(c)}}{t^{\eta(c)}}$$

where  $\eta(c)$  is the solution of the equation (16). Applying the Markov property at the time  $\tau_1$ , we then deduce that

$$\mathbb{P}_{(x,u)}^{(c)}(\tau_\infty > t) = \mathbb{E}_{(x,u)} \left[ \mathbb{P}_{(0,c|U_{\tau_1^-}^-)}^{(c)}(\tau_\infty > t - \tau_1) \right].$$

To apply the dominated convergence theorem, let us fix some deterministic  $A > 0$  such that, for any  $t \geq A$ , we have  $t^{\eta(c)} \mathbb{P}_{(0,1)}^{(c)}(\tau_\infty > t) \leq 2\kappa$ . Then, by scaling and since  $0 < \eta(c) < 1$  :

$$\begin{aligned} & t^{\eta(c)} \mathbb{P}_{(0,c|U_{\tau_1^-}^-)}^{(c)}(\tau_\infty > t - \tau_1) \\ & \leq (t - \tau_1)^{\eta(c)} \mathbb{P}_{(0,1)}^{(c)} \left( \tau_\infty > \frac{t - \tau_1}{c^\alpha |U_{\tau_1^-}^-|^\alpha} \right) + \tau_1^{\eta(c)} \\ & \leq A^{\eta(c)} c^{\alpha\eta(c)} |U_{\tau_1^-}^-|^{\alpha\eta(c)} \mathbb{I}_{\left\{ \frac{t - \tau_1}{c^\alpha |U_{\tau_1^-}^-|^\alpha} \leq A \right\}} + 2\kappa c^{\alpha\eta(c)} |U_{\tau_1^-}^-|^{\alpha\eta(c)} \mathbb{I}_{\left\{ \frac{t - \tau_1}{c^\alpha |U_{\tau_1^-}^-|^\alpha} \geq A \right\}} + \tau_1^{\eta(c)} \\ & \leq (A^{\eta(c)} + 2\kappa) c^{\alpha\eta(c)} |U_{\tau_1^-}^-|^{\alpha\eta(c)} + \tau_1^{\eta(c)} \end{aligned}$$

which is integrable since  $\eta(c) < \eta$ . The dominated convergence theorem then yields :

$$\mathbb{P}_{(x,u)}^{(c)}(\tau_\infty > t) \underset{t \rightarrow +\infty}{\sim} \kappa \frac{h^c(x, u)}{t^{\eta(c)}}.$$

Next, applying the Markov property at time  $s$ , we deduce that

$$\mathbb{P}_{(x,u)}^{(c)}(\tau_\infty > t | \mathcal{F}_s) \xrightarrow{t \rightarrow +\infty} \frac{h^c(X_s, U_s)}{h(x, u)} \mathbb{I}_{\{\tau_\infty > s\}}$$

and the result (i.e. the  $L^1$  convergence) will follow from Scheffé's lemma, once we have proven that

$$\mathbb{E}_{(x,u)}^{(c)} [h^c(X_s, U_s) \mathbb{I}_{\{\tau_\infty > s\}}] = h^c(x, u).$$

Observe that by definition of  $\eta(c)$ :

$$\begin{aligned} \mathbb{E}_{(x,u)}^{(c)} [|U_{\tau_n^-}^-|^{\alpha\eta(c)}] &= \mathbb{E}_{(x,u)}^{(c)} \left[ \mathbb{E}_{(0,c|U_{\tau_{n-1}^-}^-)}^{(c)} [|U_{\tau_1^-}^-|^{\alpha\eta(c)}] \right] \\ &= \mathbb{E}_{(x,u)}^{(c)} \left[ (c|U_{\tau_{n-1}^-}^-|)^{\alpha\eta(c)} \mathbb{E}_{(0,1)}^{(c)} [|U_{\tau_1^-}^-|^{\alpha\eta(c)}] \right] = \mathbb{E}_{(x,u)}^{(c)} [|U_{\tau_{n-1}^-}^-|^{\alpha\eta(c)}]. \end{aligned}$$

By iteration, we deduce that

$$\begin{aligned} h^c(x, u) &= c^{\alpha\eta(c)} \mathbb{E}_{(x,u)}^{(c)} [|U_{\tau_n^-}^-|^{\alpha\eta(c)}] \\ &= c^{\alpha\eta(c)} \mathbb{E}_{(x,u)}^{(c)} [|U_{\tau_n^-}^-|^{\alpha\eta(c)} \mathbb{I}_{\{\tau_n \leq s\}}] + \mathbb{E}_{(x,u)}^{(c)} [h^c(X_s, U_s) \mathbb{I}_{\{\tau_n > s\}}]. \end{aligned} \tag{20}$$

It remains to prove that the first term converges towards 0 as  $n \rightarrow +\infty$ . By the Markov property :

$$\begin{aligned} I_n &:= \mathbb{E}_{(x,u)}^{(c)} \left[ |U_{\tau_n^-}|^{\alpha\eta(c)} \mathbb{I}_{\{\tau_n \leq s\}} \right] \\ &= \underbrace{\mathbb{E}_{(x,u)}^{(c)} \left[ |U_{\tau_n^-}|^{\alpha\eta(c)} \mathbb{I}_{\{\tau_n \leq s\}} \mathbb{I}_{\{|U_{\tau_n^-}| \leq 1\}} \right]}_{J_n} + \underbrace{\mathbb{E}_{(x,u)}^{(c)} \left[ \mathbb{E}_{(0,c|U_{\tau_{n-1}^-}|)} \left[ |U_{\tau_1^-}|^{\alpha\eta(c)} \mathbb{I}_{\{\tau_1 + \tau_{n-1} \leq s\}} \mathbb{I}_{\{|U_{\tau_1^-}| \geq 1\}} \right] \right]}_{K_n}. \end{aligned}$$

Observe first that by dominated convergence,  $J_n \xrightarrow[n \rightarrow +\infty]{} 0$ . Next, by scaling, the second term  $K_n$  may be written

$$K_n = \mathbb{E}_{(x,u)}^{(c)} \left[ (c|U_{\tau_{n-1}^-}|)^{\alpha\eta(c)} \mathbb{E}_{(0,1)} \left[ |U_{\tau_1^-}|^{\alpha\eta(c)} \mathbb{I}_{\{\tau_1 c^\alpha |U_{\tau_{n-1}^-}|^\alpha + \tau_{n-1} \leq s\}} \mathbb{I}_{\{c|U_{\tau_{n-1}^-}| |U_{\tau_1^-}| \geq 1\}} \right] \right].$$

Applying the inequality  $\mathbb{I}_{\{a+b \leq s\}} \leq \mathbb{I}_{\{a \leq s\}} \mathbb{I}_{\{b \leq s\}}$  which is valid for positive  $a$  and  $b$ , we obtain

$$K_n \leq \mathbb{E}_{(x,u)}^{(c)} \left[ |U_{\tau_{n-1}^-}|^{\alpha\eta(c)} \mathbb{I}_{\{\tau_{n-1} \leq s\}} \mathbb{E}_{(0,1)} \left[ (c|U_{\tau_1^-}|)^{\alpha\eta(c)} \mathbb{I}_{\{\tau_1 c^\alpha |U_{\tau_{n-1}^-}|^\alpha \leq s\}} \mathbb{I}_{\{c|U_{\tau_{n-1}^-}| |U_{\tau_1^-}| \geq 1\}} \right] \right].$$

Then, since

$$\mathbb{I}_{\{\tau_1 c^\alpha |U_{\tau_{n-1}^-}|^\alpha \leq s\}} \mathbb{I}_{\{c|U_{\tau_{n-1}^-}| |U_{\tau_1^-}| \geq 1\}} \leq \mathbb{I}_{\{\tau_1 / |U_{\tau_1^-}|^\alpha \leq s\}}$$

we deduce that

$$K_n \leq \mathbb{E}_{(x,u)}^{(c)} \left[ |U_{\tau_{n-1}^-}|^{\alpha\eta(c)} \mathbb{I}_{\{\tau_{n-1} \leq s\}} \right] \mathbb{E}_{(0,1)} \left[ (c|U_{\tau_1^-}|)^{\alpha\eta(c)} \mathbb{I}_{\{\tau_1 / |U_{\tau_1^-}|^\alpha \leq s\}} \right] = I_{n-1} \times r$$

where we have set  $r = \mathbb{E}_{(0,1)} \left[ (c|U_{\tau_1^-}|)^{\alpha\eta(c)} \mathbb{I}_{\{\tau_1 / |U_{\tau_1^-}|^\alpha \leq s\}} \right] \in (0, 1)$ . By iteration, we obtain for  $n \geq 2$ ,

$$I_n \leq \sum_{k=0}^{n-2} J_{n-k} r^k + I_1 r^{n-1}$$

and the result follows by letting  $n \rightarrow +\infty$  and using dominated convergence.  $\square$

### 3.2 The case $c = 0$

We are now interested in letting  $c \rightarrow 0$ , in order to obtain the law of a (free) stable Langevin process conditioned on not hitting 0. In this case, notice from (16) that :

$$\lim_{c \rightarrow 0} h^c(x, u) = \lim_{\nu \rightarrow 1 + \alpha\eta} \frac{\sin(\pi\nu(1-\gamma))}{\sin(\pi\gamma\nu)} \mathbb{E}_{(x,u)} [|U_{\tau_1^-}|^{\nu-1}].$$

Passing to the limit in  $h^c$  we deduce that

$$h^0(x, u) = \alpha\eta \kappa_{\alpha,\rho}^0 \int \int_0^{+\infty} \lambda^{-\alpha\eta-1} \left( q \left( -\frac{1}{t^{1+1/\alpha}} \right) - q \left( -\frac{1+x\lambda^{1+\alpha}+ut\lambda}{t^{1+1/\alpha}} \right) \right) t^{-1-1/\alpha} d\lambda dt.$$

**Corollary 3.2.1.** *The law of an integrated  $\alpha$ -stable Lévy process conditioned to stay positive is given by*

$$\forall \Lambda_s \in \mathcal{F}_s, \quad \mathbb{P}_{(x,u)}^\uparrow(\Lambda_s) = \frac{1}{h^0(x, u)} \mathbb{E}_{(x,u)} \left[ \mathbb{I}_{\Lambda_s} h^0(X_s, U_s) \mathbb{I}_{\{s < \tau_1\}} \right]$$

*Proof.* To show that this definition makes sense, we shall prove that  $\mathbb{P}^\dagger$  may be obtained by a penalization procedure, i.e. that :

$$\mathbb{P}_{(x,u)} \left( \Lambda_s | U_{\tau_1^-} > t \right) \xrightarrow{t \rightarrow +\infty} \mathbb{P}_{(x,u)}^\dagger (\Lambda_s).$$

Indeed, observe first that :

$$\begin{aligned} t^{\alpha\eta} \mathbb{P}_{(x,u)} \left( U_{\tau_1^-} > t | \mathcal{F}_s \right) &= t^{\alpha\eta} \mathbb{P}_{(x,u)} \left( \{U_{\tau_1^-} > t\} \cap \{\tau_1 \leq s\} | \mathcal{F}_s \right) + t^{\alpha\eta} \mathbb{P}_{(x,u)} \left( \{U_{\tau_1^-} > t\} \cap \{\tau_1 > s\} | \mathcal{F}_s \right) \\ &= t^{\alpha\eta} \mathbb{I}_{\{\{U_{\tau_1^-} > t\} \cap \{\tau_1 \leq s\}\}} + t^{\alpha\eta} \mathbb{I}_{\{\tau_1 > s\}} \mathbb{P}_{(X_s, U_s)} \left( U_{\tau_1^-} > t \right) \\ &\xrightarrow{t \rightarrow +\infty} \mathbb{I}_{\{\tau_1 > s\}} h^0(X_s, U_s) \end{aligned}$$

hence, as before, the  $L^1$ -convergence will follow from Scheffé's lemma once we have proven that

$$\mathbb{E}_{(x,u)} \left[ h^0(X_s, U_s) \mathbb{I}_{\{s < \tau_1\}} \right] = h^0(x, u).$$

Going back to Formula (20) with  $n = 1$ , we obtain that :

$$h^c(x, u) = c^{\alpha\eta(c)} \mathbb{E}_{(x,u)} \left[ |L_{\tau_1}|^{\alpha\eta(c)} \mathbb{I}_{\{\tau_1 \leq s\}} \right] + \mathbb{E}_{(x,u)} \left[ h^c(X_s, U_s) \mathbb{I}_{\{\tau_1 > s\}} \right]. \quad (21)$$

Now, from Joulin [17], for any  $0 < \varepsilon < \alpha$ :

$$\begin{aligned} \mathbb{E}_{(x,u)} \left[ |L_{\tau_1}|^{\alpha-\varepsilon} \mathbb{I}_{\{\tau_1 \leq s\}} \right] &= \mathbb{E}_{(x,u)} \left[ |L_{s \wedge \tau_1}|^{\alpha-\varepsilon} \right] - \mathbb{E}_{(x,u)} \left[ |L_s|^{\alpha-\varepsilon} \mathbb{I}_{\{s \leq \tau_1\}} \right] \\ &\leq \mathbb{E}_{(x,u)} \left[ \sup_{0 \leq r \leq s} |L_r|^{\alpha-\varepsilon} \right] + \mathbb{E}_{(x,u)} \left[ |L_s|^{\alpha-\varepsilon} \right] < +\infty \end{aligned}$$

hence, since  $\eta\alpha < \alpha$  :

$$\lim_{c \rightarrow 0} c^{\alpha\eta(c)} \mathbb{E}_{(x,u)} \left[ |L_{\tau_1}|^{\alpha\eta(c)} \mathbb{I}_{\{\tau_1 \leq s\}} \right] = 0.$$

Next, fix  $0 < \delta < c_{\text{crit}}$ . Since  $(c, u) \mapsto h^c(1, u)$  is continuous on  $[0, \delta] \times \mathbb{R}$  and  $u \mapsto h^c(1, u)$  is increasing and such that  $h^c(1, u) \underset{u \rightarrow +\infty}{\sim} u^{\alpha\eta(c)}$ , we deduce that we may find two constants  $A_\delta$  and  $B_\delta$ , such that, for any  $c \in [0, \delta]$ , we have :

$$h^c(1, u) \leq A_\delta + B_\delta (|u| \wedge 1)^{\alpha\eta}$$

By the scaling property of  $h^c$ , we then obtain :

$$h^c(x, u) \leq A_\delta (x \wedge 1)^{\frac{\alpha\eta}{1+\alpha}} + B_\delta (|u| \wedge 1)^{\alpha\eta}.$$

The result finally follows by passing to the limit in (21) and using the dominated convergence, since  $X_s$  and  $L_s$  admits moments of order  $\alpha - \varepsilon$  under  $\mathbb{P}$ . □

**Corollary 3.2.2.** *Assume that  $\{x = 0 \text{ and } u > 0\}$  or  $\{x > 0 \text{ and } u \geq 0\}$ . Then, there exists a constant  $\kappa > 0$  such that :*

$$\mathbb{P}_{(x,u)}(\tau_1 > t) \underset{t \rightarrow +\infty}{\sim} \kappa \frac{h^0(x, u)}{t^\eta}.$$

*Proof.* Using Corollary 3.2.1, we first have :

$$h^0(x, u) \mathbb{E}_{(x,u)}^\dagger \left[ \frac{1}{h^0(X_t, U_t)} \right] = \mathbb{P}_{(x,u)}(\tau_1 > t).$$



By the scaling property of  $h^0$  and  $(X_t, U_t)$ , we deduce that :

$$\mathbb{P}_{(x,u)}(\tau_1 > t) = \frac{h^0(x, u)}{t^\eta} \mathbb{E}^\uparrow_{\left(\frac{x}{t^{1+1/\alpha}}, \frac{u}{t^{1/\alpha}}\right)} \left[ \frac{1}{h^0(X_1, U_1)} \right].$$

Looking at this expression, it only remains to prove that the function

$$\Psi(t) = \mathbb{E}^\uparrow_{\left(\frac{x}{t^{1+1/\alpha}}, \frac{u}{t^{1/\alpha}}\right)} \left[ \frac{1}{h^0(X_1, U_1)} \right]$$

admits a finite limit as  $t \rightarrow +\infty$ . Observe first that, since  $x$  and  $u$  are positive and  $h^0$  is increasing in both variables, the monotony (with respect to the initial conditions) of  $(X, U)$  under  $\mathbb{P}^\uparrow$  implies that  $\Psi$  is increasing. The result then follows from [22, Theorem A], in which it is proven that there exists a constant  $\kappa > 0$  such that :

$$\limsup_{t \rightarrow +\infty} \Psi(t) \leq \kappa.$$

□

**Remark 3.2.3.** In the case of integrated Brownian motion, (i.e.  $\alpha = 2$  and  $\rho = 1/2$ ) we obtain with our normalization  $U = \sqrt{2}B$  :

$$h^0(x, u) = \frac{1}{6} \sqrt{\frac{3}{\pi}} \int_0^{+\infty} \mathbb{E} \left[ \left( \int_0^t \sqrt{2} B_s ds \right)_-^{-\frac{1}{2}} \right] - \mathbb{E} \left[ \left( x + ut + \int_0^t \sqrt{2} B_s ds \right)_-^{-\frac{1}{2}} \right] dt.$$

We briefly check that this expression agrees with the one of [12], that is,

$$\widehat{h}(x, u) = x^{-1/6} u \left( \frac{2}{9} \right)^{1/6} U \left( \frac{1}{6}, \frac{4}{3}; \frac{2}{9} \frac{u^3}{x} \right), \quad \text{for } x, u > 0$$

where  $U(a, b; z)$  denote the usual confluent hypergeometric function, see [1, Chapter 13]. Let  $B$  be a standard Brownian motion and define

$$\mathbb{P} \left( x + ut + \int_0^t B_s ds \in dy, u + B_t \in dv \right) = q_t(x, u; y, v) dy dv.$$

Writing  $h^0$  in terms of  $q_t$ , we obtain :

$$h^0(x, u) = \frac{2^{-1/4}}{6} \sqrt{\frac{3}{\pi}} \int_0^\infty \int_{-\infty}^\infty \int_{\mathbb{R}} \frac{1}{\sqrt{z}} \left( q_t(0, 0; z, v) - q_t \left( \frac{x}{\sqrt{2}}, \frac{u}{\sqrt{2}}; z, v \right) \right) dt dz dv.$$

Denote by  $\mathcal{D}_{(x,u)}$  the generator of  $B$  and its integral :

$$\mathcal{D}_{(x,u)} = \frac{1}{2} \frac{\partial^2}{\partial u^2} + u \frac{\partial}{\partial x}.$$

Since  $q_t$  is a solution of the Kolmogorov backward equation, we obtain that :

$$\mathcal{D}_{(x,u)} h^0(\sqrt{2}x, \sqrt{2}u) = -\frac{2^{-1/4}}{6} \sqrt{\frac{3}{\pi}} \int_0^\infty \int_{-\infty}^\infty \int_{\mathbb{R}} \frac{1}{\sqrt{z}} \frac{\partial}{\partial t} q_t(x, u; z, v) dt dz dv = 0.$$

Using the scaling of  $h^0$ , we deduce that the function  $u \rightarrow h^0(\sqrt{2}, \sqrt{2}u)$  is a solution of the ODE

$$g''(u) = \frac{2}{3}u^2g'(u) - \frac{1}{3}ug(u).$$

Since  $h^0(\sqrt{2}, \sqrt{2}u) \underset{u \rightarrow +\infty}{\sim} 2^{1/4}\sqrt{u}$ , we finally conclude, as in [12] that :

$$\widehat{h}(x, u) = \frac{1}{2^{1/4}}h^0(\sqrt{2}x, \sqrt{2}u).$$

Using the explicit Gaussian density and the complement formula for the Gamma function, we may also check that

$$\frac{1}{2^{1/4}}h^0(\sqrt{2}, 0) = \left(\frac{9}{2}\right)^{1/6} \frac{\Gamma(1/3)}{\Gamma(1/6)} = \lim_{u \rightarrow 0^+} \widehat{h}(1, u).$$

## 4 Link with kinetic equations and a probabilistic approach for related trace problems

In this section, we apply the results of Section 2 in order to exhibit the link between (1) and the trace problems related to kinetic equations endowing Maxwellian boundary condition (see e.g. [7], Mischler [20]).

### 4.1 The Brownian case

The link between the sequence of zero times of the integrated Brownian motion, the modeling of boundary condition for Langevin dynamics and trace problems for kinetic equations was previously exploited in Bossy and Jabir [5] (see also [6] for the multi-dimensional case) in order to show the well-posedness of some Lagrangian Stochastic model related to wall-bounded fluid flows. The trace problem related to Langevin models driven by a one-dimensional Brownian diffusion ( $L = \sqrt{2}B$ ) and endowing purely reflective boundary conditions ( $p = c = 1$ ) concerns the existence, in an appropriate sense, of a solution to the boundary value problem:

$$\partial_t \rho(t, x, u) + u \partial_x \rho(t, x, u) - \partial_u^2 \rho(t, x, u) = 0, \quad (t, x, u) \in (0, \infty) \times (0, \infty) \times \mathbb{R}, \quad (22a)$$

$$\rho(t, 0, u) = \rho(t, 0, -u), \quad (t, u) \in (0, \infty) \times \mathbb{R}, \quad (22b)$$

where  $\rho(t)$  represents the probability density function of  $(X_t, U_t)$ . In a rigorous way, the variational formulation of (22a)-(22b) consists in the existence of  $\rho$  and the existence of a pair of trace functions  $\gamma^+(\rho)$  and  $\gamma^-(\rho)$  defining the value of  $\rho(t, 0, u)$  along the respective boundary sets

$$\Sigma^+ = \{(t, u) \in (0, \infty) \times \mathbb{R} \mid u < 0\} \text{ and } \Sigma^- = \{(t, u) \in (0, \infty) \times \mathbb{R} \mid u > 0\}$$

and such that: for all  $0 \leq T < \infty$  and for all  $f \in \mathcal{C}_c^\infty((0, T) \times [0, \infty) \times \mathbb{R})$ ,

$$\begin{aligned} & \int_0^T \iint_{(0, \infty) \times \mathbb{R}} (\partial_t f(t, x, u) + u \partial_x f(t, x, u) + \partial_u^2 f(t, x, u)) \rho(t, x, u) dt dx du \\ &= - \iint_{\Sigma^+} u \gamma^+(\rho)(t, 0, u) f(t, 0, u) \mathbb{I}_{\{0 \leq t \leq T\}} dt du - \iint_{\Sigma^-} u \gamma^-(\rho)(t, 0, u) f(t, 0, u) \mathbb{I}_{\{0 \leq t \leq T\}} dt du \end{aligned} \quad (23)$$

From a PDE point of view, the existence of trace functions can be handled in a classical sense by showing the continuity of  $x \mapsto \rho(t, x, u)$  up to the axis  $x = 0$  or in a weak sense by showing some appropriate

Sobolev estimates for  $\rho$ . As noticed in [5], the trace functions  $\gamma^+$  and  $\gamma^-$  have also a natural probabilistic interpretation as density functions related to  $\sum_{n \geq 1} \mathbb{P} \circ (\tau_n, U_{\tau_n})^{-1}$  for the solution of the SDE :

$$\begin{cases} X_t = X_0 + \int_0^t U_s ds, & U_t = U_0 + \sqrt{2}B_t - 2 \sum_{n \geq 1} U_{\tau_n^-} \mathbb{I}_{\{\tau_n \leq t\}}, \\ \tau_n = \inf\{t \geq \tau_{n-1}; X_t = 0\}, & \tau_0 = 0. \end{cases}$$

## 4.2 The stable Langevin case

In the more general case of a stable Langevin model (1) and assuming for simplicity that  $\theta = 1$ , the probabilistic interpretation of the trace functions  $\gamma^\pm$  in terms of the SDE

$$\begin{cases} X_t = X_0 + \int_0^t U_s ds, \\ U_t = U_0 + L_t + \sum_{n \geq 1} \left( (1 - \beta_n)(M_n - U_{\tau_n^-}) - (1 + c)\beta_n U_{\tau_n^-} \right) \mathbb{I}_{\{\tau_n \leq t\}}, \\ \tau_n = \inf\{t \geq \tau_{n-1}; X_t = 0\}, & \tau_0 = 0, \end{cases} \quad (24)$$

proceeds as follows. For all  $C_c^\infty([0, T] \times [0, \infty) \times \mathbb{R})$ -scalar function  $f$ , Itô's formula yields that

$$\begin{aligned} 0 &= \iint f(0, x, u) \mu_0(dx, du) + \int_0^T \iint (\partial_t f(t, x, u) + u \partial_x f(t, x, u) + \partial_u^\alpha f(t, x, u)) \mu_t(dx, du) dt \\ &+ \mathbb{E} \left[ \sum_{0 \leq t \leq T, \Delta L_t \neq 0} (f(t, X_t, L_t) - f(t, X_t, L_{t-})) + \sum_{n \geq 1} (f(\tau_n, X_{\tau_n}, U_{\tau_n^+}) - f(\tau_n, X_{\tau_n}, U_{\tau_n^-})) \mathbb{I}_{\{\tau_n \leq T\}} \right], \end{aligned}$$

where  $\mu_t(dx, du) = \mathbb{P}(X_t \in dx, U_t \in du)$  and where  $\partial^\alpha$  is the fractional Laplace operator:

$$\partial_u^\alpha f(x) := C(\alpha) \int_{\{y \neq 0\}} \frac{f(y+x) - f(x) - y f'(x) \mathbb{I}_{\{|y| \leq 1\}}}{|y|^{\alpha+1}} dy.$$

Assume now that

$$\mathbb{P}(M_1 \in du) = um(u) du, \quad (25)$$

and that the following properties hold true :

- (P<sub>1</sub>)  $\forall n \geq 1, \mathbb{P} \circ (\tau_n, U_{\tau_n^-})^{-1}$  is absolutely continuous w.r.t. the measure  $(\mathbb{I}_{\{0 \leq t \leq T\}} dt) \otimes (u \mathbb{I}_{\{u \leq 0\}} du)$ ,
- (P<sub>2</sub>)  $\sum_{n \geq 1} \mathbb{P}(\tau_n \leq T) < \infty$ .

Therefore there exists a non-negative integrable Borel function  $\gamma^+$  defined on  $\Sigma^+$  such that

$$\mathbb{E} \left[ \sum_{n \geq 1} f(\tau_n, X_{\tau_n}, U_{\tau_n^-}) \mathbb{I}_{\{\tau_n \leq T\}} \right] = - \iint_{(0, T) \times \mathbb{R}^-} u \gamma^+(t, 0, u) f(t, 0, u) dt du, \quad (26)$$

and since

$$U_{\tau_n} = U_{\tau_n^-} + \Delta U_{\tau_n} = U_{\tau_n^-} + (1 - \beta_n)(M_n - U_{\tau_n^-}) - (1 + c)\beta_n U_{\tau_n^-} = (1 - \beta_n)M_n - c\beta_n U_{\tau_n^-},$$

this implies that  $\mathbb{P} \circ (\tau_n, U_{\tau_n})^{-1}$  is also absolutely continuous w.r.t.  $(\mathbb{I}_{\{0 \leq t \leq T\}} dt) \otimes (u \mathbb{I}_{\{u \leq 0\}} du)$ . Denoting the related density by  $\gamma^-$ , we observe that, for all  $f \in \mathcal{C}_c((0, T) \times (0, \infty))$ ,

$$\begin{aligned} \mathbb{E} \left[ \sum_{n \geq 1} f(\tau_n, X_{\tau_n}, U_{\tau_n}) \mathbb{I}_{\{\tau_n \leq T\}} \right] &= \iint_{(0, T) \times \mathbb{R}^+} u \gamma^-(t, 0, u) f(t, 0, u) dt du \\ &= \sum_{n \geq 1} \mathbb{E} \left[ f(\tau_n, X_{\tau_n}, (1 - \beta_n) M_n - \beta_n c U_{\tau_n^-}) \mathbb{I}_{\{\tau_n \leq T\}} \right] \\ &= p \sum_{n \geq 1} \mathbb{E} [f(\tau_n, X_{\tau_n}, M_n) \mathbb{I}_{\{\tau_n \leq T\}}] + (1 - p) \sum_{n \geq 1} \mathbb{E} \left[ f(\tau_n, X_{\tau_n}, -c U_{\tau_n^-}) \mathbb{I}_{\{\tau_n \leq T\}} \right]. \end{aligned}$$

Then, since, for any  $n$ , the r.v.'s  $\tau_n$  and  $M_n$  are independent, we obtain, with  $m$  the distribution of  $M_1$ ,

$$\begin{aligned} \sum_{n \geq 1} \mathbb{E} [f(\tau_n, X_{\tau_n}, M_n) \mathbb{I}_{\{\tau_n \leq T\}}] &= \sum_{n \geq 1} \left( \int_0^{+\infty} u \mathbb{E} [f(\tau_n, X_{\tau_n}, u) \mathbb{I}_{\{\tau_n \leq T\}}] m(u) du \right) \\ &= \int_0^{\infty} u \left( \sum_{n \geq 1} \mathbb{E} [f(\tau_n, X_{\tau_n}, u) \mathbb{I}_{\{\tau_n \leq T\}}] \right) m(u) du \\ &= \int_0^{\infty} u \left( - \iint_{(0, T) \times \mathbb{R}^-} f(t, 0, u) v \gamma^+(t, 0, v) dt dv \right) m(u) du, \end{aligned}$$

which implies, using (26),

$$\begin{aligned} \iint_{(0, T) \times \mathbb{R}^+} u \gamma^-(t, 0, u) f(t, 0, u) dt du &= p \left( \iint_{(0, T) \times \mathbb{R}^+} u m(u) \left( - \int_{-\infty}^0 v \gamma^+(t, 0, v) dv \right) f(t, 0, u) dt du \right) \\ &\quad + \frac{(1-p)}{c^2} \iint_{(0, T) \times \mathbb{R}^+} u \gamma^+ \left( t, 0, \frac{-u}{c} \right) f(t, 0, u) dt du. \end{aligned} \tag{27}$$

Combining (23) and (27), we deduce that the time marginal distribution  $(\mu_t, 0 \leq t \leq T)$  satisfies the variational equation: for all  $f \in \mathcal{C}_c^{\infty}([0, T] \times [0, \infty) \times \mathbb{R})$ ,

$$\begin{aligned} &\int_0^T \iint_{(0, \infty) \times \mathbb{R}} (\partial_t f(t, x, u) + u \partial_x f(t, x, u) + \partial_u^{\alpha} f(t, x, u)) \mu_t(dx, du) dt \\ &= - \iint_{(0, \infty) \times \mathbb{R}} f(0, x, u) \mu_0(dx, du) du - \iint_{\Sigma^+} u \gamma^+(t, 0, u) f(t, 0, u) \mathbb{I}_{\{0 \leq t \leq T\}} dt du \\ &\quad - \iint_{\Sigma^-} u \gamma^-(t, 0, u) f(t, 0, u) \mathbb{I}_{\{0 \leq t \leq T\}} dt du \end{aligned} \tag{28}$$

with the boundary condition

$$\gamma^-(t, 0, u) = \frac{1-p}{c^2} \gamma^+ \left( t, 0, \frac{-u}{c} \right) + p m(u) \left( - \int_{\{v \leq 0\}} v \gamma^+(t, 0, v) dv \right).$$

The trace problem related to (28) is then reduced to the verification that  $(P_1)$  and  $(P_2)$  hold true.

**Theorem 4.2.1.** *Assume that  $(L_t, 0 \leq t \leq T)$  is symmetric, that  $(X_0, U_0)$  is distributed according to a probability measure  $\mu_0$  defined on  $(0, \infty) \times \mathbb{R}$ , that (25) hold true,  $\theta = 1$  and*

- either  $0 \leq p < 1$
- or  $p = 1$  and  $c > c_{crit}$  and there exists  $0 < \nu < 1$  such that, for any  $\delta \in (0, \nu)$ ,

$$\iint \mathbb{E}_{(x,u)} \left[ |U_{\tau_1^-}|^{-\delta} \right] \mu_0(dx, du) < \infty.$$

Then  $(P_1)$  and  $(P_2)$  hold true.

As a preliminary result for the proof of Theorem 4.2.1, let us show that

**Lemma 4.2.2.**  $\mathbb{P} \circ (\tau_1, U_{\tau_1^-})^{-1}$  is absolutely continuous with respect to the measure  $u \mathbb{1}_{\{u < 0\}} du \otimes dt$ .

*Proof of Lemma 4.2.2.* The idea of the proof relies on showing that (24) in the case of a purely reflecting wall ( $p = 1, c = 1$ ) admits trace functions (in a classical sense) and to deduce from (26) that, for all  $n$ ,  $\mathbb{P} \circ (\tau_n, U_{\tau_n^-})^{-1}$  is absolutely continuous with respect to  $u \mathbb{1}_{\{u < 0\}} du \otimes dt$ .

First, let us consider the distribution  $\mu_t^f$  of the (free) Langevin processes

$$V_t = U_0 + L_t \quad \text{and} \quad Y_t = X_0 + \int_0^t V_s ds.$$

For all  $t > 0, \lambda, \omega \in \mathbb{R}$ , we have :

$$\begin{aligned} \widehat{\mu}_t^f(\omega, \lambda) &:= \mathbb{E} \left[ e^{i\omega Y_t + i\lambda V_t} \right] = \iint_{\mathbb{R} \times \mathbb{R}} e^{i\omega(x+ut) + i\lambda u} \mathbb{E} \left[ e^{i\omega \int_0^t L_s ds + i\lambda L_t} \right] \mu_0(dx, du) \\ &= \iint_{\mathbb{R} \times \mathbb{R}} e^{i\omega(x+ut) + i\lambda u} e^{-t \int_0^1 |t\omega r + \lambda|^\alpha dr} \mu_0(dx, du) \end{aligned}$$

Then, the successive changes of variables  $\tilde{\lambda} = \lambda/\omega, \tilde{\omega}^\alpha = \omega^\alpha \int_0^1 |tr + \tilde{\lambda}|^\alpha dr$  yields

$$\begin{aligned} &\iint_{\mathbb{R} \times \mathbb{R}} |\widehat{\mu}_t^f(\omega, \lambda)| d\lambda d\omega \\ &\leq \iint_{\mathbb{R} \times \mathbb{R}} e^{-t \int_0^1 |t\omega r + \lambda|^\alpha dr} d\lambda d\omega = \left( \int_{\mathbb{R}} \frac{1}{\left( \int_0^1 |rt + \tilde{\lambda}|^\alpha dr \right)^{\frac{2}{\alpha}}} d\tilde{\lambda} \right) \left( \int_{\mathbb{R}} |\tilde{\omega}| e^{-t|\tilde{\omega}|^\alpha} d\tilde{\omega} \right) < \infty, \end{aligned}$$

hence, for all  $0 < t \leq T$ , the Fourier transform  $\widehat{\mu}_t^f$  is integrable on  $\mathbb{R} \times \mathbb{R}$ . This implies (see e.g. Jacob and Protter [16], Theorem 13.1) that the distribution  $\mu_t^f$  of  $(Y_t, V_t)$  admits a bounded continuous Lebesgue density  $\rho^f(t)$  on  $\mathbb{R} \times \mathbb{R}$  given by

$$\rho^f(t, y, v) = \frac{1}{(2\pi)^2} \iint_{\mathbb{R} \times \mathbb{R}} e^{-i\omega y - i\lambda v} \left( \iint_{\mathbb{R} \times \mathbb{R}} e^{i\omega(x+ut) + i\lambda u} e^{-t \int_0^1 |t\omega r + \lambda|^\alpha dr} \mu_0(dx, du) \right) d\lambda d\omega.$$

Additionally, for all  $k, l > 1$ , by applying the same change of variables as above,

$$\begin{aligned} |\partial_y^k \partial_v^l \rho^f(t, y, v)| &\leq \iint_{\mathbb{R} \times \mathbb{R}} |\omega|^k |\lambda|^l e^{-t \int_0^1 |t\omega r + \lambda|^\alpha dr} d\lambda d\omega \\ &\leq \left( \int_{\mathbb{R}} \frac{|\tilde{\lambda}|^l}{\left( \int_0^1 |rt + \tilde{\lambda}|^\alpha dr \right)^{\frac{k+l+2}{\alpha}}} d\tilde{\lambda} \right) \left( \int_{\mathbb{R}} |\tilde{\omega}|^{k+l+1} e^{-t|\tilde{\omega}|^\alpha} d\tilde{\omega} \right) < \infty, \end{aligned}$$

from which we deduce that, for all  $t > 0$ ,  $\rho^f(t)$  is  $\mathcal{C}^\infty$  on  $\mathbb{R} \times \mathbb{R}$ .

Next, define

$$X_t^r = |Y_t|, \quad U_t^r = V_t \operatorname{sign}(Y_t)^+, \quad t \geq 0,$$

where  $(\operatorname{sign}(Y_t)^+, t \geq 0)$  is the càdlàg modification of  $(\operatorname{sign}(Y_t), t \geq 0)$ . Since, for any  $n$ ,  $\zeta_{n+1} = \inf\{t > \zeta_n, Y_t = 0\}$  (with  $\zeta_0 = 0$ ) is a predictable stopping time,  $t \mapsto L_t$  never jumps a.s. at  $\zeta_n$  (see e.g. Blumenthal [4], Theorem 5.1). Itô's formula then yields that

$$\begin{aligned} X_t^r &= X_0 + \int_0^t U_s^r ds, \\ U_t^r &= U_0 + \int_0^t \operatorname{sign}(Y_s)^- dL_s + \sum_{0 < s \leq t} V_{s-} \Delta \operatorname{sign}(Y_s)^- \mathbb{I}_{\{\Delta \operatorname{sign}(Y_s)^- \neq 0\}} + \sum_{0 < s \leq t} \Delta L_s \Delta \operatorname{sign}(Y_s)^- \\ &= U_0 + \int_0^t \operatorname{sign}(Y_s)^- dL_s - 2 \sum_{0 < s \leq t} U_{s-}^r \mathbb{I}_{\{\Delta X_s^r \neq 0\}}. \end{aligned}$$

Thanks to the symmetric property of  $(L_t, t \geq 0)$  and the fact the  $L$  and  $\operatorname{sign}(Y)$  a.s. do not jump at the same time,  $(\int_0^t \operatorname{sign}(Y_s)^- dL_s, t \geq 0)$  is also a symmetric  $\alpha$ -stable Lévy process, and  $((X_t^r, V_t^r), t \geq 0)$  is a weak solution to the Langevin model (1) with purely elastic reflection. Therefore  $\mathbb{P}(X_t^r \in dx, U_t^r \in du)$  admits a smooth density function  $\rho^r$  given by

$$\mathbb{P}(X_t^r \in dx, U_t^r \in du) = (\rho^f(t, x, u) + \rho^f(t, -x, -u)) \mathbb{I}_{\{x \geq 0\}} dx du.$$

Owing to the smoothness of  $\mu_t^f$  and replicating the arguments of [5, Theorem 2.3], we deduce that the natural trace functions satisfying (28) in the purely reflective case are given by

$$\gamma^\pm(\rho^r)(t, 0, u) = (\rho^f(t, 0, u) + \rho^f(t, -0, -u)) \mathbb{I}_{\{x \geq 0, \pm u < 0\}}.$$

According to (26), this is enough to ensure that

$$\mathbb{P} \circ (\tau_1^r, U_{\tau_1^-}^r)^{-1} = \mathbb{P} \circ (\tau_1, U_0 + L_{\tau_1^-})^{-1}$$

admits a density with respect to  $(\mathbb{I}_{\{0 \leq t \leq T\}} dt) \otimes (u \mathbb{I}_{\{u \leq 0\}} du)$ .  $\square$

*Proof of Theorem 4.2.1.* For  $(P_1)$ , applying Lemma 4.2.2 and using (25) and the Markov property, we immediately deduce that for all  $n \in \mathbb{N}$

$$\mathbb{P}(\tau_n \in dt, U_{\tau_n^-} \in du)$$

admits a density with respect to the measure  $u \mathbb{I}_{\{u < 0\}} du \otimes dt$ , and that  $(P_1)$  is satisfied.

For  $(P_2)$ , assuming that  $0 \leq p < 1$ , by the Markov property, we have

$$\begin{aligned} \mathbb{E}[e^{-\tau_{n+1}}] &= \mathbb{E}[e^{-\tau_n} \mathbb{E}_{(0, U_{\tau_n})}[e^{-\tau_1}]] = \mathbb{E}\left[e^{-\tau_n} \left( (1-p) \mathbb{E}_{(0, M_n)}[e^{-\tau_1}] + p \mathbb{E}_{(0, -cU_{\tau_n^-})}[e^{-\tau_1}] \right)\right] \\ &\leq \mathbb{E}[e^{-\tau_n}] \mathbb{E}\left[\left( (1-p) \mathbb{E}_{(0, M_1)}[e^{-\tau_1}] + p \right)\right]. \end{aligned}$$

Therefore, setting  $\varrho := \mathbb{E}\left[\left( (1-p) \mathbb{E}_{(0, M_1)}[e^{-\tau_1}] + p \right)\right]$  which is strictly smaller than 1,

$$\mathbb{E}[e^{-\tau_{n+1}}] \leq \varrho^n \mathbb{E}[e^{-\tau_1}].$$

For any given  $0 < T < +\infty$ , choosing  $c_T > 0$  such that  $\mathbb{I}_{\{r \leq T\}} \leq c_T e^{-r}$ , it follows that

$$\sum_{n \geq 1} \mathbb{P}(\tau_n \leq T) \leq c_T \sum_{n \geq 1} \mathbb{E}[e^{-\tau_n}] \leq c_T \mathbb{E}[e^{-\tau_1}] \sum_{n \geq 0} \varrho^n < \infty.$$

In the case where  $p = 1$  and  $c > c_{\text{crit}}$ , we first write, for  $n \geq 2$  :

$$\sum_{n \geq 1} \mathbb{P}(\tau_n \leq T) \leq \sum_{n \geq 1} \mathbb{P}(\tau_{n+1} - \tau_n \leq T).$$

Then, using the Markov's inequality with  $\delta > 0$  and the decomposition (9),

$$\begin{aligned} \mathbb{P}(\tau_{n+1} - \tau_n \leq T) &= \iint_{(0, \infty) \times \mathbb{R}} \mathbb{P}_{(x, u)}(\tau_{n+1} - \tau_n \leq T) \mu_0(dx, du) \leq \iint_{(0, \infty) \times \mathbb{R}} \mathbb{E}_{(x, u)} \left[ \frac{T^\delta}{(\tau_{n+1} - \tau_n)^\delta} \right] \mu_0(dx, du) \\ &\leq T^\delta \mathbb{E}[\xi_1^{-\delta}] \mathbb{E} \left[ \left( \prod_{j=1}^n |c\ell_j|^\alpha \right)^{-\delta} \right] \iint_{(0, \infty) \times \mathbb{R}} \mathbb{E}_{(x, u)} \left[ |U_{\tau_1^-}|^{-\delta} \right] \mu_0(dx, du) \\ &\leq T^\delta \mathbb{E}[\xi_1^{-\delta}] \left( \mathbb{E}[|c\ell_1|^{-\alpha\delta}] \right)^n \iint_{(0, \infty) \times \mathbb{R}} \mathbb{E}_{(x, u)} \left[ |U_{\tau_1^-}|^{-\delta} \right] \mu_0(dx, du). \end{aligned}$$

According to Lemma 2.2.2, taking  $\delta < \nu$  immediately ensures that  $\mathbb{E}[\xi_1^{-\delta}]$  is finite. Next, since  $f(\delta) = \mathbb{E}[|c\ell_1|^{-\alpha\delta}]$  is such that  $f(0) = 1$  and

$$f'(0) = (\mathbb{E}[\exp(-\alpha\delta \ln |c\ell_1|)])'_{\delta=0} = -\alpha \mathbb{E}[\ln |c\ell_1|] < -\alpha (\ln(c_{\text{crit}}) + \mathbb{E}[\ln |\ell_1|]) = 0,$$

$f$  is decreasing near 0. Hence, choosing  $\delta > 0$  small enough,  $\varrho = \mathbb{E}[|c\ell_1|^{-\alpha\delta}] < 1$ , and we get  $\mathbb{P}(\tau_{n+1} - \tau_n \leq T) \leq C\varrho^n$  with  $\varrho < 1$ . This enables to conclude  $(P_2)$ .  $\square$

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