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► **To cite this version:**

Y.Y. Papageorgiou. Spatial public goods. II: Applications. [Research Report] Institut de mathématiques économiques (IME). 1985, 48 p. hal-01543601

HAL Id: hal-01543601

<https://hal.science/hal-01543601v1>

Submitted on 21 Jun 2017

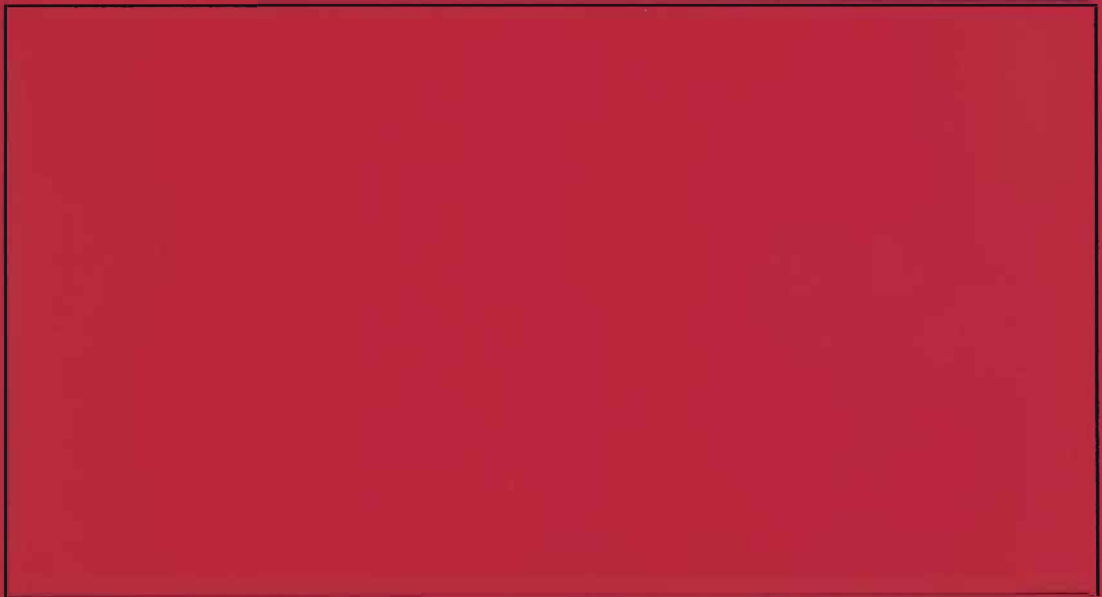
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N° 80

SPATIAL PUBLIC GOODS

II - APPLICATIONS

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October 1985

Most of the work which led to this paper was done at the Institut de Mathématiques Economiques of the Université Dijon, France, under a sabbatical grant provided by the Social Sciences and Humanities Research Council of Canada. I would like to thank both institutions for their generous support.

This paper is the second part of a two-part series. Part 1 was about how the spatial diffusion process of a public good interacts with a theory of distributive justice to produce an optimal urban form.¹ The terminology and ideas of Part 1 are assumed to be known. This paper begins by applying the framework of Part 1 to develop a classification of spatial public goods, and to order existing literature on the basis of this classification. In addition to clubs, urban contact fields, pure and local public goods, the first section discusses a new model of spatial public good -- the agora. The second section examines the problem of allocating land between private and public use in the case of transportation and agora models. The third section deals with certain aspects of optimal public finance which is known in the literature as the Mohring paradox. It is seen that optimal taxes on the use of land do not, in general, match the optimal level of public investment. In this sense, some public projects should operate under deficit while some others should generate a surplus. Generating a surplus or deficit at the optimum crucially depends on the nature of returns to scale in the maintenance of a given environmental quality. Finally, section four displays a number of aggregate relationships which express a balance between costs and benefits of an optimal agglomeration. Costs include urban transportation while benefits include the surplus value of urban production. It is seen that the nature of such balance will be determined by the shape of the city and the technology of transportation.

CLASSIFICATION

A way in which environmental quality can be determined by the spatial distribution of population and public investment over an urban area has been given in (3) of Part 1. Here it is convenient to specialize this equation by distinguishing explicitly between distance and density effects. Thus (3) of Part 1 is now written as

$$E_{ij} = \zeta_{ij} \xi_{ij} [n, \omega_i], \quad (1)$$

where ζ is a distance-response function and ξ is a density-response function; and where the explicit dependence of ξ on both i and j reflects the truth that public goods in zones differing with respect to relative location and size are congested differently by the same population.

The structure of (1) is precisely analogous to that adopted in an earlier study concerning spatial externalities.² In that study, three spatial externality cases were discussed, namely, urban contact fields, density models and distance models. An urban contact field is a spatial externality such that the density-response function depends only on magnitudes at the origin of the externality. A density model of a spatial externality implies that the distance-response function is trivial. There are two polar cases of a density model: (1) one with no spill-over effects, i.e. $\xi_{ij} = 1$ for $i = j$ and zero otherwise, which represents purely local externality effects; and (2) one with no distance response, i.e. $\zeta_{ij} = 1$ for all i and j , which represents no friction of distance. Finally, a distance

model of a spatial externality implies that the density-response function has a trivial spatial distribution. Absolute concentration at the centre, or uniform distribution over the urban area provide examples of trivial spatial distributions. Using these concepts, we may classify the main types of spatial public goods in the literature as follows.

CLUBS

The origin of clubs here is to be found in a density model with no distance-response. Since $\zeta_{ij} = 1$ for $i, j \in J$, it must also be that $\xi_{ij}[n, \omega_i] = g_i[N, \omega_i]$, i.e. that the spatial distribution of population is immaterial to the production of the public good. Thus the quality of the environment experienced in zone j is

$$E_j = \sum_{k=-b}^b g_k[N, \omega_k] = 2 \sum_{k=1}^b g_k[N, \omega_k] \equiv \bar{E} \quad (2)$$

using (4) of Part 1. That is, in the absence of distance-response, the quality of the environment is invariant over the landscape. Since both the optimal utility differences and the marginal location rates are proportional to environmental quality differences ((42) and (43) of Part 1), we conclude that $\Delta u = 0$ and $\phi_i^1 = 0$ for $i = 1, \dots, b$ irrespectively of the degree of aversion to inequality. On the other hand, from (39) of Part 1 and (2),

$$\phi_i^n = -\frac{1}{n_j} \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta \sum_{k=1}^b g_k$$

$$= - \frac{2}{n_i} \sum_k \frac{\Delta}{N} g_k \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j}. \quad (3)$$

Assuming, as in Part 1, that the total value of the urban product is distributed to the inhabitants of the optimal city, the planner must impose a tax or subsidy on the residents of zone i equal to

$$\phi_i - \bar{\phi} = n_i \phi_i^n + i \phi_i^i - \bar{\phi} = - \left(2 \sum_k \frac{\Delta}{N} g_k \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} + \bar{\phi} \right) = 0 \quad (4)$$

because $n_i \phi_i^n$ is invariant over the landscape. Hence there is no need for income re-distribution under any degree of aversion to inequality. It follows that, in this case, the theory of justice espoused by the planner is immaterial. Absence of the friction of distance, a fundamental geographical characteristic, equalizes the quality of the environment which, in turn, dictates equal treatment of equals in any case. Therefore, the optimal decentralization policy of Part 1 reduces to the problem of producing efficiently the private and public goods. Efficient production of the private good implies, as in Part 1, that the planner must impose an optimal population size N^* . Efficient production of the public good, on the other hand, hinges upon returns to scale in its production: if under decreasing returns, Ω should be uniformly distributed over \mathcal{J} ; if under increasing returns, it should be concentrated somewhere; and if increasing returns are exhausted at a size smaller than Ω , another multitude of solutions will emerge. It seems that this context trivializes such spatial issues. Once efficient production and optimal population size are ascertained, competition for land between those admitted will lead to the optimum. In conse-

quence, absence of the friction of distance eliminates the need for any land controls.

When space is completely abstracted, $\bar{E} = g[N, \Omega]$ and the model acquires the standard form of a club. Then the marginal rate on the use of land is simply

$$\phi_i^n = - \lambda \frac{\partial u}{\partial E} \Delta g. \quad (5)$$

Following Buchanan's original contribution, a growing literature has focussed on the interaction between the size of a club and the corresponding level of public investment.³ In this context, which is more general because Ω is endogenous rather than exogenous, the optimal size reflects a balance between the economies of increasing population size and the diseconomies of congestion.

URBAN CONTACT FIELDS

Whereas in the case of clubs the impact of distance is eliminated, what is missing in the case of urban contact fields is the impact of congestion on public goods. Since ζ_{ij} is non-trivial, the quality of the environment varies over the landscape. In consequence, contrary to the case of clubs, both the optimal utility differences and the marginal location rates vary, in general, with the degree of aversion to inequality as in (42) and (43) of Part 1. On the other hand, since $\xi_{ij} = g_i[\omega_i]$, it is true that $\Delta E_j = 0$. Consequently, $\phi_i^n = 0$ ((39) of Part 1). In other words, since n_j

congestion does not affect the quality of the environment, there is no reason to correct for it. This is the only simplification to be made with regard to the optimal decentralization policy. Everything else, including the need to control everything that has to do with the use of land under a finite aversion to inequality, still holds.⁴

PURE PUBLIC GOODS

A good that has the distance-response of a club and the density response of an urban contact field is a pure public good. Then, following (2),

$$\bar{E} = 2 \sum_{k=1}^b g_k[\omega_k]. \quad (6)$$

Clearly, $\Delta u_i = 0$ and $\phi_i^i = 0$ as in clubs; and $\phi_i^n = 0$ as in urban contact fields. Therefore, once more, the optimal decentralization policy of Part 1 reduces to the problem of producing efficiently the private and public goods. Since, as in clubs, issues of distributing Ω over \mathcal{J} become rather trivial, space is typically abstracted from such models. Thus $\bar{E} = g[\Omega]$ simply. Of course, the trade-off between population size and the corresponding level of public investment, which is the central issue in the theory of clubs, also disappears.⁵

LOCAL PUBLIC GOODS

These are represented by a density model with no spill-over effects, i.e. $\zeta_{ij} = 1$ for $i = j$ and zero otherwise, leading immediately to $\xi_{ij}[n, \omega_i] = g_i[n_i, \omega_i]$, hence to

$$E_j = g_j[n_j, \omega_j]. \quad (7)$$

Under these circumstances, the marginal rate on the use of land ((39) of Part 1) is simplified as

$$\phi_i^n = \lambda_i \frac{\partial u}{\partial E_i} \Delta g_i n_i \quad (8)$$

On the other hand, the variation in optimal utilities and marginal location rates, as described in (42) and (43) of Part 1, remains: here, as in the case of urban contact fields, the question of an optimal distribution of public expenditure over \mathcal{J} becomes central. Such optimal policy, together with the corresponding distribution of population, would determine the optimal distribution of environmental quality over \mathcal{J} . There are several papers dealing with these questions.⁶ In particular, Helpman, Pines and Boruchov provide the analysis of a model where policy variables are the price system, the production and distribution of public services, and taxation. They examine the case where the resulting allocation is a competitive equilibrium with equal incomes and equal utilities: in our terms, there is an infinite aversion to inequality. It is seen that the spatial distribution of population and land values adheres to the standard pattern, i.e.

both population density and land values decrease away from the centre. Furthermore, that if q_i and E_i are net substitutes (complements) then the quality of the environment decreases (increases) away from the centre. In the former case, the density of public investment, $\omega_i / \Delta Q$, also decreases. In the latter, it remains ambiguous.⁷

AGORA MODELS

Suppose that all public investment is concentrated at the centre. The centre now becomes a public good, an agora, enjoyed by all; and it can be described by a distance model such that

$$E_j = \zeta_j g[n, \Omega]. \quad (9)$$

Here ζ is a function of distance between the centre and zone j . Since this function decreases with increasing distance from the centre, the quality of the environment for more distant zones also decreases. Thus, in the case of finite aversion to inequality, optimal utilities decrease away from the centre according to (42) of Part 1. As the aversion to inequality increases, following (43) of Part 1, subsidization of more distant zones relatively increases in order to reduce optimal inequalities. Finally, the marginal rate on the use of land in (39) of Part 1 becomes

$$\phi_i^n = - \frac{1}{n_i} \frac{\Delta g}{n_i} \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \zeta_j. \quad (10)$$

It is perhaps useful at this point to clarify the explicit dependence of g in (9) on the spatial distribution of population, rather than on the total population. A distance model of this kind suggests that individuals in more distant zones enjoy relatively less the agora because distance is an impediment to interaction. At the same time, for the same reason, individuals in more distant zones impose relatively less congestion. The former is captured by ζ_j ; the latter, by the explicit dependence of g on n_i : for more distant zones, Δg is expected to become less strongly negative.

Since the value of the summation on the RHS of (10) is the same for all i (once at the centre you congest everybody there irrespectively of the zone you originate), the portion of the optimal tax attributed to the use of land, $n_i \phi_i^n$, is proportional to $-\Delta g$. In particular, individuals in zones closer to the centre pay more because, by using the agora often, they congest more. Clearly, this framework requires a distinction between time spent at the centre for work and for enjoyment. If both are combined then the frequency of interaction with the centre, hence the levels of enjoyment and congestion, are the same for everyone. Under these circumstances, $E = \zeta g[N, \Omega]$ and the model can be treated as a club.

PUBLIC GOODS THAT OCCUPY LAND

TRANSPORTATION

Suppose that travel occurs along a given dense, radial transportation network. Under these circumstances, additional congestion at i is

generated by all those further away from the centre. Combining (54) of Part 1 and (1), the additional congestion at i generated by those at k is

$$E_{ki} = \zeta_{ki} \xi_{ki} [n, G[H_i, \omega_i]] \text{ with } \zeta_{ki} = \begin{cases} -1 & \text{for } i < k \\ 0 & \text{for } i > k, \end{cases} \quad (11)$$

where the minus sign determines congestion as a negative spatial externality. The total congestion at i is created by commuters at and beyond i :

$$e_i = - \sum_{k=1}^b \xi_{ki} [n, G[H_i, \omega_i]] \equiv - g_i [n, H_i, \omega_i] < 0 \quad (12)$$

with $\Delta e_i < 0$, $\partial e / \partial H_i > 0$ and $\partial e / \partial \omega_i > 0$. Someone at j experiences

congestion over the entire ray between j and the centre. Therefore

$$E_j = \sum_{i=1}^j e_i = - \sum_{i=1}^j g_i [n, H_i, \omega_i]. \quad (13)$$

Since

$$\begin{aligned} \Delta E_j &= - \left(\sum_{i=1}^{j+1} - \sum_{i=1}^j \right) g_i [n, H_i, \omega_i] \\ &= - g_{j+1} [n, H_{j+1}, \omega_{j+1}] = - \sum_{k=j+1}^b \xi_{k,j+1} [n, G[H_{j+1}, \omega_{j+1}]] < 0, \quad (14) \end{aligned}$$

optimal utilities decrease away from the centre under finite aversion to inequality according to (42) of Part 1. This happens because more distant locations imply longer congested trips, hence lower quality of the environment there from that particular point of view. Finally, in order to determine the marginal rate on the use of land, notice that

$$\Delta E_j = \begin{cases} \sum_{\ell=1}^J \Delta g_{\ell} & \text{for } i > j \\ n_i & \\ \sum_{\ell=1}^i \Delta g_{\ell} & \text{for } i < j \end{cases} \quad (15)$$

because someone added in zone i congests others only between this zone and the centre. Therefore, using (39) of Part 1, the marginal rate on the use of land becomes

$$\phi_i^n = \frac{1}{n_i} \left(\sum_{j=1}^i \sum_{\ell=1}^j + \sum_{j=i+1}^b \sum_{\ell=1}^i \right) n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta g_{\ell}. \quad (16)$$

The literature on transportation is extensive.⁸ Within this, Strotz (1965) remains a classic reference which includes most of the fundamental intuitions now available on optimal transportation policy. Taking into account what has been said in Part 1, the optimal transportation policy can be summarized here as follows. Suppose that there is a fixed amount of public funds available for transportation and that the recovery of these funds is not an issue.⁹ Furthermore, suppose that the aggregate income of urban residents is required to equal the total value of the urban product.

The planner must organize the production of the urban good efficiently. However, although individuals should receive the marginal product of their labour, the planner must re-distribute income through congestion tolls and location taxes. The congestion toll $n_i \phi_i$, based on the marginal congestion rate (16), equals the total damage someone imposes on the other drivers as he travels to and from work in the centre. The location tax $i \phi_i^i$, based on the marginal location rate (43) of Part 1, reflects environmental differences weighted by the theory of justice adopted by the planner. Given that individuals take these rates parametrically, excepting infinite aversion to inequality, the resulting optimum will be such that some zones in the city will become more attractive than others — utility differences between zones being associated with corresponding spatial externality differences. Hence the planner will find it necessary to control everything that has to do with the use of land under a finite aversion to inequality.¹⁰

In contrast, when there is an infinite aversion to inequality, it is sufficient for the planner to fix the urban population at the optimal size and to announce the spatial distribution of optimal tax and subsidy rates ϕ_i^n and ϕ_i^i .

It seems worth pointing out that

"(t)he burden of congestion on motorists is dimensionally a product of two factors: congestion as a characteristic of driving conditions at a point and distance travelled. The burden of congestion may be reduced either by reducing congestion along the route or by reducing the length of the route. It is worse to drive ten miles

in congested traffic than one mile. Suppose a man occupies an estate near the center of the city and that all those who live farther out and who commute to work to the city center must every day drive past this estate in congested traffic. Their burden is greater, the greater the front footage the estate occupies because they must drive farther in congestion.

It would appear that the occupancy of space along a route imposes, in measure with its front footage, an external real diseconomy on those who travel past. Optimal pricing therefore suggests that not only should there (a) be a toll to discourage motorists from making too many trips and imposing external diseconomies of congestion (at every point at which they travel) on other motorists, but that (b) there should be a land rent (based on front footage or area) to discourage excessive space occupancy and thereby to reduce the mileage of congested driving for any given traffic flow. We might expect this to be in addition to the rent that, in the absence of anybody wanting to travel, would still be paid for the use of the limited desired space itself."

(Strotz op. cit., footnote 8). Surprisingly enough, we know that there is no need to impose such a tax under any circumstances. That is, for arbitrary degree of aversion to inequality, the previous analysis suggests that there is no need to apply policies which specifically encourage or discourage any particular pattern of land occupancy at the optimum.

AGORA MODELS

Let the city be partitioned in two concentric parts, $[1,a]$ and $[a+1,b]$ occupied by the agora and residences respectively. The inner border between the two land uses becomes a decision variable for the planner. For the production of the private good, when the agora expands, one must take into account the loss in manpower that the corresponding displacement of individuals entails. Thus the production function must now be written as $X[N, N[a]]$, with the function $N[a]$ such that $\Delta X = -n_{a+1}$.¹¹ Since both land at the centre and public funds are explicitly related to the production of the public good, we may express the spatial externality as

$$E_j = \zeta_j g[n, G[a, \Omega]] \text{ for } j = a+1, \dots, b. \quad (17)$$

Given that Ω is fixed and spent for the agora, the problem of the planner in (57) of Part 1 is now written as

$$D[r, z; \alpha, \Omega] = \text{minimum} \left\{ \sum_{j=1}^a r_j^0 \Delta Q + \sum_{j=a+1}^b n_j \psi_j - X[N, N[a]] \right\}$$

$$z_i \leq v[u[r_i, \psi_i]; \alpha] \text{ for } i = a+1, \dots, b, \quad (18)$$

where the first term on the RHS represents public expenditure on land.

The optimal decentralization policy holds as before — provided that adjustment is made for the new residential domain. For example, the marginal rate on the use of residential land in (10) now becomes

$$\phi_i^n = -\frac{1}{n_i} g \sum_{j=a+1}^b n_j \lambda_j \frac{\partial u}{\partial E_j} \zeta_j \text{ for } i = a+1, \dots, b. \quad (19)$$

On the other hand, the new condition regulating the spatial extent of the agora is

$$\begin{aligned} r_{a+1}^0 \Delta Q_a + \sum_{j=1}^a \Delta r_a^j \Delta Q_{j-1} - n_{a+1} \psi_{a+1} + \sum_{j=a+2}^b n_j \Delta \psi_j \\ - \Delta X_a - \sum_{j=a+2}^b n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_j = 0. \end{aligned} \quad (20)$$

The intuitive meaning of (20) is as follows. When the boundary of the agora is moved one zone outwards, thereby extending the agora, the public sector must pay for the use of the additional land (first term) and for the possible difference that this extension has imposed on existing land values in the agora (second term). As land is taken away from residences, people in the marginal zone are displaced elsewhere in the outside world. Thus there is a saving from the corresponding reduction in the number of salaries required (third term). At the same time, since the smaller workforce might induce a difference in the marginal product of labour, extending the agora may necessitate adjustments in the optimal salaries paid to the remaining individuals (fourth term). As less people remain to work in the production of the private good, the value of the urban product will normally be reduced (fifth term). Finally, as the agora extends, the remaining individuals will benefit both from increasing accessibility to and from better quality of public services (sixth term). The optimal extent of

the agora will be found where the MSB are balanced by the corresponding MSC as in (20).

For simplicity of exposition assume that $\Delta r_a^0 = 0$ and $\Delta \psi_a = 0$ in (20), that is, neither land values at the centre nor individual expenditures are influenced by an expansion of the agora. Solving (20) for the land value per unit of public land at the border of the agora,

$$r_{a+1}^0 = \frac{1}{\Delta Q_a} \left(\frac{\Delta X}{N} + n_{a+1} \psi_{a+1} \right) + \frac{1}{\Delta Q_a} \sum_{j=a+2}^b n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_j$$

$$= \frac{1}{\Delta Q_a} n_{a+1} \left(\psi_{a+1} - \frac{\Delta X}{N} \right) + \frac{1}{\Delta Q_a} \sum_{j=a+2}^b n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_j. \quad (21)$$

The first term on the RHS of (21) represents the effect of expanding the agora on the aggregate income deficit generated by the displacement of individuals in the marginal zone, expressed per unit of land. If the per capita expenditure of those in zone $a+1$ is greater (smaller) than the corresponding value of their marginal product then expanding the agora decreases (increases) the aggregate income deficit. Under these circumstances, if the spatial extent of the agora did not effect environmental quality, the value of public land in zone $a+1$ would be negative (positive). However, since the spatial extent of the agora does affect environmental quality, the value of public land in that zone is adjusted by the second term on the RHS of (21). The quantity $\lambda_j (\partial u / \partial E_j) \Delta E_j$ represents the utility gain, expressed in money terms, generated by the expansion of the agora. Thus the expression under the sum on the RHS of (21) represents the total

increase in social welfare, expressed in money terms, which stems from the expansion of the agora by one unit. Stated otherwise, it represents the total amount individuals would be prepared to pay per unit of residential land converted to public use in order to push the inner border one zone closer, for given levels of utility.

Since residential and public land values must be equal at the inner border, adjusting the value of public land by the second term on the RHS of (21) raises the public bid-rents, hence encourages the expansion of the agora at the optimum. Were the centre deleterious to the quality of the environment, as in the case of industrial pollution where $\Delta E_j < 0$, the corresponding adjustment would lower central bid-rents to generate a contraction of the centre at the optimum. In this case, the adjustment would represent the total amount individuals are prepared to pay per unit of central land converted to residential use in order to push the inner border one zone away, for given levels of utility. These adjustments, for both positive and negative spatial externalities, represent a general principle of segregated land uses at the optimum.

THE MOHRING PARADOX

Until now, the total amount of public investment was treated as given exogenously. In this section I consider Ω endogenous. In consequence, the issue of public finance is bound to surface: although one could still maintain that public funds somehow appear from the outside, it is far more convincing to admit that what benefits those in the city should be supported at the optimum by the same. Suppose that the location tax $i\phi_1^i$

still serves to re-distribute income, while the tax on the use of land $n_1 \phi_1^n$ is now collected to help finance the provision of the public good. The fundamental question is, will this revenue be sufficient at the optimum? or will it be necessary to further apply lump-sum taxes or subsidies, as those represented by A in the section "optimal decentralization policy" of Part 1, in order to balance the budget? An answer to this question is provided by the Mohring paradox.¹³ I consider first the case where the public good is produced only with capital.

Since what has been said previously holds for any feasible value of Ω , it must also hold for the optimal level of public expenditure. What is a reasonable principle to determine the optimal level of public expenditure? In the context of the Mohring paradox, it is that the public sector should pay competitive prices. When there is a competitive equilibrium between sectors for the use of resource services, the value of the marginal product of a resource service unit must equal its price. Since for public expenditure the price is simply one dollar, we obtain using (50) of Part 1

$$\sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\partial E_j}{\partial \omega_i} = - \frac{dD}{d\Omega} = 1 \quad (22)$$

as a condition which determines the level of public expenditure when the public sector pays competitive prices. This is a reasonable principle to admit. If, for example, $-dD/d\Omega > 1$ then spending an additional dollar on the public good would save more than one dollar at the optimum, for given levels of utility. Therefore it would seem proper that public expenditure be augmented at the expense of something else. If, on the other hand,

$-dD/d\Omega < 1$ then public expenditure should be reduced to provide individuals with higher disposable income. In this sense, the principle of competitive pricing seems to be perfectly compatible with the aim to optimize. It can however break down where optimality itself requires non-competitive pricing -- as will be seen later on when land is introduced in the production of the public good.

Further to competitive pricing, the Mohring paradox requires (1) that the spatial diffusion pattern of the public good be trivial, in the sense of ζ_{ij} being either zero or one everywhere in \mathcal{J} ; and (2) that the components of environmental quality g_i be homogeneous functions with respect to their arguments. The second requirement implies that the functions $e_i \equiv g_i[N, \omega_i]$ obey

$$\lambda^\beta e_i = g_i[\lambda N, \lambda \omega_i] \text{ for } \lambda > 0. \quad (23)$$

When the parameter $\beta = 0$, doubling population and public expenditure everywhere ($\lambda = 2$) implies no change in the quality of the environment. On the other hand, when $\beta > (<) 0$, doubling population and public expenditure everywhere implies an improved (deteriorated) quality of the environment. It follows that $\beta > (=, <) 0$ signifies increasing (constant, decreasing) returns to scale in the maintenance of a given environmental quality. Under these circumstances, the Mohring paradox states that

1. If these are increasing (constant, decreasing) returns to scale in the maintenance of a given environmental quality then there is a

deficit (balance, surplus) in the budget allocated for the production of the public good at the optimum.

As already mentioned, proposition 1 applies where the spatial diffusion pattern of the public good is trivial, i.e. to clubs, local public goods, spaceless transportation, and spaceless agora models without distance-response.¹⁴

I now consider the case where land enters as a factor of production. In this case, when areas are partitioned between residential and public land-use, one cannot impose a priori a condition on land analogous to (22). This happens because (61) of Part 1 fixes the value of public land at the same level as the value of residential land which, in turn, is also determined at the optimum. To compute this value, notice that (59) and (60) of Part 1 imply

$$\lambda_i \frac{\partial u}{\partial q_i} = \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\partial E_j}{\partial H_i} \quad (24)$$

This, in conjunction with (26), (28), (38) and (39) of Part 1, leads to

$$\sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\partial E_j}{\partial H_i} - \frac{\phi_i^n}{\Delta q} = r_i \quad (25)$$

It follows immediately that only when there is no population externality, i.e. $\phi_i^n = 0$ for $i \in J$, will competitive pricing in the market for land represent an optimal policy. When there is a negative (positive) population

externality, the public sector is bound to pay a price for land greater (smaller) than the value of its marginal product. In particular, since from (39) and (56) of Part 1

$$-\frac{\phi_i^n}{\Delta q} = -\frac{1}{\Delta Q - H_i} (n_i + 1) \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\Delta E_j}{n_i}, \quad (26)$$

the competitive price of land should be adjusted at the optimum by the externality effect of those in zone i distributed per unit of public land. When this is taken into account, proposition 1 is modified as

2. If returns to scale in the maintainance of a given environmental quality are larger (equal, smaller) than

$$\frac{1}{E} \sum_i \frac{H_i}{\Delta Q - H_i} (\Delta \bar{E}) (n_i + 1)$$

then there is a deficit (balance, surplus) in the budget allocated for the production of the public good at the optimum.

Thus when zones are partitioned between residential and public land-use, in the presence of a congestion externality, the Mohring paradox is deformed by a negative quantity. Therefore a balanced budget will occur at decreasing returns to scale in the maintenance of a given environmental quality: when land enters as a factor of production, the likelihood of a deficit increases.

The previous conclusions hold for any integrated land-use pattern associated with trivial distance-response, such as local public goods and transportation. When though the land-use pattern is segregated, as in agora models, the Mohring paradox can be restored. This happens because, now, it is possible to require that the value of the marginal product of land be equal to its price at the optimum. However, the Mohring paradox further requires constant returns to scale for land in the production of the public good.¹⁵ It seems that the Mohring paradox in its pure form is quite restrictive in matters concerning land; and that the same observation holds for the spatial diffusion of the externality in both the Mohring paradox and its variations.

It is worth noting that our conclusions hold for any degree of aversion to inequality. Further, the principal virtue of the Mohring paradox rests with the intuition that, in general, the budget allocated for the production of the public good does not balance at the optimum with the associated congestion tax receipts. In general, therefore, it will be necessary to further apply lump-sum taxes or subsidies, as those represented by A in the section "optimal decentralization policy" of Part 1, in order to balance the budget. As long as a particular A applies to everyone, and everyone takes it parametrically, the optimum can be retained at a balanced budget. But the optimum requires, in addition to lump-sum taxes or subsidies, the detailed tax or subsidy schemes based on ϕ_1^n and ϕ_1^i . Indeed, the context of this work, lump-sum taxes or subsidies are meaningless independently of Pigouvian taxation. We know already the nature of difficulties associated with the application of such schemes. Thus, in general, reality forces the

imposition of taxes or subsidies which distort optimality. The tax on drivers, for example, follows their consumption of gasoline and remains insensitive to the spatial distribution of other drivers; and homeowners pay a tax which reflects the spatial distribution of property values, rather than the spatial distribution of externality effects as required for optimality.

AGGREGATE RELATIONSHIPS AT THE OPTIMUM¹⁶

TRANSPORTATION, URBAN SHAPE AND THE VALUE OF URBAN LAND

We shall now abandon rotational symmetry and consider a sequence of urban shapes from the linear to the circular. Chicago, for example, could be placed on the middle of this sequence, the semi-circle. We shall describe a class of remarkably simple relationships between the total cost of urban transportation and the total value of urban land net of its opportunity cost, both estimated at the optimum. In order to specify the latter, name R the total value of urban land at the optimum and \check{R} the total opportunity cost of urban land.¹⁷ Then $R - \check{R}$ represents the total value of urban land net of its opportunity cost at the optimum, which corresponds to the shaded area of figure 1. Intuitively, whereas the total cost of urban transportation, C , represents disbenefits created by agglomeration, the total value of urban land net of its opportunity cost must reflect corresponding benefits. The relationship between these two aggregate measures of the city at the optimum is directed by two elasticities, the elasticity of transportation costs with respect to distance, $c:1$, and the elasticity of the urban area with respect to distance,

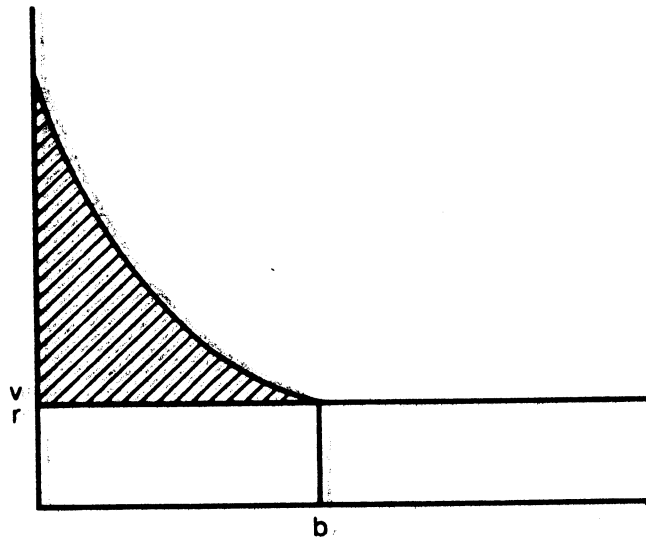


Figure 1. The value of urban land net of its opportunity cost.

$\eta_{Q:l}$.¹⁸ The second elasticity is a measure of shape. For a linear city, where the size of areas is one unit,

$$\eta_{Q:l} \equiv \frac{\Delta Q}{Q} \frac{Q_1}{l} = 1/(l/l) = 1 \quad (27)$$

whereas for a circular city

$$\eta_{Q:l} \equiv \frac{\Delta Q}{Q} \frac{Q_1}{l} = \pi(i^2 - (i-1)^2)/(\pi i^2/l) = (2i-1)/i \approx 2 \quad (28)$$

provided that distance partitions are small. The linear city corresponds to a measure of zero radians and the circular city corresponds to a measure of 2π radians. These are the two extremes of shape. In-between there is a continuum of urban shapes and related elasticities: as the measure of radians gradually increases from zero to 2π , the elasticity of the urban area with respect to distance gradually increases from one to two.¹⁹

3. If the elasticity of the urban area with respect to distance is constant over distance, then the ratio between the total cost of urban transportation and the total value of urban land net of its opportunity cost is smaller than (equal to, greater than) the elasticity of the urban area with respect to distance if the elasticity of transportation costs with respect to distance is greater than (equal to, smaller than) unity.²⁰

Applying proposition 3 to the case of a linear city one obtains

$$R - R \begin{cases} > \\ < \end{cases} C \text{ if } \eta_{c:i} \begin{cases} > \\ < \end{cases} 1; \quad (29)$$

and applying it to the case of a circular city

$$R - R \begin{cases} > \\ < \end{cases} \frac{1}{2} C \text{ if } \eta_{c:i} \begin{cases} > \\ < \end{cases} 1. \quad (30)$$

Over the continuum of urban shapes between the linear and the circular, as the measure of radians gradually increases from zero to 2π , the coefficient that multiplies the total cost of urban transportation gradually decreases from one to one-half. Suppose now that the elasticity of transportation costs with respect to distance is equal to unity. Then, for a linear optimal city, the total value of urban land net of its opportunity cost is precisely equal to the total cost of urban transportation; and, for a circular optimal city, the total value of urban land net of its opportunity cost equals one-half the total cost of urban transportation. Over the continuum of urban shapes, the total value of urban land net of its opportunity cost gradually decreases from being equal to the total cost of urban transportation to one-half of that cost.

The simplicity of this class of relationships is, indeed, striking. As with the Mohring paradox, they are strong in the sense that they are independent of the theory of justice employed by the planner. Intuitively, variations within this class must relate to variations in the efficiency of shapes. One for example may observe that the linear, being here the least efficient of urban shapes because it entails the highest average per capita transportation costs, corresponds to the highest total cost of urban transportation relative to the total value of urban land net of its opportunity

cost; and that the circular, being here the most efficient of urban shapes, corresponds to the lowest total cost of urban transportation relative to the total value of urban land net of its opportunity cost. As efficiency gradually increases, the total cost of urban transportation gradually decreases relative to the total value of urban land net of its opportunity cost.

The same intuitive observations can be extended to cases where the elasticity of transportation costs with respect to distance is other than unity. It remains now to interpret this elasticity. What it actually measures is economies of scale in transportation with respect to distance. A value greater than unity implies decreasing returns to scale in transportation with respect to distance because costs increase relatively faster than distance. Similarly, a value equal to (smaller than) unity implies constant (increasing) returns to scale in transportation with respect to distance. Consider the case of a linear city at the optimum. When there are decreasing returns to scale in transportation, the value of centrality is relatively high. Since this is reflected upon the total value of urban land net of its opportunity cost, $R - \check{R} > C$ at the optimum. On the other hand, when there are increasing returns to transportation, the value of centrality is relatively low and, consequently, $R - \check{R} < C$.

SURPLUS VALUE AND THE VALUE OF URBAN LAND

Recalling (5) and (33) of Part 1, (26) of Part 1 may be written as

$$- n_i \lambda_i \frac{\partial u}{\partial q_i} \Delta q_i - \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_j =$$

$$- n_i \lambda_i \frac{\partial u}{\partial q_i} \Delta Q_{i-1} \left(\frac{1}{n_i + 1} \right) - \frac{1}{n_i} \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_j =$$

$$\frac{\Delta X}{N} - (\psi_i + (n_i + 1) \Delta \psi) \quad (31)$$

or

$$\lambda_i \frac{\partial u}{\partial q_i} \Delta Q_{i-1} - (n_i + 1) \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_j =$$

$$(n_i + 1) \left(\frac{\Delta X}{N} - (\psi_i + (n_i + 1) \Delta \psi) \right). \quad (32)$$

Upon aggregation over the city

$$\sum_i \lambda_i \frac{\partial u}{\partial q_i} \Delta Q_{i-1} - \sum_i (n_i + 1) \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_j =$$

$$\sum_i (n_i + 1) \left(\frac{\Delta X}{N} - (\psi_i + (n_i + 1) \Delta \psi) \right). \quad (33)$$

The shadow price of residential land is the utility gain, expressed in money terms and evaluated at the optimum, of consuming an additional unit of land, i.e. $(\lambda_i \partial u / \partial q_i) |_{u=U}$. Since (33) is evaluated at the optimum, the first term on the LHS can be interpreted as the total shadow value of urban land \tilde{R} . Secondly, since $\sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_j$ denotes the total change in social

welfare stemming from changes in the quality of the environment caused by an individual in area i , the second term in (33) represents the total effect E on social welfare of the spatial externality at the optimum. Consider now the RHS of (33). Since $\frac{\Delta X}{N} - (\psi_i + (n_i + 1) \frac{\Delta \psi}{n_i})$ is the net contribution of someone in zone i to production, $(n_i + 1) \left(\frac{\Delta X}{N} - (\psi_i + (n_i + 1) \frac{\Delta \psi}{n_i}) \right)$ is the total net contribution of those in zone i to production. Thus the RHS of (33) denotes the total surplus value S of urban production. Summarizing,

4. The difference between the total shadow value of urban land and the total externality effect equals the total surplus value of urban production at the optimum.

When the spatial externality is positive (negative), proposition 4 implies that the total shadow price of urban land is larger (smaller) than the total surplus value of urban production at the optimum. Thus urban shadow prices compensate for the presence of the spatial externality in a way dictated by intuition: a negative spatial externality lowers shadow prices, and vice-versa.

It is possible to relate optimal and shadow land values through the externality. Taking into account (38) of Part 1, (32) can be written as

$$\left(r_i - \lambda_i \frac{\partial u}{\partial q_i} \right) \Delta Q = (n_i + 1) \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\Delta E_j}{n_j} \quad (34)$$

which, upon aggregation, implies

5. The difference between aggregate optimal and shadow land values equals the total externality effect at the optimum.

Thus only when there are no spatial externality effects will optimal land values equal their corresponding shadow prices.

COMBINATIONS

Combining proposition 3 with propositions 4 and 5, we obtain

6. If the elasticity of the urban area with respect to distance is constant over distance then

$$R = S + 2E \succ(=, \prec) \overset{\vee}{R} + \frac{1}{n_{Q:i}} C$$

according to whether the elasticity of transportation costs with respect to distance is greater than (equal to, smaller than) unity.

As with proposition 3, the simplicity of relationships described by its subsequent propositions is indeed striking. Here the total surplus value of production represents benefits of urbanization, while the total value of urban land removed from production in the alternative sector and the total cost of urban transportation represent costs of urbanization. On the other hand, the spatial externality enters this balance as a benefit or cost of urbanization according to whether it is positive or negative.

Following the comments on proposition 3, the nature of this balance depends on the shape of the city and on the technology of transportation.

Notice that the total externality effect E in propositions 4-6 equals $-\phi^n$, where ϕ^n is the total public revenue from congestion taxes. This, together with proposition 6 implies

$$\phi^n \begin{matrix} < \\ > \end{matrix} \frac{1}{2} (S - (\check{R} + \frac{1}{\eta_{Q:i}} C)) \text{ if } \eta_{c:i} \begin{matrix} > \\ < \end{matrix} 1. \quad (35)$$

Suppose that the public good is produced only with capital. Then, combining proposition 1 with (35),

$$\beta \begin{matrix} < \\ > \end{matrix} \underline{\text{implies}} \Omega \begin{matrix} < \\ > \end{matrix} \phi^n \begin{matrix} < \\ > \end{matrix} \frac{1}{2} (S - (\check{R} + \frac{1}{\eta_{Q:i}} C)) \text{ if } \eta_{c:i} \begin{matrix} > \\ < \end{matrix} 1. \quad (36)$$

It follows immediately that if returns to scale in both the maintenance of a given environmental quality and transportation are decreasing (constant, increasing) then

$$S > (=, <) 2\Omega + \check{R} + \frac{1}{\eta_{Q:i}} C \quad (37)$$

provided that the elasticity of the urban area with respect to distance is constant over distance. Therefore, under decreasing (constant, increasing) returns to scale, the total surplus value of production, which represents the benefits of urbanization, should be greater (equal, smaller) than the corresponding costs of urbanization. Returns to scale refer here to the two main components of urban infra-structure, that is, the production of public

goods and the nature of transportation costs. In this sense, (37) implies that weaker returns to scale associated with an urban infra-structure require stronger benefits of urbanization at the optimal size of city.

When the public good occupies land, the extreme LHS of (36) should be modified according to proposition 2. Finally, as with proposition 3, all our conclusions subsequent to this proposition are independent of the theory of justice employed by the planner.

APPENDIX 1

This appendix gives the details of the Mohring paradox and its variations. The first part is associated with proposition 1, the second with proposition 2, and the third with the application of the Mohring paradox in the case of agora models with land.

I

Taking into account (38) and (45) of Part 1 with $A = \bar{\phi}$, the tax someone in zone k pays for his damage to those in zone j is given by $-n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_j$. The distribution of this damage between the various zones those in j interact with is, using (4) of Part 1, $-n_j \lambda_j \frac{\partial u}{\partial E_j} \sum_i \Delta E_{ij}$. Hence, the portion of the tax someone in k pays for his damage to those in j as they visit together zone i is $-n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_{ij}$. Aggregating over all his victims, $-\sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_{ij}$ is the portion of the tax someone in k pays for his damage to those over \mathcal{J} as he visits zone i . The total portion of the tax paid by those in k for their damage related to zone i is $-(n_k + 1) \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_{ij}$. It follows immediately that

$$\phi_i^n = - \sum_k (n_k + 1) \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta E_{ij} =$$

$$- \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \sum_k (\Delta E_{ij})(n_k + 1) \quad (1.1)$$

is the total tax raised for zone i .

The net revenues of the public sector associated with zone i are

$$\phi_i^n - \omega_i = - \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \sum_k (\Delta E_{ij})(n_k + 1) - \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\partial E_j}{\partial \omega_i} \omega_i \quad (1.2)$$

using (1.2) and (1.1). Suppose now that the public good is a club, so that $E_{ij} = g_i [N, \omega_i] \equiv e_i$ and $E_j = \bar{E}$ for $j \in \mathcal{J}$. Let $\phi^n \equiv \sum_i \phi_i^n$ be the total revenue of the public sector from congestion taxes. Aggregating (1.2), we have

$$\begin{aligned} \phi^n - \Omega &= - \sum_i \sum_j n_j \lambda_j \frac{\partial u}{\partial \bar{E}} \left(\sum_k (\Delta g_i)(n_k + 1) + \frac{\partial g_i}{\partial \omega_i} \omega_i \right) \\ &= - \sum_i \sum_j n_j \lambda_j \frac{\partial u}{\partial \bar{E}} (\beta e_i) \quad (1.3) \end{aligned}$$

provided that the functions g_i are β -homogeneous with respect to their arguments. (A β -homogeneous function $y = f[x_1, \dots, x_n]$ obeys $\beta y = \sum_i (\partial f / \partial x_i) x_i$.) Notice that the step from (1.2) to (1.3) uses $\partial \bar{E} / \partial \omega_i = \sum_j \partial \sum_j / \partial \omega_i = \partial g_i / \partial \omega_i$, and $\Delta g_i = \Delta g_i \Delta N = \Delta g_i$.

The Mohring paradox holds only if ζ_{ij} can be either zero or one everywhere in \mathcal{J} . Then it is possible to re-order (1.2) as in (1.3). On

the other hand, if distance-response is non-trivial, the transition from (1.2) to (1.3) becomes impossible and the Mohring paradox collapses.

In the case of local public goods where $E_i = g_i [n_i, \omega_i]$ the tax paid by someone in i for his damage to those over J is simply

$- n_i \lambda_i \frac{\partial u}{\partial E_j} \Delta E_i$. Thus the net revenues of the public sector become

$$\begin{aligned} \Phi^n - \Omega &= - \sum_i n_i \lambda_i \frac{\partial u}{\partial E_i} \left((\Delta g_i) \frac{(n_i + 1)}{n_i} + \frac{\partial g_i}{\partial \omega_i} \omega_i \right) \\ &= - \sum_i n_i \lambda_i \frac{\partial u}{\partial E_i} (\beta E_i), \end{aligned} \quad (1.4)$$

using (2.) in a form obviously simplified. For transportation, observe from (12) that those who damage others in zone i must be located at or beyond i . In consequence (1.1) is now written as

$$\begin{aligned} \Phi_i^n &\equiv \sum_{k=i}^b (n_k + 1) \sum_{j=1}^b n_j \lambda_j \frac{\partial u}{\partial E_j} \Delta g_i \frac{1}{n_k} \\ &= \sum_{j=1}^b n_j \lambda_j \frac{\partial u}{\partial E_j} \sum_{k=i}^b (\Delta g_i) \frac{(n_k + 1)}{n_k}. \end{aligned} \quad (1.5)$$

Given that (22) also holds and that transportation is spaceless, we have

$$\Phi^n - \Omega = \sum_{i=1}^b \sum_{j=i}^b n_j \lambda_j \frac{\partial u}{\partial E_j} \left(\sum_{k=i}^b (\Delta g_i) \frac{(n_k + 1)}{n_k} + \frac{\partial g_i}{\partial \omega_i} \omega_i \right)$$

$$= \sum_{i=1}^b \sum_{j=i}^b n_j \lambda_j \frac{\partial u}{\partial E_j} (\beta e_i) \quad (1.6)$$

where the negative signs in (1.5) and (1.6) disappear because e_i is negative. Finally, in the case of agora models

$$\begin{aligned} \phi_i^n &= - \sum_{k=a+1}^b (n_k + 1) n_i \lambda_i \frac{\partial u}{\partial E_i} \frac{\Delta E_i}{n_k} \\ &= - n_i \lambda_i \frac{\partial u}{\partial E_i} \sum_{k=a+1}^b \frac{(\Delta E_i)(n_k + 1)}{n_k}. \end{aligned} \quad (1.7)$$

If the public good is also a club, so that $E_i = g[n, \Omega] \equiv \bar{E}$, then

$$\begin{aligned} \phi_i^n - \Omega &= - \sum_{i=a+1}^b n_i \lambda_i \frac{\partial u}{\partial \bar{E}} \left(\sum_{k=a+1}^b \frac{(\Delta g)(n_k + 1)}{n_k} + \frac{\partial g}{\partial \Omega} \Omega \right) \\ &= - \sum_{i=a+1}^b n_i \lambda_i \frac{\partial u}{\partial \bar{E}} (\beta \bar{E}). \end{aligned} \quad (1.8)$$

II

When land is a factor of production, the net revenues of the public sector become

$$\phi_i^n - (r_i H_i + \omega_i) = - \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \sum_k \frac{(\Delta E_{ij})(n_k + 1)}{n_k}$$

$$\begin{aligned}
& - \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\partial E_j}{\partial H_i} H_i + \frac{H_i}{\Delta Q - H_i} (n_i + 1) \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\Delta E_j}{n_i} \\
& - \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\partial E_j}{\partial \omega_i} \omega_i, \tag{1.9}
\end{aligned}$$

using (25) and (26). If, further, the public good is a club, so that $E_{ij} = g_i[n, H_i, \omega_i] \equiv e_i$ and $E_j = \bar{E}$ for $j \in \mathcal{J}$, the total net revenue is

$$\begin{aligned}
\phi^n - (\sum_i r_i H_i + \Omega) &= - \sum_i \sum_j n_j \frac{\partial u}{\partial \bar{E}} (\beta e_i) \\
&+ \sum_i \frac{H_i}{\Delta Q - H_i} (n_i + 1) \sum_j n_j \lambda_j \frac{\partial u}{\partial \bar{E}} \frac{\Delta \bar{E}}{n_i} \\
&= - \sum_j n_j \lambda_j \frac{\partial u}{\partial \bar{E}} (\beta \bar{E} - \sum_i \frac{H_i}{\Delta Q - H_i} \frac{(\Delta \bar{E})}{n_i} (n_i + 1)). \tag{1.10}
\end{aligned}$$

The remaining cases are similar.

III

Competitive pricing of public land in an agora model implies

$$\frac{1}{\Delta Q} \sum_{i=a+1}^b n_i \lambda_i \frac{\partial u}{\partial \bar{E}} \frac{\Delta g}{j-1} = r_j^0 \text{ for } 1 < j < a, \tag{1.11}$$

provided that the public good is also a club, so that $E_i = g[n, a, \Omega] \equiv \bar{E}$. Notice that (1.11) is entirely compatible with (21): for zones in the

interior of the agora, the first term on the RHS of (21) vanishes because there is no displacement of individuals. Under these circumstances, the total net revenues of the public sector are:

$$\begin{aligned} \phi^n - \left(\sum_{j=1}^a r_j^0 \Delta Q_{j-1} + \Omega \right) &= - \sum_{i=a+1}^b n_i \lambda_i \frac{\partial u}{\partial \bar{E}} \sum_{k=a+1}^b (\Delta g) \frac{1}{n_k} (n_k + 1) \\ &- \left(\sum_{j=1}^a \sum_{i=a+1}^b n_i \lambda_i \frac{\partial u}{\partial \bar{E}} \Delta g_{j-1} + \sum_{i=a+1}^b n_i \lambda_i \frac{\partial u}{\partial \bar{E}} \frac{\partial g}{\partial \Omega} \Omega \right). \end{aligned} \quad (1.12)$$

Suppose that there are constant returns to scale in land used for the production of the public good. Then

$$\Delta g_{j-1} = \frac{\partial g}{\partial Q_{j-1}} \Delta Q_{j-1} = \frac{\partial g}{\partial Q} \Delta Q_{j-1}. \quad (1.13)$$

Replacing (1.13) in (1.12).

$$\begin{aligned} \phi^n - \left(\sum_{j=1}^a r_j^0 \Delta Q_{j-1} + \Omega \right) &= \\ &- \sum_{i=a+1}^b n_i \lambda_i \frac{\partial u}{\partial \bar{E}} \left(\sum_{k=a+1}^b (\Delta g) \frac{1}{n_k} (n_k + 1) + \frac{\partial g}{\partial Q} \sum_{j=1}^a \Delta Q_{j-1} + \frac{\partial g}{\partial \Omega} \Omega \right) = \\ &- \sum_{i=a+1}^b n_i \lambda_i \frac{\partial u}{\partial \bar{E}} (\beta \bar{E}) \end{aligned} \quad (1.14)$$

because $\sum_{j=1}^a \Delta Q_{j-1} = Q_a$.

APPENDIX 2

This appendix contains a proof of proposition 3. It is true that

$$\Delta_i rQ = r_{i+1} \Delta_i Q + Q_i \Delta_i r. \quad (2.1)$$

Aggregating over the city

$$\sum_i \Delta_i rQ = \sum_i r_{i+1} \Delta_i Q + \sum_i Q_i \Delta_i r, \quad (2.2)$$

where the LHS of (2.2) may be written as

$$\begin{aligned} \sum_i \Delta_i rQ &\equiv r_2 Q_2 - r_1 Q_1 + \dots + r_{b+1} Q_{b+1} - r_b Q_b \\ &= -r_1 Q_1 + r_{b+1} Q_{b+1} = -r_1 \Delta_0 Q + r_{b+1} Q_{b+1}; \end{aligned} \quad (2.3)$$

the first term on the RHS of (2.2) may be written as

$$\sum_{i=1}^b r_{i+1} \Delta_i Q = \sum_{i=2}^b r_i \Delta_{i-1} Q + r_{b+1} \Delta_b Q; \quad (2.4)$$

and the second term on the RHS of 2.2 may be written as

$$\sum_i Q_i \Delta_i r = - \sum_i Q_i n_i \frac{\Delta c}{I} \frac{\Delta Q}{i-1} \quad (2.5)$$

because (5) of Part 1 is true by definition and, at the optimum, (4) of Part 1 holds. Upon substitution of (2.3), (2.4) and (2.6) into (2.21) and rearrangement,

$$r_1 \frac{\Delta Q}{Q_0} + \sum_{i=2}^b r_i \frac{\Delta Q}{Q_{i-1}} - r_{b+1} (Q_{b+1} - \frac{\Delta Q}{Q_b}) =$$

$$\sum_i r_i \frac{\Delta Q}{Q_{i-1}} - r_{b+1} Q_b = \sum_i Q_i n_i \frac{\Delta c}{Q_{i-1}} \quad (2.6)$$

Now $r_i \frac{\Delta Q}{Q_{i-1}}$ is the total value of urban land in zone i at the optimum. Hence $\sum_i r_i \frac{\Delta Q}{Q_{i-1}}$ is the total value of urban land at the optimum. Moreover, since r_{b+1} is approximately equal to the bid-rent for agriculture, $r_{b+1} Q_b$ is approximately equal to the total opportunity cost of urban land. Then (2.4) may be written as

$$R - \check{R} = \sum_i Q_i n_i \frac{\Delta c}{Q_{i-1}} \quad (2.7)$$

On the other hand the total cost of urban transportation is

$$C \equiv \sum_i n_i c_i \quad (2.8)$$

Comparisons between the total value of urban land net of its opportunity cost and the total cost of urban transportation can be made through the RHS of (2.7) and (2.8). Now

$$\eta_{c:i} \begin{matrix} > \\ < \end{matrix} 1 \quad (2.9)$$

may be written as

$$\frac{\Delta c / (c_i / i)}{i} \begin{matrix} > \\ < \end{matrix} \frac{1}{\eta_{Q:i}} \frac{\Delta Q / (Q_i / i)}{i-1}, \quad (2.10)$$

or as

$$Q_i n_i \frac{\Delta c / \Delta Q}{i \ i-1} \begin{matrix} > \\ < \end{matrix} \frac{1}{\eta_{Q:i}} n_i c_i. \quad (2.11)$$

Then, since it is true that

$$Q_i n_i \frac{\Delta c / \Delta Q}{i \ i-1} \begin{matrix} > \\ < \end{matrix} \frac{1}{\eta_{Q:i}} n_i c_i \text{ if and only if } \eta_{c:i} \begin{matrix} > \\ < \end{matrix} 1 \quad (2.12)$$

it must also be true that

$$\sum_i Q_i n_i \frac{\Delta c / \Delta Q}{i \ i-1} \begin{matrix} > \\ < \end{matrix} \sum_i \frac{1}{\eta_{Q:i}} n_i c_i \text{ if } \eta_{c:l} \begin{matrix} > \\ < \end{matrix} 1 \quad (2.13)$$

which, upon comparison with (2.7) and (2.8) implies

$$C / (R - \check{R}) \begin{matrix} < \\ > \end{matrix} \eta_{Q:l} \text{ if } \eta_{c:i} \begin{matrix} > \\ < \end{matrix} 1 \quad (2.14)$$

because the elasticity of the urban area with respect to distance is constant over distance, hence it may be taken out of the summation symbol in (2.13).

FOOTNOTES

1. See Y.Y. Papageorgiou, "Spatial Public Goods. I: Theory".
2. See Y.Y. Pagageorgiou, "Spatial Externalities I: Theory, "Annals, Association of American Geographers, Vol. 68(1978), pp. 465-76.
3. See J.M. Buchanan, "An Economic Theory of Clubs," Economica Vol. 32(1965), pp. 1-14. Examples of papers on clubs include M.J. Pauly, "Optimality, 'Public' Goods and Local Governments: a General Theoretical Analysis," Journal of Political Economy, Vol. 78(1970), pp. 572-585; W.H. Oakland, "Congestion, Public Goods and Welfare," Journal of Public Economics, Vol. 1(1972), pp. 339-357; B. Ellickson, "A Generalization of the Pure Theory of Public Goods," American Economic Review, Vol. 63(1973), pp. 417-32; Y.-K. Ng, "The Economic Theory of Clubs: Pareto Optimality Conditions," Economica, Vol. 40(1973), pp. 291-98; J.R. Chamberlin, "Provision of Collective Goods as a Function of Group Size," American Political Science Review, Vol. 68(1974), pp. 707-16; M. McGuire, "Group Segregation and Optimal Jurisdictions," Journal of Political Economy, Vol. 80(1974), pp. 112-32; O. Fisch, "Optimal City Size. The Economic Theory of Clubs and Exclusionary Zoning," Public Choice, Vol. 24(1975), pp. 59-70; E. Berglas, "On the Theory of Clubs," American Economic Review, Papers and Proceedings, Vol. 66(1976), pp. 116-21; K. Lancaster, "The Pure Theory of Impure Public Goods," in R.E. Grierson, ed., Public and Urban Economics (Lexington, Mass.: Lexington Books, 1976); and J.V. Henderson, "Theories of Groups, Jurisdictions and City Size," in P. Mieszkowski and M. Straszheim, eds., Current Issues in Urban Economics (Baltimore: Johns Hopkins University

- Press, 1979). A good source of literature on public goods in general is A.C. Lea, "Public Facility Location Models and the Theory of Impure Public Goods," Sistemi Urbani, Vol. 3(1981), pp. 345-90.
4. Models in this area include A. Williams, "The Optimal Provision of Public Goods in a System of Local Government," Journal of Political Economy, Vol. 74(1966), pp. 18-33; and C. Wright, "Financing Public Goods and Residential Location," Urban Studies, Vol. 14(1977), pp. 51-58.
 5. Models in this area include J.L. Barr, "City Size, Land Rent and The Supply of Public Goods," Regional and Urban Economics. Vol. 2(1972), pp. 67-103; and E. Helpman and D. Pines, "Optimal Public Investment and Dispersion Policy in a System of Open Cities," Working Paper 97, The Foerder Institute for Economic Research, Tel-Aviv University, 1975.
 6. See, for example, E. Boruchov, "Optimal Provision and Financing of Local Public Goods," Public Finance, Vol. 27(1972), pp. 267-81; R. E. Schuler, "The Interaction Between Local Government and Urban Residential Location," American Economic Review, Vol. 64(1974), pp. 682-96; J.H. Wile, "On the Interaction Between Local Government and Urban Location," Working Paper 139, Department of Economics, State University of New York at Stony Brook, 1975; E. Helpman, D. Pines and E. Boruchov, "The Interaction Between Local Government and Urban Residential Location: Comment," American Economic Review, Vol. 66(1977), pp. 961-67; and R.E. Schuler, "The Interaction Between Local Government and Urban Residential Location: Reply and Further Analysis," American Economic Review, Vol. 66(1977), pp. 968-75.

7. Contrary to equilibrium studies where the spatial structure of the city is established in some detail, little is known about the ways optimal public controls are distributed over the city. In this sense the study of Helpman, Pines and Boruchov is significant. (The only other analogous example, to my knowledge, is provided by the literature on the optimal allocation of land to streets.) The same questions, in the context of a general model, remain unanswered. Also unanswered remain questions about comparative statics, including how the spatial distribution of public controls is affected by the theory of justice employed.
8. See, among others, A.A. Walters, "The Theory and Measurement of Private and Social Cost of Highway Congestion," Econometrica, Vol. 29(1961), pp. , R.H. Strotz, "Urban Transportation Parables," in J. Margolis, ed., The Public Economy of Urban Communities (Baltimore: Johns Hopkins University Press, 1965); W.S. Vickrey, "Congestion Theory and Transport Investment," American Economic Review, Vol. 59(1969), pp. 251-60; L.B. Lave, "Congestion and Urban Location," Papers of the Regional Science Association, Vol. 25(1970), pp. 133-50; E. Boruchov, "Diseconomies of Scale in Urban Transportation," Southern Economic Journal, Vol. 38(1971), pp. 79-82; E.S. Mills and D.M. de Ferranti, "Market Choices and Optimum City Size," American Economic Review, Papers and Proceedings, Vol. 61(1971), pp. 340-45; R.M. Solow and W.S. Vickrey, "Land Use in a Long Narrow City," Journal of Economic Theory, Vol. 3(1971), pp. 430-47; R.M. Solow, "Congestion, Density, and the Use of Land in Transportation," The Swedish Journal of Economics, Vol. 74(1972), pp. 161-73; C.P. Averous and D.B. Lee Jr., "Land Allocation and Transportation Pricing in a Mixed Urban Economy," Journal of

Regional Science, Vol. 13(1973), pp. 173-85; A. Dixit, "The Optimum Factory Town," Bell Journal of Economics and Management Science, Vol. 4(1973), pp. 637-51; L. Legey, M. Ripper and P. Varaiya, "Effects of Congestion on the Shape of a City," Journal of Economic Theory, Vol. 6(1973), pp. 162-79; D.A. Livesey, "Optimum City Size: A Minimum Congestion Cost Approach," Journal of Economic Theory, Vol. 6(1973), pp. 144-61; J.G. Riley, "Gammaville: An Optimal Town," Journal of Economic Theory, Vol. 6(1973), pp. 471-82; E. Sheshinski, "Congestion and Optimum City Size," American Economic Review, Papers and Proceedings, Vol. 63(1973), pp. 61-66; R.M. Solow, "Congestion Cost and the Use of Land for Streets," Bell Journal of Economics and Management Science, Vol. 4(1973), pp. 602-18; J.V. Henderson, "Road Congestion: A Reconsideration of Pricing Theory," Journal of Urban Economics, Vol. 3(1974), pp. 346-65; M. Kraus "Land Use in a Circular City," Journal of Economic Theory, Vol. 8(1974), pp. 440-57; J.G. Riley, "Optimal Residential Density and Road Transportation," Journal of Urban Economics, Vol. 1(1974), pp. 230-49; E. Boruchov, "The Effects of Public Provision of Roads on the Structure and Size of Cities," Environment and Planning A, Vol. 7(1975), pp. 349-55; J.V. Henderson, "Congestion and Optimum City Size," Journal of Urban Economics, Vol. 2(1975), pp. 48-62; O. Hochman, "Market Equilibrium Versus Optimum in a Model with Congestion: Note," American Economic Review, Vol. 65(1975), pp. 992-96; J.L. Barr, "Decentralized Urban Resource Allocation," Journal of Regional Science, Vol. 16(1976), pp. 35-43; A.J. Robson, "Cost-Benefit Analysis and the Use of Urban Land for Transportation," Journal of Urban Economics, Vol. 3(1976), pp. 180-91; R.J. Arnott and J.G. McKinnon, "Market and Shadow Land Rents with Congestion,"

American Economic Review, Vol. 67(1977), pp. 588-600; R.J. Arnott and J.G. MacKinnon, "The Effects of Urban Transportation Changes. A General Equilibrium Simulation," Journal of Public Economics, Vol. 8(1977), pp. 19-36; A.E. Boardman and L.B. Lave, "Highway Congestion and Congestion Tolls," Journal of Urban Economics, Vol. 4(1977), pp. 340-59; Y. Kanemoto, "Cost Benefit Analysis and the Second-Best Land Use for Transportation," Journal of Urban Economics, Vol. 4(1977), pp. 483-503; W.C. Wheaton, "Price-Induced Distortions in Urban Highway Investment," Bell Journal of Economics, Vol. 9(1978), pp. 622-32.

9. These assumptions will be relaxed in the following sections.
10. Consider the principle of partitioning land between housing and roads. This principle, represented by (59) of Part 1, dictates that the marginal product of land should be the same for the two uses. Since fewer people cross the road in areas further away from the centre, the marginal benefit of land used to streets falls with distance from the centre. To the extent that per capita consumption of land increases away from the centre, (59) of Part 1 implies that the corresponding marginal cost of land used to streets also falls with distance from the centre. Thus it is not generally possible to determine how the proportion of land allocated to streets behaves with distance from the centre. However, under specific conditions which include that g depends only on land, one may conclude the following. The proportion of land allocated to streets decreases at an increasing rate away from the centre. In absolute terms, the same is true for small cities, i.e. the amount of land allocated to streets also decreases at an increasing rate away from the centre. For larger cities, we observe an initial

increase followed by a decrease. There may also be some very large cities where there is an area around the centre which is completely allocated to transportation. In any case, the amount of land allocated to streets appears to be a concave function of distance which becomes zero at the outer city-border. The original contributions dealing with this problem are Mills and de Ferranti and Solow and Vickrey (op. cit. footnote 8). Dixit (op. cit., footnote 8) placed the Mills and de Ferranti results within a properly optimal framework. Subsequent generalizations include Legey, Ripper and Varaiya, and Sheshinski (op. cit., footnote 8).

11. For simplicity, I still assume that the private good is produced only with labour.
12. The lagrangean function of problem (18) is

$$\sum_{j=1}^a r_j^0 \Delta Q + \sum_{j=a+1}^b n_j \psi_j - X + \sum_{j=a+1}^b \mu_j (z_j - v_j).$$

Partially differencing with respect to a , we have

$$r_{a+1}^0 \Delta Q + \sum_{j=1}^a \Delta r_j^0 - n_{a+1} \psi_{a+1} + \sum_{j=a+2}^b n_j \Delta \psi_j - \Delta X -$$

$$\mu_{a+1} (z_{a+1} + v_{a+1}) + \sum_{j=a+2}^b (\Delta z_j - \mu_j \frac{dv_j}{du_j} \frac{\partial u}{\partial E_j} \Delta E_j) = 0.$$

Equation (20) is then obtained by noting that $\Delta z_j = 0$, that $z_{a+1} = v_{a+1}$ according to (30) of Part 1, and using (33) of Part 1. Notice

that the rules

$$\Delta_x \sum_{j=1}^{g[x]} f_j[x] = \sum_{j=1}^{g[x]} \Delta_x f_j[x] + \sum_{j=g[x]+1}^{g[x+1]} f_j[x+1]$$

$$\Delta_x \sum_{j=g[x]}^n f_j[x] = \sum_{j=g[x+1]}^n f_j[x+1] - \sum_{j=g[x]}^n f_j[x]$$

have also been used in the derivation.

13. The original source is H.D. Mohring and M. Harwitz, Highway Benefits (Evanston, Ill.: Northwestern University Press, 1962), chapter two. Although Strotz, op. cit., footnote 8, mentions this explicitly, it is frequently referred to as the "Strotz result". Other contributions include F.R. Forsund, "Externalities, Environmental Pollution and Allocation in Space: A General Equilibrium Approach," Regional and Urban Economics, Vol. 3 (1973), pp. 3-32; S.C. Kolm, "Rendement Qualitatif et Financement Optimal des Politiques, d'Environnement," Econometrica, Vol. 43(1975), pp. 93-115; B. Greenwald, "Conventional Returns to Scale and Financing Pollution Control," Econometrica, Vol.44(1976), pp. 811-14; and E. Helpman, D. Pines and E. Boruchov, op. cit., footnote 6.
14. For details see appendix 1.
15. For details see appendix 1.
16. This section is based on some ideas in R. Arnott, "Optimal City Size in a Spatial Economy," Journal of Urban Economics, Vol. 6(1979), pp. 65-89.

17. The opportunity cost per unit of land equals the bid rent of agricultural activities.
18. The elasticity of the urban area with respect to distance is the ratio between the relative change in the urban area enclosed within a certain distance from the centre and the relative change in that distance. Similarly, the elasticity of transportation costs with respect to distance is the ratio between the relative change in transportation costs and the relative change in the corresponding distance.
19. The classification based on radians implies urban areas of regular shape. Then the elasticity of the urban area with respect to distance is constant over distance. This is not the case with irregular shapes where the above elasticity varies over distance.
20. For a proof see appendix 2.

September 12, 1985
G/GJP.8/A/GJP4.0/pc