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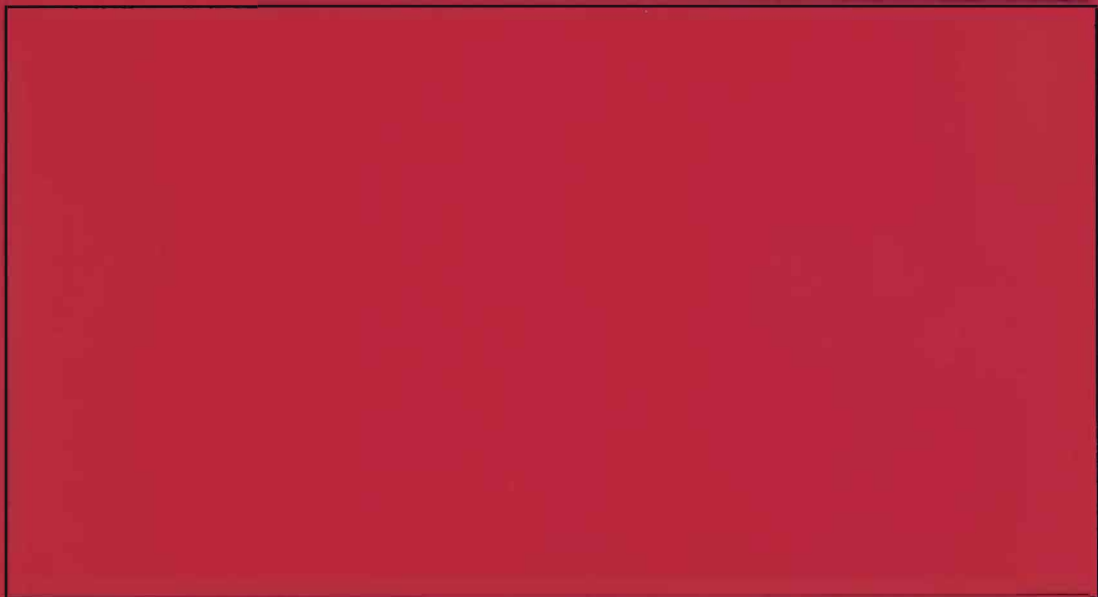
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SPATIAL PUBLIC GOODS

I - THEORY

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INTRODUCTION

Spatial interdependence is a fundamental characteristic of human geography. This happens because, very often, aspects of human activity in one place diffuse their impact elsewhere. Pollution, congestion, prejudice and the human propensity to interact provide examples of spatial interdependence. One way of modelling such phenomena is through the idea of a spatial externality. Years ago, the Annals published a sequence of two articles in which some general aspects of the idea were first developed, and then applied to specific types.¹ The entire sequence was based on the following abstract conceptualization of spatial interdependence.

"Consider a distribution of agents over some landscape \mathcal{L} . Every agent emits an externality which diffuses somehow its impact to other agents in \mathcal{L} . Thus every agent experiences a composite of externalities emitted by agents over \mathcal{L} . We name this composite a spatial externality. If the externality diffusion process is non-trivial (in the sense that the level of a contribution to the externality changes with distance and/or direction from the source) then the level of the spatial externality at $s \in \mathcal{L}$ (hence the entire distribution of the spatial externality in \mathcal{L}) depends on the distribution of agents in \mathcal{L} . Further, a distribution of the spatial externality in \mathcal{L} prompts adjustments which alter the associated distribution of agents. In this manner one obtains two interacting surfaces unfolding over the landscape — a population surface

and an externality surface. The nature of such interaction depends on the nature of the externality diffusion process, in other words on the structure of the spatial externality."²

In the present sequence of two articles, this idea is generalized to account for spatial public goods. As before, the entire sequence will be based on the following abstract conceptualization of spatial interdependence. Consider a distribution of individuals and a distribution of public expenditure over some landscape \mathcal{J} . Every unit of public expenditure emits an externality which diffuses somehow its impact to individuals in \mathcal{J} . The externality, in turn, is affected by the very distribution of individuals over \mathcal{J} . In this sense every individual also emits an externality to other individuals in \mathcal{J} . Thus every individual experiences a composite of externalities emitted by individuals and by public expenditure over \mathcal{J} . We name this composite a spatial public good. If the externality diffusion processes involved are non-trivial (in the sense that the level of a contribution to each externality changes with distance and/or direction from the source) then the level of the spatial public good anywhere in the landscape (hence the entire distribution of the spatial public good in \mathcal{J}) depends on the distribution of individuals and of public expenditure in \mathcal{J} . In this manner one obtains three interacting surfaces unfolding over the landscape -- a population surface, a public expenditure surface and a consequent public good surface. There is a planner who seeks to determine the optimal form of these surfaces. The nature of his solution depends on the nature of the externality diffusion processes involved, in other words on the structure of the spatial public good.

We have thus arrived at the point where externalities and public goods appear to be different faces of the same coin. Indeed there is every reason to perceive spatial public goods as intentional spatial externalities. Thus, in general, it is fair to say that the quality of the urban environment is structured from a large number of comprehensively interdependent, intentional and unintentional, spatial externality effects. This intricate pattern of spatial interdependence is the essence of our problem; and the problem of the planner, in these terms, is nothing much more than to manipulate the elements that compose the quality of the urban environment in some optimal way.

Unlike the sequence on spatial externalities, which included both positive and normative aspects, the present sequence is directed exclusively on the second type: our question is how public goods should be distributed over the city, and what is the structure of city that supports such an optimal distribution. Our study is performed on the assumption that all final decisions are taken by the planner who personifies public institutions. To fix ideas, one could think about the design of a new town where everything that pertains to the use of land is perfectly controlled. Once the planner begins to consider how the city should be, subjectivity enters via the need to decide what constitutes distributive justice. Different theories of distributive justice, in turn, will generate different optimal city concepts. Our main, geographical, question now becomes about how the spatial diffusion process of the public good interacts with a theory of distributive justice to produce an optimal urban form.

The backbone around which most arguments of the type presented here develop is a constrained optimization problem. In general, the problem may be posed either in the context of positive or normative analysis. In the

former case it pertains to the decisions of a "representative" individual while, in the latter, to both the decisions of individuals and the planner. For individual decisions, the solution of the constrained optimization problem can be obtained through standard Kuhn-Tucker theory.³ On the other hand, for planning decisions, more advanced methods may be required. These can be circumvented if one discretizes space, recognizes population as discrete, and applies Kuhn-Tucker theory in an approximate sense. One could, therefore, either promote exactitude at the expense of technical simplicity, or elect simplicity at some loss of exactitude. In this sequence, following the now classic paper of Strotz, I elected the second alternative.⁴ Thus one can follow the arguments throughout if one knows elementary constrained optimization. The same approach to the planning problem at hand has been used in an earlier paper.⁵ Since the model used here is also the same, the discussion draws freely upon that earlier paper.

The paper is partitioned into four main sections. The first section describes the landscape, the spatial structure of the public good and the decision framework, both for the individuals involved and for the planner. The second section deals with the issue of decentralization, i.e. with the search for a minimum combination of public controls sufficient to maintain the particular kind of optimal form which is determined through the theory of justice espoused by the planner. This section includes an explicit comparison of individual and public decisions, a discussion of optimal inequalities induced by the geography of a place, and a detailed description of optimal decentralization policy. The third section compares private goods, public goods and spatial externalities. Finally, the fourth section deals with the problem of allocating land to public goods. All these, in turn, provide the basis for the applications and issues examined in the second paper of the series.

THE MODEL

LANDSCAPE

We shall consider a circular city possessing rotational symmetry. In the simplest case the urban area \mathcal{J} , the total population N and the total amount of public funds Ω to be distributed over the city are fixed. The urban area is partitioned by concentric circles into a fixed number of unit-width zones and represented by the vector $(1, \dots, b)$, where zone one is innermost and zone b marks the border of the city. Thus zone i is i units of distance from the centre. Zones are perfectly homogeneous. If Q_i is the total area enclosed between the centre and zone i , $\Delta Q \equiv Q_i - Q_{i-1}$ is the size of zone i . Feasible spatial population distributions over the urban area are represented by the vectors

$$n \equiv (n_1, \dots, n_b) \in \{n \mid \sum_{i=1}^b n_i = N \text{ and } n_i > 0, i = 1, \dots, b\} \quad (1)$$

where n_i , an integer, represents the population of zone i . Similarly, feasible spatial distributions of public investment are represented by the vectors

$$\omega \equiv (\omega_1, \dots, \omega_b) \in \{\omega \mid \sum_{i=1}^b \omega_i = \Omega \text{ and } \omega_i > 0, i = 1, \dots, b\} \quad (2)$$

where ω_i , a real, represents public investment in zone i .

PUBLIC GOOD

The level of the public good experienced by someone in zone i stems from potential interaction with all zones. The contribution of public expenditure in zone i to the level of the public good experienced by someone in zone j is

$$E_{ij} = f_{ij}[n, \omega_i]. \quad (3)$$

Equation (3) reflects the idea that E_{ij} depends on the distance between the two zones (through the spatial diffusion of the public good), on the spatial distribution of population (which, together with the relative position of zone i determines the level of congestion of the public good in that zone), and on the public expenditure realized in zone i .⁶

For analytical simplicity, we shall assume throughout that spatial externality effects are only transmitted along the diameters of our circular city, thereby underlining the essentially uni-dimensional character of our constructs. We shall describe any diameter as $(-b, \dots, -1, 0, 1, \dots, b)$, with the understanding that $(-i, i)$ belong to the same zone i and $n_0 = 0$. Then the level of the public good experienced by someone in zone j is

$$E_j[n, \omega] = \sum_{i=-b}^b E_{ij} \quad (4)$$

with $\Delta_{n_i} E_j < 0$ and $\partial E_j / \partial \omega_i > 0$.

INDIVIDUALS

Individuals in zone i have preferences defined over the consumption of a numéraire private good (x_i), land ($q[n_i]$) and the public good, where

$$q[n_i] \equiv q_i = \frac{\Delta Q}{n_i} \quad (5)$$

Thus the inverse of q_i denotes gross population density in zone i . Since individuals have identical tastes, utility levels are determined by a common utility function

$$u_i = u[x_i, q_i, E_i] \quad (6)$$

which is differentiable, increasing in its arguments and strictly quasi-concave. Furthermore, since the expenditure on the public good is exogenous, individuals face costs related to the consumption of the private good, land, and transportation. In general, individuals should be self-sufficient, that is, they should generate at least what they spend. Therefore

$$x_i + r_i q_i + c[i] \leq y_i \quad (7)$$

is a constraint corresponding to individuals in zone i , where r_i is the land-rent charged per unit area, $c[i]$ is the cost of transportation represented by a known function of i , and y_i is income. The level of the public good experienced is parametric, that is, individuals believe that they cannot affect its magnitude.

Individuals aim to optimize the use of their limited resources. This, for someone in zone i , can be expressed as the problem of determining a feasible consumption mix (x_i, q_i) such that

$$v[r_i, y_i] = \text{maximum } \{u_i | x_i + r_i q_i + c[i] \leq y_i\} \quad (8)$$

where v is the indirect utility. Nevertheless, for every particular zone, it is true that to maximize utility subject to an income constraint is to minimize expenditure subject to a utility constraint; and that to minimize expenditure subject to a utility constraint is to maximize utility subject to an income constraint -- provided that the given level of utility requires an expenditure above the minimum possible for sustenance.⁷ Thus the two approaches are almost equivalent. In particular, the problem of someone in zone i can be expressed as finding a feasible consumption mix (x_i, q_i) such that

$$\psi[r_i, z_i] = \text{minimum } \{x_i + r_i q_i + c[i] | z_i \leq u_i\} \quad (9)$$

provided that the given level of utility z_i requires an expenditure above the minimum possible for sustenance. Under these circumstances,

$$v[r_i, \psi[r_i, z_i]] = z_i \quad (10)$$

i.e. the maximal utility from income $\psi[r_i, z_i]$ is z_i ; and

$$\psi[r_i, v[r_i, y_i]] = y_i \quad (11)$$

i.e. the minimal expenditure to achieve utility $u[r_i, y_i]$ is y_i . In order to describe everything in terms of money, rather than utils, individual behaviour will be described using expenditure minimization, rather than utility maximization. Within this context, (r_i, z_i) are parameters varying only with respect to location.

If solutions are interior, the necessary and sufficient conditions for a solution to problem (9) are

$$\lambda_i (z_i - u_i) = 0 \quad (12)$$

$$1 - \lambda_i \frac{\partial u}{\partial x_i} = 0 \quad (13)$$

$$(r_i - \lambda_i \frac{\partial u}{\partial q_i}) \Delta q = 0 \quad (14)$$

n_i

where $\lambda_i \equiv d\psi/dz_i$ is the Lagrangean multiplier of problem (9).⁸ Since $\lambda_i > 0$, (12) implies

$$u_i = z_i. \quad (15)$$

Combining (13) and (14) we conclude, as usual, that the marginal rate of substitution between the private good and land equals the corresponding ratio of prices. Combining (13), (14) and (15) yields the compensated demand functions $x[r_i, z_i]$ and $q[r_i, z_i]$ which determine $\psi[r_i, z_i]$. Therefore (r_i, z_i) , $i = 1, \dots, b$ determines the behaviour of individuals every-

where in \mathcal{J} . Specifying the optimal distribution of (r_i, z_i) over \mathcal{J} is a central concern of this paper.

Equilibrium analysis is confined to spatial structures which eliminate advantages of location, in other words, which make utility invariant over \mathcal{J} . In the case of optimal cities, on the other hand, it is not known a priori whether utility should be invariant. Therefore, in order to account for possible inequalities at the optimum, one must be confined to spatial structures which support a given distribution of utility. Stated otherwise, $\psi[r_i, z_i]$ must be a local minimum with respect to location:

$$q_i \Delta r_i + \Delta c_i + \lambda_i \left(\Delta z_i - \frac{\partial u}{\partial E_i} \Delta E_i \right) = 0. \quad (16)$$

In order to gain some intuition about the meaning of (16), consider the case of two adjacent zones with a difference in utility level $\Delta z_i \neq 0$. If the income differences between the two zones, adjusted to account for existing environmental differences, were smaller than $\lambda_i \Delta z_i$, an individual in zone i with fixed income ψ_i would improve his utility by moving to zone $i + 1$. If, on the other hand, the difference between the two zones were greater than $\lambda_i \Delta z_i$, an individual in zone $i + 1$ with fixed income ψ_{i+1} would improve his utility by moving to zone i . Thus (16) conveys the idea that the movement between adjacent zones will stop when the established differences in locational benefits and costs between adjacent zones precisely offset the income differences necessary to attain the corresponding exogenous utility differences.

THE PLANNER

There is a planner who serves as the composite of institutions in society. He aims to optimize the resources available in the sense of determining what, for him, is the best feasible distribution of welfare between the individuals. This implies a stance concerning distributive justice. For example, given a pattern of inequalities, differences of opinion might arise about how important is to raise the welfare of the least advantaged in society. Such differences may be accounted for through a distinction between utility (u_i) and corresponding social valuation of utility (v_i). The relationship between u_i and v_i under a theory of distributive justice is essential for the analysis of optima, as it provides us with an explicit link between the individual and society. The relationship adopted here is based on the following premise. As the degree of aversion to inequality increases, the social valuation of individual welfare decreases relatively faster for relatively higher utility: there is an increasing bias in favour of the less advantaged. This can be expressed as

$$-\frac{d}{du_i} \left(\frac{dv}{du_i} \right) / \left(\frac{1}{u_i} \frac{dv}{du_i} \right) \equiv -\eta_{v':u_i} = \alpha \quad (17)$$

where $\eta_{v':u}$ is the utility elasticity of the change in the social valuation of individual welfare and α is the degree of aversion to inequality. Social valuations consistent with this definition are represented by

$$v_i = \begin{cases} \frac{1}{1-\alpha} u_i^{1-\alpha} & \text{for } \alpha \neq 1 \\ \ln u_i & \text{for } \alpha = 1 \end{cases} \quad (18)$$

where $u_i > 1$ and, without loss of generality, the constants of integration have been omitted. Notice that

$$\frac{dv}{du_i} = u_i^{-\alpha}, \quad (19)$$

an expression which will be used extensively in the ensuing analysis.

The planner faces the following constraints. Firstly, the city should be self-sufficient, in other words, it should generate at least the income spent by its inhabitants. This income is obtained by working at the centre to produce the private good. Thus the technology of producing the private good imposes limits on production which, in turn, affects the income available. In the simplest case, production will depend only on employment, so that

$$\sum_{j=1}^b n_j \psi_j \leq X[N] \text{ with } N = \sum_{j=1}^b n_j \quad (20)$$

expresses the idea that the net value of what is produced in the city cannot be less than the total income spent by its inhabitants. Secondly, the constraint

$$\sum_{j=1}^b \omega_j \leq \Omega \quad (21)$$

which expresses the idea that public expenditure cannot exceed a fixed amount.

As in the case of individuals, optimal planning will be expressed in terms of expenditure minimization, rather than welfare maximization.

Namely, the problem of the planner is to determine a feasible consumption mix (x_i, q_i) and a feasible level of public expenditure ω_i for every zone such that

$$D[r, z; \alpha, \Omega] = \text{minimum} \left\{ \sum_{j=1}^b n_j \psi_j - X[N] \mid z_i < v[u[r_i, \psi_i]; \alpha] \right. \\ \left. \text{for } i = 1, \dots, b \text{ and } \sum_{j=1}^b \omega_j < \Omega \right\} \quad (22)$$

where D represents an aggregate income deficit, $r \equiv (r_1, \dots, r_b)$ and $z \equiv (z_1, \dots, z_b)$. As in the case of individuals, (r_i, z_i) are parameters varying only with respect to location. Since (22) can be solved for any feasible (r, z) , it can also be solved for the optimal (r, z) . This, in turn, will be specified using the concept of an optimal decentralization policy.

If solutions are interior, the necessary conditions for an optimum are

$$\mu_i (z_i - v_i) = 0 \text{ for } i = 1, \dots, b \quad (23)$$

$$v \left(\sum_j \omega_j - \Omega \right) = 0 \quad (24)$$

$$\left(n_i \frac{\partial \psi}{\partial x_i} - \mu_i \frac{dv}{du_i} \frac{\partial u}{\partial x_i} \right) \Big|_{u_i=v_i} = 0 \text{ for } i = 1, \dots, b \quad (25)$$

$$\left(\psi_i + (n_i + 1) \frac{\Delta \psi_i}{n_i} - \frac{\Delta X}{N} - \mu_i \frac{dv}{du_i} \frac{\partial y}{\partial q_i} \frac{\Delta q}{n_i} \right)$$

$$- \sum_j \mu_j \frac{dv}{du_j} \frac{\partial u}{\partial E_j} \Delta E_j \Big|_{u=v} = 0 \text{ for } i = 1, \dots, b \quad (26)$$

$$- \left(\sum_j \mu_j \frac{dv}{du_j} \frac{\partial u}{\partial E_j} \frac{\partial E_j}{\partial \omega_i} + v \right) \Big|_{u=v} = 0 \text{ for } i = 1, \dots, b \quad (27)$$

where μ_i and v are Lagrangean multipliers associated with utility and expenditure constraints; and $u \equiv (u_1, \dots, u_b)$, $v \equiv (v_1, \dots, v_b)$.¹⁰

Using the envelope theorem on problem (22)

$$\mu_i = \frac{\partial D}{\partial z_i} = n_i \frac{\partial \psi}{\partial z_i} > 0 \text{ for } i = 1, \dots, b \quad (28)$$

$$v = - \frac{\partial D}{\partial \Omega} > 0. \quad (29)$$

Thus μ_i represents the additional funds necessary to raise the level of socially evaluated utility in zone i by one unit. On the other hand, the negative sign in (29) occurs because the aggregate income necessary to support a given distribution of utility decreases as the funds available for the public good increase. Using (28) and (29) in conjunction with (23) and (24),

$$z_i = v_i \quad (30)$$

$$\sum_j \omega_j = \Omega. \quad (31)$$

As in the case of individuals, analysis is confined to spatial structures which support a given distribution of utility. In other words, for every zone i , it is required that $D[r, z; \alpha, \Omega]$ is a local minimum with respect to location:

$$\left(n_i \frac{\Delta \psi}{i} + \mu_i \left(\Delta z - \frac{dv}{du_i} \frac{\partial u}{\partial E_i} \Delta E \right) \right) \Big|_{u_i = u_i} = 0 \text{ for } i = 1, \dots, b. \quad (32)$$

The necessary conditions for an optimum are different manifestations of the marginal principle. The first term in (25) represents the marginal social cost (MSC) of the aggregate change in income required to increase the consumption of the private good in zone i by one unit. The second term represents the corresponding marginal social benefit (MSB). With respect to condition (26), when the planner decides to add another individual in zone i , the consequences are as follows. Firstly, the newcomer must be provided with income (first term). Moreover, the associated reduction of the land consumed per capita in zone i will cause a reduction to the expenditure required per capita. This appears in the second term which, unlike the previous, represents a MSB. Another MSB appears in the third term: the newcomer works at the centre and produces as everyone else. The fourth term is the loss in social welfare due to the increased congestion in zone i . Finally, the interaction of the newcomer with public facilities in the city raises congestion with respect to the public good, hence lowers the levels of the public good experienced over the city and, therefore, results in a further loss of social welfare (fifth term). With respect to condition (27), the quantity $\partial E_j / \partial \omega_i$ represents the gain in the level of the public good experienced in zone j because the public expenditure in zone i has been

raised by one dollar. The quantity $(\partial u / \partial E_j)(\partial E_j / \partial \omega_i)$ represents the value of this gain to someone in zone j . The quantity $(dv / du_j)(\partial u / \partial E_j)(\partial E_j / \partial \omega_i)$ represents the social evaluation of this gain. Hence the first term in (27) represents the MSB of public expenditure in zone i . The second term, which is the marginal social value of a dollar spent, represents a MSC. This condition says that the MSB of public expenditure should be the same in all zones. With respect to condition (32), when the planner decides to change the location of those in zone i to zone $i + 1$, the consequences refer to the change in the aggregate income necessary to support the migrants which stems from changes in locational costs (first term); to the social value of related utility differences (first component of the second term); and to the social value of the difference between the levels of the public good experienced in the two areas (second component of the second term). The partitioning of these into MSC and MSB will vary between cases.

DECENTRALIZATION

CONCEPTUAL FRAMEWORK

Given that (r, z) are chosen optimally, a solution to problem (22) prescribes exactly how the optimal city should be. In particular, it prescribes the distribution of public investment over \mathcal{J} and, for everyone in the city, consumption of the private and public goods, consumption of land, and location. The question now arises about how an optimum is to be achieved. One way is of course by fiat. But is such an extreme approach necessary to achieve an optimum? In other words, is it true that all private decisions necessarily differ from the corresponding public decisions at

the optimum? If not then the planner should concentrate only where private and the corresponding public decisions deviate. This would not only save him some effort but also would, to some extent, resolve the problem of individual freedom which is bound to be acute in a highly centralized system.

The previous remarks lead to the issue of decentralization, i.e. to identifying the lowest level of public controls sufficient to support an optimum. Such public controls form an optimal decentralization policy. In order to define an optimal decentralization policy, individual decisions concerning the private good, land and location must be composed with corresponding public decisions. If decisions belonging to a pair are equivalent, there is no need for public intervention in this type of decision. If, on the other hand, corresponding decisions differ then there is a need for public intervention. This can be achieved either by imposing the optimal decision directly, or indirectly through a well-known system of taxes and subsidies.¹¹

COMPARISON OF INDIVIDUAL AND PUBLIC DECISIONS

Individual decisions concerning the private good, land and location are represented by conditions (13), (14) and (16) respectively. Corresponding public decisions are represented by (25), (26) and (32). Thus, in order to define an optimal decentralization policy, one must compare (13) with (25), (14) with (26), and (16) with (32). Comparability dictates that parameters which appear in both problems (9) and (22) must be the same, i.e. for comparison purposes, individuals and the planner to be faced with the same parameter set (r, z) in the context of problem (9) and (22) respectively. Using (15), (19), (28) and (30),

$$\frac{\partial D}{\partial z_i} \frac{\partial v}{\partial u_i} \Big|_{z_i=v_i} = n_i \frac{\partial \psi}{\partial z_i} \frac{\partial v}{\partial u_i} \Big|_{z_i=v_i} = n_i \frac{\partial \psi}{\partial u_i} \quad (33)$$

by the chain rule. Since u_i is evaluated at v_i in the solution of problem (22), $\partial \psi / \partial u_i = \lambda_i$. This, together with (19), implies

$$\mu_i = n_i \lambda_i z_i^\alpha. \quad (34)$$

Applying (34) on (25), (26) and (32), and dividing throughout by n_i , we obtain

$$n_i \left(1 - \lambda_i \frac{\partial u}{\partial x_i} \right) \Big|_{u_i=z_i} = 0 \quad (35)$$

$$\begin{aligned} & \left((r_i - \lambda_i \frac{\partial u}{\partial q_i} \frac{\Delta q}{n_i}) - \frac{1}{n_i} \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\Delta E_j}{n_i} \right) \Big|_{u=z} \\ & + \frac{1}{n_i} \left((\psi_i + \frac{\Delta \psi}{n_i}) - \frac{\Delta X}{N} \right) = 0 \end{aligned} \quad (36)$$

$$\left((q_i \frac{\Delta r}{i} + \frac{\Delta c}{i} + \lambda_i (\Delta z - \frac{\partial u}{\partial E_i} \frac{\Delta E}{i})) + (\frac{\mu_i}{n_i} - \lambda_i) \Delta z \right) \Big|_{u_i=z_i} = 0. \quad (37)$$

Notice that, because of (15) and (30), both individual and public decisions are evaluated at $u = z$ during comparisons. From now on, for simplicity, we shall avoid the explicit notation $\Big|_{u_i=z_i}$.

Comparison between (13) and (35) indicates that the choice of individuals regarding the consumption of the private good coincides with that of the planner. Thus there is no need to interfere in that type of decision.

In contrast, comparison between (14) and (36) indicates that the choice of individuals regarding congestion differs from that of the planner. Notice that $\psi_i + \frac{\Delta \psi}{n_i}$ represents the per capita income in zone i adjusted for the presence of an additional person at the optimum. Suppose that the planner distributes an income equal to the value of the marginal product of individuals,

$$\psi_i + \frac{\Delta \psi}{n_i} \equiv y \equiv \frac{\Delta X}{N}, \quad (38)$$

in other words, production is organized efficiently. Under these circumstances, individual and public decisions about the use of land differ only because of the second term in (36). Indeed, if the planner sets a marginal rate on the use of land equal to

$$\phi_i^n = -\frac{1}{n_i} \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\Delta E_j}{n_i} \text{ for } i = 1, \dots, b \quad (39)$$

and if this marginal rate is taken parametrically by the individuals then (14) would be augmented by ϕ_i^n and would become equivalent to (36) at the optimum. Now $\frac{\Delta E_j}{n_i}$ is the effect of the marginal individual in zone i upon the public good experienced in zone j . The quantity $\lambda_j \frac{(\partial u / \partial E_j)}{n_i} \frac{\Delta E_j}{n_i}$ is the value of this effect on someone in j . Thus the expression under the sum in (39) is the total value of this effect. Since individuals do not take into account the effect of their presence upon the level of the public good experienced over the city, the expression under the sum in (39) represents

the net marginal social costs (NMSC) of congestion in zone i . Since there is no distinction between individuals in zone i , this marginal cost must be equally borne by everyone there — hence the division by n_i . Finally, the minus sign ascertains that if ΔE_j is negative (that is, if further congestion is a cost) then ϕ_i^n will be positive (a tax).

Consider now the decision where to locate. As with congestion, if the planner sets a marginal rate on location equal to

$$\phi_i^i = \left(\frac{\mu_i}{n_i} - \lambda_i \right) \Delta z \text{ for } i = 1, \dots, b \quad (40)$$

and if this marginal rate is taken parametrically by the individuals then (16) would be augmented by ϕ_i^i and would become equivalent to (37) at the optimum. An individual who moves from zone i to zone $i + 1$ experiences a change in utility valued as $\lambda_i \Delta z$. For the planner, however, this change is valued as $(\mu_i/n_i) \Delta z$. Since all other consequences of a move are properly taken into account by individuals, the difference in (40) represents the NMSC or net marginal social benefits (NMSB) of location in zone i .

Using (38), in the context of problem (22),

$$\begin{aligned} \Delta(\psi_i + \Delta \psi) &= \Delta(x_i + r_i q_i + c[i] + r_i \Delta q) \\ &= \Delta(x_i + r_i q_i |_{n_i+1} + c[i]) \\ &= q_i |_{n_i+1} \Delta r + \Delta c \approx q_i \Delta r + \Delta c = \Delta \psi = 0, \end{aligned} \quad (41)$$

provided that the number of individuals in zone i at the optimum is large enough.¹² Introducing this result to (32) and taking into account (19) and (28),

$$\Delta u_i = u_i^{-\alpha} \frac{\partial u}{\partial E_i} \Delta E_i \quad (42)$$

Furthermore, using (28) and (42) on (40), if the planner sets a marginal rate on location equal to

$$\phi_i^i = (1 - u_i^{-\alpha}) \lambda_i \frac{\partial u}{\partial E_i} \Delta E_i \quad (43)$$

and if this marginal rate is taken parametrically by the individuals then (16) would be augmented by ϕ_i^i and would become equivalent to (37) at the optimum.

The perfect asymmetry between (42) and (43) is noteworthy. The former prescribes optimal inequalities associated with a given degree of aversion to inequality. At zero aversion, optimal utility differences should precisely equal the value of the corresponding environmental differences. Increasing aversion to inequality suppresses such differences until, at infinite aversion, optimal utility is constant over the city. The instrument for obtaining optimal inequalities is provided by (43). At zero aversion, there is no need to intervene. This happens because, at zero aversion, individual and social valuations of utility coincide. Increasing aversion to inequality generates increasingly differentiated marginal tax or subsidy rates which reduce, in turn, every optimal location advantage.

UNEQUAL TREATMENT OF EQUALS

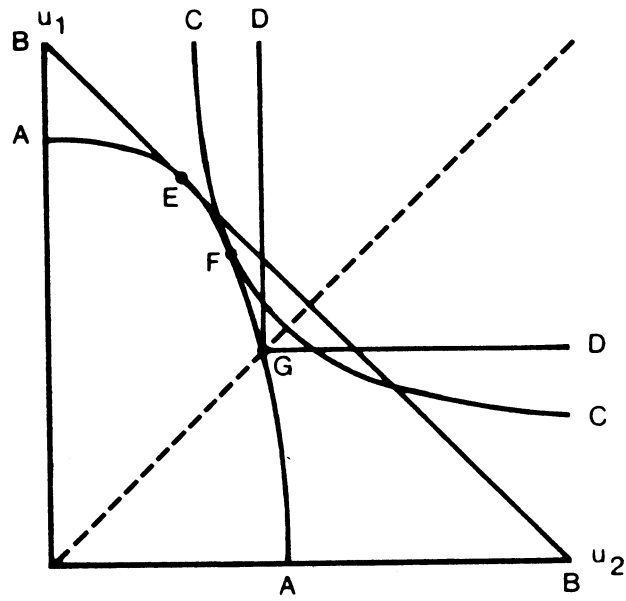
Equation (42) is a generalized rule of Mirrlees.¹³ The following intuitive explanation of the rule has been provided by Levhari, Oron and Pines.¹⁴

We concentrate on how environmental quality affects the distribution of utilities at the optimum. In the simplest possible case, there are only two identical individuals and two locations of fixed unit-size. The quantity of private good available is also fixed. One of the two locations available has an intrinsic advantage over the other, say, a superior quality of the environment. The planner decides where to locate whom and how to distribute the private good available. Suppose he places the first individual in the good location. Then all the possible distribution of utilities generated by all the possible partitions of the private good between individuals are shown as the line AA in figure 1(1). The crucial observation is that unequal quality of the environment induces an asymmetry of the utility-possibility frontier AA: the individual in the better location can enjoy higher levels of utility per unit of the private good.

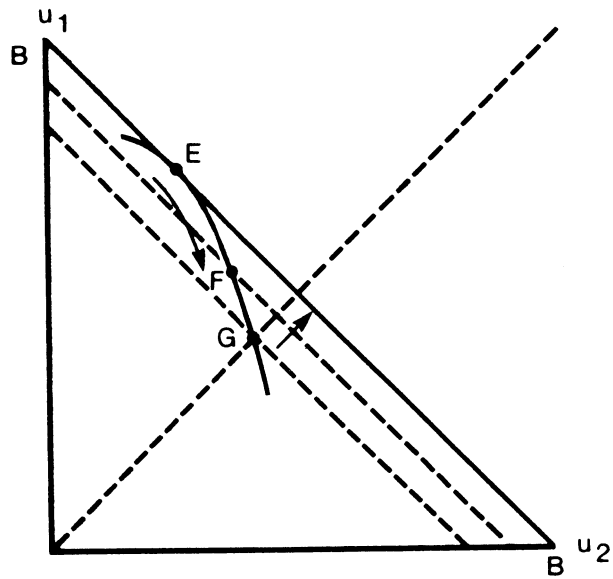
Within this framework, the level of social welfare is simply $W = v_1 + v_2$, with social indifference curves determined by

$$dW = \frac{dv}{du_1} du_1 + \frac{dv}{du_2} du_2 = u_1^{-\alpha} du_1 + u_2^{-\alpha} du_2 = 0, \quad (44)$$

using (19), which implies



(1)



(2)

Figure 1 Unequal treatment of equals.

$$\frac{du_1}{du_2} = - \left(\frac{u_1}{u_2}\right)^\alpha. \quad (45)$$

Under zero aversion to inequality, social indifference curves are represented by straight lines of slope minus one. In figure 1(1), the highest possible such line having a point E in common with the utility-possibility frontier AA is BB. Hence point E corresponds to the distribution of utilities associated with the highest possible level of social welfare, in other words, with the highest possible average utility. This is also the point of highest inequality. Increasing the degree of aversion to inequality generates curved social indifference lines, for example CC with optimal solution F. At the limit, where aversion to inequality becomes infinite, the social indifference curve DD implies equal treatment of individuals at the optimal solution G. This is also the point of lowest average utility within the domain of optimal solutions EG. No allocation within this domain is Pareto-inferior to any other allocation. In other words, the planner cannot shift to another allocation without a loss to someone.

Clearly, an asymmetric utility-possibility frontier forces a trade-off between equity and average utility at the optimum. In particular, increasing degree of aversion to inequality imposes increasing limitations on how far Pareto-efficiency can push the level of average utility (figure 1(2)). In this manner, a fundamental planning dilemma may arise, especially when further reducing inequality necessitates a strong reduction in average utility. A balance between the two opposites, i.e. the choice of a point belonging to the optimal domain EG, is a matter of philosophical argument rather than of a neutral, positive analysis.

OPTIMAL DECENTRALIZATION POLICY

Comparison between individual and public decisions implies that, in order to achieve an optimum, the planner must impose a tax or subsidy on the residents of zone i equal to

$$\phi_i - A \equiv n_i \phi_i^n + i \phi_i^i - A \quad (46)$$

where A represents any constant which allows an expenditure above the minimum possible for sustenance.¹⁵ There is an infinite number of such taxes or subsidies. This provides considerable flexibility to the planner because it permits transfers of money, hence of welfare, between the city and the rest of the world. In consequence, different values of A imply different optimal utility levels for the city.

Suppose that the planner aims for self-sufficiency at the optimum, i.e. $D = 0$. This, in conjunction with (38), implies

$$N_y = \frac{N \Delta X}{N} = X[N], \quad (47)$$

in other words, that the planner must determine the urban population size in a way that the marginal product of labour equals the corresponding average product at the optimum. A geometric interpretation of (47) appears in figure 2. Returns to scale in production first increase and then decrease. The optimal population size will be found at N^* , where (47) is satisfied and the average product of labour is maximized. Now, introducing the optimal taxes or subsidies and taking into account (47), the new aggregate income

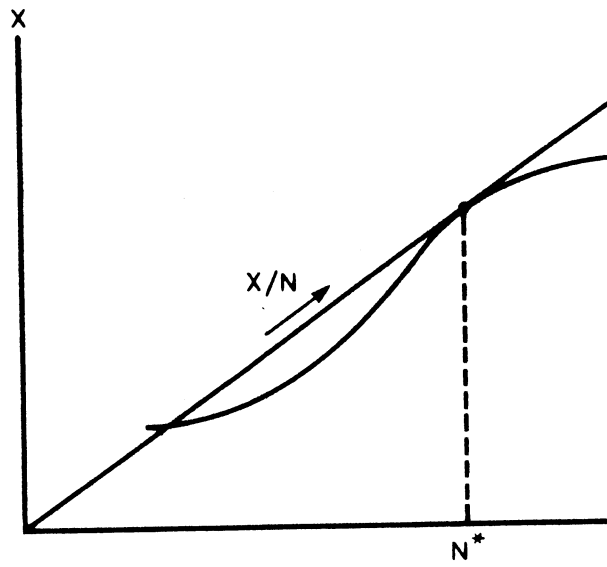


Figure 2: Determination of optimal city size.

$Ny - \sum_j n_j \phi_j + NA$ will not balance the total value of the urban product unless $A = \bar{\phi}$, the average of optimal taxes and subsidies. In this case

$$- \sum_j n_j \phi_j + NA = - \sum_j n_j \phi_j + N\bar{\phi} = 0 \quad (48)$$

and $\phi_1 - \bar{\phi}$ serves to re-distribute the original aggregate income Ny in the way required to attain an optimum.

We are now ready to sketch a solution procedure for the problem of the planner (22). According to (27) and (31), the planner must spend the entire amount of public funds available, and distribute these over the city in a way that the marginal social benefit of public investment is the same in all zones at the optimum. Let $\mathcal{E} \equiv \{x_1, q_1; i = 1, \dots, b\}$. For any feasible \mathcal{E} , (27) gives $\omega[i, \alpha, \Omega; \mathcal{E}]$. This, in conjunction with (4), determines $E[i, \alpha, \Omega; \mathcal{E}]$. By comparing first-order conditions, we have $\phi[i, \alpha, \Omega; \mathcal{E}] \equiv \phi_1 - \bar{\phi}$ (using (39), (43) and (46)), and $\Delta z[i, \alpha, \Omega; \mathcal{E}] = \Delta u|_{u_1=z_1}$ (using (42)). The former, together with $y = \Delta X$, gives $y[i, \alpha, \Omega; \mathcal{E}]$; the latter, together with $D = 0$, gives $z[i, \alpha, \Omega; \mathcal{E}]$. Solving $y[i, \alpha, \Omega; \mathcal{E}] = \psi[r_1, z[i, \alpha, \Omega; \mathcal{E}]$ establishes $r[i, \alpha, \Omega; \mathcal{E}]$. Now, the distribution of optimal individual allocations $\mathcal{E}^*[i, \alpha, \Omega]$ is determined as a solution to either $v[r[i, \alpha, \Omega; \mathcal{E}], y[i, \alpha, \Omega; \mathcal{E}]] = \text{maximum } \{u_1 | x_1 + r[i, \alpha, \Omega; \mathcal{E}]q_1 + c[i] < y[i, \alpha, \Omega; \mathcal{E}]\}$ or $\psi[r[i, \alpha, \Omega; \mathcal{E}], z[i, \alpha, \Omega; \mathcal{E}]] = \text{minimum } \{x_1 + r[i, \alpha, \Omega; \mathcal{E}]q_1 + c[i] | z[i, \alpha, \Omega; \mathcal{E}] < u_1\}$. Once \mathcal{E}^* has been established, the optimal distributions of public investment, income and land values can be computed as $\omega^*[i, \alpha, \Omega] = \omega[i, \alpha, \Omega; \mathcal{E}^*]$, $y^*[i, \alpha, \Omega] = y[i, \alpha, \Omega; \mathcal{E}^*]$ and $r^*[i, \alpha, \Omega] = r[i, \alpha, \Omega; \mathcal{E}^*]$.

The previous remarks suggest the following guidelines for planning an optimal city. The planner organizes production efficiently and requires that the aggregate income be equal to the total value of the urban product, thereby determining the optimal size N^* . He must distribute the public funds available for the provision of public goods over the city following $\omega^*[i, \alpha, \Omega]$. Although individuals should receive the marginal product of their labour, the planner should, excepting zero aversion to inequality, redistribute income as $y^*[i, \alpha, \Omega]$. This, together with the optimal distribution of land values, generates an optimal distribution of utilities $z^*[i, \alpha, \Omega] = z[i, \alpha, \Omega; S^*]$ which, excepting infinite aversion to inequality, is structured according to optimal environmental differences between zones. Therefore, excepting infinite aversion to inequality, some zones in the city will become more attractive than others at the optimum. In consequence, the planner will find it necessary to allocate individuals to zones, thereby determining their place into the chosen scheme of optimal inequality. This could be based on a lottery, a first-come first-serve principle, or whatever. Since he must pre-determine how many to allow in high-utility zones, how many to force in low-utility zones, and so on, he essentially partitions zones into lot sizes. In consequence, the planner will find it necessary to control everything that has to do with the use of land under a finite aversion to inequality. According to (37), once income and location (hence utility) have been determined for everyone, under the optimal distribution of land values, the established differences in locational benefits and costs between adjacent areas precisely offset the income differences necessary to attain the corresponding exogenous utility differences. Finally, since everything else is determined, the consumption of the private good is also

determined here. Nevertheless, since there is no need to correct for the consumption of the private good and since the same holds even if the single composite good is decomposed into many then, in general, decisions of this nature are always left free of interference -- contrary to any decisions about the use of land under a finite aversion to inequality.

Under an infinite aversion to inequality, according to (43), the redistribution of income is such that every inherent location advantage is eliminated. Then it is sufficient for the planner to announce the spatial distribution of optimal tax and subsidy rates, and to keep the urban population size at N^* . Competition between those admitted will ensure (41), together with a uniform distribution of utility. The difference between this optimum and a corresponding equilibrium is the following. Whereas in the latter environmental differences are absorbed by bid-rents through competition, in the former they are absorbed through appropriate income differences imposed before competition.

PRIVATE GOODS, PUBLIC GOODS AND EXTERNALITIES

The optimal conditions (35) for the consumption of the private good may be written as

$$l = \lambda_i \frac{\partial u}{\partial x_i}. \quad (49)$$

On the other hand, using (29) and (33) on (27),

$$-\frac{dD}{d\Omega} = \sum_j n_j \lambda_j \frac{\partial u}{\partial E_i} \frac{\partial E_j}{\partial \omega_i} \equiv \sum_j n_j \phi_{ij}^{\omega} \quad (50)$$

where

$$\phi_{ij}^{\omega} = \lambda_j \frac{\partial u}{\partial E_j} \frac{\partial E_j}{\partial \omega_i}. \quad (51)$$

Now the LHS of (49) denotes the amount of money an individual in zone i is prepared to pay at the optimum for increasing his consumption of the private good by one unit with utility held constant. On the other hand, ϕ_{ij}^{ω} in (51) is called a Lindahl price and denotes the amount of money an individual in zone j is prepared to pay for increasing public investment in zone i by one dollar, thereby increasing the experienced level of spatial public good, with utility held constant. Taking this into account, (50) can be interpreted as follows. The planner, who is an externality generator, may be thought of as being "compensated" (for slightly increasing public investment) on the basis of Lindahl prices at the optimum, in the sense that he can now reduce the aggregate income deficit by an amount equal to the marginal social value of a dollar spent in zone i .

A crucial difference between private and public goods is the following. Under optimal conditions, whereas in the case of a private good individuals with different tastes consume different quantities at the same price, (one dollar), in the case of a public good individuals consume the same quantity (ω_i) at different Lindahl prices. This is essentially the distinction between (49) and (50), illustrated in figures 3 and 4. Consider two individuals, each in a different zone. The demands of these individuals

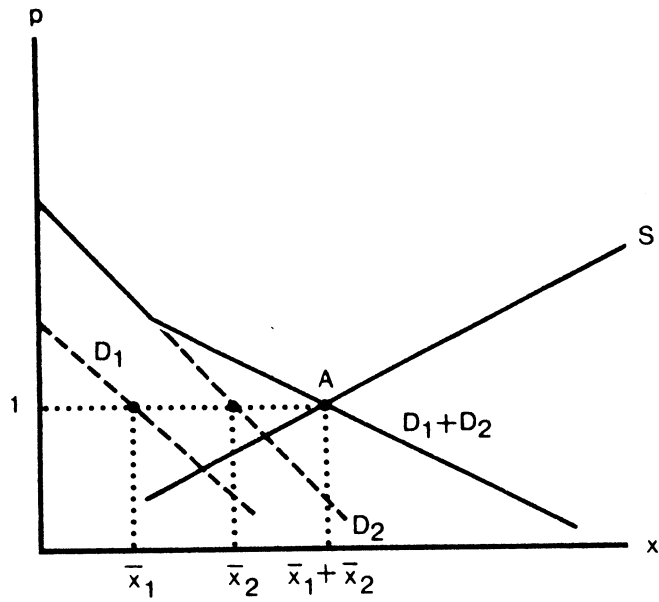


Figure 3: Aggregation of individual demands for a private good.

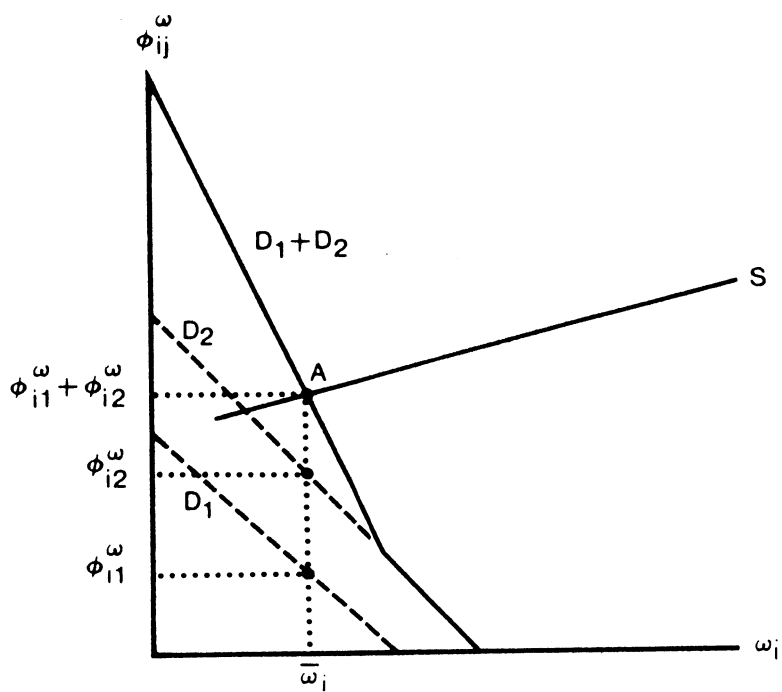


Figure 4: Aggregation of individual demands for public investment.

for the private good appear in figure 3 as lines D_1 and D_2 . Aggregate demand, $D_1 + D_2$, is obtained by horizontal addition of individual demands because the total amount consumed is the sum of individual amounts. Equality of supply S and demand at A determines a common price, l , (which is treated here as exogenous throughout), individual consumptions \bar{x}_1 and \bar{x}_2 and, therefore, total consumption. Notice that the vertical distance under each individual's demand curve reflects the social valuation of the marginal benefit which he gains from consumption, i.e. $\lambda_i \partial u / \partial x_i$ for $i = 1, 2$, and this equals the corresponding marginal cost, l , as in (49). Thus efficiency is ascertained without any need for further correction — as it has already been established by the optimal decentralization rules. Indeed

"... perfect competition among productive enterprises would ensure that goods are produced at minimum costs and are sold at proper marginal costs, with all factors receiving their proper marginal productivities; and ... each individual, in seeking as a competitive buyer to get to the highest level of indifference subject to given prices and tax, would be led as if by an Invisible Hand to the grand solution of the social maximum position. Of course the institutional framework of competition would have to be maintained, and political decision-making would still be necessary, but of a computationally minimum type: namely, algebraic taxes and transfers ... would have to be varied until society is swung to the ethical observer's optimum. The servant of the ethical observer would not have to make explicit decisions about each person's detailed consumption and work; he need only decide about generalized purchasing

power, knowing that each person can be counted to allocate it optimally. In terms of communication theory and game terminology, each person is motivated to do the signalling of his tastes needed to define and reach the attainable bliss point."¹⁶

Similarly to the case of the private good, the individual demands for public expenditure in zone i are represented in figure 4 by lines D_1 and D_2 . Aggregate demand, $D_1 + D_2$, is obtained by vertical addition of individual demands because each consumes the same amount. Individual demand here is a schedule of public expenditure in zone i required at different Lindahl prices. Supply S , on the other hand, is a schedule of optimal public expenditure at different levels of externality benefit. Equality of supply and demand at A determines a level of public expenditure $\bar{\omega}_i$, Lindahl prices ϕ_{i1}^ω and ϕ_{i2}^ω and, therefore, total revenue. This last may be used to "compensate" the externality generator at i , that is, individuals at 1 and 2 would be prepared to pay a total amount equal to $\bar{\omega}_i(\phi_{i1}^\omega + \phi_{i2}^\omega)$ in order to enjoy the effects of public expenditure at i . Notice that the vertical distance below each individual's demand curve reflects the social valuation of the marginal benefit which he incurs from consumption, i.e. $\lambda_j(\partial u/\partial E_j)(\partial E_j/\partial \omega_i)$ for $j = 1, 2$, and this equals the corresponding marginal cost ϕ_{ij}^ω as in (51). Thus efficiency is ascertained. At this point, the sum of social valuations of marginal benefits of individuals equals the total reduction in the aggregate income deficit necessary to maintain utility.

Contrary to the case of the private good, where every individual exposes the relevant part of his tastes by seeking to satisfy (49), individual tastes in the case of the spatial externality are hidden under the

summation symbol of (50). Computing (50) imposes staggering information requirements about the tastes of all individuals. What compounds difficulty even further is the incentive for wrong signalling by the individuals under the summation symbol. To see this, consider the case where the planner collects Lindahl prices in order to raise funds for public investment. Each individual, by understating the effects of the public good, may hope to pay less without reducing significantly the level at which the public good is provided. This is the celebrated "free rider" problem.¹⁷ Of course

"one could imagine every person in the community being indoctrinated to behave like a 'parametric decentralized bureaucrat' who reveals his preferences by signalling in response to price parameters or Lagrangean multipliers, to questionnaires, or to other devices. But there is still this fundamental technical difference going to the heart of the whole problem of social economy: by departing from his indoctrinated rules, any one person can hope to snatch some selfish benefit in a way not possible under the self-policing competitive pricing of private goods; and the 'external economies' or 'jointness of demand' intrinsic to the very concept of collective goods and governmental activities makes it impossible for the grand ensemble of optimizing equations to have that special pattern of zeros which makes laissez-faire competition even theoretically possible as an analogue computer."¹⁸

According to (4), the level of the public good experienced in a zone does not only depend on the spatial distribution of public investment, but also on the spatial distribution of population which determines how con-

gestion affects the public good in that zone. Congestion is a negative spatial externality. Using (39) and (46) and assuming no re-distribution, the optimal tax to be paid by someone in zone i for his contribution to the spatial externality is

$$n_i \phi_i^n = - \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\Delta E_j}{n_i} \equiv - \sum_j n_j \phi_{ij}^n, \quad (52)$$

where

$$\phi_{ij}^n = \lambda_j \frac{\partial u}{\partial E_j} \frac{\Delta E_j}{n_i} \quad (53)$$

denotes the amount of money an individual in zone j is prepared to pay at the optimum for increasing the number of individuals in zone i by one, thereby altering the experienced level of spatial externality, with utility held constant. There are strong similarities between (51) and (52). Indeed, the quantities ϕ_{ij}^n may be thought of as negative Lindahl prices. Therefore an individual in zone i , as externality generator, may be thought of as being "compensated" by the aggregate sum that these prices generated at the optimum.

Figure 5 describes the case of two individuals. The individual "demands" for a negative spatial externality are represented by lines D_1 and D_2 and, as in figure 4, aggregate "demand" $D_1 + D_2$ is obtained by vertical addition. Individual "demand" here is a schedule of potential compensations per externality generator necessary to maintain utility at different levels of land-use intensity in zone i . "Supply" S , on the other hand, is a schedule of optimal land-use intensity in zone i at different levels of externality damage. Equality of "supply" and "demand" at A determines a

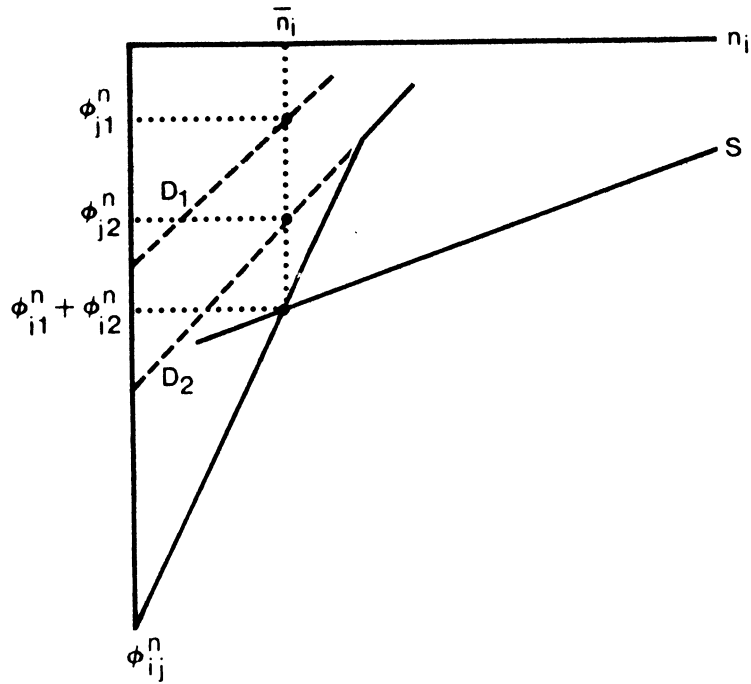


Figure 5: Aggregation of individual "demands" for a negative externality.

"consumption" \bar{n}_i , individual "prices" ϕ_{i1}^n and ϕ_{i2}^n and, therefore, total "revenue". This last is used to "compensate" the externality generator at i , that is, individuals at i must pay each a total amount equal to $-(\phi_{i1}^n + \phi_{i2}^n)$ in order to account for their damage on the two others. Notice that the vertical distance above each individual's "demand" curve reflects the social valuation of the marginal cost which he incurs from "consumption", i.e. $\lambda_j (\partial u / \partial E_j) \Delta E_j$ for $j = 1, 2$, and this equals the corresponding potential marginal benefit ϕ_{ij}^n as in (53). Thus efficiency is ascertained. At this point, the sum of social valuations of net marginal costs of individuals equals the total potential compensation to be paid, one for which the individual at i must be accountable irrespectively of whether the others are compensated or not.¹⁹ As in the case of public investment, there is an incentive for wrong signalling by the individuals under the summation symbol of (52). For example, suppose that the recipients of an externality are not compensated. Then they must have an incentive to overstate the effects of the spatial externality. In this manner one hopes that he will cause an increase (decrease) in the output of the positive (negative) externality, thereby increasing his welfare. False signalling here produces an output more (less) than the optimal.

PUBLIC GOODS THAT OCCUPY LAND

Until now the analysis has been simplified by assuming that only investment was necessary for the production of the public good. When the need for land is recognized, (3) may be modified as

$$E_{ij} = f_{ij}[n, G[H_i, \omega_i]] \quad (54)$$

where G is a production function for the public good which, for zone i , depends on the amount of land allocated to the public good in this zone, H_i , and on the corresponding public investment. Clearly, $\partial f_{ij} / \partial G > 0$, $\partial G / \partial H_i > 0$, $\partial G / \partial \omega_i > 0$ and $0 < H_i < \Delta Q_{i-1}$. That is, the spatial externality emitted from zone i to those in zone j increases when the level of the public good in zone i increases; the level of the public good, in turn, increases when land and/or investment allocated to its production increase; and the (non-negative) amount of land allocated to the production of a public good in zone i cannot be larger than the total available in that zone. Aggregating over i , we obtain

$$E_j[n, G[H, \omega]] = \sum_{i=-b}^b E_{ij} \quad (55)$$

where $G[H, \omega] \equiv (G[H_{-b}, \omega_{-b}], \dots, G[H_b, \omega_b])$.

Every residential zone in \mathcal{J} may now contain two, rather than a single land-use. Consequently, the amount of land consumed per capita in zone i becomes

$$q_i = (\Delta Q_{i-1} - H_i) / n_i. \quad (56)$$

Furthermore, since land occupied by the public good over \mathcal{J} is costly, the problem of the planner is modified as

$$D[r, z; \alpha, \Omega] = \text{minimum} \left\{ \sum_{j=1}^b (r_j^0 H_j + n_j \psi_j) - X[N] | z_i \right. \\ \left. < v[v[r_i, \psi_i]; \alpha] \text{ for } i = 1, \dots, b \text{ and } \sum_{j=1}^b \omega_j < \Omega \right\}, \quad (57)$$

where r_j^0 is the land rent associated with the production of the public good at j . Clearly, in addition to the other variables of problem (22), the aggregate income deficit must now be minimized with respect to $H \equiv (H_1, \dots, H_b)$. The new conditions regulating the allocation of land between residential and public land-use read

$$r_i^0 + n_i r_i \frac{\partial q}{\partial H_i} - n_i \lambda_i \frac{\partial u}{\partial q_i} \frac{\partial q}{\partial H_i} - \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\partial E_j}{\partial H_i} = 0 \\ \text{for } i = 1, \dots, b. \quad (58)$$

When a unit of land in zone i is taken from residential land-use and given to public land-use, there is a marginal cost from payment for the use of this additional unit by the public sector (first term in (58)); a corresponding marginal benefit from savings in the residential sector -- now that the per capita consumption of land there has been reduced by $\partial q / \partial H_i$, hence that the savings in residential rent per capita is $r_i \partial q / \partial H_i$ (second term); a marginal cost of reducing utility for everyone in zone i because of increased congestion generated by the residential land-use contraction there (third term); and a corresponding marginal benefit of increasing utility for everyone in \mathcal{J} because of the improvements in the quality of the environment generated by the public land-use expansion in zone i (fourth term).

At the optimum, the marginal social cost of reducing utility for everyone in zone i because of increased congestion should equal the corresponding marginal social benefit of increasing utility for everyone in \mathcal{J} because of improved quality of the environment. Otherwise, social welfare would not be maximized at the optimum — a contradiction. Therefore, at the optimum, it must be that

$$- n_i \lambda_i \frac{\partial u}{\partial q_i} \frac{\partial q}{\partial H_i} - \sum_j n_j \lambda_j \frac{\partial u}{\partial E_j} \frac{\partial E_j}{\partial H_i} = 0. \quad (59)$$

Moreover, using (56), notice that

$$n_i \frac{\partial q}{\partial H_i} = -1 \quad (60)$$

i.e. that the loss of land in the residential sector because of transferring one unit to the public sector is precisely one unit. Taking into account (59) and (60), (58) implies

$$r_i^0 = r_i, \quad (61)$$

in other words, the two types of land-use should be charged the same land rents everywhere in \mathcal{J} . For arbitrary degree of aversion to inequality, we know that such rents must be imposed by the planner. In the special case of infinite aversion to inequality, however, it is sufficient for the planner to fix the urban population at the optimal size and to announce the spatial distribution of optimal tax and subsidy rates ϕ_i^n and ϕ_i^i . Since the optimal

re-distribution of income created by these rates is such that every inherent location advantage is eliminated, competition between those admitted will ensure bid-rents that satisfy both (26) and (41) — as required for optimality. These bid-rents, in turn, will guide the public land-use as well because of (61). That is, in the case of infinite aversion to inequality, the optimal urban form is established by competition for land between the residential and public sectors everywhere over J .

FOOTNOTES

1. See Y.Y. Papageorgiou, "Spatial Externalities I: Theory", Annals, Association of American Geographers, Vol. 68(1978), pp. 465-76 and Y.Y. Papageorgiou, "Spatial Externalities II: Applications", Vol. 68(1978), pp. 477-92.
2. Papageorgiou, op.cit., footnote 1, p. 465.
3. H.W. Kuhn and A.W. Tucker, "Nonlinear Programming", Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability (Berkeley: University of California Press, 1950), pp. 481-92.
4. R.H. Strotz, "Urban Transportation Parables", in J. Margolis, ed., The Public Economy of Urban Communities (Baltimore: Johns Hopkins University Press, 1965).
5. Y.Y. Papageorgiou and M. O'Kelly, "Guidelines for Designing an Abstract New Town", Geographical Analysis, Vol. 16(1984), pp. 97-120.
6. It is also possible to define E_{ij} as depending upon ω rather than merely upon ω_i . In other words to recognize the role of the other potential opportunities on how important is public investment in zone i . Since however no further intuition emerges, the simpler (3) is chosen for convenience.
7. For a proof see K.J. Arrow, F. Hahn, General Competitive Analysis (San Francisco: Holden-Day, 1971), p. 81.
8. Condition (12) is the Kuhn-Tucker condition for inequality constraints. The Lagrangean function of problem (9) is

$$L_i = x_i + r_i q_i + c[i] + \lambda_i (z_i - u_i).$$

This function depends on both continuous and discrete variables. Condition (13) is obtained in the standard manner from $\partial L_i / \partial x_i = 0$ because x_i is continuous. However, for discrete variables, the difference $\Delta f[x] \equiv f[x+1] - f[x]$ replaces an ordinary derivative df/dx , while $\Delta f[x, y] \equiv f[x+1, y] - f[x, y]$ replaces a partial derivative $\partial f / \partial x$. Thus condition (14) is obtained from $\Delta L = 0$ because n_i is discrete. Partial differencing, instead of partial differentiation, renders the last condition only approximate, the approximation improving as the number of individuals in zone i becomes larger. Finally notice that (14) implies a parametric treatment of the public good by individuals.

9. This is obtained from $\Delta L = 0$ because location is a discrete variable. As with (14), this condition is only approximate, the approximation improving with finer partitioning of the urban area.
10. Conditions (23) and (24) are the Kuhn-Tucker conditions for inequality constraints. The Lagrangean function of problem (22) is

$$L = \sum_j n_j \psi_j - X + \sum_j \mu_j (z_j - v_j) + v (\sum_j \omega_j - \Omega).$$

Conditions (25) and (27) are obtained upon partial differentiation of the Lagrangean function with respect to x_i and ω_i respectively. Condition (26), on the other hand, is obtained upon partial differencing with respect to n_i and is therefore only approximate. This condition uses the rule

$$\Delta X = \Delta X \Delta N = \Delta X$$

$$n_i \quad N \quad n_i \quad N$$

because $\Delta N = 1$; and the rule

$$\Delta f[x]g[x] = g[x]\Delta f[x] + f[x + 1]\Delta g[x].$$

$$x \quad x \quad x$$

11. The general principles governing these corrective taxes and subsidies have been proposed by A.C. Pigou, The Economics of Welfare (London: MacMillan, 1920).
12. The actual difference is $\Delta q \Delta r$. As the partitioning of population n_i becomes finer, this quantity tends to zero.
13. J.A. Mirrlees, "The Optimum Town", Swedish Journal of Economics, Vol. 74(1972), pp. 114-35.
14. D. Levhari, Y. Oron and D. Pines, "A Note on Unequal Treatment of Equals in an Urban Setting", Journal of Urban Economics, Vol. 5(1978), pp. 278-84.
15. Lagrangean function of the problem of the individual now becomes

$$L_i = x_i + r_i q_i + c[i] + n_i \phi_i^n + i \phi_i^i - A + \lambda_i (z_i - u_i).$$

Differencing with respect to n_i and i yields

$$(r_i - \lambda_i \frac{\partial u}{\partial q}) \Delta q + \phi_i^n = 0$$

$$i \quad n_i$$

$$q_i \Delta r + \Delta c + \phi_i^i + \lambda_i (\Delta z - \frac{\partial u}{\partial E} \Delta E) = 0$$

$$i \quad i \quad i \quad i \quad i$$

respectively, because individuals take both marginal rates

parametrically by assumption. These are equivalent to the corresponding optimal decisions (36) and (37).

16. P. Samuelson, "The Pure Theory of Public Expenditure", Review of Economics and Statistics, Vol. 36(1954), p. 388.
17. There are ingenious rules in existence, called "optimal planning procedures", aiming to bypass the problem of false signalling. For a good review see H. Tulken, "Dynamic Processes for Public Goods: An Institution-Oriented Survey", Journal of Public Economics, Vol. 9(1978), pp. 163-201.
18. Samuelson, op.cit., footnote 16.
19. It should be remembered that the correction principle using optimal taxes and subsidies pertains only to those who generate the externality. Whether the recipients of the externality should or should not be compensated is irrelevant to the correction principle. Compensation is only relevant to distributive issues, and is therefore implicitly subsumed in the structure of the marginal rate (43) which determines the degree of optimal inequality corresponding to the theory of justice professed by the planner.
20. These conditions are obtained upon partial differentiation of the Lagrangean function

$$L = \sum_j (r_j^0 H_j + n_j \psi_j) - X + \sum_j \mu_j (z_j - v_j) + v(\sum_j \omega_j - \Omega)$$

with respect to H_i and taking into account (33).