



HAL
open science

Some inverse problem remarks of a continuous-in-time financial model in $L^1([t_1, \Theta_{\max}])$

Tarik Chakkour

► **To cite this version:**

| Tarik Chakkour. Some inverse problem remarks of a continuous-in-time financial model in $L^1([t_1, \Theta_{\max}])$. 2017. hal-01543421v1

HAL Id: hal-01543421

<https://hal.science/hal-01543421v1>

Preprint submitted on 20 Jun 2017 (v1), last revised 24 Aug 2023 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Some inverse problem remarks of a continuous-in-time financial model in $\mathbb{L}^1([t_I, \Theta_{\max}])$

Tarik Chakkour

Univ. Bretagne-Sud, UMR 6205, LMBA, F-56000 Vannes, France

May 31, 2017

Abstract. In the paper we are going to introduce operator that is involved in the inverse problem of the continuous-in-time financial model. We discuss this inverse problem in the space of integrable functions over \mathbb{R} having their support. Then we will give some remarks for not functionality of the model for a given Repayment Pattern Density γ .

Keywords: Inverse problem; integral operators; financial model, densities and measures.

1 Introduction

Over the last two decades, we have seen significant progress in econometric theory, computational and estimation methods to test and implement continuous-time models. The continuous-time approach in these areas has produced models with a rich variety of testable implications. It is useful to begin our survey with an overview of some of the major developments in the field. There are many references that deal with research in finance and the way in which finance research was viewed by the practitioners. The book [7] by Merton laid the foundations for the development of intertemporal asset pricing theory. The purpose of the paper [4] is to build a bridge between continuous time models, which are central in the modern finance literature, and (weak) GARCH processes in discrete time, which often provide parsimonious descriptions of the observed data.

In the paper [5], we build a continuous-in-time model which is designed to be used for the finances of public institutions. This model is based on using measures over time interval to describe loan scheme, reimbursement scheme and interest payment scheme; and, on using mathematical operators. At the same time, we implement in [2] some mathematical operators as convolution and primitive for continuous-in-time financial model. This development is given in form of API (Application Programming Interface). The originality of this API lie in the fact that it will be used by the company MGDIS.

The understanding of most inverse financial problems is in so rapid a state of development that a review of the subject will inevitably date rapidly. Moreover, it is important to quote some contributions in the field. For example, the paper [6] presents a phase retrieval problem and a specific inverse option pricing problem. In the paper [8], the author deals with the time-harmonic acoustic obstacle scattering problem with Dirichlet boundary condition. In the paper [1], the authors have developed a series of novel variational source conditions for Linear Inverse Problems in Hilbert Spaces.

The questions of the inverse problem in Schwartz space are currently investigated in paper [3] for ill-posed problems. We investigate also in the paper [3] the properties of operators with determining them spectrum. For completing the history of the inverse problem of the model, we discuss in [3] the inverse problem stability in Hilbert space $\mathbb{L}^2([t_I, \Theta_{\max}])$. The aim of this paper is to formulate, and to attempt to construct a solution in the space of integrable functions $\mathbb{L}^1([t_I, \Theta_{\max}])$ having their support in $[t_I, \Theta_{\max}]$. In the sequel, we show in examples that Repayment Pattern Density γ does not hold the expectation given in theory.

2 Inverse problem of the model in $\mathbb{L}^1([t_I, \Theta_{\max}])$

Denoting $\mathcal{C}_c([t_I, \Theta_{\max}])$ the continuous functions space $\mathcal{C}_c([t_I, \Theta_{\max}])$ defined over a time interval $[t_I, \Theta_{\max}]$ with the usual norm $\|\cdot\|_{L^\infty([t_I, \Theta_{\max}])}$ is given by:

$$\|\psi\|_{L^\infty([t_I, \Theta_{\max}])} = \sup_{t \in [t_I, \Theta_{\max}]} \{|\psi(t)|\}. \quad (1)$$

For any interval $[t_1, t_2]$, $t_2 > t_1$, $\mathbb{L}^1([t_1, t_2])$ stands for the space of integrable functions over \mathbb{R} having their support in $[t_1, t_2]$. For a positive number Θ_γ such that $\Theta_\gamma < \Theta_{\max} - t_I$, we set the Repayment Pattern γ such that:

$$\gamma \in \mathbb{L}^\infty([0, \Theta_\gamma]). \quad (2)$$

Linear operator \mathcal{L} acting on Loan Density $\kappa_E \in \mathbb{L}^1([t_I, \Theta_{\max} - \Theta_\gamma]) \cap \mathcal{C}_c([t_I, \Theta_{\max} - \Theta_\gamma])$ is defined as:

$$\mathcal{L}[\kappa_E](t) = \kappa_E(t) - (\kappa_E \star \gamma)(t) - \alpha \int_{t_I}^t (\kappa_E - \kappa_E \star \gamma)(s) ds. \quad (3)$$

Lemma 2.1. *Assuming that density γ satisfies (2), and following assumption*

$$\left(\frac{1}{1 - \mathcal{F}(\gamma)} \right) \in \mathbb{L}^\infty(\mathbb{R}). \quad (4)$$

Let $\sigma - \mathcal{D}[\rho_K^I]$ be a positive function and rate α be a negative real. Density κ_E is expressed in terms of σ as:

$$\kappa_E = \mathcal{F}^{-1} \left(\frac{\mathcal{F} \left(\sigma - \mathcal{D}[\rho_K^I] + \alpha \int_{t_I}^{\bullet} (\sigma(s) - \mathcal{D}[\rho_K^I](s)) e^{\alpha(\bullet-s)} ds \right)}{1 - \mathcal{F}(\gamma)} \right). \quad (5)$$

Proof. We recall that we have shown in [5] on the space of square-integrable functions over \mathbb{R} having their support in $[t_I, \Theta_{\max}]$ the following equality:

$$\kappa_E(t) - \kappa_E \star \gamma(t) = \mathcal{L}[\kappa_E](t) + \alpha \int_{t_I}^t \mathcal{L}[\kappa_E](s) e^{\alpha(t-s)} ds, \quad (6)$$

which stays true in $\mathbb{L}^1([t_I, \Theta_{\max}])$ due to the integration by parts (see equality (3.12) of Lemma 3.2). Furthermore, since the Repayment Pattern Density γ is a non-negative function which support is included in $[0, \Theta_\gamma]$ with total mass equals 1, we get:

$$\kappa_E(t) - (\kappa_E \star \gamma)(t) = \int_0^{\Theta_\gamma} \gamma(y) (\kappa_E(t) - \kappa_E(t-y)) dy. \quad (7)$$

Using Heine theorem which states that a continuous function κ_E on compact interval $[t_I, \Theta_{\max} - \Theta_\gamma]$ is uniformly continuous:

$$\begin{aligned} |\kappa_E(t) - (\kappa_E \star \gamma)(t)| &\leq \sup_{y \in [0, \Theta_\gamma]} \left\{ |\gamma(y)| \right\} \int_0^{\Theta_\gamma} y dy, \\ &\leq \frac{\Theta_\gamma^2}{2} \|\gamma\|_{L^\infty([0, \Theta_\gamma])}. \end{aligned} \quad (8)$$

Inequality (8) means that density $\kappa_E - \kappa_E \star \gamma$ is bounded on interval $[t_I, \Theta_{\max}]$. From this and according to equality (6) it follows that:

$$\sigma(t) - \mathcal{D}[\rho_{\mathcal{K}}^I](t) + \alpha \int_{t_I}^t (\sigma - \mathcal{D}[\rho_{\mathcal{K}}^I])(s) e^{\alpha(t-s)} ds \leq \frac{\Theta_{\max}^2}{2} \|\gamma\|_{L^\infty([0, \Theta_\gamma])}. \quad (9)$$

Multiplying inequality (9) by $e^{-\alpha t}$, we get:

$$(\sigma - \mathcal{D}[\rho_{\mathcal{K}}^I])(t) e^{-\alpha t} \leq \frac{\Theta_{\max}^2}{2} \|\gamma\|_{L^\infty([0, \Theta_\gamma])} e^{-\alpha t} - \alpha \int_{t_I}^t (\sigma - \mathcal{D}[\rho_{\mathcal{K}}^I])(s) e^{-\alpha s} ds. \quad (10)$$

Applying Gronwall lemma to inequality (10), we get:

$$\begin{aligned} \forall t \in [t_I, \Theta_{\max}], (\sigma - \mathcal{D}[\rho_{\mathcal{K}}^I])(t) e^{-\alpha t} &\leq \frac{\Theta_{\max}^2}{2} \|\gamma\|_{L^\infty([0, \Theta_\gamma])} e^{-\alpha t} + \int_{t_I}^t (-\alpha) a e^{-\alpha s} \exp\left(-\int_s^t \alpha du\right) ds, \\ &\leq \frac{\Theta_{\max}^2}{2} \|\gamma\|_{L^\infty([0, \Theta_\gamma])} e^{-\alpha t} - a\alpha \int_{t_I}^t \exp\left(-\alpha s - \alpha(t-s)\right) ds, \\ &\leq \frac{\Theta_{\max}^2}{2} \|\gamma\|_{L^\infty([0, \Theta_\gamma])} e^{-\alpha t} - a\alpha e^{-\alpha t} (t - t_I), \\ &\leq \frac{\Theta_{\max}^2}{2} \|\gamma\|_{L^\infty([0, \Theta_\gamma])} e^{-\alpha t} (1 - \alpha(t - t_I)). \end{aligned} \quad (11)$$

That is,

$$\forall t \in [t_I, \Theta_{\max}], (\sigma - \mathcal{D}[\rho_{\mathcal{K}}^I])(t) \leq \frac{\Theta_{\max}^2}{2} \|\gamma\|_{L^\infty([0, \Theta_\gamma])} (1 - \alpha(t - t_I)). \quad (12)$$

The function defined by $t \rightarrow 1 - \alpha(t - t_I)$ is in $\mathbb{L}^2([t_I, \Theta_{\max}])$ because of:

$$\begin{aligned} \|t \rightarrow 1 - \alpha(s - t_I)\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} &= \int_{t_I}^{\Theta_{\max}} (1 - \alpha(s - t_I))^2 ds, \\ &= (\Theta_{\max} - t_I) \left[1 + \frac{(\alpha(\Theta_{\max} - t_I))^2}{3} - \alpha(\Theta_{\max} - t_I) \right]. \end{aligned} \quad (13)$$

From this and according to inequality (12), operator \mathcal{L} is in $\mathbb{L}^2([t_I, \Theta_{\max}])$. As $\mathcal{L} \in \mathbb{L}^2([t_I, \Theta_{\max}])$, we will show that:

$$\mathcal{L}[\kappa_E] + \alpha \int_{t_I}^{\bullet} \mathcal{L}[\kappa_E](s) e^{\alpha(\bullet-s)} ds \in \mathbb{L}^2([t_I, \Theta_{\max}]). \quad (14)$$

Indeed, for all reals $t \leq \Theta_{\max}$ we have:

$$\begin{aligned} \left\| \int_{t_I}^{\bullet} \mathcal{L}[\kappa_E](s) e^{\alpha(\bullet-s)} ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} &= \sqrt{\int_{t_I}^{\Theta_{\max}} \left(\int_{t_I}^t \mathcal{L}[\kappa_E](s) e^{\alpha(t-s)} ds \right)^2 dt}, \\ &\leq \sqrt{\Theta_{\max} - t_I} \times \left| \int_{t_I}^{\Theta_{\max}} \mathcal{L}[\kappa_E](s) e^{\alpha(\Theta_{\max}-s)} ds \right|, \end{aligned} \quad (15)$$

In addition we get following inequality $|e^{\alpha(\Theta_{\max}-s)}| \leq e^{|\alpha|(\Theta_{\max}-t_I)}$ for all $s \in [t_I, \Theta_{\max}]$. This yields,

$$\left\| \int_{t_I}^{\bullet} \mathcal{L}[\kappa_E](s) e^{\alpha(\bullet-s)} ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq \sqrt{\Theta_{\max} - t_I} \times e^{|\alpha|(\Theta_{\max} - t_I)} \times \|\mathcal{L}[\kappa_E]\|_{\mathbb{L}^1([t_I, \Theta_{\max}])}. \quad (16)$$

Next, we use Cauchy-Schwarz inequality to obtain the following inequality:

$$\|\mathcal{L}[\kappa_E]\|_{\mathbb{L}^1([t_I, \Theta_{\max}])} \leq \sqrt{\Theta_{\max} - t_I} \times \|\mathcal{L}[\kappa_E]\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}. \quad (17)$$

Inequalities (16) and (17) imply:

$$\left\| \mathcal{L}[\kappa_E] + \alpha \int_{t_I}^{\bullet} \mathcal{L}[\kappa_E](s) e^{\alpha(\bullet-s)} ds \right\|_{\mathbb{L}^2([t_I, \Theta_{\max}])} \leq (1 + |\alpha|(\Theta_{\max} - t_I)) \|\mathcal{L}[\kappa_E]\|_{\mathbb{L}^2([t_I, \Theta_{\max}])}, \quad (18)$$

achieving the proof of (14). Applying Fourier Transform to relation (6) and using assumption (4), Loan Density κ_E is given in terms of Algebraic Spending Density σ according to relation (27), completing the proof of the Lemma. \square

Remark 2.2. We will show in this remark that even Repayment Pattern Density γ get a lower bound, we could not apply Lemma 2.1. Indeed, if density γ admits a lower bound over its support, then there exists a positive real M satisfying:

$$\inf_{z \in [0, \Theta_\gamma]} \left\{ |\gamma(z)| \right\} = M. \quad (19)$$

We will show that under assumption (19),

$$\left(\frac{1}{1 - \mathcal{F}(\gamma)} \right) \notin \mathbb{L}^\infty(\mathbb{R}). \quad (20)$$

We have,

$$\begin{aligned} \left| \frac{1}{1 - \mathcal{F}(\gamma)(\xi)} \right|^2 &= \frac{1}{\left| \int_0^{\Theta_\gamma} \gamma(x) (1 - e^{-ix\xi}) dx \right|^2}, \\ &\leq \frac{1}{M^2} \times \frac{1}{\left| \int_0^{\Theta_\gamma} (1 - e^{-ix\xi}) dx \right|^2}. \end{aligned} \quad (21)$$

The integration of the exponential function $x \rightarrow e^{-ix\xi}$ states that:

$$\int_0^{\Theta_\gamma} (1 - e^{-ix\xi}) dx = \Theta_\gamma + \frac{(e^{-i\Theta_\gamma\xi} - 1)}{i\xi}. \quad (22)$$

We take the square root of the modulus

$$\left| \int_0^{\Theta_\gamma} (1 - e^{-ix\xi}) dx \right|^2 = \Theta_\gamma^2 + \frac{2(1 - \cos(\Theta_\gamma\xi))}{\xi^2} - \frac{2\Theta_\gamma \sin(\Theta_\gamma\xi)}{\xi} \quad (23)$$

Noticing that this function

$$\xi \rightarrow \frac{1}{\Theta_\gamma^2 + \frac{2(1-\cos(\Theta_\gamma\xi))}{\xi^2} - \frac{2\Theta_\gamma \sin(\Theta_\gamma\xi)}{\xi}}$$

is pair. For that we will show that is $\notin \mathbb{L}^\infty(\mathbb{R}^+)$ in order to prove (20). Functions $\sin(\Theta_\gamma\xi)$ and $\cos(\Theta_\gamma\xi)$ are Taylor expanded in 0 respectively until the order 3 and 2 to obtain following inequalities:

$$\sin(\Theta_\gamma\xi) \leq \Theta_\gamma\xi - \frac{(\Theta_\gamma\xi)^3}{6}, \cos(\Theta_\gamma\xi) \leq 1 - \frac{(\Theta_\gamma\xi)^2}{2}. \quad (24)$$

From this, we get for all positive real ξ :

$$\frac{1}{\Theta_\gamma^2 + \frac{2(1-\cos(\Theta_\gamma\xi))}{\xi^2} - \frac{2\Theta_\gamma \sin(\Theta_\gamma\xi)}{\xi}} \leq \frac{3}{\Theta_\gamma^4 \xi^2}. \quad (25)$$

Since function $\xi \rightarrow \frac{3}{\Theta_\gamma^4 \xi^2}$ is not bounded, then it is not in $\mathbb{L}^\infty(\mathbb{R}^+)$, completing the proof of (20).

Lemma 2.3. If density γ satisfies (2), and satisfies following assumption

$$\left(\frac{1}{1 - \mathcal{F}(\gamma)} \right) \in \mathbb{L}^2(\mathbb{R}), \quad (26)$$

then, Loan Density κ_E can be expressed in terms of σ as:

$$\kappa_E = \mathcal{F}^{-1} \left(\frac{\mathcal{F} \left(\sigma - \mathcal{D}[\rho_K^I] + \alpha \int_{t_I}^\bullet (\sigma(s) - \mathcal{D}[\rho_K^I](s)) e^{\alpha(\bullet-s)} ds \right)}{1 - \mathcal{F}(\gamma)} \right). \quad (27)$$

Proof. First, showing that function $\int_{t_I}^\bullet (\sigma(s) - \mathcal{D}[\rho_K^I](s)) e^{\alpha(\bullet-s)} ds$ is integrable over \mathbb{R} having their support in $[t_I, \Theta_{\max}]$. On other words,

$$\int_{t_I}^\bullet (\sigma(s) - \mathcal{D}[\rho_K^I](s)) e^{\alpha(\bullet-s)} ds \in \mathbb{L}^1(\mathbb{R}). \quad (28)$$

For all reals $t \leq \Theta_{\max}$, we have:

$$\begin{aligned} \left\| \int_{t_I}^\bullet \mathcal{L}[\kappa_E](s) e^{\alpha(\bullet-s)} ds \right\|_{\mathbb{L}^1([t_I, \Theta_{\max}])} &= \int_{t_I}^{\Theta_{\max}} \left| \int_{t_I}^t \mathcal{L}[\kappa_E](s) e^{\alpha(t-s)} ds \right| dt, \\ &\leq (\Theta_{\max} - t_I) \times e^{|\alpha|(\Theta_{\max}-s)} \|\mathcal{L}[\kappa_E]\|_{\mathbb{L}^1([t_I, \Theta_{\max}])}. \end{aligned} \quad (29)$$

Noticing that Linear operator \mathcal{L} acting on Loan Density $\kappa_E \in \mathbb{L}^1([t_I, \Theta_{\max} - \Theta_\gamma]) \cap \mathcal{C}_c([t_I, \Theta_{\max} - \Theta_\gamma])$ is defined on $\mathbb{L}^1([t_I, \Theta_{\max} - \Theta_\gamma])$. From this the Fourier transform of function in $\mathbb{L}^1([t_I, \Theta_{\max} - \Theta_\gamma])$ is in $\mathbb{L}^\infty(\mathbb{R})$. Formally, we get

$$\mathcal{F} \left(\sigma - \mathcal{D}[\rho_K^I] + \alpha \int_{t_I}^\bullet (\sigma(s) - \mathcal{D}[\rho_K^I](s)) e^{\alpha(\bullet-s)} ds \right) \in \mathbb{L}^\infty(\mathbb{R}). \quad (30)$$

Under assumption (26), since the product of this function in $\mathbb{L}^\infty(\mathbb{R})$ by function $\frac{1}{1 - \mathcal{F}(\gamma)}$ which is in $\mathbb{L}^2(\mathbb{R})$, is in space $\mathbb{L}^2(\mathbb{R})$. Hence, the proof of the lemma is achieved. \square

Remark 2.4. In this remark, we check lemmas 2.1 and 2.3 for a constant piecewise density γ , which is $\frac{1}{\Theta_\gamma}$ over $[0, \Theta_\gamma]$ and is zero outside, i.e:

$$\gamma = \frac{1}{\Theta_\gamma} \mathbb{1}_{[0, \Theta_\gamma]}. \quad (31)$$

We recall that Repayment Pattern Density γ is a non-negative density with total mass which equals 1:

$$\int_{-\infty}^{+\infty} \gamma = 1. \quad (32)$$

We notice that Density γ given by (31) satisfies definition (32). The aim here is to show that this density γ does not satisfy assumption (4). Indeed, when ξ goes to 0, modulus of $\frac{1}{1 - \mathcal{F}(\gamma)}$

$$\left| \frac{1}{1 - \mathcal{F}(\gamma)(\xi)} \right| = \frac{|\xi \Theta_\gamma|}{\sqrt{(\xi \Theta_\gamma - \sin(\xi \Theta_\gamma))^2 + (\cos(\xi \Theta_\gamma) - 1)^2}} \quad (33)$$

converges to

$$\frac{1}{\frac{(\xi \Theta_\gamma)^2}{4} \left(1 + \frac{(\xi \Theta_\gamma)^4}{9} \right)}. \quad (34)$$

Consequently, $\left| \frac{1}{1 - \mathcal{F}(\gamma)(\xi)} \right|$ diverges around origine, proving that:

$$\left(\frac{1}{1 - \mathcal{F}(\gamma)} \right) \notin \mathbb{L}^\infty(\mathbb{R}). \quad (35)$$

Taking now the same Repayment Pattern Density γ defined by equality (31). This density γ does not satisfy assumption (26). Indeed, around infinity, we get the following equivalence:

$$\left| \frac{1}{1 - \mathcal{F}(\gamma)(\xi)} \right|^2 = \frac{(\xi \Theta_\gamma)^2}{(\xi \Theta_\gamma - \sin(\xi \Theta_\gamma))^2 + (\cos(\xi \Theta_\gamma) - 1)^2} \simeq 1, \quad (36)$$

which is not integrable between a positive number A and infinity. Consequently, we get:

$$\left(\frac{1}{1 - \mathcal{F}(\gamma)} \right) \notin \mathbb{L}^2(\mathbb{R}). \quad (37)$$

Conflict of Interests

The authors declare that there is no conflict of interests.

References

- [1] Roman Andreev, Peter Elbau, Maarten V de Hoop, Lingyun Qiu, and Otmar Scherzer. Generalized convergence rates results for linear inverse problems in hilbert spaces. *Numerical Functional Analysis and Optimization*, 36(5):549–566, 2015.
- [2] Tarik Chakkour. Implementing some mathematical operators for a continuous-in-time financial model. *Engineering Mathematics Letters*, 2017:Article-ID, 2017.

- [3] Tarik Chakkour. Some notes about the continuous-in-time financial model. *Abstract And Applied Analysis*, 2017, 2017.
- [4] Feike C Drost and Bas JM Werker. Closing the garch gap: Continuous time garch modeling. *Journal of Econometrics*, 74(1):31–57, 1996.
- [5] Emmanuel Frenod and Tarik Chakkour. A continuous-in-time financial model. *Mathematical Finance Letters*, 2016:Article–ID, 2016.
- [6] Bernd Hofmann, Barbara Kaltenbacher, Christiane Poeschl, and Otmar Scherzer. A convergence rates result for tikhonov regularization in banach spaces with non-smooth operators. *Inverse Problems*, 23(3):987, 2007.
- [7] R.C. Merton. *Continuous-Time Finance*. Blackwell, 1st edition, 1992.
- [8] Roland Potthast. Fréchet differentiability of boundary integral operators in inverse acoustic scattering. *Inverse Problems*, 10(2):431, 1994.