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► **To cite this version:**

Bernadette Mathieu. Fuzzy expected utility. [Research Report] Institut de mathématiques économiques (IME). 1984, 16 p., bibliographie. hal-01542423

HAL Id: hal-01542423

<https://hal.science/hal-01542423>

Submitted on 19 Jun 2017

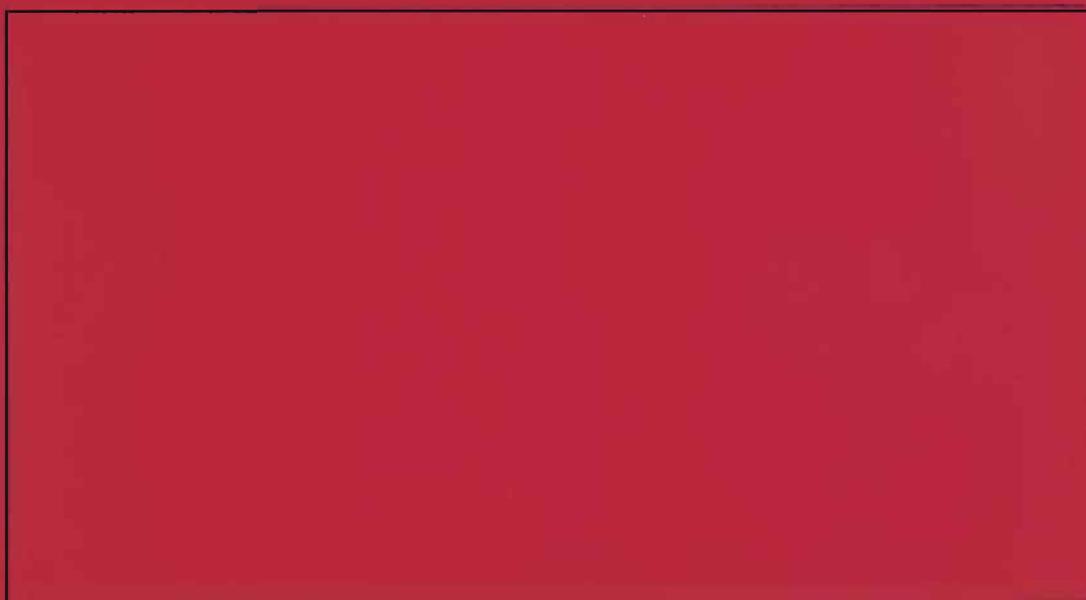
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N° 67

FUZZY EXPECTED UTILITY

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June 1984

Cette étude a fait l'objet d'une Communication au Vith European Congress on Operations Research, Vienna, Austria, july, 19 - 22, 1983, au XVème Colloque annuel de l'Institut de Mathématiques Economiques, Dijon, le 25 novembre 1983.

1 - INTRODUCTION

Decision making under uncertainty requires not only measures of the uncertainty of situations that we try to recognize , but also an estimate of the imprecision from which they are determined.

This imprecision can be the result either of a lack of exactness in the measure of the elements which are necessary to the determination of the states of nature or the purely subjective interpretation of these states.

Through a subjective measure of the non-measurable imprecision, the purpose of the fuzzy expected utility, which is investigated, is to translate with a great accuracy the imprecise behaviour of the decision-maker in an uncertain world. Consequently we propose to introduce first the probability of a fuzzy subset, or fuzzy event, to the theory of fuzzy utility, also called "fuzzy stochastic utility".

On the other hand the uncertainty can correspond to an information state where a decision can lead to alternative results from which the probability may be unknown. Moreover most of the information of the decision-maker is often imprecise, incomplete and may be inaccessible on an exact number form. For these reasons, it is difficult for an individual to define a probability law of a fuzzy subset, so that we shall use later on a weaker structure than the one of probabilities, where we no longer study what can happen but what could happen. The change from probabilities to possibilities allows the decision-maker to choose what he does prefer from what is possible. We will speak in this case of "fuzzy possibilistic utility". Using a set of well defined axioms (such as the existence of a fuzzy complete preordered structure on the set of actions ; the continuity of preferences ; the existence of a mixing operation on the set of actions...) it is possible first to determine a function of fuzzy expected utility. More exactly, according the information available, the decision-maker can assign a probability or a possibility to a given event, and define an additive fuzzy utility on actions. We shall see later that the choice behaviour in uncertainty will thus be rationalized by a fuzzy expected function, which is continuous in the intrinsic topology.

2 - THE PROBLEM

In order to determine the axiomatic of a fuzzy decision problem in uncertainty, we will apply Bernoulli's utility theory to a fuzzy case.

Let these be the following four elements :

$$\langle \underline{S}, \underline{D}, P(\underline{S}_k), U(H_j, \underline{S}_k) \rangle$$

\underline{S} represents the set of states of nature also called world states. This set gives a full specification of the different stochastic factors of environment.

$$\underline{S} = \{ \underline{S}_1, \underline{S}_2, \dots, \underline{S}_k, \dots, \underline{S}_m \}.$$

\underline{S}_k subset of \underline{S} is called an event. $\underline{S}_k \subset \underline{S}$.

The data belonging to \underline{S} have an empty intersection and their union is \underline{S} .

$P(\underline{S}_k)$ is the probability of \underline{S}_k event.

\underline{D} represents the set of all actions possible a priori.

$$\underline{D} = \{ \underline{D}_1, \underline{D}_2, \underline{D}_j, \dots, \underline{D}_r \}$$

An act is a mapping from the states of nature to the set of results

$$\text{denoted: } X = \{ \underline{X}_1, \underline{X}_2, \dots, \underline{X}_n \}$$

$U(H_j, \underline{S}_k)$ is a fuzzy utility function. This represents the utility of selecting a fuzzy H_j in an imprecise context, when \underline{S}_k event appears. Unlike the usual case, the axiomatic of the fuzzy expected utility theory does not rest on a \mathcal{B} Borel's field, but on a weak \mathcal{S} Borel's field. Indeed, we will admit that for a given fuzzy decision problem, an economic agent can determine a weak \mathcal{S} Borel's field of events including all the information he has at his disposal.

Consider a referential denoted by \underline{D} and a family of subsets denoted by \mathcal{S} .

$\mathcal{S} \subset \mathcal{L}^{\underline{D}}$; in general we represent \underline{L} by a lattice or by an $[0, 1]$ interval.

\mathcal{S} is called a weak \mathcal{S} Borel's field, if it verifies the following properties :

- a) $\emptyset \in \mathcal{S}, \underline{D} \in \mathcal{S}$
- b) $(A \in \mathcal{S}, B \in \mathcal{S}) \implies A \cap B \in \mathcal{S}$
- c) $(A_1, A_2, \dots, A_i, \dots) \in \mathcal{S} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{S}$.

3 - FUZZY STOCHASTIC UTILITY

Consider $\underline{D} = \{ \underline{D}_1, \underline{D}_2, \dots, \underline{D}_m \}$. We introduce the weak \mathcal{G} Borel's field.

$$\mathcal{G} = \{ H_1, H_2, \dots, H_j, \dots, H_r \} \quad H_j \subset \mathcal{G}.$$

We can associate a function d from \underline{S} to \underline{X} with every fuzzy act $d \in H_j$, $H_j \subset \underline{D}$, such that

$$d : s \longrightarrow d(s)$$

This function characterizes the decision of the individual, so that we can also write $H_j \subset \underline{D} = \underline{X} \underline{S}$.

Confronted by a risky environment, the consequence of a fuzzy decision is the image on \underline{X} of P probability law by d mapping.

$$\text{We obtain : } d^{-1}(H_j) \in \mathcal{G} \quad \forall H_j \in \mathcal{G}.$$

The measure P^* defined on $(\underline{D}, \mathcal{G})$ structure by $P^*(H_j) = P(d^{-1}(H_j))$, is a probability law of a fuzzy decision on \underline{D} .

P^* is called image by d of P law.

$P^*(H_j)$ is the probability that the consequence of the fuzzy decision belongs to \underline{X} .

The comparison between decisions will finally amount to that between the associated P^* , so we can eventually compare directly between them the probability laws on the structure of $(\underline{D}, \mathcal{G})$. The fuzzy expected utility of a decision H_j is thus defined by the following formula :

$$U(H_j) = \sum_{k=1}^m U(H_j, \underline{S}_k) \cdot P(\underline{S}_k) = P(H_j).$$

This equals the sum of fuzzy utilities to the conditional decisions H_j , weighted by the probability of these events. It also corresponds to the probability of a fuzzy subset H_j . This latter is expressed by the fuzzy utility of a fuzzy decision .

3.1 - System of Axioms for the fuzzy expected utility and Interpretation.

The choice criterion of the decision-maker is explained by the preference concept relative to the various actions. We express this preference or indifference by a binary relation denoted by \mathcal{R} , defined on \underline{D} .

$\forall d_1 \in \underline{D}, \forall d_2 \in \underline{D}$ we read,

$d_1 \prec d_2$ by " d_2 is preferred to d_1 ", and

$d_1 \sim d_2$ by "the action d_1 is indifferent to the d_2 action".

Let \underline{H} be the set of acts made possible for technical reasons.

AXIOM 1 : A fuzzy complete preorder exists on the set of actions $\underline{H} \subset \underline{D}$.

The set \underline{H} has a fuzzy complete preordered structure if the fuzzy relation \mathcal{R} is reflexive and Max-Min Transitive.

a) the relation \mathcal{R} is reflexive :

$$\forall (d_1, d_1) \in \underline{H}^2, \mu_{\mathcal{R}}(d_1, d_1) = 1$$

b) the relation \mathcal{R} is $\wedge \vee$ transitive* :

$$\forall (d_1, d_3) \in \underline{H}^2.$$

$$\mu_{\mathcal{R}}(d_1, d_3) \geq \bigvee_{d_2 \in \underline{H}} [\mu_{\mathcal{R}}(d_1, d_2) \wedge \mu_{\mathcal{R}}(d_2, d_3)]$$

In the imprecise context, the preference or indifference structure denoted by $(\underline{H}, \mathcal{R})$ has the property of completeness. In classical theory this latter condition requires that the economic agent should always be able to compare two actions according to his preferences.

More precisely :

$$\forall d_1, d_2 \in \underline{H}^2, \mu_{\mathcal{R}}(d_1) \geq \mu_{\mathcal{R}}(d_2) \text{ or} \\ \mu_{\mathcal{R}}(d_2) \geq \mu_{\mathcal{R}}(d_1).$$

Nevertheless we have $\mu_{\mathcal{R}}(d_1, d_2) = 0$ if $(d_1, d_2) = \emptyset$ (being a consequence of the fuzzy measure definition). This in fact, is the non comparability assumption. (PONSARD 1979). Unlike the usual theory which requires the complementary concept, this latter is not constraining.

Axiom 1 states that the choice behaviour of an individual can be rationalized by a complete fuzzy preorder denoted $(\underline{H}, \succsim)$ in a fuzzy context.

A fuzzy utility U defined on a preference space $(\underline{H}, \succsim)$ is called fuzzy expected utility if the following definition holds.

Definition 3.1 : For any fuzzy decision H_1, H_2 , and for any $\alpha \in [0, 1]$, the fuzzy expected utility is defined by :

$$U \left[(\alpha \cdot \mu_{H_1}(s)) + (1 - \alpha) \mu_{H_2}(s) \right] = \alpha U(\mu_{H_1}(s)) + (1 - \alpha) U(\mu_{H_2}(s))$$

The second part of the above formula represents the fuzzy mathematical expectation of the fuzzy utility. So the definition of the fuzzy expected utility has a meaning only if the composition $\alpha \mu_{H_1}(s) + (1 - \alpha) \mu_{H_2}(s)$ is introduced in the set $\underline{H} \subset \underline{D}$.

* where \vee denotes the maximum of the membership functions and \wedge denotes the minimum of the membership functions.

As in the usual theory, this composition is described by an imprecise lottery which admits H_1 or H_2 as a possible outcome with α and $(1 - \alpha)$ respective probabilities.

To obtain it we propose the following axiom :

AXIOM 2 : The set $\underline{H} \subset \underline{D}$ is called a fuzzy mixture operation towards \mathcal{S} (\mathcal{S} is a weak Borel's field). Consider a set $\underline{H} \subset \underline{D}$ and $[0, 1]$ interval, a fuzzy mixture operation, denoted by M' on \mathcal{S} is a binary operation on \underline{H} such that :

$$\alpha \in [0, 1] \quad H \subset \underline{H}$$

$$H \times H \longrightarrow H$$

$$d_1, d_2 \longrightarrow \alpha d_1 + (1 - \alpha) d_2. \text{ We have :}$$

$$1) \forall d_1, d_2 \in H^2, \alpha = 1$$

$$1 d_1 + (1 - 1) d_2 = d_1.$$

2) the property of Commutativity :

$$\forall d_1, d_2 \in H^2, \alpha \in [0, 1]$$

$$\alpha d_1 + (1 - \alpha) d_2 = (1 - \alpha) d_2 + \alpha d_1$$

3) the property of distributivity :

$$\forall d_1, d_2 \in H^2, \forall \alpha, \beta \in (0, 1)^2$$

$$\beta(\alpha d_1 + (1 - \alpha) d_2) + (1 - \beta) d_2 = \beta \alpha d_1 + (1 - \beta \alpha) d_2.$$

A convex non-empty part of R^n is a mixture set if the general expression $\alpha d_1 + (1 - \alpha) d_2$ is also convex. We say that we have the same result for a non empty set of probability measures which are all defined on a Borel's field. For the economist M' is both a mixture set and a fuzzy complete preordered set. The mixture operation in this case becomes a "probabilized mixture operation".

We also note that property (3) agrees with VON NEUMANN and MORGENSTERN's principle (1953) of composed probability.

Axiom 2 establishes all possible acts described from events included in \mathcal{Y} weak Borel's field. Moreover, the acts are generally conditioned by events.

This conditioning of acts allows us to define how the information is integrated in the decision-maker's choice.

Given an act d ; $d \in H$, $H \subset \underline{H}$, we call fuzzy conditional decision to \underline{S}_k event, $\underline{S}_k \subset \underline{S}$, the restriction d denoted by $d_{/\underline{S}_k}$ to \underline{S}_k event.

$\underline{H}/\underline{S}_k$ represents the decision set conditional to \underline{S}_k .

We have : $\forall (d_1, d_2) \in \underline{H}^2, \underline{S}_k \subset \underline{S}$

$$d_{1/\underline{S}_k} \cup d_{2/\underline{S}_k} \in \underline{H}$$

\underline{H} set is closed to a mixture of conditionnal act which is build from the event.

AXIOM 3 : Independance axiom.

$\forall \underline{S}_k \in \mathcal{B}, \mathcal{B}$ is a Borel's field.

$$\forall d_1, d_2 \in \underline{H}^2, \exists d_3 \in \underline{H} / d_{1/\underline{S}_k} \cup d_{3/\underline{S}_k} \preccurlyeq d_{2/\underline{S}_k} \cup d_{3/\underline{S}_k}$$

$$\implies \forall d'_3 \in \underline{H}, d_{1/\underline{S}_k} \cup d'_{3/\underline{S}_k} \preccurlyeq d_{2/\underline{S}_k} \cup d'_{3/\underline{S}_k}$$

When axiom III is verified in the preordered structure $(\underline{H}, \succcurlyeq)$, we say that it is independent from \mathcal{G} .

We can interpret this axiom in the following way :

"If two acts conditional to the same event are combined with a third act conditional to the complementary event, then the direction of preferences remains the same."

The following lemma is an immediate consequence of the independence axiom.

Lemma 3.1.: If $(\underline{H}, \succcurlyeq)$ is a fuzzy complete preordered structure and if we have axiom 1 and 2 then $\forall \underline{S}_k \in \underline{S}, (\underline{H}/\underline{S}_k, \succcurlyeq)$ is a fuzzy complete preordered structure.

AXIOM 4 : Continuity axiom.

Consider any three acts $d_1, d_2, d_3 \in \underline{H}^3$, if $d_1 \succcurlyeq d_2 \succcurlyeq d_3$, then there exists a number $\alpha, 0 \leq \alpha \leq 1$ such that $\alpha d_1 + (1 - \alpha) d_3 \sim d_2$.

More specifically, we can say that a lottery ticket exists. This one associates the two extreme acts respectively with the probabilities α and $(1 - \alpha)$ and it is equivalent to the intermediate act. These last two axioms are necessary to build a fuzzy utility function of a fuzzy decision, so that any available information is taken into account.

3.2 - Preference structure and uncertainty in a fuzzy context.

The decision-maker sets up a comparison between two fuzzy actions at a more or less strong level of preference or indifference. This level expresses the imprecision of his decision.

Hence a fuzzy binary relation denoted \mathcal{R} between the elements of \underline{H}^2 is defined by :

$$d_1 \mathcal{R} d_2 = \{ (d_1, d_2), \mu_{\mathcal{R}}, \forall d_1 \in \underline{H}, \forall d_2 \in \underline{H}, \mu_{\mathcal{R}}(d_1, d_2) \in \underline{M} \}$$

where $\mu_{\mathcal{R}}(d_1, d_2)$ denotes the level of preference and indifference between the two actions and \underline{M} is a complete preordered membership set. In other words, to each given preference or indifference system, a fuzzy subset denoted H is associated.

H is an element of the set of applications from \underline{H}^2 to \underline{M} on the following pattern :

$$\underline{H} = \{ (d_1, d_2), \mu_H ; \forall d_1 \in \underline{H}, \forall d_2 \in \underline{H}, \mu_H(d_1, d_2) \in \underline{M} \}$$

$(\underline{H}, \mathcal{R})$ is called an imprecise individual preference or indifference structure. It is compounded by two structures denoted respectively (\underline{H}, \succ) , (\underline{H}, \sim) according to whether the relation \mathcal{R} is a strict fuzzy preference relation or a fuzzy indifference relation.

The decision-maker prefers more or less an action d_1 to another action d_2 if :

$$\mu_{\mathcal{R}}(d_1, d_2) \succ \mu_{\mathcal{R}}(d_2, d_1).$$

$\mu_{\mathcal{R}}(d_1, d_2)$ is called the degree of strong preference for action d_1 relative to action d_2 .

$\mu_{\mathcal{R}}(d_2, d_1)$ is called the degree of weak preference for act d_2 relative to act d_1 .

The determination process of a fuzzy utility function is similar to the one explained by PONSARD (1979-1984), we will build however a fuzzy utility function, where the preference structure is imprecise in the choice of an act conditional to an event.

The uncertainty can be found in the description of a fuzzy act.

3.3 - Existence and continuity of the fuzzy utility function.

Given the $(\underline{H}/\underline{S}_k, \succ)$ structure, the fuzzy utility is the numerical translation of the qualitative description of the conditional choice which is contained

in the definition of imprecise preference structure.

We call fuzzy utility denoted U : any homomorphism from $(\underline{H}/\underline{S}_k, \succsim)$ to $([0,1], \succsim)$.

i.e. any application :

$$\begin{aligned} u : \underline{H} &\longrightarrow [0,1] \\ d_1 &\longrightarrow u(d_1) \end{aligned}$$

$$\forall (d_1, d_2) \in \underline{H}_2, d_1/\underline{S}_k \preceq d_2/\underline{S}_k \iff u(d_1) \leq u(d_2)$$

If a fuzzy utility function exists for $(\underline{H}/\underline{S}_k, \succsim)$ structure we also check that :

$$\begin{aligned} \forall (d_1, d_2) \in \underline{H}^2, d_1 \prec d_2 &\iff u(d_1) < u(d_2) \\ d_1 \sim d_2 &\iff u(d_1) = u(d_2) \end{aligned}$$

If U is a fuzzy utility on fuzzy conditional acts on a $(\underline{H}/\underline{S}_k, \succsim)$ structure, it remains the same for the o composed function (in the usual meaning). $u = f \circ g$; where f is an increasing homomorphism for $([0,1], \succsim)$ to $([0,1], \succsim)$.

$$\begin{aligned} \forall d_1, d_2 \in \underline{H}^2, d_1 \preceq d_2 &\iff u(d_1) \leq u(d_2) \\ &\iff f[g(d_1)] \leq f[g(d_2)] \end{aligned}$$

This function is defined on $(\underline{H}/\underline{S}_k, \succsim)$ up to an increasing monotonic transformation.

Theorem 3.1 : If the imprecise structure $(\underline{H}/\underline{S}_k, \succsim)$ is a complete preordered set, there exists a fuzzy utility function.

Definition 3.2 : A fuzzy utility function is a real-valued function such that if :

$$\begin{aligned} U(d_j) &= u(d_{j/\underline{S}_1}) + u(d_{j/\underline{S}_2}) + \dots + u(d_{j/\underline{S}_m}) \\ &= u_{j_1} + u_{j_2} + \dots + u_{j_m} \\ &= \sum_{k=1}^m u_{j_k} \end{aligned}$$

$$\text{Then } d_j \preceq d_1 \iff \sum_{k=1}^m u_{j_k} \leq \sum_{k=1}^m u_{1_k}$$

$$\iff U(d_j) \leq U(d_1)$$

From the above theorem and definition, we can deduce the following theorem :

Theorem 3.2 : If the imprecise preference or indifference structure $(\underline{H}/\underline{S}_k, \succsim)$ is a complete preordered set, then there exists a fuzzy utility function additive on conditional acts.

AXIOM 4 : Allows us to prove the continuity of the fuzzy utility function.

$\forall d_1, d_2 \in \underline{H}$ the fuzzy subsets :

$$H^* = \left\{ d_2 \in \underline{H} / \mu_H(d_2) \leq \mu_H(d_1) \right\}$$

$$H_* = \left\{ d_2 \in \underline{H} / \mu_H(d_2) > \mu_H(d_1) \right\}$$

are pseudo closed into \underline{H} .

The topological space $(\underline{H}, \succsim, \mathcal{C})$ is an imprecise preference structure ; the preferences are continuous for \mathcal{C} if and only if $\forall d_1 \in \underline{H}, H^* \in \mathcal{C}, H_* \in \mathcal{C}$.

Using Uzawa's theorem (1960) we can also prove that the $(\underline{H}/\underline{S}_k, \succsim, \mathcal{C})$ space is a complete preordered topological space. We can then define an additive fuzzy utility function on the conditional acts as being any continuous homomorphism from $(\underline{H}, \succsim, \mathcal{C}) \longrightarrow ([0,1], \succsim, \mathcal{D})$ where \mathcal{C} and \mathcal{D} are the respective intrinsic topologies. Thus, given axioms 1 to 4, we can rationalize a choice behaviour in uncertainty by a fuzzy utility on $(\underline{H}/\underline{S}_k, \succsim)$.

This fuzzy utility is continuous in its intrinsic topology, it is defined this way :

$$U(d) = \sum_{k=1}^m u(d/\underline{S}_k)$$

3.4 - Utility of fuzzy decision using fuzzy numbers.

We can further introduce a second type of imprecision in the fuzzy utility. Indeed, to our imprecise preference structure we associate a fuzzy numerical translation of a qualitative description of the decision-maker's choice. In this way, the utility of a fuzzy decision is no longer expressed by a precise number but by a fuzzy number. Therefore :

$$\text{when } \underline{H} = \mathbb{R} \quad \mathcal{C} \subset \mathcal{M}^{\underline{H}}$$

\mathcal{M} is the fuzzy numbers set.

The fuzzy utility of a fuzzy decision d_{j/\underline{S}_k} will be defined by :

$$\tilde{U}_j = \tilde{u}_{j_1} \oplus \tilde{u}_{j_2} \oplus \dots \oplus \tilde{u}_{j_k} \oplus \dots \oplus \tilde{u}_{j_m} = \sum_{k=1}^m \tilde{u}_{j_k}$$

The memberships functions of fuzzy numbers satisfy the properties of continuity, normalization, and quasi convexity.

However, in order that the fuzzy utility functions should retain the same characteristics, we show easily that the additivity of fuzzy numbers always possesses the same properties as those.

3.5 - Existence and continuity of a fuzzy stochastic utility.

The last two theorems allow us to define a choice behaviour in uncertainty. This latter is rationalized by our axioms. At this level, we introduce the probability of an event \underline{S}_k , $\underline{S}_k \in \mathcal{B}$.

This probability is considered as a behaviour coefficient, and it is defined on a \mathcal{B} Borel's field.

Theorem 3.3 : Existence of a fuzzy stochastic utility function.

Given a choice behaviour under uncertainty, when axioms 1 to 4 are satisfied, we show that, for an event $\underline{S}_k \subset \underline{S}$, there exists a fuzzy expected utility which is continuous in its intrinsic topology such that :

$$\forall H_j \in \underline{H} \quad U(H_j) = \sum_{k=1}^m \tilde{U}(H_j/\underline{S}_k) \cdot P(\underline{S}_k).$$

This theorem is a consequence of the previous theorems, where we have demonstrated the existence and continuity of a fuzzy utility function which is additive on the conditional acts.

4 - FUZZY POSSIBILISTIC UTILITY.

A fuzzy decision problem in uncertainty is defined by using the possibility of an event \underline{S}_k .

Let the following four elements :

$$\langle \underline{S}, \underline{U} \pi(\underline{S}_k), U(H_j, \underline{S}_k) \rangle$$

The axiomatic of the fuzzy possibilistic utility also rests on a weak \mathcal{F} Borel's field.

$\pi(\underline{S}_k)$ is the possibility of \underline{S}_k event.

As in the usual theory, the possibility is defined on a \mathcal{B} Borel's field.

$U(H_j, \underline{S}_k)$ is the fuzzy expected utility function.

The fuzzy possibilistic utility of H_j fuzzy decision is :

$$\begin{aligned} U(H_j) &= \Pi(U_{j_k}) = \bigvee_{\underline{S}_k} \left[\mu_{H_j}(\underline{S}_k) \wedge \pi(\underline{S}_k) \right] \\ &= \bigvee_{\underline{S}_k} \left[U(H_j, \underline{S}_k) \wedge \pi(\underline{S}_k) \right] \\ &= \bigvee_{\underline{S}_k} \left[U(x_{jk}) \wedge \pi(\underline{S}_k) \right] \end{aligned}$$

with $\text{Max}_{\underline{S}_k} \pi(\underline{S}_k) = 1$

$\Pi(U_{j_k})$ is the possibility of a fuzzy decision H_j . Knowing the \underline{S}_k event, this latter is expressed by a fuzzy utility.

$\Pi(U_{j_k})$ also corresponds to the possibility of fuzzy subsets.

Consider again the axioms system :

AXIOM 1 : A fuzzy complete preorder exists on the set of actions \underline{H} , $\underline{H} \subset \underline{D}$.

\underline{H} is the set of actions made possible for technical reasons.

AXIOM 2 : The set $\underline{H} \subset \underline{D}$ is called fuzzy $\forall \wedge$ mixture operation towards \mathcal{S} .

Condition $\underline{H} \subset \underline{D}$. We call fuzzy $\forall \wedge$ mixture operation (denoted M'') towards \mathcal{S} the following convex arrangement.

Consider the three H_1, H_2, \mathcal{L} fuzzy subsets.

$$\forall \underline{S}_k \in \underline{S} \quad \mu_{(H_1, H_2, \mathcal{L})}(\underline{S}_k) = \mu_{\mathcal{L}}(\underline{S}_k) \cdot \mu_{H_1}(\underline{S}_k) + \left[1 - \mu_{\mathcal{L}}(\underline{S}_k) \cdot \mu_{H_2}(\underline{S}_k) \right]$$

This latter is also denoted by :

$$(H_1, H_2, \mathcal{L}) = \mathcal{L}H_1 + \bar{\mathcal{L}}H_2$$

and we have the following properties :

a) In the particular case where $\mathcal{L} = 1$ we have :

$$\mathcal{L}H_1 + \bar{\mathcal{L}}H_1 = H_1$$

b) The property of Commutativity.

$$\mathcal{L}H_1 + \bar{\mathcal{L}}H_2 = \bar{\mathcal{L}}H_2 + \mathcal{L}H_1$$

c) The property of distributivity.

\mathcal{X} is a fuzzy subset.

$$\alpha [\alpha H_1 + \bar{\alpha} H_2] + \bar{\alpha} H_2 = \alpha \alpha H_1 + \alpha \bar{\alpha} H_2 + \bar{\alpha} H_2 = \alpha \alpha H_1 + (1 - \alpha \alpha) H_2$$

Axioms 3 and 4, respectively about absolute preference and continuity, are the same as those which are defined in the theory of fuzzy stochastic utility. The interpretation of these axioms are of course identical. The fuzzy possibilistic utility expectation of H_j fuzzy decision is :

$$U(H_j) = \Pi(U_{j_k}) = \bigvee_{\underline{S}_k} [U(H_j, \underline{S}_k) \wedge \pi(\underline{S}_k)]$$

with $\text{Max}_{\underline{S}_k} \pi(\underline{S}_k) = 1$

Knowing \underline{S}_k events, we demonstrate the existence and continuity of such a fuzzy utility function of a fuzzy decision. This is done in the same way as the axiomatic of the fuzzy stochastic utility.

However the existence and continuity proof of the fuzzy possibilistic utility is different.

Indeed, the mathematical expectation of the fuzzy utility corresponds with the possibility of a fuzzy event, also called possibility of a sensation by KAUFMANN (1977). This sensation represents the conditional utility of a fuzzy decision.

When $\underline{S} = \mathbb{R}$, $\pi(\underline{S}_k)$ is called density of possibility.

When \underline{S} is finite, $\pi(\underline{S}_k)$ is called \underline{S}_k possibility.

We write also the fuzzy possibilistic utility as :

$$U(H_j) = \sup_{\underline{S}_k \subset \underline{S}} [\text{Min } U(H_j, \underline{S}_k), \pi(\underline{S}_k)] = \Pi(U_{j_k})$$

We prove the existence and the continuity of such a function with the possibility subsets.

$$\text{Indeed : } \Pi(U_{j_k}) = \sup_{\underline{S}_k \in U_{j_k}} \pi(\underline{S}_k)$$

$\Pi(U_{j_k})$ is a fuzzy measure. So a fuzzy integral in SUGENO's meaning (1974) exists.

$$\Pi(U_{j_k}) = \int_{\underline{S}} U_{j_k} d\Pi$$

Let the U utility be a fuzzy variable denoted by X , we have then $X : \underline{S} \longrightarrow \mathbb{R}$ is a fuzzy variable.

Its density of possibility is $\varphi : \mathbb{R} \longrightarrow [0,1]$ and $f(x) = \text{Poss} \{ X = x \}$

Consider the F fuzzy subset of \mathbb{R} . We have :

$$\text{Poss} \{ X = F \} = \sup_{x \in \mathbb{R}} \{ \text{Min} (F(x)), \varphi (x) \}$$

$$\text{and Poss} \{ X = F \} = \sup_F \varphi(x).$$

More precisely, this latter expression means that we have maximized φ on the fuzzy constraint F . Moreover for any $A \subset \mathbb{R}$, if $\Pi_X(A) = \sup_{x \in A} \varphi(x)$ is the possibility measure of φ .

Then, we can write :

$$\text{Poss} \{ X = F \} = \int_{\mathbb{R}} F d\Pi_X$$

$$\text{where } \int_{\mathbb{R}} F d\Pi = \sup_{\alpha \geq 0} [\text{Min} (\alpha, \Pi \{ F \geq \alpha \})]$$

Hence if φ is continuous, F is superlatively half continuous, strictly convexe ; and $\text{Sup} F$ is compact.

$$\text{Then Poss} \{ X = F \} = \text{Poss} X \in A \text{ where } \forall \alpha \in [0,1]$$

$$A = \{ x \in \mathbb{R} : F(x) \geq \alpha_0 \}.$$

This result is proved with the SUGENO's theorem (1974) which is applied to fuzzy subsets :

If $F_\alpha = v_f \{ f \geq \alpha \}$ is continuous, v_f is a fuzzy measure of fuzzy subsets, then there exists $\alpha_0 \geq 0$ such as $\int_A f dv_f = v_f \{ f \geq \alpha_0 \}$.

5 - CONCLUSION

The fuzzy expected utility gives first a precise measure of the preferences of an individual by taking into account his satisfaction curve, and translate a set of decision rules.

However the introduction of fuzzy subsets weakens the rational criteria of decision, so that these latters are considered as reasonable and gives a good representation of the decision-maker's psychology.

More precisely the real behaviour of an individual is expressed by a more or less thick, convex or concave curve. This latter translates respectively the more or less strong preference or dislike for the risk. This thick curves express the imprecision which characterises individual behaviour, sensitive to the vagueness and the dispersion of utility values.

They give a generalization of the usual theories. In particular, we find once more the single linear curve, which is representative of the indifference behaviour of the decision-maker.

On the other hand, the uncertainty is translated through the description of actions, thus the fuzzy stochastic and possibilistic subsets enlarge the view of uncertainty .

We note again that the axiomatic structure of possibility is weaker than the probability structure. Therefore the former can take into account unforeseeable factors, and is well defined for the domain of uncertainty.

It is indeed more realistic and easier to evaluate what can happen than what must happen.

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