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STURM'S THEOREM ON ZEROS OF LINEAR COMBINATIONS OF EIGENFUNCTIONS

PIERRE BÉRARD AND BERNARD HELFFER

ABSTRACT. Motivated by recent questions about the extension of Courant's nodal domain theorem, we revisit a theorem published by C. Sturm in 1836, which deals with zeros of linear combination of eigenfunctions of Sturm-Liouville problems. Although well known in the nineteenth century, this theorem seems to have been ignored or forgotten by some of the specialists in spectral theory since the second half of the twentieth-century. Although not specialists in History of Sciences, we have tried to replace these theorems into the context of nineteenth century mathematics.

1. INTRODUCTION

In this paper, we are interested in the following one-dimensional eigenvalue problem, where r denotes the spectral parameter.

$$(1.1) \quad \frac{d}{dx} \left(K \frac{dV}{dx} \right) + (rG - L)V = 0, \text{ for } x \in]\alpha, \beta[,$$

$$(1.2) \quad \left(K \frac{dV}{dx} - hV \right) (\alpha) = 0,$$

$$(1.3) \quad \left(K \frac{dV}{dx} + HV \right) (\beta) = 0.$$

Here,

$$(1.4) \quad K, G, L : [\alpha, \beta] \rightarrow \mathbb{R} \text{ are positive functions,}$$

$$(1.5) \quad h, H \in [0, \infty] \text{ are non negative constants, possibly infinite.}$$

Remark 1.1. When $h = \infty$ (resp. $H = \infty$), the boundary condition should be understood as the Dirichlet boundary condition $V(\alpha) = 0$ (resp. as the Dirichlet boundary condition $V(\beta) = 0$).

Precise assumptions on K, G, L are given below.

Note that when $K = G \equiv 1$, (1.1)–(1.3) is an eigenvalue problem for the classical operator $-\frac{d^2V}{dx^2} + LV$.

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This eigenvalue problem, in the above generality (K, G, L functions of x), was first studied by Charles Sturm in a Memoir presented to the Paris Academy of sciences in September 1833, summarized in [33, 34], and published in [35, 36].

Remark 1.2. In this paper, we have mainly retained the notation of [35], except that we use $[\alpha, \beta]$ for the interval, instead of Sturm's notation $[x, X]$. We otherwise use today notation and vocabulary. Note that in [36], Sturm uses lower case letters for the functions K, G, L . Sturm uses the same notation as Joseph Fourier in [11].

As far as the eigenvalue problem (1.1)–(1.3) is concerned, Sturm's results can be roughly summarized in the following theorems.

Theorem 1.3 (Sturm, 1836). *Under the assumptions (1.4)–(1.5), the eigenvalue problem (1.1)–(1.3) admits an increasing infinite sequence $\{\rho_i, i \geq 1\}$ of positive simple eigenvalues, tending to infinity. Furthermore, the associated eigenfunctions V_i have the following remarkable property: the function V_i vanishes, and changes sign, precisely $(i - 1)$ times in the open interval $] \alpha, \beta [$.*

Theorem 1.4 (Sturm, 1836). *Let $Y = A_m V_m + \dots + A_n V_n$ be a non trivial linear combination of eigenfunctions of the eigenvalue problem (1.1)–(1.3), with $1 \leq m \leq n$, and $\{A_j, m \leq j \leq n\}$ real constants such that $A_m^2 + \dots + A_n^2 \neq 0$. Then, the function Y has at least $(m - 1)$, and at most $(n - 1)$ zeros in the open interval $] \alpha, \beta [$.*

The first theorem today appears in most textbooks on Sturm-Liouville theory. Although well known in the nineteenth century, the second theorem seems to have been ignored or forgotten by some of the specialists in spectral theory since the second half of the twentieth-century, as the following chronology indicates.

- 1833:** Sturm's Memoir presented to the Paris Academy of sciences in September, and [33, 34]
- 1836:** Sturm's papers [35, 36] published. The main results, Theorems 1.3 and 1.4, are summarized by Joseph Liouville in [20, § III, p. 257].
- 1877:** Lord Rayleigh writes “a beautiful theorem has been discovered by Sturm” as he mentions Theorem 1.4 in [31, Section 142].
- 1891:** F. Pockels [27, pp. 68-73] gives a summary of Sturm's results, including Theorem 1.4, and mentions the different proofs provided by Sturm, Liouville and Rayleigh. On the basis of a note of Sturm in Férussac's Bulletin [32], Pockels (p. 71, lines 12-17) also suggests that Sturm may have looked for a statement in higher dimension as well, without success. Sturm indeed mentions studying an example with spherical symmetry in 3 dimensions (leading to an ordinary differential equation with singularity), to which he may have applied Theorem 1.4.

1903: Hurwitz [16] gives a lower bound for the number of zeros of the sum of a trigonometric series with a spectral gap and refers, somewhat inaccurately, to Sturm's Theorems. This result, known as the Sturm-Hurwitz theorem, already appears in a more general framework in Liouville's paper [20].

See [10, § 2] for a generalization, and [25] for geometric applications.

1931: Courant and Hilbert [8, 9] extensively mention the Sturm-Liouville problem. They do not refer to the original papers of Sturm, but to Bôcher's book [6] which does not include Theorem 1.4. They state an extension of Courant's nodal domain theorem to linear combination of eigenfunctions, [8, footnote, p. 394] and [9, footnote, p. 454], and refer to the dissertation of H. Herrmann [15]. It turns out that neither Herrmann's dissertation, nor his later papers, consider this extension of Courant's Theorem.

1956: Pleijel mentions Sturm's Theorem 1.4, somewhat inaccurately, in [26, p. 543 and 550].

1973: V. Arnold [1] points out that an extension of Courant's theorem to linear combinations of eigenfunctions cannot be true in general. Counterexamples were first given by O. Viro for the 3-sphere (with the canonical metric) [37] and, more recently in the paper [4], see also [14].

It seems to us that Arnold was not aware of Theorem 1.4. In [3], see also Supplementary problem 9 in [2, p. 327], he mentions a proof, suggested by I. Gelfand, of the upper bound in Theorem 1.4. Gelfand's idea is to "use fermions rather than bosons", and to apply Courant's nodal domain theorem. However, Arnold concludes by writing [3, p. 30], "*the arguments above do not yet provide a proof*". As far as we know, Gelfand's idea has so far not yielded any complete proof of Sturm's result. It is interesting to note that Liouville's and Rayleigh's proofs of the lower bound in Theorem 1.4 use an idea similar to Gelfand's, see the proof of Claim 3.5.

Remark 1.5. In [36], Theorem 1.4 first appears as a corollary to a much deeper theorem [36, § XXIV], in which Sturm describes the time evolution of the x -zeros of a solution $u(x, t)$ of the heat equation. We shall not consider this topic here, and we refer to [12, 23] for modern formulations and a historical analysis.

Our interest in Theorem 1.4 arose from reading [17], and investigating Courant's nodal domain theorem and its extension to linear combination of eigenfunctions.

The main purpose of this paper is to popularize Sturm's originality and ideas, and to provide an accessible ordinary differential equation proof

of Theorem 1.4, meeting today standards of rigor. More precisely, we revisit Sturm's original statement and proof, see Theorem 2.14 and [36, § XXVI], as well as Liouville's, see Theorems 3.2 and [21]. We also clarify some technical points and provide alternative proofs.

We make the following strong assumptions.

$$(1.6) \quad \begin{cases} [\alpha, \beta] \subset]\alpha_0, \beta_0[, \\ K, G, L \in C^\infty(] \alpha_0, \beta_0[), \\ K, G, L > 0 \text{ on }] \alpha_0, \beta_0[. \end{cases}$$

Remark 1.6. Neither Sturm nor Liouville make explicit regularity assumptions, see Subsection 4.3 and Remark 3.7 for more details.

Organization of the paper. In Section 2, we prove Theorem 2.14, Sturm's refined version of Theorem 1.4, following the ideas of [36, § XXVI]. In Section 3, we prove Theorem 3.2, Liouville's version of Theorem 1.4, following [20, 21]. In Section 4, we describe the context of Sturm's papers and his ideas. Appendix A provides the detailed proof of a technical argument. Appendix B provides the translations into English of the citations in French.

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2. STURM'S O.D.E. PROOF OF THEOREM 1.4

2.1. Preliminary lemmas and notation.

2.1.1. Recall that $\{(\rho_j, V_j), j \geq 1\}$ are the eigenvalues and eigenfunctions of the eigenvalue problem (1.1)–(1.3).

By our assumption $L > 0$, the eigenvalues are positive, $\rho_j > 0$. Under the Assumptions (1.6), the functions V_j are C^∞ on $] \alpha_0, \beta_0[$. This follows from Cauchy's existence and uniqueness theorem, or from Liouville's existence proof [20].

In this section, we fix

$$(2.1) \quad Y = \sum_{j=m}^n A_j V_j,$$

a linear combination of eigenfunctions of the eigenvalue problem (1.1)–(1.3), where $1 \leq m \leq n$, and where the A_j are real constants.

Remark 2.1. We shall always assume that $Y \neq 0$, which is equivalent to assuming that $\sum_m^n A_j^2 \neq 0$. As far as the statement of Theorem 1.4 is concerned, the numbers which actually matter are

$$(2.2) \quad n_e = \sup\{p \mid m \leq p \leq n \text{ such that } A_p \neq 0\},$$

$$(2.3) \quad m_e = \inf\{p \mid m \leq p \leq n \text{ such that } A_p \neq 0\}.$$

Without loss of generality, we may assume that $A_m A_n \neq 0$.

We also introduce the associated family of functions, $\{Y_k, k \in \mathbb{Z}\}$, where

$$(2.4) \quad Y_k = (-1)^k \sum_{j=m}^n A_j \rho_j^k V_j.$$

Note that Y_0 is the original linear combination Y , and that $Y_k \equiv 0$ if and only if $Y \equiv 0$.

Roughly speaking, Sturm's idea is to show that the number of zeros of Y_k , in the interval $]\alpha, \beta[$, is non-decreasing with respect to k , and then to take the limit when k tends to infinity, see Subsection 2.3. Up to changing the constants A_j , it suffices to compare the number of zeros of Y and Y_1 . For this purpose, Sturm compares the signs of Y and Y_1 near the zeros of Y (Lemma 2.4), and at the non-zero local extrema of Y (Lemmas 2.6 and 2.8). The main ingredient for this purpose is the differential relation (2.8). In the sequel, we indicate the pages in Sturm's papers corresponding to the different steps of the proof.

2.1.2. For $m \leq p \leq n$, write the equations satisfied by the eigenfunction V_p ,

$$(2.5) \quad \frac{d}{dx} \left(K \frac{dV_p}{dx} \right) + (\rho_p G - L)V_p = 0,$$

$$(2.6) \quad \left(K \frac{dV_p}{dx} - hV_p \right) (\alpha) = 0,$$

$$(2.7) \quad \left(K \frac{dV_p}{dx} + HV_p \right) (\beta) = 0,$$

and multiply the p -th equation by $A_p \rho_p^k$. Summing up from $p = m$ to n , yields the following lemma.

Lemma 2.2. *Assume that (1.6) holds. Let $k \in \mathbb{Z}$.*

- (1) *The function Y_k satisfies the boundary conditions (1.2) and (1.3).*
- (2) *The functions Y_k and Y_{k+1} satisfy the differential relation*

$$(2.8) \quad G Y_{k+1} = K \frac{d^2 Y_k}{dx^2} + \frac{dK}{dx} \frac{dY_k}{dx} - L Y_k.$$

- (3) *Under the Assumptions (1.6), the function Y_k cannot vanish at infinite order at a point $\xi \in [\alpha, \beta]$, unless $Y \equiv 0$.*

Proof. [36, p. 437] Assertions (1) and (2) are clear by linearity.

For Assertion (3), assume that $Y_k \not\equiv 0$, and that it vanishes at infinite order at some ξ . Then, according to (2.8) and its successive derivatives, the function Y_{k+1} also vanishes at infinite order at ξ , and so does Y_ℓ for

any $\ell \geq k$. Assume, as indicated in Remark 2.1, that $A_n \neq 0$. Fixing some $p \geq 0$, we can write, for any $\ell \geq k$,

$$\frac{d^p V_n}{dx^p}(\xi) + \sum_{j=m}^{n-1} \frac{A_j}{A_n} \left(\frac{\rho_j}{\rho_n} \right)^\ell \frac{d^p V_j}{dx^p}(\xi) = 0.$$

Since $\rho_n > \rho_j$ for $m \leq j \leq n-1$, letting ℓ tend to infinity, we conclude that $\frac{d^p V_n}{dx^p}(\xi) = 0$. This would be true for all p , which is impossible by Cauchy's uniqueness theorem, or by Sturm's argument [35, § II]. \square

Remark 2.3. Assertion (3), and the fact that the zeros of Y are isolated, with finite multiplicities, are implicit in [36].

Lemma 2.4. *Assume that (1.6) holds. Let U denote any Y_k , and $U_1 = Y_{k+1}$. Let $\xi \in [\alpha, \beta]$ be a zero of U , of order $p \geq 2$. Then, there exist constants B_ξ and $B_{1,\xi}$, and smooth functions R_ξ and $R_{1,\xi}$, such that*

$$(2.9) \quad \begin{cases} U(x) = B_\xi(x - \xi)^p + (x - \xi)^{p+1}R_\xi(x), \\ U_1(x) = B_{1,\xi}(x - \xi)^{p-2} + (x - \xi)^{p-1}R_{1,\xi}(x), \\ \text{with } B_\xi B_{1,\xi} > 0. \end{cases}$$

Proof. [36, p. 439] Assume that ξ is a zero of order $p \geq 2$ of U , so that

$$U(\xi) = \dots = \frac{d^{p-1}U}{dx^{p-1}}(\xi) = 0$$

and

$$\frac{d^p U}{dx^p}(\xi) \neq 0.$$

Taylor's formula with integral remainder term, see Laplace [18], gives the existence of some function R_ξ such that

$$U(x) = B_\xi(x - \xi)^p + (x - \xi)^{p+1}R_\xi(x),$$

where

$$B_\xi = \frac{1}{p!} \frac{d^p U}{dx^p}(\xi) \neq 0.$$

Equation (2.8) implies that

$$(GU_1)(x) = p(p-1)B_\xi(x - \xi)^{p-2}K(x) + (x - \xi)^{p-1}S_\xi(x),$$

for some smooth function S_ξ . It follows that

$$U_1(x) = B_{1,\xi}(x - \xi)^{p-2} + (x - \xi)^{p-1}R_{1,\xi}(x),$$

for some function $R_{1,\xi}$, with $B_{1,\xi} = p(p-1)\frac{K(\xi)}{G(\xi)}B_\xi$.

In particular, $B_{1,\xi}B_\xi > 0$ and this proves the lemma. \square

Lemma 2.5. *Assume that (1.6) holds. Assume that $h \in [0, \infty[$, i.e., that the boundary condition at α is not the Dirichlet boundary condition. Let U denote any Y_k , $U_1 = Y_{k+1}$, and assume that $U(\alpha) = 0$. Then, α is a zero of U of even order, i.e., there exists $n_U \in \mathbb{N} \setminus \{0\}$ such that $\frac{d^p U}{dx^p}(\alpha) = 0$ for $0 \leq p \leq 2n_U - 1$ and $\neq 0$ for $p = 2n_U$.*

When $H \in [0, \infty[$, a similar statement holds at the boundary β .

Proof. [36, p. 440-441] Assume that $U(\alpha) = 0$. By Lemma 2.2, U does not vanish at infinite order at α , so that there exists $p \geq 1$ with

$$U(\alpha) = \dots = \frac{d^{p-1} Y_k}{dx^{p-1}}(\alpha) = 0$$

and

$$\frac{d^p U}{dx^p}(\alpha) \neq 0.$$

Taylor's formula with integral remainder term gives

$$U(x) = B_\alpha(x - \alpha)^p + (x - \alpha)^{p+1} R_\alpha(x),$$

where $B_\alpha = \frac{1}{p!} \frac{d^p U}{dx^p}(\alpha) \neq 0$.

The boundary condition at α implies that $\frac{dU}{dx}(\alpha) = 0$, and hence that $p \geq 2$. By Lemma 2.4, we can write

$$U_1(x) = B_{1,\alpha}(x - \alpha)^{p-2} + (x - \alpha)^{p-1} R_{1,\alpha}(x),$$

with $B_{1,\alpha} B_\alpha > 0$.

If $p = 2$, then $U_1(\alpha) \neq 0$. If $p > 2$, one can continue. If $p = 2q$, one arrives at

$$Y_{k+q}(x) = B_{k+q,\alpha} + (x - \alpha) R_{k+q,\alpha}(x),$$

with $Y_{k+q}(\alpha) = B_{k+q,\alpha}$ and $B_{k+q,\alpha} B_{k,\alpha} > 0$.

If $p = 2q + 1$, one arrives at

$$Y_{k+q}(x) = B_{k+q,\alpha}(x - \alpha) + (x - \alpha)^2 R_{k+q,\alpha}(x),$$

with $B_{k+q,\alpha} B_{k,\alpha} > 0$ and $\frac{dY_{k+q}}{dx}(\alpha) = B_{k+q,\alpha} \neq 0$. On the other-hand, since Y_{k+q} satisfies (1.2) and $Y_{k+q}(\alpha) = 0$, we must have $\frac{dY_{k+q}}{dx}(\alpha) = 0$, because $h > 0$. This yields a contradiction and proves that the case $p = 2q + 1$ cannot occur. The lemma is proved. \square

2.2. Counting zeros. Assume that (1.6) holds. Let U denote any Y_k , and $U_1 = Y_{k+1}$. They satisfy the relation (2.8).

From Lemma 2.2, we know that U cannot vanish at infinite order at a point $\xi \in [\alpha, \beta]$. If $\xi \in]\alpha, \beta[$ and $U(\xi) = 0$, we define the *multiplicity* $m(U, \xi)$ of the zero ξ by

$$(2.10) \quad m(U, \xi) = \min\{p \mid \frac{d^p U}{dx^p}(\xi) \neq 0\}.$$

From Lemma 2.5, we know that the multiplicity $m(U, \alpha)$ is even if $h \in [0, \infty[$, and that the multiplicity $m(U, \beta)$ is even if $H \in [0, \infty[$. We define the *reduced multiplicity* of α by

$$(2.11) \quad \bar{m}(U, \alpha) = \begin{cases} \frac{1}{2}m(U, \alpha) & \text{if } h \in [0, \infty[, \\ 0 & \text{if } h = \infty, \end{cases}$$

and a similar formula for the reduced multiplicity of β .

By Lemma 2.2, the function U has finitely many distinct zeros $\xi_1(U) < \xi_2(U) < \dots < \xi_p(U)$ in the interval $] \alpha, \beta[$. We define the *number of zeros of U in $] \alpha, \beta[$, counted with multiplicities*, by

$$(2.12) \quad N_m(U,] \alpha, \beta[) = \sum_{j=1}^p m(U, \xi_j(U)),$$

and use the notation $N_m(U)$ whenever the interval is clear.

We define the *number of zeros of U in $[\alpha, \beta]$, counted with multiplicities*, by

$$(2.13) \quad \bar{N}_m(U, [\alpha, \beta]) = \sum_{j=1}^p m(U, \xi_j(U)) + \bar{m}(U, \alpha) + \bar{m}(U, \beta),$$

and use the notation $\bar{N}_m(U)$ whenever the interval is clear.

We define the *number of zeros of U in $] \alpha, \beta[$ (multiplicities not accounted for)* by

$$(2.14) \quad N(U,] \alpha, \beta[) = p,$$

and use the notation $N(U)$ whenever the interval is clear.

Finally, we define the *number of sign changes of U in the interval $] \alpha, \beta[$* by

$$(2.15) \quad N_v(U,] \alpha, \beta[) = \sum_{j=1}^p \frac{1}{2} \left[1 - (-1)^{m(U, \xi_j(U))} \right].$$

2.3. Comparing the number of zeros of Y_k and Y_{k+1} . Assume that (1.6) holds. Let U be some Y_k and $U_1 = Y_{k+1}$. In this subsection, we show that the number of zeros of U_1 is not smaller than the number of zeros of U .

Lemma 2.6. *Let $\xi < \eta$ be two zeros of U in $[\alpha, \beta]$. Then, there exists some $a_{\xi, \eta} \in] \xi, \eta[$ such that $U(a_{\xi, \eta})U_1(a_{\xi, \eta}) < 0$.*

Remark 2.7. We do not assume that ξ, η are consecutive zeros. The point $a_{\xi, \eta}$ is a point at which U achieves a non-zero local extremum of U .

Proof. [36, p. 437] Since U cannot vanish identically in $]\xi, \eta[$ (see Lemma 2.2), there exists some $x_0 \in]\xi, \eta[$ such that $U(x_0) \neq 0$. Let $\varepsilon_0 = \text{sign}(U(x_0))$. Then $\varepsilon_0 U$ takes a positive value at x_0 , and hence $M := \sup\{\varepsilon_0 U(x) \mid x \in [\xi, \eta]\}$ is positive and achieved at some $a_{\xi, \eta} \in]\xi, \eta[$. Denote this point by a for short, then,

$$\varepsilon_0 U(a) > 0, \quad \frac{dU}{dx}(a) = 0 \quad \text{and} \quad \varepsilon_0 \frac{d^2U}{dx^2}(a) \leq 0.$$

It follows from (2.8) that $\varepsilon_0 U_1(a) < 0$, or equivalently, that $U(a)U_1(a) < 0$. The lemma is proved. \square

Lemma 2.8. *Let $\xi \in]\alpha, \beta[$. Assume that $U(\xi) = 0$, and that U does not change sign in $]\alpha, \xi[$. Then, there exists some $a_\xi \in [\alpha, \xi[$ such that $U(a_\xi)U_1(a_\xi) < 0$.*

Let $\eta \in]\alpha, \beta[$. Assume that $U(\eta) = 0$, and that U does not change sign in $]\eta, \beta[$. Then, there exists some $b_\eta \in]\eta, \beta[$ such that $U(b_\eta)U_1(b_\eta) < 0$.

Proof. [36, p. 438] Since U cannot vanish identically in $]\alpha, \xi[$ (see Lemma 2.2), there exists $x_0 \in]\alpha, \xi[$ such that $U(x_0) \neq 0$. Let $\varepsilon_\xi = \text{sign}(U(x_0))$. Since U does not change sign in $]\alpha, \xi[$, $\varepsilon_\xi U(x) \geq 0$ in $]\alpha, \xi[$. Then,

$$M_\xi := \sup\{\varepsilon_\xi U(x) \mid x \in [\alpha, \xi]\} > 0.$$

Let

$$a_\xi := \inf\{x \in [\alpha, \xi] \mid \varepsilon_\xi U(x) = M_\xi\}.$$

Then $a_\xi \in [\alpha, \xi[$.

If $a_\xi \in]\alpha, \xi[$, then $\varepsilon_\xi U(a_\xi) > 0$, $\frac{dU}{dx}(a_\xi) = 0$, and $\varepsilon_\xi \frac{d^2U}{dx^2}(a_\xi) \leq 0$. By (2.8), this implies that $\varepsilon_\xi U_1(a_\xi) < 0$. Equivalently, $U(a_\xi)U_1(a_\xi) < 0$.

Claim 2.9. *If $a_\xi = \alpha$, then $\varepsilon_\xi U(\alpha) > 0$, $h = 0$, $\frac{dU}{dx}(\alpha) = 0$, and $\varepsilon_\xi \frac{d^2U}{dx^2}(\alpha) \leq 0$.*

Proof of the claim. Assume that $a_\xi = \alpha$, then $\varepsilon_\xi U(\alpha) > 0$, and hence $h \neq \infty$. If h were in $]0, \infty[$, we would have $\varepsilon_\xi \frac{dU}{dx}(\alpha) = h\varepsilon_\xi U(\alpha) > 0$, and hence $a_\xi > \alpha$. It follows that the assumption $a_\xi = \alpha$ implies that $h = 0$ and $\frac{dU}{dx}(\alpha) = 0$. If $\varepsilon_\xi \frac{d^2U}{dx^2}(\alpha)$ were positive, we would have $a_\xi > \alpha$. Therefore, the assumption $a_\xi = \alpha$ also implies that $\varepsilon_\xi \frac{d^2U}{dx^2}(\alpha) \leq 0$. The claim is proved.

If $a_\xi = \alpha$, then by Claim 2.9 and (2.8), we have $\varepsilon_\xi U_1(\alpha) < 0$. Equivalently, $U(\alpha)U_1(\alpha) < 0$. The first assertion of the lemma is proved. The proof of the second assertion is similar. \square

Proposition 2.10. *Assume that (1.6) holds, and let $k \in \mathbb{Z}$. Then,*

$$(2.16) \quad N_v(Y_{k+1},]\alpha, \beta]) \geq N_v(Y_k,]\alpha, \beta]),$$

i.e., in the interval $]\alpha, \beta[$, the function Y_{k+1} changes sign at least as many times as the function Y_k .

Proof. [36, p. 437-439]

We keep the notation $U = Y_k$ and $U_1 = Y_{k+1}$. By Lemma 2.2, the functions U and U_1 have finitely many zeros in $]α, β[$, with finite multiplicities. Note that the sign changes of U correspond to the zeros of U with odd multiplicities. Since $α$ and $β$ are fixed, we skip the mention to the interval $]α, β[$ in the proof, and we examine several cases.

Case 1. If $N_v(U) = 0$, there is nothing to prove.

Case 2. Assume that $N_v(U) = 1$. Then U admits a unique zero $ξ ∈]α, β[$ having odd multiplicity. Without loss of generality, we may assume that $U ≥ 0$ in $]α, ξ[$ and $U ≤ 0$ in $]ξ, β[$. By Lemma 2.8, there exist $a ∈ [α, ξ[$ and $b ∈]ξ, β]$ such that $U_1(a) < 0$ and $U_1(b) > 0$.

It follows that the function U_1 vanishes and changes sign at least once in $]α, β[$, so that $N_v(U_1) ≥ 1 = N_v(U)$, which proves the lemma in Case 2.

Case 3. If $N_v(U) = 2$, the function U has exactly two zeros, having odd multiplicities, $ξ$ and $η$ in $]α, β[$, $α < ξ < η < β$, and we may assume that $U|_{]α, ξ[} ≥ 0$, $U|_{]ξ, η[} ≤ 0$, and $U|_{]η, β[} ≥ 0$. The arguments given in Case 2 imply that there exist $a ∈ [α, ξ[$ such that $U_1(a) < 0$ and $b ∈]η, β]$ such that $U_1(b) < 0$. In $]ξ, η[$ the function U does not vanish identically and therefore achieves a global minimum at a point c such that $U(c) < 0$, $\frac{dU}{dx}(c) = 0$, and $\frac{d^2U}{dx^2}(c) ≥ 0$. Equation(2.8) then implies that $U_1(c) > 0$.

We can conclude that the function U_1 vanishes and changes sign at least twice in $]α, β[$, so that $N_v(U_1) ≥ 2 = N_v(U)$.

Case 4. Assume that $N_v(U) = p ≥ 3$. Then, U has exactly p zeros, with odd multiplicities, in $]α, β[$, $α < ξ_1 < ξ_2 < \dots < ξ_p < β$, and one can assume that

$$\begin{aligned} U|_{]α, ξ_1[} &\geq 0, \quad (-1)^p U|_{]ξ_p, β[} \geq 0, \quad \text{and} \\ (-1)^i U|_{]ξ_i, ξ_{i+1}[} &\geq 0 \quad \text{for } 1 \leq i \leq p-1. \end{aligned}$$

One can repeat the arguments given in the Cases 2 and 3, and conclude that there exist a_0, \dots, a_p with $a_0 ∈ [α, ξ_1[$, $a_i ∈]ξ_i, ξ_{i+1}[$ for $1 ≤ i ≤ p-1$, and $a_p ∈]ξ_p, β]$ such that $(-1)^i U_1(a_i) < 0$.

We can then conclude that the function U_1 vanishes and changes sign at least p times in $]α, β[$, i.e. that $N_v(U_1) ≥ p = N_v(U)$.

This concludes the proof of Proposition 2.10. □

Proposition 2.11. *Assume that (1.6) holds. For any $k ∈ \mathbb{Z}$,*

$$(2.17) \quad N_m(Y_{k+1},]α, β[) \geq N_m(Y_k,]α, β[),$$

i.e., in the interval $]α, β[$, counting multiplicities of zeros, the function Y_{k+1} vanishes at least as many times as the function Y_k .

Proof. [36, p. 439-442] Let $U = Y_k$ and $U_1 = Y_{k+1}$. If U does not vanish in $] \alpha, \beta [$, there is nothing to prove. We now assume that U has at least one zero in $] \alpha, \beta [$. By Lemma 2.2, U and U_1 have finitely many zeros in $] \alpha, \beta [$. Let

$$\alpha < \xi_1 < \cdots < \xi_k < \beta$$

be the distinct zeros of U , with multiplicities $p_i = m(U, \xi_i)$ for $1 \leq i \leq k$. Let σ_0 be the sign of U in $] \alpha, \xi_1 [$, σ_i the sign of U in $] \xi_i, \xi_{i+1} [$ for $1 \leq i \leq k-1$, and σ_k the sign of U in $] \xi_k, \beta [$. Note that

$$\sigma_i = \text{sign} \left(\frac{d^{p_i} U}{dx^{p_i}}(\xi_i) \right) \text{ for } 1 \leq i \leq k.$$

By Lemma 2.8, there exist $a_0 \in [\alpha, \xi_1[$ and $a_k \in] \xi_k, \beta]$ such that $U(a_0)U_1(a_0) < 0$ and $U(a_k)U_1(a_k) < 0$. By Lemma 2.6, there exists $a_i \in] \xi_i, \xi_{i+1} [$, $1 \leq i \leq k-1$, such that $U(a_i)U_1(a_i) < 0$.

Summarizing, we have obtained:

$$(2.18) \quad \text{For } 0 \leq i \leq k, \quad U(a_i)U_1(a_i) < 0.$$

We have the relation

$$(2.19) \quad N_m(U,] \alpha, \beta [) = \sum_{i=1}^k N_m(U,] a_{i-1}, a_i [) = \sum_{i=1}^k p_i.$$

Indeed, for $1 \leq i \leq k$, the interval $] a_{i-1}, a_i [$ contains precisely one zero ξ_i of U , with multiplicity p_i .

For U_1 , we have the inequality

$$(2.20) \quad N_m(U_1,] \alpha, \beta [) \geq \sum_{i=1}^k N_m(U_1,] a_{i-1}, a_i [),$$

because U_1 might have zeros in the interval $] \alpha, a_0 [$ if $a_0 > \alpha$ (resp. in the interval $] a_k, \beta [$ if $a_k < \beta$).

Claim 2.12. For $1 \leq i \leq k$,

$$N_m(U_1,] a_{i-1}, a_i [) \geq N_m(U,] a_{i-1}, a_i [) = p_i.$$

To prove the claim, we consider several cases.

- If $p_i = 1$, then $U(a_{i-1})U(a_i) < 0$ and, by (2.18), $U_1(a_{i-1})U_1(a_i) < 0$, so that $N_m(U_1,] a_{i-1}, a_i [) \geq 1$.

- If $p_i \geq 2$, we apply Lemma (2.4) at ξ_i : there exist real numbers B, B_1 and smooth functions R and R_1 , such that, in a neighborhood of ξ_i ,

$$(2.21) \quad \begin{cases} U(x) &= B(x - \xi_i)^{p_i} + (x - \xi_i)^{p_i+1}R(x), \\ U_1(x) &= B_1(x - \xi_i)^{p_i-2} + (x - \xi_i)^{p_i-1}R_1(x), \end{cases}$$

where $\text{sign}(B) = \text{sign}(B_1) = \sigma_i$.

We now use (2.18) and the fact that $\text{sign}(U(a_i)) = \sigma_i$.

- ◊ If $p_i \geq 2$ is odd, then $\sigma_{i-1}\sigma_i = -1$. It follows that

$$\sigma_i U(a_i) < 0 \text{ and } \sigma_i U(a_{i-1}) > 0.$$

By (2.21), for ε small enough, we also have

$$\sigma_i U(\xi_i + \varepsilon) > 0 \text{ and } \sigma_i U(\xi_i - \varepsilon) < 0.$$

This means that U_1 vanishes at order $p_i - 2$ at ξ_i , and at least once in the intervals $]a_{i-1}, \xi_i - \varepsilon[$ and $]\xi_i + \varepsilon, a_i[$, so that

$$N_m(U_1,]a_{i-1}, a_i]) \geq p_i - 2 + 2 = p_i = N_m(U,]a_{i-1}, a_i]).$$

◇ If $p_i \geq 2$ is even, then $\sigma_{i-1}\sigma_i = 1$. It follows that

$$\sigma_i U(a_i) < 0 \text{ and } \sigma_i U(a_{i-1}) < 0.$$

By (2.21), for ε small enough, we also have

$$\sigma_i U(\xi_i + \varepsilon) > 0 \text{ and } \sigma_i U(\xi_i - \varepsilon) > 0.$$

This means that U_1 vanishes at order $p_i - 2$ at ξ_i , and at least once in the intervals $]a_{i-1}, \xi_i - \varepsilon[$ and $]\xi_i + \varepsilon, a_i[$, so that

$$N_m(U_1,]a_{i-1}, a_i]) \geq p_i - 2 + 2 = p_i = N_m(U,]a_{i-1}, a_i]).$$

The claim is proved, and the proposition as well. \square

Proposition 2.13. *Assume that (1.6) holds. For any $k \in \mathbb{Z}$,*

$$(2.22) \quad \overline{N}_m(Y_{k+1}, [\alpha, \beta]) \geq \overline{N}_m(Y_k, [\alpha, \beta]),$$

i.e., in the interval $[\alpha, \beta]$, counting multiplicities of interior zeros, and reduced multiplicities of α and β , the function Y_{k+1} vanishes at least as many times as the function Y_k .

Proof. [36, p. 440-442] Recall that the reduced multiplicity of α (resp. β) is zero if Dirichlet condition holds at α (resp. at β) or if $U(\alpha) \neq 0$ (resp. $U(\beta) \neq 0$). Furthermore, according to Lemma 2.5, if $h \in [0, \infty[$ and $U(\alpha) = 0$ (resp. if $H \in [0, \infty[$ and $U(\beta) = 0$), then $m(U, \alpha) = 2p$ (resp. $m(U, \beta) = 2q$).

Case 1. Assume that $N_m(U,]\alpha, \beta]) = 0$. Without loss of generality, we may assume that $U > 0$ in $]\alpha, \beta[$.

- If $U(\alpha) \neq 0$ and $U(\beta) \neq 0$, there is nothing to prove.
- Assume that $U(\alpha) = U(\beta) = 0$. Then, there exists $a \in]\alpha, \beta[$ such that

$$U(a) = \sup\{U(x) \mid x \in [\alpha, \beta]\},$$

with

$$U(a) > 0, \quad \frac{dU}{dx}(a) = 0, \quad \text{and} \quad \frac{d^2U}{dx^2}(a) \leq 0.$$

It follows from (2.8) that $U_1(a) < 0$, and that

$$(2.23) \quad \overline{N}_m(U_1, [\alpha, \beta]) = \overline{N}(U_1, [\alpha, a]) + \overline{N}(U_1, [a, \beta]).$$

It now suffices to look separately at the intervals $[\alpha, a]$ and $[a, \beta]$.

◇ Interval $[\alpha, a]$. If the Dirichlet condition holds at α , there is nothing to prove. If $h \in [0, \infty[$, $m(U, \alpha) = 2p \geq 2$ and, by Lemma 2.4,

$$(2.24) \quad \begin{aligned} U(x) &= B(x - \alpha)^{2p} + (x - \alpha)^{2p+1}R(x), \\ U_1(x) &= B_1(x - \alpha)^{2p-2} + (x - \alpha)^{2p-1}R_1(x), \\ &\text{with } B > 0 \text{ and } B_1 > 0. \end{aligned}$$

It follows that $U_1(\alpha + \varepsilon) > 0$ for any positive ε small enough so that $N_m(U_1,]\alpha, a]) \geq 1$. It follows that

$$(2.25) \quad \begin{aligned} \overline{N}_m(U_1, [\alpha, a]) &= \overline{m}(U_1, \alpha) + N_m(U_1,]\alpha, a]) \geq p - 1 + 1, \\ \text{i.e.} \\ \overline{N}_m(U_1, [\alpha, a]) &\geq \overline{N}(U, [\alpha, a]). \end{aligned}$$

◇ Interval $[a, \beta]$. The proof is similar.

- Assume that $U(\alpha) = 0$ and $U(\beta) \neq 0$. The proof is similar to the previous one with $a \in]a, \beta]$.
- Assume that $U(\alpha) \neq 0$ and $U(\beta) = 0$. The proof is similar to the previous one with $a \in [\alpha, a[$.

Case 2. Assume that $N_m(U,]\alpha, \beta]) \geq 1$.

- If $U(\alpha) \neq 0$ (resp. $U(\beta) \neq 0$), there is nothing to prove for the boundary α (resp. β).
- If $U(\alpha) = 0$ (resp. $U(\beta) = 0$), the number a_0 (resp. a_k) which appears in the proof of Proposition 2.11 belongs to the open interval $]\alpha, \xi_1[$ (resp. to the open interval $]\xi_k, \beta[$), where ξ_1 (resp. ξ_k) is the smallest (resp. largest) zero of U in $]\alpha, \beta[$. We can then apply the proof of Step. 1 to the interval $[\alpha, a_0]$ (resp. to the interval $[a_k, \beta]$) to prove that $\overline{N}_m(U_1, [\alpha, a_0]) \geq \overline{N}_m(U, [\alpha, a_0])$ (resp. to prove that $\overline{N}_m(U_1, [a_k, \beta]) \geq \overline{N}_m(U, [a_k, \beta])$). This proves Proposition 2.13. \square

We can now prove Sturm's refined version of Theorem 1.4.

Theorem 2.14. *Assume that (1.6) holds. With the notation of Sections 1 and 2.2, let Y be the linear combination*

$$(2.26) \quad Y = \sum_{p=m}^n A_p V_p,$$

where $1 \leq m \leq n$, and where $\{A_p, m \leq p \leq n\}$ are real constants such that $A_m^2 + \dots + A_n^2 \neq 0$, which is equivalent to saying that $Y \neq 0$.

Then, with the notation of Subsection 2.2,

$$(2.27) \quad N_v(Y,]\alpha, \beta]) \leq N_m(Y,]\alpha, \beta]) \leq \overline{N}_m(Y, [\alpha, \beta]),$$

$$(2.28) \quad (m - 1) \leq N_v(Y,]\alpha, \beta]) \text{ and } \overline{N}_m(Y, [\alpha, \beta]) \leq (n - 1).$$

Proof. [36, p. 442] Let $N(V)$ be any of the above functions. We may of course assume that $A_m \neq 0$ and $A_n \neq 0$. In the preceding lemmas, we have proved that $N(Y_{k+1}) \geq N(Y_k)$ for any $k \in \mathbb{Z}$, which can also be rewritten as

$$(2.29) \quad N(Y_{(-k)}) \leq N(Y) \leq N(Y_k) \quad \text{for any } k \geq 1.$$

Letting k tend to infinity, we see that

$$(2.30) \quad N(V_m) \leq N(Y) \leq N(V_n),$$

and we can apply Theorem 1.3. \square

Remark. For a complete proof of the limiting argument when k tends to infinity, we refer to Appendix A.

3. LIOUVILLE'S APPROACH TO THEOREM 1.4

3.1. Main statement. We keep the notation of Section 2. Starting from a linear combination Y as in (2.1), Liouville also considers the family Y_k given by (2.4) and shows that the number of zeros of Y_{k+1} is not smaller than the number of zeros of Y_k . His proof is based on a generalization of Rolle's theorem.

Remark 3.1. In his proof, Liouville [21] only considers the zeros in the open interval $] \alpha, \beta [$.

As in Section 2, for $1 \leq m \leq n$, we fix $Y = \sum_{j=m}^n A_j V_j$, a linear combination of eigenfunctions of the eigenvalue problem (1.1)–(1.3), and we assume that $A_m A_n \neq 0$, see Remark 2.1.

Theorem 3.2. *Counting multiplicities, the function Y has, in the interval $] \alpha, \beta [$,*

- (1) *at most $(n - 1)$ zeros,*
- and*
- (2) *at least $(m - 1)$ zeros.*

Proof. Liouville uses the following version of Rolle's theorem (Michel Rolle (1652-1719) was a French mathematician). This version of Rolle's theorem seems to go back to Cauchy and Lagrange.

Lemma 3.3. *Let f be a function in $] \alpha_0, \beta_0 [$. Assume that*

$$f(x') = f(x'') = 0 \quad \text{for some } x', x'', \alpha_0 < x' < x'' < \beta_0.$$

- (1) *If the function f is differentiable, and has $\nu - 1$ distinct zeros in the interval $] x', x'' [$, then the derivative f' has at least ν distinct zeros in $] x', x'' [$.*
- (2) *If the function f is smooth, and has $\mu - 1$ zeros counted with multiplicities in the interval $] x', x'' [$, then the derivative f' has at least μ zeros counted with multiplicities in $] x', x'' [$.*

Proof of the lemma. Call $x_1 < x_2 < \cdots < x_{\nu-1}$ the distinct zeros of f in $]x', x''[$. Since $f(x') = f(x'') = 0$, by Rolle's theorem [29], the function f' vanishes at least once in each open interval determined by the x_j , $1 \leq j \leq \nu - 1$, as well as in the intervals $]x', x_1[$ and $]x_{\nu-1}, x''[$. It follows that f' has at least ν distinct zeros in $]x', x''[$, which proves the first assertion.

Call m_j the multiplicity of the zero x_j , $1 \leq j \leq \nu - 1$. Then f' has at least ν zeros, one in each of the open intervals determined by x', x'' and the x_j 's, and has a zero at each x_j with multiplicity $m_j - 1$, provided that $m_j > 1$. It follows that the number of zeros of f' in $]x', x''[$, counting multiplicities, is at least

$$\sum_{j=1}^{\nu-1} (m_j - 1) + \nu = \sum_{j=1}^{\nu-1} m_j + 1,$$

which proves the second assertion. \square

3.2. Proof of the assertion “ Y has at most $(n - 1)$ zeros in $]\alpha, \beta[$, counting multiplicities”.

Write (1.1) for V_1 and for V_p , for some $m \leq p \leq n$. Multiply the first equation by $-V_p$, the second by V_1 , and add the resulting equations. Then

$$(3.1) \quad V_1 \frac{d}{dx} \left(K \frac{dV_p}{dx} \right) - V_p \frac{d}{dx} \left(K \frac{dV_1}{dx} \right) + (\rho_p - \rho_1) G V_1 V_p = 0.$$

Use the identity

$$(3.2) \quad V_1 \frac{d}{dx} \left(K \frac{dV_p}{dx} \right) - V_p \frac{d}{dx} \left(K \frac{dV_1}{dx} \right) = \frac{d}{dx} \left(V_1 K \frac{dV_p}{dx} - V_p K \frac{dV_1}{dx} \right),$$

and integrate from α to t to get the identity

$$(3.3) \quad (\rho_1 - \rho_p) \int_{\alpha}^t G V_1 V_p dx = K(t) \left(V_1(t) \frac{dV_p}{dx}(t) - V_p(t) \frac{dV_1}{dx}(t) \right).$$

Here we use the boundary condition (1.2) which implies that

$$\left(V_1(\alpha) \frac{dV_p}{dx}(\alpha) - V_p(\alpha) \frac{dV_1}{dx}(\alpha) \right) = 0.$$

Multiplying the preceding identity by A_p , and summing for p from m to n , we obtain

$$(3.4) \quad \int_{\alpha}^t G V_1 \sum_{p=m}^n (\rho_1 - \rho_p) A_p V_p dx = K(t) \left(V_1 \frac{dY}{dx} - Y \frac{dV_1}{dx} \right) (t).$$

or

$$(3.5) \quad \int_{\alpha}^t G V_1 \sum_{p=m}^n (\rho_1 - \rho_p) A_p V_p dx = K(t) V_1^2(t) \frac{d}{dt} \left(\frac{Y}{V_1} \right) (t),$$

where we have used the fact that the function V_1 does not vanish in the interval $] \alpha, \beta [$.

Let $\Psi(x) = \frac{Y}{V_1}(x)$. The zeros of Y in $] \alpha, \beta [$ are the same as the zeros of Ψ . Let μ be the number of zeros of Y , counted with multiplicities. By Lemma 3.3, Assertion (2), $\frac{d\Psi}{dx}$ has at least $\mu - 1$ zeros in $] \alpha, \beta [$, and hence so does the left-hand side of (3.5),

$$\int_{\alpha}^t GV_1 \sum_{p=m}^n (\rho_1 - \rho_p) A_p V_p dx.$$

On the other hand, this function vanishes at α and β (because of the boundary condition (1.3) or orthogonality). By Lemma 3.3, its derivative,

$$(3.6) \quad V_1 \sum_{p=m}^n (\rho_1 - \rho_p) A_p V_p$$

has at least μ zeros counted with multiplicities in $] \alpha, \beta [$. We have proved the following

Lemma 3.4. *If the function $Y = \sum_{p=m}^n A_p V_p$ has at least μ zeros counted with multiplicities in the interval $] \alpha, \beta [$, then the function $Y_1 = \sum_{p=m}^n (\rho_1 - \rho_p) A_p V_p$ has at least μ zeros, counted with multiplicities, in $] \alpha, \beta [$.*

Applying this lemma iteratively, we deduce that if Y has at least μ zeros counted with multiplicities in $] \alpha, \beta [$, then, for any $k \geq 1$, the function

$$(3.7) \quad Y_k = \sum_{p=m}^n (\rho_1 - \rho_p)^k A_p V_p$$

has at least μ zeros, counted with multiplicities, in $] \alpha, \beta [$.

We may of course assume that the coefficient A_n is non-zero. The above assertion can be rewritten as the statement:

For all $k \geq 0$, the equation

$$(3.8) \quad A_m \left(\frac{\rho_m - \rho_1}{\rho_n - \rho_1} \right)^k V_m + \cdots + A_{n-1} \left(\frac{\rho_{n-1} - \rho_1}{\rho_n - \rho_1} \right)^k V_{n-1} + A_n V_n = 0$$

has at least μ solutions in $] \alpha, \beta [$, counting multiplicities.

Letting k tend to infinity, and using the fact that V_n has exactly $(n - 1)$ zeros in $] \alpha, \beta [$, this implies that $\mu \leq (n - 1)$. This proves the first assertion. \square

3.3. Proof of the assertion “ Y has at least $(m - 1)$ zeros in $] \alpha, \beta[$, counting multiplicities”.

We have seen that the number of zeros of Y_k is less than or equal to the number of zeros of the function Y_{k+1} . This assertion actually holds for any $k \in \mathbb{Z}$, and can also be rewritten as, for any $k \geq 0$,

$$(3.9) \quad N_m(Y_{-k}) \leq N_m(Y),$$

where, for $k \geq 0$,

$$(3.10) \quad Y_{-k} = A_m(\rho_m - \rho_1)^{-k} V_m + \cdots + A_n(\rho_n - \rho_1)^{-k} V_n,$$

and we can again let k tend to infinity. The second assertion is proved and Theorem 3.2 as well. \square

3.4. Liouville’s 2nd approach to the 2nd part of Theorem 3.2.

If the function Y has μ_1 distinct zeros, then it changes sign μ times, at points a_i which satisfy $\alpha < a_1 < \cdots < a_\mu < \beta$, with $\mu_1 \geq \mu$.

Claim 3.5. *The function Y changes sign at least $(m - 1)$ times in the interval $] \alpha, \beta[$.*

Proof of the claim. Assume, by contradiction, that $\mu \leq (m - 2)$. Consider the function

$$(3.11) \quad x \mapsto W(x) := \Delta(a_1, \dots, a_\mu; x),$$

where the function Δ is defined as the determinant

$$(3.12) \quad \begin{vmatrix} V_1(a_1) & V_1(a_2) & \cdots & V_1(a_\mu) & V_1(x) \\ V_2(a_1) & V_2(a_2) & \cdots & V_2(a_\mu) & V_2(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{\mu+1}(a_1) & V_{\mu+1}(a_2) & \cdots & V_{\mu+1}(a_\mu) & V_{\mu+1}(x) \end{vmatrix}.$$

The function W vanishes at the points $a_i, 1 \leq i \leq \mu$. According to the first part in Theorem 3.2, W being a linear combination of the first $\mu + 1$ eigenfunctions, vanishes at most μ times in $] \alpha, \beta[$, counting multiplicities. This implies that each zero a_i of W has order one, and that W does not have any other zero in $] \alpha, \beta[$. It follows that the function YW vanishes only at the points $\{a_i\}, 1 \leq i \leq \mu$, and that it does not change sign. We can assume that $YW \geq 0$. On the other hand, we have

$$(3.13) \quad \int_\alpha^\beta GYW \, dx = 0,$$

because Y involves the functions V_p with $p \geq m$ and W the functions V_q with $q \leq \mu + 1 \leq m - 1$. This gives a contradiction. \square

Remark 3.6. Liouville does actually not use the determinant (3.12), but a similar approach, see [20, p. 259], Lemme 1^{er}. The determinant Δ appears in [31, Section 142].

Remark 3.7. The arguments in Subsection 3.2, using Assertion (1) of Lemma 3.3, instead of Assertion (2), yield an upper bound on the number of zeros of Y , multiplicities not accounted for. This estimate holds under weaker regularity assumptions, namely only assuming that the functions G, L are continuous, and that the function K is C^1 .

4. MATHEMATICAL CONTEXT OF STURM'S PAPERS. STURM'S MOTIVATIONS AND IDEAS

4.1. **On Sturm's style.** Sturm's papers [35, 36] are written in French, and quite long, about 80 pages each. One difficulty in reading them is the lack of layout structure. The papers are divided into sequences of sections, without any title. Most results are stated without tags, "Theorem" and the like, and only appear in the body of the text. For example, [35] only contains one theorem stated as such, see § XII, p. 125. In order to have an overview of the results contained in [35], the reader should look at the announcement [33]. Theorem 1.4 is stated in [34].

For a more thorough analysis of Sturm's papers on differential equations, we refer to [23, 12]. We refer to [5, 30] for the relationships between Theorem 1.3 and Sturm's theorem on the number of real roots of real polynomials.

4.2. **Sturm's motivations.** Sturm's motivations come from mathematical physics, and more precisely, from the problem of heat diffusion in a non-homogeneous bar. He considers the heat equation,

$$(4.1) \quad G \frac{\partial u}{\partial t} = \frac{\partial}{\partial x} \left(K \frac{\partial u}{\partial x} \right) - Lu, \text{ for } (x, t) \in]\alpha, \beta[\times \mathbb{R}_+,$$

with boundary conditions

$$(4.2) \quad \begin{cases} K(\alpha) \frac{\partial u}{\partial x}(\alpha, t) - h u(\alpha, t) = 0, \\ K(\beta) \frac{\partial u}{\partial x}(\beta, t) + H u(\beta, t) = 0, \end{cases}$$

for all $t > 0$, and the initial condition

$$(4.3) \quad u(x, 0) = f(x), \text{ for } x \in]\alpha, \beta[,$$

where f is a given function.

The functions K, G, L and the constants h, H describe the physical properties of the bar, see [36, Introduction, p. 376]. Sturm refers to the book of Siméon Denis Poisson [28], rather than to Fourier's book [11], because Poisson's equations are more general, see [30, Chap. III].

The boundary conditions (1.2)-(1.3) and (4.2) first appeared in the work of Fourier [11] but are called "Robin's condition" in the recent literature. Victor Gustave Robin (1855-1897) was a French mathematician.

As was popularized by Fourier and Poisson, in order to solve (4.1), Sturm uses the method of separation of variables, and is therefore led to the eigenvalue problem (1.1)–(1.3).

4.3. Sturm’s assumptions. In [35, 36], Sturm implicitly assumes that the functions K, G, L are C^∞ and, explicitly, that K is positive, see [35, p. 108]. For the eigenvalue problem, he also assumes that G, L are positive, see [36, p. 381]. In [36, p. 394], he mentions that L could take negative values, and implicitly assumes, in this case, that $\frac{L}{G}$ is bounded from below.

In [21], Liouville does not mention any regularity assumption on the functions G, K, L . He however indicates a regularity assumption (piecewise C^2 functions) in a previous paper, [20, Footnote (*), p. 256].

4.4. Sturm’s originality. Before explaining Sturm’s proofs, we would like to insist on the *originality* of his approach. Indeed, unlike his predecessors, Sturm does not look for explicit solutions of the differential equation (4.4) (i.e., solutions in closed form, or given as sums of series or as integrals), but he rather looks for *qualitative properties* of the solutions, properties which can be deduced directly from the differential equation itself. The following excerpts are taken from [35, Introduction]¹.

On ne sait [ces équations] les intégrer que dans un très petit nombre de cas particuliers hors desquels on ne peut pas même en obtenir une intégrale première ; et lors même qu’on possède l’expression de la fonction qui vérifie une telle équation, soit sous forme finie, soit en série, soit en intégrales définies ou indéfinies, il est le plus souvent difficile de reconnaître dans cette expression la marche et les propriétés caractéristiques de cette fonction. . . .

S’il importe de pouvoir déterminer la valeur de la fonction inconnue pour une valeur isolée quelconque de la variable dont elle dépend, il n’est pas moins nécessaire de discuter la marche de cette fonction, ou en d’autres termes, d’examiner la forme et les sinuosités de la courbe dont cette fonction serait l’ordonnée variable, en prenant pour abscisse la variable indépendante. Or on peut arriver à ce but par la seule considération des équations différentielles elles-mêmes, sans qu’on ait besoin de leur intégration. Tel est l’objet du présent mémoire. . . .

4.5. Sturm and the existence and uniqueness theorem for ordinary differential equation. In [35, p. 108], Sturm considers the differential equation

$$\frac{d}{dx} \left(K \frac{dV}{dx} \right) + GV = 0, \quad (I)$$

¹See Appendix B for the English translations of the citations in French.

and takes the existence and uniqueness theorem for granted. More precisely, he claims [35, p. 108], without any reference whatsoever, *L'intégrale complète de l'équation (I) doit contenir deux constantes arbitraires, pour lesquelles on peut prendre les valeurs de V et de $\frac{dV}{dx}$ correspondantes à une valeur particulière de x . Lorsque ces valeurs sont fixées, la fonction V est entièrement définie par l'équation (I), elle a une valeur déterminée et unique pour chaque valeur de x .*

On the other hand, he gives two arguments for the fact that a solution of (I) and its derivative cannot vanish simultaneously at a point without vanishing identically, see [35, § II]. When the coefficients K, G of the differential equation depend upon a parameter m , e.g. continuously, Sturm also takes for granted the fact that the solution $V(x, m)$, and its zeros, depend continuously on m .

In [36, § II], Sturm mentions the existence proof given by Liouville in [20], see also [19]. According to [13], Augustin-Louis Cauchy may have presented the existence and uniqueness theorem for ordinary differential equations in his course at École polytechnique as early as in the year 1817-1818. Following a recommendation of the administration of the school, Cauchy delivered the notes of his lectures in 1824, see [7] and, in particular, the introduction by Christian Gilain who discovered these notes in 1974. These notes apparently had a limited distribution. Liouville entered the École polytechnique in 1825, and there attended the mathematics course given by Ampère² (as a matter of fact Ampère and Cauchy gave the course every other year, alternatively). Liouville's proof of the existence theorem for differential equations in [19], à la Picard but before Picard, though limited to the particular case of 2nd order linear equations, might be the first well circulated proof of an existence theorem for differential equations, see [22, § 34]. Cauchy's theorem was later popularized in the second volume of Moigno's book, published in 1844, see [24], "Vingt-sixième Leçon" § 159, pp. 385–396.

4.6. Sturm's proof of Theorem 1.3. Theorem 1.3 is proved in [36]. For the first assertion, see § III (p. 384) to VII; for the second assertion, see § VIII (p. 396) to X.

The proof is based on the paper [35] in which Sturm studies the zeros of the solution of the initial value problem,

$$(4.4) \quad \frac{d}{dx} \left(K(x, m) \frac{dV}{dx}(x, m) \right) + G(x, m)V(x, m) = 0,$$

$$(4.5) \quad \left(K \frac{dV}{dx} - hV \right) (\alpha, m) = 0.$$

Here K, G are assumed to be functions of x depending on a real parameter m , with K positive (the constants h and H may also depend

²We are grateful to J. Lützen for providing this information.

on the parameter m). The solution $V(x, m)$ is well defined up to a scaling factor. The main part of [35] is devoted to studying how the zeros of the function $V(x, m)$ (and other related functions) depend on the parameter m , see [35, § XII, p. 125]. While developing this program, Sturm proves the *oscillation, separation and comparison theorems* which nowadays bear his name, [35, § XV, XVI and XXXVII].

The eigenvalue problem (1.1)–(1.3) itself is studied in [36]. For this purpose, Sturm considers the functions

$$K(x, r) \equiv K(x) \text{ and } G(x, r) = rG(x) - L(x),$$

the solution $V(x, r)$ of the corresponding initial value problem (4.4)–(4.5), and applies the results and methods of [35].

The spectral data of the eigenvalue problem (1.1)–(1.3) are determined by the following transcendental equation in the spectral parameter r ,

$$(4.6) \quad K(\beta) \frac{dV}{dx}(\beta, r) + HV(\beta, r) = 0.$$

4.7. Sturm’s two proofs of Theorem 1.4. Theorem 1.4 appears in [36, § XXV, p. 431], see also the announcement [34].

Sturm’s general motivation, see the introductions to [35] and [36], was the investigation of heat diffusion in a (non-homogeneous) bar, whose physical properties are described by the functions K, G, L . He first obtained Theorem 1.4 as a corollary of a much deeper theorem which describes the behaviour, as time varies, of the x -zeros of a solution $u(x, t)$ of the heat equation (4.1)–(4.3). When the initial temperature $u(x, 0)$ is given by a linear combination of simple states,

$$(4.7) \quad u(x, 0) = Y(x) = \sum_{j=m} A_j V_j$$

the function $u(x, t)$ is given by

$$(4.8) \quad u(x, t) = \sum_{j=m}^n e^{-t\rho_j} A_j V_j.$$

When t tends to infinity, the x -zeros of $u(x, t)$ approach those of V_p , where p is the least integer $j, m \leq j \leq n$ such that $A_j \neq 0$.

J. Liouville, who was aware of Theorem 1.4, made use of it in [20], and provided a purely “ordinary differential equation” (o.d.e.) proof in [21], a few months before the actual publication of [36]. This induced Sturm to provide two proofs of Theorem 1.4 in [36], his initial proof using the heat equation, and another proof based only on the sole ordinary differential equation. The proofs of Sturm actually give a more precise result. In [36, p. 379], Sturm writes,

M. Liouville a démontré directement ce théorème [Theorem 1.4], qui n’était pour moi qu’un corollaire du précédent, sans s’occuper du cas

particulier où la fonction serait nulle à l'une des extrémités de la barre. J'en ai aussi trouvé après lui une autre démonstration directe que je donne dans ce mémoire. M. Liouville a fait usage du même théorème dans un très beau Mémoire qu'il a publié dans le numéro de juillet de son journal et qui a pour objet le développement d'une fonction arbitraire en une série composée de fonctions V que nous avons considérées.

The time independent analog to studying the behaviour of the x -zeros of (4.8) is to study the behaviour of the zeros of the family of functions $\{Y_k\}_{k \in \mathbb{Z}}$, where

$$(4.9) \quad Y_k(x) = \sum_{j=m} A_j \rho_j^k V_j,$$

as k tends to infinity.

APPENDIX A. THE LIMITING ARGUMENT IN (3.8)

Recall that we assume that $A_n \neq 0$. Define

$$(A.1) \quad \omega = \left(\frac{\rho_{n-1} - \rho_1}{\rho_n - \rho_1} \right)^k.$$

One can rewrite (3.8) as

$$V_n(x) + \omega \Pi(x) = 0,$$

where

$$(A.2) \quad \Pi(x) = \sum_{p=m}^{n-1} \frac{A_p}{A_n} \left(\frac{\rho_p - \rho_1}{\rho_{n-1} - \rho_1} \right)^k V_p.$$

It follows that Π is uniformly bounded by

$$(A.3) \quad |\Pi(x)| \leq M := n \max_p \left| \frac{A_p}{A_n} \right| \max_{[\alpha, \beta]} \sup |V_p|.$$

Similarly,

$$(A.4) \quad \left| \frac{d\Pi}{dx}(x) \right| \leq N := n \max_p \left| \frac{A_p}{A_n} \right| \max_{[\alpha, \beta]} \sup \left| \frac{dV_p}{dx} \right|.$$

Call $\xi_1 < \xi_2 < \dots < \xi_{n-1}$ the zeros of the function V_n in the interval $] \alpha, \beta [$.

- Assume that $V_n(\alpha) \neq 0$ and $V_n(\beta) \neq 0$.

Since $\frac{dV_n}{dx}(\xi_i) \neq 0$, there exist $\delta_1, \varepsilon_1 > 0$ such that $|\frac{dV_n}{dx}(x)| \geq \varepsilon_1$ for $x \in [\xi_i - \delta_1, \xi_i + \delta_1]$, and $|V_n(x)| \geq \varepsilon_1$ in $[\alpha, \beta] \setminus \cup [\xi_i - \delta_1, \xi_i + \delta_1]$.

For k large enough, we have $\omega M, \omega N \leq \varepsilon_1/2$. It follows that in the interval $[\xi_i - \delta_1, \xi_i + \delta_1]$,

$$\left| \frac{d}{dx}(V_n + \omega \Pi) \right| \geq \left| \frac{dV_n}{dx} \right| - \omega N \geq \varepsilon_1/2.$$

Furthermore,

$$V_n(\xi_i \pm \delta_1) + \omega\Pi(\xi_i \pm \delta_1) \geq |V_n(\xi_i \pm \delta_1)| - \omega M \geq \varepsilon_1/2.$$

Since $V_n(\xi_i + \delta_1)V_n(\xi_i - \delta_1) < 0$, we can conclude that the function $V_n + \omega\Pi$ has exactly one zero in each interval $]\xi_i - \delta_1, \xi_i + \delta_1[$.

In $[\alpha, \beta] \setminus \cup]\xi_i - \delta_1, \xi_i + \delta_1[$, we have

$$|V_n(x) + \omega\Pi(x)| \geq |V_n(x)| - \omega M \geq \varepsilon_1/2,$$

which implies that $V_n(x) + \omega\Pi(x) \neq 0$.

- Assume that $V_n(\alpha) = 0$ and $V_n(\beta) \neq 0$. This corresponds to the case $h = +\infty$ and $H \neq +\infty$. Hence the V_j verify Dirichlet at α and Π verifies Dirichlet at α . Observing that $V_n'(\alpha) \neq 0$, it is immediate to see that there exists $\delta_1 > 0$, such that, for k large enough, $V_n(x) + \omega\Pi(x)$ has only α as zero in $[\alpha, \alpha + \delta_1]$.

- The other cases are treated in the same way. □

APPENDIX B. TRANSLATION FOR THE CITATIONS FROM STURM'S PAPERS [35, 36]

Citation from [35, Introduction].

On ne sait [ces équations] les intégrer que dans un très petit nombre de cas particuliers hors desquels on ne peut pas même en obtenir une intégrale première ; et lors même qu'on possède l'expression de la fonction qui vérifie une telle équation, soit sous forme finie, soit en série, soit en intégrales définies ou indéfinies, il est le plus souvent difficile de reconnaître dans cette expression la marche et les propriétés caractéristiques de cette fonction. ...

S'il importe de pouvoir déterminer la valeur de la fonction inconnue pour une valeur isolée quelconque de la variable dont elle dépend, il n'est pas moins nécessaire de discuter la marche de cette fonction, ou en d'autres termes, d'examiner la forme et les sinuosités de la courbe dont cette fonction serait l'ordonnée variable, en prenant pour abscisse la variable indépendante. Or on peut arriver à ce but par la seule considération des équations différentielles elles-mêmes, sans qu'on ait besoin de leur intégration. Tel est l'objet du présent mémoire. ...

One only knows how to integrate these equations in a very small number of particular cases, and one can otherwise not even obtain a first integral; even when one knows the expression of the function which satisfies such an equation, in finite form, as a series, as integrals either definite or indefinite, it is most generally difficult to recognize in this expression the behaviour and the characteristic properties of this function. ...

Although it is important to be able to determine the value of the unknown function for an isolated value of the variable it depends upon, it is not less necessary to discuss the behaviour of this function, or otherwise stated, the form and the twists and turns of the curve whose ordinate would be the function, and the abscissa the independent variable. It turns out that one can achieve this goal by the sole consideration of the differential equation themselves, without having to integrate them. This is the purpose of the present memoir. ...

Citation from [35, p. 108].

L'intégrale complète de l'équation (I) doit contenir deux constantes arbitraires, pour lesquelles on peut prendre les valeurs de V et de $\frac{dV}{dx}$ correspondantes à une valeur particulière de x . Lorsque ces valeurs sont fixées, la fonction V est entièrement définie par l'équation (I), elle a une valeur déterminée et unique pour chaque valeur de x .

The complete integral of equation (I) must contain two arbitrary constants, for which one can take the values of V and of $\frac{dV}{dx}$ corresponding to some particular value of x . Once these values are fixed, the function V is fully determined by equation (I), it has a uniquely determined value for each value of x .

Citation from [36, p. 379].

M. Liouville a démontré directement ce théorème, qui n'était pour moi qu'un corollaire du précédent, sans s'occuper du cas particulier où la fonction serait nulle à l'une des extrémités de la barre. J'en ai aussi trouvé après lui une autre démonstration directe que je donne dans ce mémoire. M. Liouville a fait usage du même théorème dans un très beau Mémoire qu'il a publié dans le numéro de juillet de son journal et qui a pour objet le développement d'une fonction arbitraire en une série composée de fonctions V que nous avons considérées.

M. Liouville gave a direct proof of this theorem, which for me was a mere corollary of the preceding one, without taking care of the particular case in which the function vanishes at one of the extremities of the bar. I have also found, after him, another direct proof which I give in this memoir. M. Liouville made use of the same theorem in a very nice memoir which he published in the July issue of his journal, and which deals with the expansion of an arbitrary function into a series made of the functions V which we have considered.

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