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# STURM'S THEOREM ON ZEROS OF LINEAR COMBINATIONS OF EIGENFUNCTIONS

PIERRE BÉRARD AND BERNARD HELFFER

ABSTRACT. Motivated by recent questions about the extension of Courant's nodal domain theorem, we revisit a theorem published by C. Sturm in 1836, which deals with zeros of linear combination of eigenfunctions of Sturm-Liouville problems. Although well known in the nineteenth century, this theorem seems to have been ignored or forgotten by some of the specialists in spectral theory since the second half of the twentieth-century. Although not specialists in History of Sciences, we have tried to replace these theorems into the context of nineteenth century mathematics.

## 1. INTRODUCTION

In this paper, we are interested in the following eigenvalue problem, where  $r$  denotes the spectral parameter.

$$(1.1) \quad \frac{d}{dx} \left( K \frac{dV}{dx} \right) + (rG - L)V = 0,$$

$$(1.2) \quad \left( K \frac{dV}{dx} - hV \right) (\alpha) = 0,$$

$$(1.3) \quad \left( K \frac{dV}{dx} + HV \right) (\beta) = 0.$$

Here,

$$(1.4) \quad K, G, L : [\alpha, \beta] \rightarrow \mathbb{R} \text{ are positive functions,}$$

$$(1.5) \quad h, H \in [0, \infty] \text{ are non negative constants, possibly infinite.}$$

*Remark 1.1.* When  $h = \infty$  (resp.  $H = \infty$ ), the boundary condition should be understood as the Dirichlet boundary condition  $V(\alpha) = 0$  (resp. as the Dirichlet boundary condition  $V(\beta) = 0$ ).

Precise assumptions on  $K, G, L$  will be given later on.

Note that when  $K = G \equiv 1$ , (1.1)–(1.3) is an eigenvalue problem for the classical operator  $-\frac{d^2V}{dx^2} + LV$ .

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The above boundary conditions first appear in the work of Joseph Fourier [9] but are called “Robin’s condition” in the recent literature. Victor Gustave Robin (1855-1897) was a French mathematician.

This eigenvalue problem, in the above generality ( $K, G, L$  functions of  $x$ ), was first studied by Charles Sturm in a Memoir presented to the Paris Academy of sciences in September 1833, summarized in [31, 32], and published in [33, 34].

*Remark 1.2.* In this paper, we have mainly retained the notation of [33], except that Sturm considers the interval  $[x, X]$  instead of  $[\alpha, \beta]$ . We otherwise use today notation and vocabulary. Note that in [34], Sturm uses lower case letters for the functions  $K, G, L$ . Sturm uses the same notation as Fourier in [9].

**1.1. Sturm’s motivations and approach.** Sturm’s motivations come from mathematical physics, and more precisely from the problem of heat diffusion in a non-homogeneous bar. The functions  $K, G, L$  actually describe the physical properties of the bar, see [34, Introduction, p. 376]. Sturm refers to the book of Siméon Denis Poisson [26], rather than to Fourier’s book [9], because Poisson’s equations are more general, see [28, Chap. III]. As was popularized by Fourier and Poisson, Sturm uses the method of separation of variables, and is therefore led to the eigenvalue problem (1.1)–(1.3). To find the spectral data (eigenvalues and eigenfunctions), Sturm begins by studying the following initial value problem.

$$(1.6) \quad \frac{d}{dx} \left( K(x, m) \frac{dV}{dx}(x, m) \right) + G(x, m)V(x, m) = 0,$$

$$(1.7) \quad \left( K \frac{dV}{dx} - hV \right) (\alpha, m) = 0,$$

where he assumes that  $K, G$  are functions of  $x$  depending on a real parameter  $m$ , with  $K$  positive (the constants  $h$  and  $H$  may also depend on the parameter  $m$ ). The solution  $V(x, m)$  is well defined up to a scaling factor. The main part of [33] is devoted to studying how the zeros of the function  $V(x, m)$  (and other related functions) depend on the parameter  $m$ , under suitable assumptions on the functions  $K$  and  $G$  ( $h$  and  $H$ ), see [33, § XII, p. 125]. While developing this program, Sturm proves the oscillation, separation and comparison theorems which nowadays bear his name, [33, § XV, XVI and XXXVII].

We would like to insist on the *originality* of Sturm’s approach. Indeed, unlike his predecessors, Sturm does not look for explicit solutions of the differential equation (1.6) (i.e., solutions in closed form, or given as sums of series or as integrals), but he rather looks for *qualitative properties* of the solutions, properties which can be deduced directly

from the differential equation itself. The following excerpts are taken from [33, Introduction]<sup>1</sup>.

*On ne sait [ces équations] les intégrer que dans un très petit nombre de cas particuliers hors desquels on ne peut pas même en obtenir une intégrale première ; et lors même qu'on possède l'expression de la fonction qui vérifie une telle équation, soit sous forme finie, soit en série, soit en intégrales définies ou indéfinies, il est le plus souvent difficile de reconnaître dans cette expression la marche et les propriétés caractéristiques de cette fonction. . . .*

*S'il importe de pouvoir déterminer la valeur de la fonction inconnue pour une valeur isolée quelconque de la variable dont elle dépend, il n'est pas moins nécessaire de discuter la marche de cette fonction, ou en d'autres termes, d'examiner la forme et les sinuosités de la courbe dont cette fonction serait l'ordonnée variable, en prenant pour abscisse la variable indépendante. Or on peut arriver à ce but par la seule considération des équations différentielles elles-mêmes, sans qu'on ait besoin de leur intégration. Tel est l'objet du présent mémoire. . . .*

The eigenvalue problem (1.1)–(1.3) itself is studied in [34]. For this purpose, Sturm considers the functions

$$K(x, r) \equiv K(x) \text{ and } G(x, r) = rG(x) - L(x),$$

the solution  $V(x, r)$  of the corresponding initial value problem (1.6)–(1.7), and applies the results and methods of [33].

The spectral data of the eigenvalue problem (1.1)–(1.3) are determined by the following transcendental equation in the spectral parameter  $r$ ,

$$(1.8) \quad K(\beta) \frac{dV}{dx}(\beta, r) + HV(\beta, r) = 0.$$

As far as the eigenvalue problem (1.1)–(1.3) is concerned, Sturm's first results can be summarized in the following theorem.

**Theorem 1.3** (Sturm, 1836). *Under the assumptions (1.4)–(1.5), the eigenvalue problem (1.1)–(1.3) admits an increasing infinite sequence  $\{\rho_i, i \geq 1\}$  of positive simple eigenvalues, tending to infinity. Furthermore, the associated eigenfunctions  $V_i$  have the following remarkable property: the function  $V_i$  vanishes, and changes sign, precisely  $(i - 1)$  times in the open interval  $] \alpha, \beta [$ .*

*Remark 1.4.* Theorem 1.3 is proved in [34]. For the first assertion, see § III (p. 384) to VII; for the second assertion, see § VIII (p. 396) to X.

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<sup>1</sup>See Appendix B for the English translations of the citations in French.

**1.2. About Sturm’s assumptions and proofs.** In [33, 34], Sturm implicitly assumes that the functions  $K, G, L$  are  $C^\infty$  and, explicitly, that  $K$  is positive, see [33, p. 108]. For the eigenvalue problem, he also assumes that  $G, L$  are positive, see [34, p. 381]. In [34, p. 394], he does however mention that  $L$  could take negative values and implicitly assumes, in this case, that  $\frac{L}{G}$  is bounded from below.

In [33, p. 108], Sturm considers the differential equation

$$\frac{d}{dx} \left( K \frac{dV}{dx} \right) + GV = 0, \quad (I)$$

and takes the existence and uniqueness theorem for granted. More precisely, he claims [33, p. 108], without any reference whatsoever, “*L’intégrale complète de l’équation (I) doit contenir deux constantes arbitraires, pour lesquelles on peut prendre les valeurs de  $V$  et de  $\frac{dV}{dx}$  correspondantes à une valeur particulière de  $x$ . Lorsque ces valeurs sont fixées, la fonction  $V$  est entièrement définie par l’équation (I), elle a une valeur déterminée et unique pour chaque valeur de  $x$ .*”

On the other hand, he gives two arguments for the fact that a solution of (I) and its derivative cannot vanish simultaneously at a point without vanishing identically, see [33, § II]. When the coefficients  $K, G$  of the differential equation depend upon a parameter  $m$ , at least continuously, Sturm also takes for granted the fact that the solution  $V(x, m)$  and its zeros depend continuously on  $m$ .

*Remark 1.5.* In [34, § II], Sturm mentions the existence proof given by Joseph Liouville in [18], which had already appeared in [17]. According to [11], Augustin-Louis Cauchy may have presented the existence and uniqueness theorem for ordinary differential equations in his course at École polytechnique as early as in the year 1817-1818. Following a recommendation of the administration of the school, Cauchy delivered the notes of his lectures in 1824, see [6] and, in particular, the introduction by Christian Gilain who discovered these notes in 1974. These notes apparently had a limited distribution. It should be mentioned that Liouville entered the École polytechnique in 1825, and attended the courses of either Ampère or Cauchy. His proof of the existence theorem for differential equations in [17], à la Picard but before Picard, though limited to the particular case of 2nd order linear differential equations, might be the first well circulated proof of an existence theorem, see [20, § 34]. Cauchy’s theorem was later popularized in the second volume of Moigno’s book, published in 1844, see [22], “Vingt-sixième Leçon” § 159, pp. 385–396.

*Remark 1.6.* Sturm’s papers [33, 34] are written in French, and quite long, about 80 pages each. One difficulty in reading them is the lack of layout structure. There is only a sequence of sections, without any title, and most results are stated without tags, “Theorem” and the

like, and only appear in the body of the text. For example, [33] only contains one theorem stated as such, see § XII, p. 125. In order to have an overview of the results contained in [33], the reader might look at the announcement [31].

For a more thorough analysis of Sturm's papers on differential equations, we refer to [21, 10]. We refer to [4, 28] for the relationships between Theorem 1.3 and Sturm's theorem on the number of real roots of real polynomials.

**1.3. Our purpose and assumptions.** In order to make the proofs easier, we make the following strong assumptions.

$$(1.9) \quad \begin{cases} [\alpha, \beta] \subset ]\alpha_0, \beta_0[, \\ K, G, L \in C^\infty(] \alpha_0, \beta_0[), \\ K, G, L > 0 \text{ on } ] \alpha_0, \beta_0[. \end{cases}$$

We refer to Section 5 for results under weaker assumptions.

*Remark 1.7.* The positivity assumption on  $L$  is actually not necessary, but makes the discussion simpler. Compare with [34, p. 394].

The purpose of this paper is to discuss a result of Sturm which does not seem to be as well known as Theorem 1.3. We begin with a rough statement, Theorem 1.8, which will be made more precise later on, see Theorems 3.10 and 4.2.

**Theorem 1.8** (Sturm, 1836). *Let  $Y = A_m V_m + \cdots + A_n V_n$  be a non trivial linear combination of eigenfunctions of the eigenvalue problem (1.1)–(1.3), with  $1 \leq m \leq n$ , and  $\{A_j, m \leq j \leq n\}$  real constants such that  $A_m^2 + \cdots + A_n^2 \neq 0$ . Then, the function  $Y$  has at least  $(m - 1)$ , and at most  $(n - 1)$  zeros in the open interval  $] \alpha, \beta[$ .*

Theorem 1.8 appears in [34, § XXV, p. 431], see also the announcement [32]. Sturm's motivation, see the introductions to [33] and [34], was the investigation of heat diffusion in a (non-homogeneous) bar, whose physical properties are described by the functions  $K, G, L$ . He first obtained Theorem 1.8 as a corollary of his investigation of the behaviour, when  $t$  tends to infinity, of the temperature  $u(t, x)$  of a bar whose initial temperature  $u(0, x)$  is given by a linear combination of simple states.

J. Liouville, who was aware of Theorem 1.8, made use of it in [18], and provided a purely "ordinary differential equation" (o.d.e.) proof in [19], a few months before the actual publication of [34]. This induced Sturm to provide two proofs of Theorem 1.8 in [34], his initial proof using the heat equation, and another proof based only on the ordinary differential equation. The proofs of Sturm actually give a more precise result. In [34, p. 379], Sturm writes,

*M. Liouville a démontré directement ce théorème, qui n'était pour moi*

*qu'un corollaire du précédent, sans s'occuper du cas particulier où la fonction serait nulle à l'une des extrémités de la barre. J'en ai aussi trouvé après lui une autre démonstration directe que je donne dans ce mémoire. M. Liouville a fait usage du même théorème dans un très beau Mémoire qu'il a publié dans le numéro de juillet de son journal et qui a pour objet le développement d'une fonction arbitraire en une série composée de fonctions  $V$  que nous avons considérées.*

In this paper, we revisit the (o.d.e.) proofs of Theorem 1.8 given by Sturm and Liouville respectively. Liouville's proof is simpler, but slightly less precise than Sturm's proof. Liouville's paper, also in French, is quite short, and very clear. As explained in Remark 1.6, Sturm's proof is more difficult to read. This might explain why Theorem 1.8 is not better known.

In this paper, we do not consider Sturm's heat equation proof of Theorem 1.8, see [34, § XVIII ff], and refer to [10, 21] for modern formulations.

*Remark 1.9.* Some remarks to conclude this introduction.

- (1) In [29, Section 142], Lord Rayleigh writes “*a beautiful theorem has been discovered by Sturm*” as he mentions Theorem 1.8.
- (2) F. Pockels [25, pp. 68-73] gives a summary of Sturm's results, including Theorem 1.8, and mentions the different proofs provided by Sturm, Liouville and Rayleigh. On the basis of a note of Sturm in Férussac's Bulletin [30], Pockels (p. 71, lines 12-17) also suggests that Sturm may have looked for a statement in higher dimension as well, without success. Sturm indeed mentions studying an example with spherical symmetry in 3 dimensions (leading to an ordinary differential equation with singularity), to which he may have applied Theorem 1.8.
- (3) Courant and Hilbert [7] extensively mention the Sturm-Liouville problem. They however do not refer to the original papers of Sturm, but rather to Bôcher's book [5] which does not include Theorem 1.8. Courant and Hilbert do not mention Theorem 1.8 either, but they do mention an analogue in higher dimensions when they discuss Courant's nodal domain theorem [7, footnote, p. 454], and refer to the dissertation of H. Herrmann [13]. As pointed out by V. Arnold [2, 15], such an extension cannot be true in general. Counter-examples were first given by O. Viro for the 3-sphere (with the canonical metric) [35] and, more recently in the paper [3], see also [12].
- (4) More recently Sturm's Theorem 1.8 is mentioned, somewhat inaccurately, in Pleijel's 1956 paper [24, p. 543 and 550].
- (5) It seems to us that Arnold was not aware of Theorem 1.8. In [2], see also Supplementary problem 9 in [1, p. 327], he mentions a proof, suggested by I. Gelfand, of the upper bound in

Theorem 1.8. Gelfand’s idea is to “use fermions rather than bosons”, and to apply Courant’s nodal domain theorem. However, Arnold concludes by writing [2, p. 30], “*the arguments above do not yet provide a proof*”. As far as we know, Gelfand’s idea has so far not yielded any complete proof of Sturm’s result. It is interesting to note that Liouville’s proof of the lower bound in Theorem 1.8 uses an idea similar to Gelfand’s, see the proof of Claim 4.6.

- (6) The lower bound in Theorem 1.8 is related to the so-called Sturm-Hurwitz theorem for Fourier series with a spectral gap, see [14], [8, § 2], and [23] for geometric applications.

1.4. **Organization of the paper.** In Section 2, we give some preliminary results. In Section 3, we prove Theorem 3.10, a refined version of Theorem 1.8, following the ideas of Sturm [34, § XXVI]. In Section 4, we prove Theorem 4.2, another refined version of Theorem 1.8, following the proof given by Liouville in [18, 19]. In Section 5 we consider results under weaker assumptions. Appendix A provides the detailed proof of a technical argument. Appendix B provides the translations into English of the citations in French.

## 2. O.D.E. PROOFS OF STURM’S THEOREM 1.8, COMMON FEATURES

### 2.1. Preliminaries.

2.1.1. Recall that  $\{(\rho_j, V_j), j \geq 1\}$  are the eigenvalues and eigenfunctions of the eigenvalue problem (1.1)–(1.3). By our assumption  $L > 0$ , the eigenvalues are positive,  $\rho_j > 0$ .

Under the Assumptions (1.9), the functions  $V_j$  are  $C^\infty$  on  $] \alpha_0, \beta_0[$ . This follows easily for example from Liouville’s existence proof [18].

In the sequel, we study the zeros of a linear combination

$$(2.1) \quad Y = \sum_{j=m}^n A_j V_j,$$

where  $1 \leq m \leq n$ , and where the  $A_j$  are real constants. We always assume that  $Y \not\equiv 0$ , which is equivalent to assuming that  $\sum_m^n A_j^2 \neq 0$ .

*Remark 2.1.* The numbers which actually matter are

$$(2.2) \quad n_e = \sup\{p \mid m \leq p \leq n \text{ such that } A_p \neq 0\},$$

$$(2.3) \quad m_e = \inf\{p \mid m \leq p \leq n \text{ such that } A_p \neq 0\}.$$

Without loss of generality, we may assume that  $A_m A_n \neq 0$ .

For any  $k \in \mathbb{Z}$ , we define the function

$$(2.4) \quad Y_k = (-1)^k \sum_{j=m}^n A_j \rho_j^k V_j.$$



Note in particular that  $Y_0$  is the original linear combination  $Y$ , and that  $Y_k \equiv 0$  if and only if  $Y \equiv 0$ .

In order to simplify notation, we shall, in the sequel, often denote a function  $Y_k$  (for some  $k \in \mathbb{Z}$ ) by  $U$  and the function  $Y_{k+1}$  by  $U_1$ , the important point being that  $Y_k$  and  $Y_{k+1}$  satisfy the relation (2.8) below.

2.1.2. We write the equations satisfied by the functions  $V_p$  for  $m \leq p \leq n$ ,

$$(2.5) \quad \frac{d}{dx} \left( K \frac{dV_p}{dx} \right) + (\rho_p G - L)V_p = 0,$$

$$(2.6) \quad \left( K \frac{dV_p}{dx} - hV_p \right) (\alpha) = 0,$$

$$(2.7) \quad \left( K \frac{dV_p}{dx} + HV_p \right) (\beta) = 0,$$

and multiply the  $p$ -th equation by  $A_p \rho_p^k$ . Summing up from  $p = m$  to  $p = n$ , we obtain the following lemma.

**Lemma 2.2.** *Let  $k \in \mathbb{Z}$ .*

- (1) *The function  $Y_k$  satisfies the boundary conditions (1.2) and (1.3).*
- (2) *The functions  $Y_k$  and  $Y_{k+1}$  satisfy the relation*

$$(2.8) \quad G Y_{k+1} = K \frac{d^2 Y_k}{dx^2} + \frac{dK}{dx} \frac{dY_k}{dx} - L Y_k.$$

- (3) *Under the Assumptions (1.9), the function  $Y_k$  cannot vanish at infinite order at a point  $\xi \in [\alpha, \beta]$ , unless  $Y_k \equiv 0$ .*

**Proof.** Assertions (1) and (2) are clear by linearity.

For Assertion (3), assume that  $Y_k \not\equiv 0$ , and that it vanishes at infinite order at some  $\xi$ . Then, according to (2.8) and its successive derivatives, the function  $Y_{k+1}$  also vanishes at infinite order at  $\xi$ , and so does  $Y_\ell$  for any  $\ell \geq k$ . Assume, as indicated above, that  $A_n \neq 0$ . Fixing some  $p \geq 0$ , we can write, for any  $\ell \geq k$ ,

$$\frac{d^p V_n}{dx^p}(\xi) + \sum_{j=m}^{n-1} \frac{A_j}{A_n} \left( \frac{\rho_j}{\rho_n} \right)^\ell \frac{d^p V_j}{dx^p}(\xi) = 0.$$

Since  $\rho_n > \rho_j$  for  $m \leq j \leq n-1$ , letting  $\ell$  tend to infinity, we conclude that  $\frac{d^p V_n}{dx^p}(\xi) = 0$ . This would be true for all  $p$ , which is impossible by Cauchy's uniqueness theorem, or by Sturm's argument [33, § II].  $\square$

**Lemma 2.3.** *Under the Assumptions (1.9), let  $\xi \in [\alpha, \beta]$  be a zero of  $Y_k$ , of order  $p \geq 2$ . Then, there exist constants  $B_{k,\xi}$  and  $B_{k+1,\xi}$ , and*

smooth functions  $R_{k,\xi}$  and  $R_{k+1,\xi}$ , such that

$$(2.9) \quad \begin{cases} Y_k(x) = B_{k,\xi}(x - \xi)^p + (x - \xi)^{p+1}R_{k,\xi}(x), \\ Y_{k+1} = B_{k+1,\xi}(x - \xi)^{p-2} + (x - \xi)^{p-1}R_{k+1,\xi}(x), \\ \text{with } B_{k,\xi} B_{k+1,\xi} > 0. \end{cases}$$

**Proof.** Assume that  $\xi$  is a zero of order  $p \geq 2$  of  $Y_k$ , so that

$$Y_k(\xi) = \cdots = \frac{d^{p-1}Y_k}{dx^{p-1}}(\xi) = 0$$

and

$$\frac{d^p Y_k}{dx^p}(\xi) \neq 0.$$

Taylor's formula with integral remainder term, see Laplace [16], gives the existence of some function  $R_{k,\xi}$  such that

$$Y_k(x) = B_{k,\xi}(x - \xi)^p + (x - \xi)^{p+1}R_{k,\xi}(x),$$

where

$$B_{k,\xi} = \frac{1}{p!} \frac{d^p Y_k}{dx^p}(\xi) \neq 0.$$

Equation (2.8) implies that

$$(GY_{k+1})(x) = p(p-1)B_{k,\xi}(x - \xi)^{p-2}K(x) + (x - \xi)^{p-1}S_{k,\xi}(x),$$

for some smooth function  $S_{k,\xi}$ . It follows that

$$Y_{k+1}(x) = B_{k+1,\xi}(x - \xi)^{p-2} + (x - \xi)^{p-1}R_{k+1,\xi}(x),$$

for some function  $R_{k+1,\xi}$ , with  $B_{k+1,\xi} = p(p-1)\frac{K(\xi)}{G(\xi)}B_{k,\xi}$ . In particular,  $B_{k+1,\xi}B_{k,\xi} > 0$ . This proves the lemma.  $\square$

**Lemma 2.4.** *Assume that  $h \in [0, \infty[$ , i.e., that the boundary condition at  $\alpha$  is not the Dirichlet boundary condition.*

*Fix some  $k$ , and assume that  $Y_k(\alpha) = 0$ . Then, under the Assumptions (1.9),  $\alpha$  is a zero of  $Y_k$  of even order, i.e., there exists  $n_k \in \mathbb{N} \setminus \{0\}$  such that  $\frac{d^p Y_k}{dx^p}(\alpha) = 0$  for  $0 \leq p \leq 2n_k - 1$  and  $\neq 0$  for  $p = 2n_k$ .*

*When  $H \in [0, \infty[$ , a similar statement holds at the boundary  $\beta$ .*

**Proof.** Assume that  $Y_k(\alpha) = 0$ . By Lemma 2.2,  $Y_k$  does not vanish at infinite order at  $\alpha$ , so that there exists  $p \geq 1$  with

$$Y_k(\alpha) = \cdots = \frac{d^{p-1}Y_k}{dx^{p-1}}(\alpha) = 0$$

and

$$\frac{d^p Y_k}{dx^p}(\alpha) \neq 0.$$

Taylor's formula with integral remainder term gives

$$Y_k(x) = B_{k,\alpha}(x - \alpha)^p + (x - \alpha)^{p+1}R_{k,\alpha}(x),$$

where  $B_{k,\alpha} = \frac{1}{p!} \frac{d^p Y_k}{dx^p}(\alpha) \neq 0$ . The boundary condition at  $\alpha$  implies that  $\frac{dY_k}{dx}(\alpha) = 0$ , and hence that  $p \geq 2$ . By Lemma 2.3, we can write

$$Y_{k+1}(x) = B_{k+1,\alpha}(x - \alpha)^{p-2} + (x - \alpha)^{p-1} R_{k+1,\alpha}(x),$$

with  $B_{k+1,\alpha} B_{k,\alpha} > 0$ .

If  $p = 2$ , then  $Y_{k+1}(\alpha) \neq 0$ . If  $p > 2$ , one can continue. If  $p = 2q$ , one arrives at

$$Y_{k+q}(x) = B_{k+q,\alpha} + (x - \alpha) R_{k+q,\alpha}(x),$$

with  $Y_{k+q}(\alpha) = B_{k+q,\alpha}$  and  $B_{k+q,\alpha} B_{k,\alpha} > 0$ .

If  $p = 2q + 1$ , one arrives at

$$Y_{k+q}(x) = B_{k+q,\alpha}(x - \alpha) + (x - \alpha)^2 R_{k+q,\alpha}(x),$$

with  $B_{k+q,\alpha} B_{k,\alpha} > 0$  and  $\frac{dY_{k+q}}{dx}(\alpha) = B_{k+q,\alpha} \neq 0$ .

On the other-hand, since  $Y_{k+q}$  satisfies (1.2) and  $Y_{k+q}(\alpha) = 0$ , we must have  $\frac{dY_{k+q}}{dx}(\alpha) = 0$ . This yields a contradiction and proves that the case  $p = 2q + 1$  cannot occur. The lemma is proved.  $\square$

**2.2. Counting zeros.** Let  $U$  denote one of the functions  $Y_k$ , and  $U_1$  the function  $Y_{k+1}$  which is related to  $U$  by (2.8).

Under the Assumptions (1.9), we can consider the multiplicities of the zeros of the functions  $U$  and  $U_1$ .

From Lemma 2.2, we know that  $U$  cannot vanish at infinite order at a point  $\xi \in [\alpha, \beta]$ . If  $U(\xi) = 0$ , we define the *multiplicity*  $m(U, \xi)$  of the zero  $\xi$  by

$$(2.10) \quad m(U, \xi) = \min\{p \mid \frac{d^p U}{dx^p}(\xi) \neq 0\}.$$

From Lemma 2.4, we know that the multiplicity  $m(U, \alpha)$  is even if  $h \in [0, \infty[$ , and that the multiplicity  $m(U, \beta)$  is even if  $H \in [0, \infty[$ . We define the *reduced multiplicity* of  $\alpha$  by

$$(2.11) \quad \bar{m}(U, \alpha) = \begin{cases} \frac{1}{2}m(U, \alpha) & \text{if } h \in [0, \infty[, \\ 0 & \text{if } h = \infty, \end{cases}$$

and a similar formula for the reduced multiplicity of  $\beta$ .

By Lemma 2.2, the function  $U$  has finitely many distinct zeros  $\xi_1(U) < \xi_2(U) < \dots < \xi_p(U)$  in the interval  $] \alpha, \beta [$ . We define the *number of zeros of  $U$  in  $] \alpha, \beta [$ , counted with multiplicities*, by

$$(2.12) \quad N_m(U, ] \alpha, \beta [) = \sum_{j=1}^p m(U, \xi_j(U)),$$

and use the notation  $N_m(U)$  whenever the interval is clear.

We define the *number of zeros of  $U$  in  $[\alpha, \beta]$ , counted with multiplicities*, by

$$(2.13) \quad \overline{N}_m(U, [\alpha, \beta]) = \sum_{j=1}^p m(U, \xi_j(U)) + \overline{m}(U, \alpha) + \overline{m}(U, \beta),$$

and use the notation  $\overline{N}_m(U)$  whenever the interval is clear.

We define the *number of zeros of  $U$  in  $] \alpha, \beta [$  (multiplicities not accounted for)* by

$$(2.14) \quad N(U, ] \alpha, \beta [) = p,$$

and use the notation  $N(U)$  whenever the interval is clear.

Finally, we define the *number of sign changes of  $U$  in the interval  $] \alpha, \beta [$*  by

$$(2.15) \quad N_v(U, ] \alpha, \beta [) = \sum_{j=1}^p \frac{1}{2} \left[ 1 - (-1)^{m(U, \xi_j(U))} \right].$$

### 3. STURM'S O.D.E. PROOF OF THEOREM 1.8

In this section, we work under the Assumptions (1.9). Sturm's proof uses a careful analysis of where  $U_1$  vanishes and possibly changes sign, in terms of the zeros of  $U$ .

We fix some  $k \in \mathbb{Z}$  and use the notation  $U = Y_k$  and  $U_1 = Y_{k+1}$ .

**Lemma 3.1.** *Let  $\xi < \eta$  be two zeros of  $U$  in  $[\alpha, \beta]$ . Then, there exists some  $a_{\xi, \eta} \in ] \xi, \eta [$  such that  $U(a_{\xi, \eta})U_1(a_{\xi, \eta}) < 0$ .*

*Remark 3.2.* We do not assume that  $\xi, \eta$  are consecutive zeros.

**Proof.** Since  $U$  cannot vanish identically in  $] \xi, \eta [$  (Lemma 2.2), there exists some  $x_0 \in ] \xi, \eta [$  such that  $U(x_0) \neq 0$ . Let  $\varepsilon_0 = \text{sign}(U(x_0))$ . Then  $\varepsilon_0 U$  takes a positive value at  $x_0$ , and hence  $M := \sup\{\varepsilon_0 U(x) \mid x \in ] \xi, \eta [\}$  is positive and achieved at some  $a_{\xi, \eta} \in ] \xi, \eta [$ . Denote this point by  $a$  for short, then,  $\varepsilon_0 U(a) > 0$ ,  $\frac{dU}{dx}(a) = 0$ , and  $\varepsilon_0 \frac{d^2U}{dx^2}(a) \leq 0$ . It follows from (2.8) that  $\varepsilon_0 U_1(a) < 0$ , or equivalently, that  $U(a)U_1(a) < 0$ . The lemma is proved.  $\square$

**Lemma 3.3.** *Let  $\xi \in ] \alpha, \beta [$ . Assume that  $U(\xi) = 0$ , and that  $U$  does not change sign in  $] \alpha, \xi [$ . Then, there exists some  $a_\xi \in ] \alpha, \xi [$  such that  $U(a_\xi)U_1(a_\xi) < 0$ .*

*Let  $\eta \in ] \alpha, \beta [$ . Assume that  $U(\eta) = 0$ , and that  $U$  does not change sign in  $] \eta, \beta [$ . Then, there exists some  $b_\eta \in ] \eta, \beta [$  such that  $U(b_\eta)U_1(b_\eta) < 0$ .*

**Proof.** Since  $U$  cannot vanish identically in  $] \alpha, \xi [$  (Lemma 2.2), there exists  $x_0 \in ] \alpha, \xi [$  such that  $U(x_0) \neq 0$ . Let  $\varepsilon_\xi = \text{sign}(U(x_0))$ . Since  $U$  does not change sign in  $] \alpha, \xi [$ ,  $\varepsilon_\xi U(x) \geq 0$  in  $] \alpha, \xi [$ . Then,

$$M_\xi := \sup\{\varepsilon_\xi U(x) \mid x \in ] \alpha, \xi [\} > 0.$$

Let

$$a_\xi := \inf\{x \in [\alpha, \xi] \mid \varepsilon_\xi U(x) = M_\xi\}.$$

Then  $a_\xi \in [\alpha, \xi[$ .

If  $a_\xi \in ]\alpha, \xi[$ , then  $\varepsilon_\xi U(a_\xi) > 0$ ,  $\frac{dU}{dx}(a_\xi) = 0$ , and  $\varepsilon_\xi \frac{d^2U}{dx^2}(a_\xi) \leq 0$ . By (2.8), this implies that  $\varepsilon_\xi U_1(a_\xi) < 0$ . Equivalently,  $U(a_\xi)U_1(a_\xi) < 0$ .

**Claim 3.4.** *If  $a_\xi = \alpha$ , then  $\varepsilon_\xi U(\alpha) > 0$ ,  $h = 0$ ,  $\frac{dU}{dx}(\alpha) = 0$ , and  $\varepsilon_\xi \frac{d^2U}{dx^2}(\alpha) \leq 0$ .*

*Proof of the claim.* Assume that  $a_\xi = \alpha$ , then  $\varepsilon_\xi U(\alpha) > 0$ , and hence  $h \neq \infty$ . If  $h$  belonged to  $]0, \infty[$  we would have  $\varepsilon_\xi \frac{dU}{dx}(\alpha) = h\varepsilon_\xi U(\alpha) > 0$  and hence  $a_\xi > \alpha$ . It follows that the assumption  $a_\xi = \alpha$  implies that  $h = 0$  and  $\frac{dU}{dx}(\alpha) = 0$ . If  $\varepsilon_\xi \frac{d^2U}{dx^2}(\alpha)$  were positive, we would have  $a_\xi > \alpha$ . Therefore, the assumption  $a_\xi = \alpha$  also implies that  $\varepsilon_\xi \frac{d^2U}{dx^2}(\alpha) \leq 0$ . The claim is proved.

If  $a_\xi = \alpha$ , then by Claim 3.4 and (2.8), we have  $\varepsilon_\xi U_1(\alpha) < 0$ . Equivalently,  $U(\alpha)U_1(\alpha) < 0$ . The first assertion of the lemma is proved. The proof of the second assertion is similar.  $\square$

*Remark 3.5.* Lemma 3.3 actually holds without the assumption that  $U$  does not change sign in  $] \alpha, \xi[$ .

**Proposition 3.6.** *Let  $k \in \mathbb{Z}$ . Then, under the Assumptions (1.9),*

$$(3.1) \quad N_v(Y_{k+1}, ]\alpha, \beta]) \geq N_v(Y_k, ]\alpha, \beta]),$$

*i.e., the function  $Y_{k+1}$  changes sign in  $] \alpha, \beta[$  at least as many times as the function  $Y_k$ .*

**Proof.** We keep the notation  $U = Y_k$  and  $U_1 = Y_{k+1}$ . By Lemma 2.2, the functions  $U$  and  $U_1$  have finitely many zeros in  $] \alpha, \beta[$ , with finite multiplicities. Note that the sign changes of  $U$  correspond to the zeros of  $U$  with odd multiplicities. Since  $\alpha$  and  $\beta$  are fixed, we skip the mention to the interval  $] \alpha, \beta[$  in the proof, and we examine several cases.

*Case 1.* If  $N_v(U) = 0$ , there is nothing to prove.

*Case 2.* Assume that  $N_v(U) = 1$ . Then  $U$  admits a unique zero  $\xi \in ] \alpha, \beta[$  having odd multiplicity. Without loss of generality, we may assume that  $U \geq 0$  in  $] \alpha, \xi[$  and  $U \leq 0$  in  $] \xi, \beta[$ . By Lemma 3.3, there exist  $a \in [\alpha, \xi[$  and  $b \in ] \xi, \beta]$  such that  $U_1(a) < 0$  and  $U_1(b) > 0$ .

It follows that the function  $U_1$  vanishes and changes sign at least once in  $] \alpha, \beta[$ , so that  $N_v(U_1) \geq 1 = N_v(U)$ , which proves the lemma in Case 2.

*Case 3.* If  $N_v(U) = 2$ , the function  $U$  has exactly two zeros, having odd multiplicities,  $\xi$  and  $\eta$  in  $] \alpha, \beta[$ ,  $\alpha < \xi < \eta < \beta$ , and we may assume that  $U|_{] \alpha, \xi[} \geq 0$ ,  $U|_{] \xi, \eta[} \leq 0$ , and  $U|_{] \eta, \beta[} \geq 0$ . The arguments

given in Case 2 imply that there exist  $a \in [\alpha, \xi[$  such that  $U_1(a) < 0$  and  $b \in ]\eta, \beta]$  such that  $U_1(b) < 0$ . In  $] \xi, \eta[$  the function  $U$  does not vanish identically and therefore achieves a global minimum at a point  $c$  such that  $U(c) < 0$ ,  $\frac{dU}{dx}(c) = 0$ , and  $\frac{d^2U}{dx^2}(c) \geq 0$ . Equation(2.8) then implies that  $U_1(c) > 0$ .

We can conclude that the function  $U_1$  vanishes and changes sign at least twice in  $] \alpha, \beta[$ , so that  $N_v(U_1) \geq 2 = N_v(U)$ .

*Case 4.* Assume that  $N_v(U) = p \geq 3$ . Then,  $U$  has exactly  $p$  zeros, with odd multiplicities, in  $] \alpha, \beta[$ ,  $\alpha < \xi_1 < \xi_2 < \dots < \xi_p < \beta$ , and one can assume that

$$\begin{aligned} U|_{] \alpha, \xi_1[} &\geq 0, \quad (-1)^p U|_{] \xi_p, \beta]} \geq 0, \quad \text{and} \\ (-1)^i U|_{] \xi_i, \xi_{i+1}[} &\geq 0 \text{ for } 1 \leq i \leq p-1. \end{aligned}$$

One can repeat the arguments given in the Cases 2 and 3, and conclude that there exist  $a_0, \dots, a_p$  with  $a_0 \in [\alpha, \xi_1[$ ,  $a_i \in ] \xi_i, \xi_{i+1}[$  for  $1 \leq i \leq p-1$ , and  $a_p \in ] \xi_p, \beta]$  such that  $(-1)^i U_1(a_i) < 0$ .

We can then conclude that the function  $U_1$  vanishes and changes sign at least  $p$  times in  $] \alpha, \beta[$ , i.e. that  $N_v(U_1) \geq p = N_v(U)$ .

This concludes the proof of Proposition 3.6.  $\square$

**Proposition 3.7.** *Let  $k \in \mathbb{Z}$ ,  $U = Y_k$  and  $U_1 = Y_{k+1}$ . Then, under the Assumptions (1.9),*

$$(3.2) \quad N_m(Y_{k+1}, ] \alpha, \beta]) \geq N_m(Y_k, ] \alpha, \beta]),$$

*i.e., counting multiplicities, the number of zeros of  $Y_{k+1}$  in  $] \alpha, \beta[$  is not less than the number of zeros of  $Y_k$  in  $] \alpha, \beta[$ .*

**Proof.** We use the same notation as in the proof of Proposition 3.6,  $U = Y_k$  and  $U_1 = Y_{k+1}$ . If  $U$  does not vanish in  $] \alpha, \beta[$ , there is nothing to prove. We now assume that  $U$  has at least one zero in  $] \alpha, \beta[$ . By Lemma 2.2,  $U$  and  $U_k$  have finitely many zeros in  $] \alpha, \beta[$ . Let

$$\alpha < \xi_1 < \dots < \xi_k < \beta$$

be the distinct zeros of  $U$ , with multiplicities  $p_i = m(U, \xi_i)$  for  $1 \leq i \leq k$ . Let  $\sigma_0$  be the sign of  $U$  in  $] \alpha, \xi_1[$ ,  $\sigma_i$  the sign of  $U$  in  $] \xi_i, \xi_{i+1}[$  for  $1 \leq i \leq k-1$ , and  $\sigma_k$  the sign of  $U$  in  $] \xi_k, \beta[$ . Note that

$$\sigma_i = \text{sign} \left( \frac{d^{p_i} U}{dx^{p_i}}(\xi_i) \right) \text{ for } 1 \leq i \leq k.$$

By Lemma 3.3, there exist  $a_0 \in [\alpha, \xi_1[$  and  $a_k \in ] \xi_k, \beta]$  such that  $U(a_0)U_1(a_0) < 0$  and  $U(a_k)U_1(a_k) < 0$ . By Lemma 3.1, there exists  $a_i \in ] \xi_i, \xi_{i+1}[$ ,  $1 \leq i \leq k-1$ , such that  $U(a_i)U_1(a_i) < 0$ . Summarizing,

$$(3.3) \quad \text{For } 0 \leq i \leq k, \quad U(a_i)U_1(a_i) < 0.$$

We have the relation

$$(3.4) \quad N_m(U, ]\alpha, \beta[ = \sum_{i=1}^k N_m(U, ]a_{i-1}, a_i]) = \sum_{i=1}^k p_i.$$

For  $U_1$ , we have the inequality

$$(3.5) \quad N_m(U_1, ]\alpha, \beta[ \geq \sum_{i=1}^k N_m(U_1, ]a_{i-1}, a_i]),$$

because  $U_1$  might have zeros in the interval  $]\alpha, a_0[$  if  $a_0 > \alpha$  (resp. in the interval  $]a_k, \beta[$  if  $a_k < \beta$ ).

For  $1 \leq i \leq k$  the interval  $]a_{i-1}, a_i[$  contains precisely one zero of  $U$ ,  $\xi_i$  with multiplicity  $p_i$ .

**Claim 3.8.** For  $1 \leq i \leq k$ ,  $N_m(U_1, ]a_{i-1}, a_i]) \geq N_m(U, ]a_{i-1}, a_i]) = p_i$ .

To prove the claim, we consider several cases.

- If  $p_i = 1$ , then  $U(a_{i-1})U(a_i) < 0$  and, by (3.3),  $U_1(a_{i-1})U_1(a_i) < 0$ , so that  $N_m(U_1, ]a_{i-1}, a_i]) \geq 1$ .

- If  $p_i \geq 2$ , we apply Lemma (2.3) at  $\xi_i$ : there exist real numbers  $B, B_1$  and smooth functions  $R$  and  $R_1$ , such that, in a neighborhood of  $\xi_i$ ,

$$(3.6) \quad \begin{cases} U(x) &= B(x - \xi_i)^{p_i} + (x - \xi_i)^{p_i+1}R(x), \\ U_1(x) &= B_1(x - \xi_i)^{p_i-2} + (x - \xi_i)^{p_i-1}R_1(x), \end{cases}$$

where  $\text{sign}(B) = \text{sign}(B_1) = \sigma_i$ .

We now use (3.3) and the fact that  $\text{sign}(U(a_i)) = \sigma_i$ .

◊ If  $p_i \geq 2$  is odd, then  $\sigma_{i-1}\sigma_i = -1$ . It follows that

$$\sigma_i U(a_i) < 0 \text{ and } \sigma_i U(a_{i-1}) > 0.$$

By (3.6), for  $\varepsilon$  small enough, we also have

$$\sigma_i U(\xi_i + \varepsilon) > 0 \text{ and } \sigma_i U(\xi_i - \varepsilon) < 0.$$

This means that  $U_1$  vanishes at order  $p_i - 2$  at  $\xi_i$ , and at least once in the intervals  $]a_{i-1}, \xi_i - \varepsilon[$  and  $]\xi_i + \varepsilon, a_i[$ , so that

$$N_m(U_1, ]a_{i-1}, a_i]) \geq p_i - 2 + 2 = p_i = N_m(U, ]a_{i-1}, a_i]).$$

◊ If  $p_i \geq 2$  is even, then  $\sigma_{i-1}\sigma_i = 1$ . It follows that

$$\sigma_i U(a_i) < 0 \text{ and } \sigma_i U(a_{i-1}) < 0.$$

By (3.6), for  $\varepsilon$  small enough, we also have

$$\sigma_i U(\xi_i + \varepsilon) > 0 \text{ and } \sigma_i U(\xi_i - \varepsilon) > 0.$$

This means that  $U_1$  vanishes at order  $p_i - 2$  at  $\xi_i$ , and at least once in the intervals  $]a_{i-1}, \xi_i - \varepsilon[$  and  $]\xi_i + \varepsilon, a_i[$ , so that

$$N_m(U_1, ]a_{i-1}, a_i]) \geq p_i - 2 + 2 = p_i = N_m(U, ]a_{i-1}, a_i]).$$

The claim is proved, and the proposition as well.  $\square$

**Proposition 3.9.** *Let  $k \in \mathbb{Z}$ . Then, under the Assumptions (1.9),*

$$(3.7) \quad \overline{N}_m(Y_{k+1}, [\alpha, \beta]) \geq \overline{N}_m(Y_k, [\alpha, \beta]),$$

*i.e., counting multiplicities of interior zeros, and reduced multiplicities of  $\alpha$  and  $\beta$ , the number of zeros of  $Y_{k+1}$  in  $[\alpha, \beta]$  is not less than the number of zeros of  $Y_k$  in  $[\alpha, \beta]$ .*

**Proof.** Recall that the reduced multiplicity of  $\alpha$  (resp.  $\beta$ ) is zero if Dirichlet condition holds at  $\alpha$  (resp. at  $\beta$ ) or if  $U(\alpha) \neq 0$  (resp.  $U(\beta) \neq 0$ ). Furthermore, according to Lemma 2.4, if  $h \in [0, \infty[$  and  $U(\alpha) = 0$  (resp. if  $H \in [0, \infty[$  and  $U(\beta) = 0$ ), then  $m(U, \alpha) = 2p$  (resp.  $m(U, \beta) = 2q$ ).

*Case 1.* Assume that  $N_m(U, ]\alpha, \beta]) = 0$ . Without loss of generality, we may assume that  $U > 0$  in  $] \alpha, \beta [$ .

- If  $U(\alpha) \neq 0$  and  $U(\beta) \neq 0$ , there is nothing to prove.
- Assume that  $U(\alpha) = U(\beta) = 0$ . Then, there exists  $a \in ]\alpha, \beta[$  such that  $U(a) = \sup\{U(x) \mid x \in [\alpha, \beta]\}$ , with  $U(a) > 0$ ,  $\frac{dU}{dx}(a) = 0$  and  $\frac{d^2U}{dx^2}(a) \leq 0$ . It follows from (2.8) that  $U_1(a) < 0$ , and that

$$(3.8) \quad \overline{N}_m(U_1, [\alpha, \beta]) = \overline{N}(U_1, [\alpha, a]) + \overline{N}(U_1, [a, \beta]).$$

It now suffices to look separately at the intervals  $[\alpha, a]$  and  $[a, \beta]$ .

◊ Interval  $[\alpha, a]$ . If the Dirichlet condition holds at  $\alpha$ , there is nothing to prove. If  $h \in [0, \infty[$ ,  $m(U, \alpha) = 2p \geq 2$  and, by Lemma 2.3,

$$(3.9) \quad \begin{aligned} U(x) &= B(x - \alpha)^{2p} + (x - \alpha)^{2p+1}R(x), \\ U_1(x) &= B_1(x - \alpha)^{2p-2} + (x - \alpha)^{2p-1}R_1(x), \end{aligned}$$

with  $B > 0$  and  $B_1 > 0$ .

It follows that  $U_1(\alpha + \varepsilon) > 0$  for any positive  $\varepsilon$  small enough so that  $N_m(U_1, ]\alpha, a]) \geq 1$ . It follows that

$$(3.10) \quad \begin{aligned} \overline{N}_m(U_1, [\alpha, a]) &= \overline{m}(U_1, \alpha) + N_m(U_1, ]\alpha, a]) \geq p - 1 + 1, \\ \text{i.e.} \end{aligned}$$

$$\overline{N}_m(U_1, [\alpha, a]) \geq \overline{N}(U, [\alpha, a]).$$

◊ Interval  $[a, \beta]$ . The proof is similar.

- Assume that  $U(\alpha) = 0$  and  $U(\beta) \neq 0$ . The proof is similar to the previous one with  $a \in ]a, \beta]$ .
- Assume that  $U(\alpha) \neq 0$  and  $U(\beta) = 0$ . The proof is similar to the previous one with  $a \in [\alpha, a[$ .

*Case 2.* Assume that  $N_m(U, ]\alpha, \beta]) \geq 1$ .

- If  $U(\alpha) \neq 0$  (resp.  $U(\beta) \neq 0$ ), there is nothing to prove for the boundary  $\alpha$  (resp.  $\beta$ ).



• If  $U(\alpha) = 0$  (resp.  $U(\beta) = 0$ ), the number  $a_0$  (resp.  $a_k$ ) which appears in the proof of Proposition 3.7 belongs to the open interval  $]\alpha, \xi_1[$  (resp. to the open interval  $]\xi_k, \beta[$ ), where  $\xi_1$  (resp.  $\xi_k$ ) is the smallest (resp. largest) zero of  $U$  in  $]\alpha, \beta[$ . We can then apply the proof of Step. 1 to the interval  $[\alpha, a_0]$  (resp. to the interval  $[a_k, \beta]$ ) to prove that  $\overline{N}_m(U_1, [\alpha, a_0]) \geq \overline{N}_m(U, [\alpha, a_0])$  (resp. to prove that  $\overline{N}_m(U_1, [a_k, \beta]) \geq \overline{N}_m(U, [a_k, \beta])$ ). This proves Proposition 3.9.  $\square$

We can now prove the following refined version of Theorem 1.8.

**Theorem 3.10.** *With the notation of Sections 1 and 2.2, let  $Y$  be the linear combination*

$$(3.11) \quad Y = \sum_{p=m}^n A_p V_p,$$

where  $1 \leq m \leq n$ , and where  $\{A_p, m \leq p \leq n\}$  are real constants such that  $A_m^2 + \dots + A_n^2 \neq 0$ , which is equivalent to saying that  $Y \neq 0$ . Then,

$$(3.12) \quad \begin{aligned} (m-1) &\leq N_v(Y, ]\alpha, \beta[) \leq (n-1), \\ (m-1) &\leq N_m(Y, ]\alpha, \beta[) \leq (n-1), \\ (m-1) &\leq \overline{N}_m(Y, [\alpha, \beta]) \leq (n-1). \end{aligned}$$

**Proof.** Let  $N(V)$  be any of the above functions. We may of course assume that  $A_m \neq 0$  and  $A_n \neq 0$ . In the preceding lemmas, we have proved that  $N(Y_{k+1}) \geq N(Y_k)$  for any  $k \in \mathbb{Z}$ , which can also be rewritten as

$$(3.13) \quad N(Y_{(-k)}) \leq N(Y) \leq N(Y_k) \quad \text{for any } k \geq 1.$$

Letting  $k$  tend to infinity, we see that

$$(3.14) \quad N(V_m) \leq N(Y) \leq N(V_n),$$

and we can apply Theorem 1.3.  $\square$

**Remark.** For a complete proof of the limiting argument when  $k$  tends to infinity, we refer to Appendix A.

#### 4. LIOUVILLE'S PROOF OF STURM'S THEOREM

We keep the notation of Section 1.

*Remark 4.1.* In his proof, Liouville [19] only considers the zeros in the open interval  $]\alpha, \beta[$ .

Let  $1 \leq m \leq n$ , and let  $Y = \sum_{j=m}^n A_j V_j$  be a linear combination of eigenfunctions of the eigenvalue problem (1.1)–(1.3).

Assume that  $Y \neq 0$ , i.e.,  $\sum_{j=m}^n A_j^2 \neq 0$ . Without loss of generality, we may assume that  $A_m A_n \neq 0$ .

**Theorem 4.2.** *Counting multiplicities, the function  $Y$  has, in the interval  $]\alpha, \beta[$ ,*

- (1) at most  $(n - 1)$  zeros and,
- (2) at least  $(m - 1)$  zeros.

**Proof.** Liouville uses the following version of Rolle's theorem (Michel Rolle (1652-1719) was a French mathematician). This version of Rolle's theorem seems to go back to Cauchy and Lagrange.

**Lemma 4.3.** *Let  $f$  be a function in  $] \alpha_0, \beta_0[$ . Assume that  $f(x') = f(x'') = 0$  for some  $x', x''$ ,  $\alpha_0 < x' < x'' < \beta_0$ .*

- (1) *If the function  $f$  is differentiable and has  $\nu - 1$  distinct zeros in the interval  $]x', x''[$ , then the derivative  $f'$  has at least  $\nu$  distinct zeros in  $]x', x''[$ .*
- (2) *If the function  $f$  is smooth and has  $\mu - 1$  zeros counted with multiplicities in the interval  $]x', x''[$ , then the derivative  $f'$  has at least  $\mu$  zeros counted with multiplicities in  $]x', x''[$ .*

**Proof of the lemma.** Call  $x_1 < x_2 < \dots < x_{\nu-1}$  the distinct zeros of  $f$  in  $]x', x''[$ . Since  $f(x') = f(x'') = 0$ , by Rolle's theorem [27], the function  $f'$  vanishes at least once in each open interval determined by the  $x_j$ ,  $1 \leq j \leq \nu - 1$ , as well as in the intervals  $]x', x_1[$  and  $]x_{\nu-1}, x''[$ . It follows that  $f'$  has at least  $\nu$  distinct zeros in  $]x', x''[$ , which proves the first assertion.

Call  $m_j$  the multiplicity of the zero  $x_j$ ,  $1 \leq j \leq \nu - 1$ . Then  $f'$  has at least  $\nu$  zeros, one in each of the open intervals determined by  $x', x''$  and the  $x_j$ 's, and has a zero at each  $x_j$  with multiplicity  $m_j - 1$ , provided that  $m_j > 1$ . It follows that the number of zeros of  $f'$  in  $]x', x''[$ , counting multiplicities, is at least

$$\sum_{j=1}^{\nu-1} (m_j - 1) + \nu = \sum_{j=1}^{\nu-1} m_j + 1,$$

which proves the second assertion. □

**4.1. Proof of the assertion “ $Y$  has at most  $(n - 1)$  zeros in  $] \alpha, \beta[$ , counting multiplicities”.** Write (1.1) for  $V_1$  and for  $V_p$ , for some  $m \leq p \leq n$ . Multiply the first equation by  $-V_p$ , the second by  $V_1$ , and add the resulting equations. Then

$$(4.1) \quad V_1 \frac{d}{dx} \left( K \frac{dV_p}{dx} \right) - V_p \frac{d}{dx} \left( K \frac{dV_1}{dx} \right) + (\rho_p - \rho_1) G V_1 V_p = 0.$$

Use the identity

$$(4.2) \quad V_1 \frac{d}{dx} \left( K \frac{dV_p}{dx} \right) - V_p \frac{d}{dx} \left( K \frac{dV_1}{dx} \right) = \frac{d}{dx} \left( V_1 K \frac{dV_p}{dx} - V_p K \frac{dV_1}{dx} \right),$$

and integrate from  $\alpha$  to  $t$  to get the identity

$$(4.3) \quad (\rho_1 - \rho_p) \int_{\alpha}^t G V_1 V_p dx = K(t) \left( V_1(t) \frac{dV_p}{dx}(t) - V_p(t) \frac{dV_1}{dx}(t) \right).$$

Here we use the boundary condition (1.2) which implies that

$$\left( V_1(\alpha) \frac{dV_p}{dx}(\alpha) - V_p(\alpha) \frac{dV_1}{dx}(\alpha) \right) = 0.$$

Multiply the preceding identity by  $A_p$  and sum for  $p$  from  $m$  to  $n$  to obtain

$$(4.4) \quad \int_{\alpha}^t G V_1 \sum_{p=m}^n (\rho_1 - \rho_p) A_p V_p dx = K(t) \left( V_1 \frac{dY}{dx} - Y \frac{dV_1}{dx} \right) (t).$$

or

$$(4.5) \quad \int_{\alpha}^t G V_1 \sum_{p=m}^n (\rho_1 - \rho_p) A_p V_p dx = K(t) V_1^2(t) \frac{d}{dt} \left( \frac{Y}{V_1} \right) (t),$$

where we have used the fact that the function  $V_1$  does not vanish in the interval  $] \alpha, \beta [$ .

Let  $\Psi(x) = \frac{Y}{V_1}(x)$ . The zeros of  $Y$  in  $] \alpha, \beta [$  are the same as the zeros of  $\Psi$ . Let  $\mu$  be the number of zeros of  $Y$ , counted with multiplicities. By Lemma 4.3, Assertion (2),  $\frac{d\Psi}{dx}$  has at least  $\mu - 1$  zeros in  $] \alpha, \beta [$ , and hence so does the left-hand side of (4.5),

$$\int_{\alpha}^t G V_1 \sum_{p=m}^n (\rho_1 - \rho_p) A_p V_p dx.$$

On the other hand, this function vanishes at  $\alpha$  and  $\beta$  (because of the boundary condition (1.3) or orthogonality). By Lemma 4.3, its derivative,

$$(4.6) \quad V_1 \sum_{p=m}^n (\rho_1 - \rho_p) A_p V_p$$

has at least  $\mu$  zeros counted with multiplicities in  $] \alpha, \beta [$ . We have proved the following

**Lemma 4.4.** *If the function  $Y = \sum_{p=m}^n A_p V_p$  has at least  $\mu$  zeros counted with multiplicities in the interval  $] \alpha, \beta [$ , then the function  $Y_1 = \sum_{p=m}^n (\rho_1 - \rho_p) A_p V_p$  has at least  $\mu$  zeros, counted with multiplicities, in  $] \alpha, \beta [$ .*

Applying this lemma iteratively, we deduce that if  $Y$  has at least  $\mu$  zeros counted with multiplicities in  $] \alpha, \beta [$ , then, for any  $k \geq 1$ , the function

$$(4.7) \quad Y_k = \sum_{p=m}^n (\rho_1 - \rho_p)^k A_p V_p$$

has at least  $\mu$  zeros, counted with multiplicities, in  $] \alpha, \beta [$ .

*Remark 4.5.* Applying Lemma 4.3, Assertion (1), we obtain a result on the number of zeros, multiplicities not accounted for.

We may of course assume that the coefficient  $A_n$  is non-zero. The above assertion can be rewritten as the statement:

For all  $k \geq 0$ , the equation

$$(4.8) \quad A_m \left( \frac{\rho_m - \rho_1}{\rho_n - \rho_1} \right)^k V_m + \cdots + A_{n-1} \left( \frac{\rho_{n-1} - \rho_1}{\rho_n - \rho_1} \right)^k V_{n-1} + A_n V_n = 0$$

has at least  $\mu$  solutions in  $] \alpha, \beta[$ , counting multiplicities.

Letting  $k$  tend to infinity, and using the fact that  $V_n$  has exactly  $(n-1)$  zeros in  $] \alpha, \beta[$ , this implies that  $\mu \leq (n-1)$ . This proves the first assertion.  $\square$

**4.2. Proof of the assertion “ $Y$  has at least  $(m-1)$  zeros in  $] \alpha, \beta[$ , counting multiplicities”.** We have seen that the number of zeros of  $Y_k$  is less than or equal to the number of zeros of the function  $Y_{k+1}$ . This assertion actually holds for any  $k \in \mathbb{Z}$  and can also be rewritten as, for any  $k \geq 0$ ,

$$(4.9) \quad N_m(Y_{-k}) \leq N_m(Y),$$

where, for  $k \geq 0$ ,

$$(4.10) \quad Y_{-k} = A_m(\rho_m - \rho_1)^{-k} V_m + \cdots + A_n(\rho_n - \rho_1)^{-k} V_n$$

and we can again let  $k$  tend to infinity. The second assertion is proved and Theorem 4.2 as well.  $\square$

**4.3. Liouville’s second approach to the second part of Theorem 4.2.** If the function  $Y$  has  $\mu_1$  distinct zeros, then it changes sign  $\mu$  times, with  $\mu_1 \geq \mu$ , at points  $a_i$  which satisfy  $\alpha < a_1 < \cdots < a_\mu < \beta$ .

**Claim 4.6.** *The function  $Y$  changes sign at least  $(m-1)$  times in the interval  $] \alpha, \beta[$ .*

**Proof of the claim.** Assume that  $\mu \leq (m-2)$ . Consider the function

$$(4.11) \quad x \mapsto W(x) := \Delta(a_1, \dots, a_\mu; x),$$

where the function  $\Delta$  is defined by the determinant

$$(4.12) \quad \begin{vmatrix} V_1(a_1) & V_1(a_2) & \cdots & V_1(a_\mu) & V_1(x) \\ V_2(a_1) & V_2(a_2) & \cdots & V_2(a_\mu) & V_2(x) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ V_{\mu+1}(a_1) & V_{\mu+1}(a_2) & \cdots & V_{\mu+1}(a_\mu) & V_{\mu+1}(x) \end{vmatrix}.$$

The function  $W$  vanishes at the points  $a_i, 1 \leq i \leq \mu$ . According to the first part Theorem 4.2,  $W$  being a linear combination of the first  $\mu+1$  eigenfunctions, vanishes at most  $\mu$  times in  $] \alpha, \beta[$ , counting multiplicities. This implies that each zero  $a_i$  of  $W$  has order one, and that  $W$  does not have any other zero in  $] \alpha, \beta[$ . It follows that the function  $YW$

vanishes only at the points  $\{a_i\}$ , and that it does not change sign. We can assume that  $YW \geq 0$ . On the other hand, we have

$$(4.13) \quad \int_{\alpha}^{\beta} GYW \, dx = 0,$$

because  $Y$  involves the functions  $V_p$  with  $p \geq m$  and  $W$  the functions  $V_q$  with  $q \leq \mu + 1 \leq m - 1$ . This gives a contradiction.  $\square$

*Remark 4.7.* Liouville does actually not use the determinant (4.12), but a similar approach, see [18, p. 259], Lemme 1<sup>er</sup>.

## 5. STURM'S RESULTS UNDER WEAKER ASSUMPTIONS

We proved Theorems 3.10 and 4.2 under the Assumptions (1.9). In this section, we consider the weaker assumptions

$$(5.1) \quad \begin{cases} [\alpha, \beta] \subset ]\alpha_0, \beta_0[, \\ K \in C^1(]\alpha_0, \beta_0[), \\ G, L \in C^0(]\alpha_0, \beta_0[), \\ K, G, L > 0 \text{ on } ]\alpha_0, \beta_0[. \end{cases}$$

Under these assumptions, the functions  $V_j$  are  $C^2$  on  $]\alpha_0, \beta_0[$ . This follows easily for example from Liouville's existence proof [18], and we have the following lemma, whose proof is analogous to the proof of Lemma 2.2.

**Lemma 5.1.** *Let  $k \in \mathbb{Z}$ .*

- (1) *The function  $Y_k$  satisfies the boundary conditions (1.2) and (1.3).*
- (2) *The functions  $Y_k$  and  $Y_{k+1}$  satisfy the relation*

$$(5.2) \quad GY_{k+1} = K \frac{d^2 Y_k}{dx^2} + \frac{dK}{dx} \frac{dY_k}{dx} - LY_k.$$

- (3) *Under the Assumptions (5.1), the function  $Y_k$  cannot vanish identically on an open interval  $]\alpha_1, \beta_1[ \subset ]\alpha_0, \beta_0[$ , unless  $Y_k \equiv 0$ .*

In Subsection 4.1, we have used Lemma 4.3 (2) which uses the fact that the functions  $V_j$  are  $C^\infty$ . If the functions  $V_j$  are only  $C^2$ , we can apply Lemma 4.3 (1). It is easy to conclude that Liouville's proofs in Subsection 4.1 and 4.2 go through, under the weaker Assumptions (5.1), if we only count distinct zeros, see (2.13). More precisely, we can prove the following claim.

**Claim 5.2.** *Under the Assertions (5.1), for any  $k \in \mathbb{Z}$ , if the function  $Y_k$  has at least  $\mu$  distinct zeros in the interval  $]\alpha, \beta[$ , then the function  $Y_{k+1}$  has at least  $\mu$  distinct zeros in the interval  $]\alpha, \beta[$ .*

We can then deduce from this claim, as in Section 4, that a linear combination  $Y = \sum_{j=m}^n A_j V_j$  has at most  $(n - 1)$  distinct zeros (in particular it has finitely many zeros).

Once this result secured, we can define zeros at which  $Y$  changes sign (without using the multiplicity) and apply Sturm's argument to conclude that the function  $Y$  must change sign at least  $(m - 1)$  times.

#### APPENDIX A. THE LIMITING ARGUMENT IN (4.8)

Recall that we assume the  $A_n \neq 0$ . Define

$$(A.1) \quad \omega = \left( \frac{\rho_{n-1} - \rho_1}{\rho_n - \rho_1} \right)^k.$$

One can rewrite (4.8) as

$$V_n(x) + \omega \Pi(x) = 0,$$

where

$$(A.2) \quad \Pi(x) = \sum_{p=m}^{n-1} \frac{A_p}{A_n} \left( \frac{\rho_p - \rho_1}{\rho_{n-1} - \rho_1} \right)^k V_p.$$

It follows that  $\Pi$  is uniformly bounded by

$$(A.3) \quad |\Pi(x)| \leq M := n \max_p \left| \frac{A_p}{A_n} \right| \max_{[\alpha, \beta]} \sup |V_p|.$$

Similarly,

$$(A.4) \quad \left| \frac{d\Pi}{dx}(x) \right| \leq N := n \max_p \left| \frac{A_p}{A_n} \right| \max_{[\alpha, \beta]} \sup \left| \frac{dV_p}{dx} \right|.$$

Call  $\xi_1 < \xi_2 < \dots < \xi_{n-1}$  the zeros of the function  $V_n$  in the interval  $] \alpha, \beta [$ .

• Assume that  $V_n(\alpha) \neq 0$  and  $V_n(\beta) \neq 0$ .

Since  $\frac{dV_n}{dx}(\xi_i) \neq 0$ , there exist  $\delta_1, \varepsilon_1 > 0$  such that  $|\frac{dV_n}{dx}(x)| \geq \varepsilon_1$  for  $x \in [\xi_i - \delta_1, \xi_i + \delta_1]$ , and  $|V_n(x)| \geq \varepsilon_1$  in  $[\alpha, \beta] \setminus \cup ]\xi_i - \delta_1, \xi_i + \delta_1 [$ .

For  $k$  large enough, we have  $\omega M, \omega N \leq \varepsilon_1/2$ . It follows that in the interval  $[\xi_i - \delta_1, \xi_i + \delta_1]$ ,

$$\left| \frac{d}{dx}(V_n + \omega \Pi) \right| \geq \left| \frac{dV_n}{dx} \right| - \omega N \geq \varepsilon_1/2.$$

Furthermore,

$$V_n(\xi_i \pm \delta_1) + \omega \Pi(\xi_i \pm \delta_1) \geq |V_n(\xi_i \pm \delta_1)| - \omega M \geq \varepsilon_1/2.$$

Since  $V_n(\xi_i + \delta_1)V_n(\xi_i - \delta_1) < 0$ , we can conclude that the function  $V_n + \omega \Pi$  has exactly one zero in each interval  $]\xi_i - \delta_1, \xi_i + \delta_1 [$ .

In  $[\alpha, \beta] \setminus \cup ]\xi_i - \delta_1, \xi_i + \delta_1 [$ , we have

$$|V_n(x) + \omega \Pi(x)| \geq |V_n(x)| - \omega M \geq \varepsilon_1/2,$$

which implies that  $V_n(x) + \omega \Pi(x) \neq 0$ .

- Assume that  $V_n(\alpha) = 0$  and  $V_n(\beta) \neq 0$ . This corresponds to the case  $h = +\infty$  and  $H \neq +\infty$ . Hence the  $V_j$  verify Dirichlet at  $\alpha$  and  $\Pi$  verifies Dirichlet at  $\alpha$ . Observing that  $V_n'(\alpha) \neq 0$ , it is immediate to see that there exists  $\delta_1 > 0$ , such that, for  $k$  large enough,  $V_n(x) + \omega\Pi(x)$  has only  $\alpha$  as zero in  $[\alpha, \alpha + \delta_1]$ .
- The other cases are treated in the same way. □

## APPENDIX B. TRANSLATIONS FOR THE CITATIONS FROM [33, 34]

**Citation from** [33, Introduction].

On ne sait [ces équations] les intégrer que dans un très petit nombre de cas particuliers hors desquels on ne peut pas même en obtenir une intégrale première ; et lors même qu'on possède l'expression de la fonction qui vérifie une telle équation, soit sous forme finie, soit en série, soit en intégrales définies ou indéfinies, il est le plus souvent difficile de reconnaître dans cette expression la marche et les propriétés caractéristiques de cette fonction. . . .

S'il importe de pouvoir déterminer la valeur de la fonction inconnue pour une valeur isolée quelconque de la variable dont elle dépend, il n'est pas moins nécessaire de discuter la marche de cette fonction, ou en d'autres termes, d'examiner la forme et les sinuosités de la courbe dont cette fonction serait l'ordonnée variable, en prenant pour abscisse la variable indépendante. Or on peut arriver à ce but par la seule considération des équations différentielles elles-mêmes, sans qu'on ait besoin de leur intégration. Tel est l'objet du présent mémoire. . . .

One only knows how to integrate these equations in a very small number of particular cases and one can otherwise not even obtain a first integral; even when one knows the expression of the function which satisfies such an equation, in finite form, as a series, as integrals either definite or indefinite, it is most generally difficult to recognize in this expression the behaviour and the characteristic properties of this function. . . .

Although it is important to be able to determine the value of the unknown function for an isolated value of the variable it depends upon, it is not less necessary to discuss the behaviour of this function, or otherwise stated, the form and the twists and turns of the curve whose ordinate would be the function and the abscissa the independent variable. It turns out that one can achieve this goal by the sole consideration of the differential equation themselves, without having to integrate them. This is the purpose of the present memoir. . . .



**Citation from** [33, p. 108].

L'intégrale complète de l'équation (I) doit contenir deux constantes arbitraires, pour lesquelles on peut prendre les valeurs de  $V$  et de  $\frac{dV}{dx}$  correspondantes à une valeur particulière de  $x$ . Lorsque ces valeurs sont fixées, la fonction  $V$  est entièrement définie par l'équation (I), elle a une valeur déterminée et unique pour chaque valeur de  $x$ .

**Citation from** [34, p. 379].

M. Liouville a démontré directement ce théorème, qui n'était pour moi qu'un corollaire du précédent, sans s'occuper du cas particulier où la fonction serait nulle à l'une des extrémités de la barre. J'en ai aussi trouvé après lui une autre démonstration directe que je donne dans ce mémoire. M. Liouville a fait usage du même théorème dans un très beau Mémoire qu'il a publié dans le numéro de juillet de son journal et qui a pour objet le développement d'une fonction arbitraire en une série composée de fonctions  $V$  que nous avons considérées.

The complete integral of equation (I) must contain two arbitrary constants, for which one can take the values of  $V$  and of  $\frac{dV}{dx}$  corresponding to some particular value of  $x$ . Once these values are fixed, the function  $V$  is fully determined by equation (I), it has a uniquely determined value for each value of  $x$ .

M. Liouville gave a direct proof of this theorem, which for me only was a corollary of the preceding one, without taking care of the particular case in which the function vanishes at one of the extremities of the bar. I have also found, after him, another direct proof which I give in this memoir. M. Liouville made use of the same theorem in a very nice memoir which he published in the July issue of his journal and which deals with the expansion of an arbitrary function into a series made of the functions  $V$  which we have considered.

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