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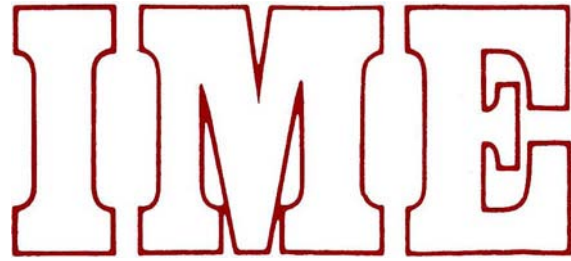
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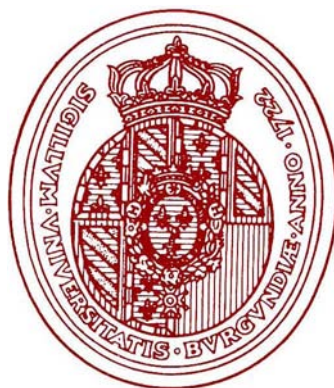
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DISTRIBUTION OF POWER AND SOCIAL UTILITY  
UNDER NONADDITIVE AGGREGATION RULES

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***Distribution of Power and Social Utility  
under Nonadditive Aggregation Rules***

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INTRODUCTION

During the last fifteen years, a lot of contributions, which follow the initiated work of Barberà & Sonnenschein 1978, investigate the structure of coalitional power under probabilistic social decision rules (Bandyopadhyay, Deb & Pattanaik 1982, Barberà & Valenciano 1983, Heiner & Pattanaik 1983). The restrictions on coalitional power derived in all of them relate to the power of coalitions to influence social probabilities over two-element feasible sets only.

As noticed by Pattanaik & Peleg 1986, all these authors adopted a formulation in terms of probabilistic social *preferences* and not in terms of probabilistic social *choice*. Basically, Barberà & Sonnenschein 1978 show that every binary and Paretian method for passing from preference profiles to lotteries over preferences is associated with a subadditive function on the set of coalitions of individuals. This function gives the power of each coalition in order to secure its preference for any *pair* of alternatives. McLennan 1980 proves this power function to be strictly additive, i.e. probabilistic, whenever there are six or more alternatives. Also, as indicated by both Barberà & Sonnenschein 1978 and McLennan 1980, Butters proves by another way the converse of Barberà & Sonnenschein's Theorem is valid when the set of alternatives does not exceed five. This methodology depends on the view that admitting as social preferences lotteries on preferences over the basic alternatives increases the possibility for satisfactory preference aggregation. Nevertheless, they show that the dimensions of this increase are limited and they come to the conclusion that the force of Arrow's Theorem is not diminished.

Originally, Arrow tries to consider the only methods which allow passing from social preference profiles to social preference relations while Barberà & Sonnenschein studies methods for passing from social preferences profiles to lotteries over social preference relations (called social welfare schemes). Thus, the latter show that allowing lotteries permits a wide variety of nondictatorial procedures. In return, their paper is a generalization of Arrow's Possibility Theorem 1963 by considering functions which map profiles on individual preference relations into lotteries on the set of preference relations. These functions translate formally the idea of *method* for passing from profiles to lotteries. They have to satisfy two axioms : binarity and a Paretian property. Binarity is simply a generalization of Arrow's condition of Independence of Irrelevant Alternatives and the Paretian property corresponds to that of Unanimity.

In this paper, we try first to shift the problem onto the agents. We want actually to consider some methods for passing from coalitions to lotteries over coalitions. These methods are associated with a function that could correspond to a social utility function. It gives the level of satisfaction coming from the anticipated "consumption" of a subset of alternatives (goods, candidates, ..) by a coalition. Second, we want to define the distributions of power regardless of those two-element feasible sets which characterize preferences in basic models on one side, and we try to exhibit a theoretical relation between the basic Barberà & Sonnenschein's model and ours thanks to the Choice Axiom of Luce 1959 on the other side. Third, McLennan's argument against the validity of subadditivity for all power functions, i.e. the number of alternatives, disappears since we can introduce nonadditive lotteries directly from a Generalized Choice Axiom that can be obtained with the introduction of Choquet capacities. These latter, which appeared in Choquet 1953, were introduced in economics by Gilboa 1987 and Schmeidler 1989. They allow to weaken the standard constraints on probabilities in putting a simple condition of monotonicity in place of additivity. Finally, we try to prove that it is possible to deduce a social or a coalitional utility function directly from lotteries over preferences in implying a Power Scheme (i.e. a distribution of power decision) from the basic Social Welfare Scheme, this last being probabilistic or non-additive.

## 1. NOTATIONS AND DEFINITIONS

Let  $X = \{x, y, z, \dots\}$  be the set of *alternatives* and  $N = \{1, 2, \dots, n\}$  be the set of *individuals*. In order to avoid McLennan's limitation of relevance for Barberà &

Sonnenschein's Theorem, we assume throughout  $X$  and  $N$  to be finite and we already suppose the number of elements of  $X$  as those of  $N$  (denoted  $\#X$  and  $\#N$ ) to be at least six. Let  $\mathcal{B} = \{B, B', \bar{B}, \dots\}$  be the set of *individual strict preference relations* (complete, asymmetric and transitive binary relations) on  $X$ , and  $\mathcal{B}^n$ , the  $n$ -fold Cartesian product of  $\mathcal{B}$ , be the set of social preferences *profiles*. A generic profile is denoted by  $\mathbf{B} = (B_1, B_2, \dots, B_n)$ .

A *probabilistic lottery* on  $\mathcal{B}$  is a probability measure on  $\mathcal{B}$ ; i.e. a function  $l(\cdot) : 2^{\mathcal{B}} \rightarrow [0,1]$  such that  $l(\emptyset) = 0$ ,  $l(\mathcal{B}) = 1$  and  $l(S \cup T) = l(S) + l(T) - l(S \cap T)$  for all  $S, T \subset \mathcal{B}$ . Let  $L(\mathcal{B})$  be the set of lotteries on  $\mathcal{B}$ . When a probabilistic lottery is defined on a subset  $T \in \mathcal{B}$ , we note it  $l_T(\cdot)$ . For  $x, y \in X$ , let  $\mathcal{B}(xBy) = \{B \in \mathcal{B} : xBy\}$ . For  $l(\cdot) \in L(\mathcal{B})$ , let  $p[l(\cdot), xBy] = l(\mathcal{B}(xBy))$ . It means, for all distinct  $x, y$ ,  $p[l(\cdot), xBy] + p[l(\cdot), yBx] = 1$ . Furthermore, given  $\mathbf{B} \in \mathcal{B}^n$ , let  $\psi(\mathcal{B}, xBy) = \{i \in N : xB_i y\}$ .

A *nonadditive lottery* on  $\mathcal{B}$  (resp.  $N$ ) is a normalized Choquet capacity on  $\mathcal{B}$  (resp.  $N$ ); i.e. a set function  $c(\cdot) : 2^{\mathcal{B}} \rightarrow [0,1]$  (resp.  $2^N$ ) such that  $c(\emptyset) = 0$ ,  $c(\mathcal{B}) = 1$  (resp.  $c(N)$ ) and  $S \supset T$  implies  $c(S) \geq c(T)$  for all  $S, T \subset \mathcal{B}$  (resp.  $N$ ). Let  $C(\mathcal{B})$  (resp.  $C(N)$ ) be the set of nonadditive lotteries on  $\mathcal{B}$  (resp.  $N$ ). Moreover, we note  $NAC(\mathcal{B})$  (resp.  $NAC(N)$ ) the subset of strict nonadditive lotteries. When a nonadditive lottery is defined on a subset  $T \in \mathcal{B}$  (resp.  $C \in N$ ), we note it  $c_T(\cdot)$  (resp.  $c_C$ ). Given any coalition  $A \in 2^N$ ,  $x(A) = \{y \in X : xB_A y\}$ . We assume " $B_A$ " to be set up from the following rule:  $x(A) = x[\cup_{i \in A} i] = \cup_{i \in A} x(i)$ . " $x(A)$ " can be also interpreted as a degree of *dispersion* for the  $A$ 's collective choices. If  $\#x(A)$  is little when  $\#A$  is big, it means that the  $A$ 's collective choices are very concentrated; if  $x(A) = x(B)$  while  $\#A > \#B$ , it means that the  $A$ 's collective choices are more concentrated than those of  $B$ . Furthermore,  $x(A) \cup \bar{x}(A) = X$ .  $\forall S \in 2^X$ ,  $\forall x \in S$ , we note  $S_x$ , the set of alternatives which belong to  $S$  except  $x$ . If  $x(A) = X_x$ , the collective choices of  $A$  are said *totally divided*.

*Remark 1* :  $L(\mathcal{B}) \subset C(\mathcal{B})$ . It means that a lottery which belongs to  $C(\mathcal{B})$  can be a probability. Actually,  $L(\mathcal{B}) \cup NAC(\mathcal{B}) = C(\mathcal{B})$ .

A *Strict Social Welfare Scheme* (SSWS) is a function  $g(\cdot) : \mathcal{B}^n \rightarrow C(\mathcal{B})$ . If  $g(\cdot)$  comes from  $\mathcal{B}^n \rightarrow L(\mathcal{B})$ , we note it  $g_+(\cdot)$  in order to designate the SSWS(s) which lead to probabilistic lotteries. It means that  $g_+(\cdot)$  corresponds exactly to a SSWS à la Barberà & Sonnenschein 1978.

A SSWS  $g(\cdot)$  is *binary* if for any  $x, y \in X$ , and any pair of profiles  $\mathbf{B}, \mathbf{B}' \in \mathcal{B}^n$ ,

$$"\psi(\mathbf{B}, xBy) = \psi(\mathbf{B}', xBy)" \Rightarrow "p[g(\mathbf{B}), xBy] = p[g(\mathbf{B}'), xBy]"$$

A SSWS  $g(\cdot)$  is *Paretian* if for any  $x, y \in X$ , and any  $\mathbf{B} \in \mathcal{B}^n$ ,

$$"\psi(\mathbf{B}, xBy) = N" \Rightarrow "p[g(\mathbf{B}), xBy] = 1"$$

A *Power Scheme (PS)* is any lottery  $h(\cdot) : 2^N \rightarrow [0,1]$  of  $C(N)$ . If  $h(\cdot) = l(\cdot)$ , where  $l(\cdot)$  is a probability measure, we note it  $h_+(\cdot)$  in order to indicate that this *PS* is additive ;  $p[h_+(A), x] = l_N(A, x)$  could be interpreted as the *part* of representativity in  $N$  of the agent  $i \in A$  who are designated to be the  $A$ -Representative ; i.e. the power-decision of  $(A, x)$  where  $(A, x)$  corresponds to this  $A$ -Representative. There exists a Representative for each feasible coalition of  $2^N$  and each element  $x \in X$  ; i.e.  $\forall A \in 2^N ; A \neq \emptyset, \forall x \in X, (A, x) \neq \emptyset$ . There is a lot of ways to define these Representatives, from preferences, alternatives or even lotteries. We do not choose here a particular way in order to focus the analysis on the only power decision which comes from leadership. We could call them *natural leaders of  $A$  for  $x$*  (cf. Billot 1991a) as they are exogenously determined. If  $h(\cdot)$  is a nonadditive probability, i.e. a Choquet capacity, then  $p[h(A), x]$  could be interpreted as a *degree* of representativity of these natural leaders. If we restrict a *PS*  $h(\cdot)$  into a subset  $C \subseteq N$ , we note it  $h_C(\cdot)$ .

*Remark 2* : Most of the time, the *degree* of representativity in any  $C$  of the Representative  $(A, x) \in C$  will be noted directly  $h_C(A, x)$ .

*Remark 3* :  $p[h(A), x]$  can be a probability or a capacity. " $p[., .]$ " is just a symbol to note any probabilistic or strictly nonadditive lottery ; but as soon as  $h(\cdot) \in L(N)$ , we note  $p[h_+(A), x]$ .

The following properties hold for any restriction of any *PS*.

A *PS*  $h(\cdot)$  is *anonymous* if for any  $A, B \in 2^N$  such that  $A, B \subseteq C$  and for any  $x \in X$ ,

$$"x(A) = x(B)" \Rightarrow "p[h_C(A), x] = p[h_C(B), x]" .$$

A *PS*  $h(\cdot)$  is *unanimous* if for any  $A \in 2^N$  and for any  $x \in X$ ,

$$"x(A) = X_x" \Rightarrow "p[h_A(A), x] = 1" .$$

The first axiom means that the induced power decision of two Representatives must be the same whenever their two coalitions draw the same set of preferred alternatives for a given one ; the coalition must stay anonymous for the distribution of power. The second axiom means that the coalition concentrates all the power on its Representative whenever its collective choices are extremely dispersed ; the leadership is strengthened up to an enforced unanimity when choices are totally divided. Another interpretation leads to consider the unanimity axiom in so far as a definition of dictatorship according to collective choices since the coalition (or the society in the whole) totally deposes its power decision onto its Representative. In and only in the particular case where  $x(N) = X_x$ ,  $(N, x)$  is exactly the Arrovian dictator : i.e.  $p[h_N(N), x] = 1$ . It means also that each time  $x(N) \neq X_x$ , then there is no Arrovian dictatorship.



Let us introduce a new axiom of choice from Luce 1959, which was generalized to nonadditive probabilities in Billot & Thisse 1991.

A PS  $h(\cdot)$  is *Lucian* if for any  $A, B \in 2^N$  such that  $A \subseteq B \subseteq C$  and for any  $x \in X$ ,

$$p[h_C(A), x] = p[h_C(B), \cdot] \times p[h_B(A), x] \text{ i.e.,}$$

$$h_C(A, x) = h_C(B) \times h_B(A, x).$$

This axiom is called the Generalized Choice Axiom (GCA). We call *class of distributions* all the lotteries which are linked thanks to GCA. We must consider it as a coherence axiom which guarantees partial distributions of power to be relevant with social ones. Furthermore, it is natural to assume the power decision of a given Representative to be weakened inside a wider coalition than his.

## 2. FROM PREFERENCES TO LOTTERIES OVER AGENTS VIA POWER FUNCTIONS AND GCA

Basically, Barberà & Sonnenschein's Theorem consists in pointing out an additive function (since  $\#X \geq 6$ ) on the set of coalitions of individuals from probabilistic lotteries over preferences :

**BARBERA & SONNENSCHN'S THEOREM :** *Given any binary and Paretian SSWS  $g_+(\cdot) : \mathcal{B}^n \rightarrow L(\mathcal{B})$ , there exists an additive function  $\mu_g(\cdot) : 2^N \rightarrow [0,1]$  such that :  $\forall \mathbf{B} \in \mathcal{B}^n, \forall x, y \in X, \mu_g(\psi(\mathbf{B}, xBy)) = p[g_+(\mathbf{B}), xBy]$ . Moreover, we have,  $\mu_g(\emptyset) = 0$ ; for all  $A \subset N, \mu_g(A) + \mu_g(N - A) = 1$ ; whenever  $A \supset B, \mu_g(A) \geq \mu_g(B)$ .*

In order to prove this theorem, Barberà & Sonnenschein 1978 established the two following lemmas of neutrality and nonperversity (pp. 247 and 248) :

**NEUTRALITY LEMMA :** *Let  $g_+(\cdot)$  be a binary and Paretian SSWS. Then, for any  $\mathbf{B}, \mathbf{B}' \in \mathcal{B}^n$ , and  $x, y, z, w \in A$ ,*

$$"\psi(\mathbf{B}, xBy) = \psi(\mathbf{B}', zBw)" \Rightarrow "p[g_+(\mathbf{B}), xBy] = p[g_+(\mathbf{B}'), zBw]".$$

**NONPERVERSITY LEMMA :** *Let  $g_+(\cdot)$  be a binary and Paretian SSWS. Then, for any  $\mathbf{B}, \mathbf{B}' \in \mathcal{B}^n$ , and  $x, y \in A$ ,*

$$"\psi(\mathbf{B}, xBy) \supset \psi(\mathbf{B}', xBy)" \Rightarrow "p[g_+(\mathbf{B}), xBy] \geq p[g_+(\mathbf{B}'), xBy]".$$

*Remark 4* : From now on, we note  $SSWS_P^b$  any SSWS which is binary and Paretian.

Because we assume  $\#X$  to be greater than six, the power function  $\mu_g$  coming from Barberà & Sonnenschein's Theorem is necessarily additive. Hence, we can leave out the term "subadditive" from the original Theorem. By Luce's Theorem, we know such a function, positive and defined upon a powerset, to be equivalent to the existence of a class of distributions of probabilities. We note  $L^g(N)$  the subset of  $L(N)$  which is induced by  $\mu_g$ , i.e. the  $g$ -class of distributions of probabilities. It means :

LUCIAN (+) LEMMA : For any given  $SSWS_P^b g_+(\cdot) : \mathcal{B}^n \rightarrow L(\mathcal{B})$ , GCA is satisfied if and only if there exists a  $g$ -class of distributions of probabilities  $L^g(N)$  such that,  $\forall A \subseteq N$ , for any lottery  $l_A^g(\cdot) \in L^g(N)$  we have :

$$l_A^g(B) = \frac{\mu_g(B)}{\mu_g(A)} \quad \text{for all } B \subseteq A \subseteq N.$$

The proof is the same as Luce's one. Hence, we can associate each power function  $\mu_g$  to a  $g$ -class of distributions of probabilities over agents. Thus, since  $g_+(\cdot) : \mathcal{B}^n \rightarrow L(\mathcal{B})$ ,  $l^g(\cdot)$  can only be interpreted as a *probabilistic* lottery over the agents. It means that we can define probabilistic lotteries over coalitions from profiles of preferences whenever there exists a power function ; i.e. each time Barberà & Sonnenschein Theorem holds. This way, GCA could be the second step of a method searching for passing from preferences to lotteries over coalitions.

*Remark 5* : If there exists an induced set of lotteries  $L^g(N)$ , it means that,  $\forall A \subseteq N$ , any  $l_A^g(\cdot) \in L^g(N)$  is Lucian since GCA holds ; i.e. each induced lottery over coalitions is Lucian.

*Remark 6* : We shall note  $C^g(N)$  the subset of  $C(N)$  which will be induced by  $\mu_g$ , i.e. the  $g$ -class of distributions of Choquet capacities, when generalizing Barberà & Sonnenschein 1978 to  $C(N)$ .

### 3. FROM AGENTS TO NONADDITIVE LOTTERIES OVER AGENTS VIA SOCIAL UTILITY FUNCTIONS

The following theorem and lemmas are completely independent from previous theorems. We try to prove the existence of a social or a coalitional utility (it depends on the support of lotteries) from lotteries over agents in order to pass from coalitions to lotteries over coalitions. This analysis is just the opposite side of the one we treated

before, i.e. Barberà & Sonnenschein's one. Actually, we study some social rules dealing with agents but based on alternatives while they study some other rules according to agents behaviors ; their paper comes from preferences to choice probabilities of preferences when ours comes from agents to power-decision probabilities.

We can notice the dominance lemma to be the dual of nonperversity one in Barberà & Sonnenschein 1978. Identically, the duality lemma is also the dual of neutrality.

*h*-THEOREM : Given an anonymous, unanimous and Lucian PS  $h(\cdot)$  of  $C(N)$ , there exists a function  $u_h(\cdot) : 2^X \rightarrow [0,1]$  such that :  $\forall A \in 2^N, \forall x \in X$ ,

$$u_h(x(A)) = p[h(A), x], \quad (1)$$

furthermore we have,

$$u_h(\emptyset) = 0, \quad (2)$$

if  $h(\cdot) : 2^N \rightarrow L(N)$ ,

$$u_h(x(A)) + u_h(\bar{x}(A)) = 1, \quad (3)$$

whenever  $x(A) \supset x(B)$ ,

$$u_h(x(A)) \geq u_h(x(B)). \quad (4)$$

*Remark 6* : From now on, we note  $PS_u^a$  any PS which is anonymous and unanimous.

*Remark 7* : The function  $u_h(\cdot)$  is not necessarily additive. This property does not depend on  $\#A$  even though we assume  $\#N$  to be at least six. Actually, if  $A \subseteq N$ ,  $\#A \leq \#N$  ; hence  $\#A$  can be lower than six and we could suppose Barberà & Sonnenschein's Theorem to apply. But,  $u_h(\cdot)$  is directly generated from a lottery. If this last belongs to  $L(N)$ , then  $u_h(\cdot)$  is additive. In return, if  $h(\cdot)$  corresponds to a lottery of  $C(N)$ , then  $u_h(\cdot)$  can be nonadditive. We want here to analyze the proper influence of the designed lottery without studying that of the set of alternatives. Even if  $\#A$  is greater than seven, twelve or seventy-two,  $u_h(\cdot)$  is strictly nonadditive if and only if  $h(\cdot) \in NAC(N)$ .

The proof of the theorem follows two lemmas.

We suppose here  $h(\cdot)$  to be a Lucian  $PS_u^a$ .

DOMINANCE LEMMA : If  $x(A) \supseteq x(B) \Rightarrow p[h_c(A), x] \geq p[h_c(B), x]$ .

*Proof:* If  $x(A) = X_x$ , the unanimity axiom yields  $p[h_C(A), x] \geq p[h_C(B), x]$ . We know that  $x(A) = x[\cup_{i \in A} i] = \cup_{i \in A} x(i)$ . Hence, we can conclude by  $x(A \cup B) = x(A) \Rightarrow p[h_C(A \cup B), x] = p[h_C(A), x]$ , since the anonymity axiom holds. By another way,  $h_C(B, x) = h_C(A \cup B, x) \times h_{A \cup B, x}(B, x)$  because  $h(\cdot)$  is a Lucian  $PS_u^a$ . It means :  $h_C(B, x) = h_C(A \cup B, x) \times h_{A \cup B, x}(B, x) = h_C(A, x) \times h_{A \cup B, x}(B, x)$ . Now,  $h_C(A \cup B, x) = h_C(B, x) / h_{A \cup B, x}(B, x)$ , thus  $h_{A \cup B, x}(B, x) \leq 1$ . Hence,  $h_C(B, x) \leq h_C(A, x)$ . So, we can deduce :  $p[h_C(A), x] \geq p[h_C(B), x]$ .

Q.E.D.

DUALITY LEMMA : If  $x(A) = y(B) \Rightarrow p[h_C(A), x] = p[h_C(B), y]$ .

*Proof:* If  $x(A) = X_x$ , the unanimity axiom yields  $p[h_C(A), x] = p[h_C(B), y]$ . A similar argument applies if  $x(A) = \emptyset$ , since the preference relations are strict. Let us suppose the only proper subsets of  $X$ . When proving duality, we have to prove the following monotony :  $x(A) \supset y(A) \Rightarrow p[h_C(A), x] \geq p[h_C(A), y]$ . If  $x(A) = y(B) \Rightarrow x(A) \subset x(B)$  and  $y(B) \subset y(A)$ . From the dominance lemma, we can deduce :  $h_C(A, x) \leq h_C(B, x)$  and  $h_C(B, y) \leq h_C(A, y)$ . Moreover,  $x(B) \supset y(B)$ . The other case (equality) is prohibited by strictness of the preference relations. Let us suppose the following proposition  $x(B) \supset y(B) \Rightarrow h_C(B, x) < h_C(B, y)$ . Thus,  $y(A) \supset x(A) \Rightarrow h_C(A, y) < h_C(A, x)$ . Now,  $h_C(A, y) < h_C(A, x) \leq h_C(B, x) < h_C(B, y)$  which implies  $h_C(A, y) < h_C(B, y)$ . This is a contradiction with the previous proposition  $h_C(B, y) \leq h_C(A, y)$ . Thus, if  $x(A) = y(B)$ ,  $x(B) \supset y(B) \Rightarrow h_C(B, x) \geq h_C(B, y)$ . In mixing monotony with anonymity, we can conclude  $h_C(A, x) = h_C(B, y)$ .

Q.E.D.

*Remark 8 :* Both propositions  $x(B) \supset y(B)$  and  $y(A) \supset x(A)$  are obviously true since  $x(A) = y(B)$  and the preference relations are strict.

Because we have proved duality, it is natural to propose a basic definition of  $u_h(\cdot) : 2^X \rightarrow [0, 1]$  that one can find in the theorem by  $u_h(S) = p[h(A), x]$  where  $x$  is an arbitrary alternative and  $A$  any coalition such that  $x(A) = S$ . This expression of  $u_h(\cdot)$  as a function of alternatives suggests that it could be interpreted as a *social* utility (if  $h_A(\cdot) = h_N(\cdot)$ ) or a *coalitional* utility (if  $h_A(\cdot) \neq h_N(\cdot), A \neq \emptyset$ ) for any subset of alternatives.

*Proof of the Theorem* : From the basic definition of  $u_h(\cdot)$ , it is clear that  $u_h(\cdot)$  satisfies (1) and (2). Furthermore, if  $h(\cdot) \in L(N)$ , (3) trivially holds because  $x(A) \cup \bar{x}(A) = X$ . Lemma 1 of dominance guarantees that (4) is always satisfied.

Q.E.D.

In order to obtain lotteries over coalitions, the method consists in defining a social or coalitional utility function which allows to pass from coalitions to lotteries. It means that if such a function exists, we can explain the existence of power distributions and then their influence on methods for passing from profiles to lotteries over preferences. This particular manner for conferring utility levels coming from the *weight* of each Representative for each alternative means all the Representatives to be the only determinants of the utility levels. Moreover, the latter are directly measured by each Representative's power rather than that of the whole coalition. The social utility function (based on the induced  $PS_u^a h_N(\cdot)$ ) is unique for each  $SSWS_p^b g(\cdot)$  even though we shall specify later (point 4.) the relation between a previous  $SSWS_p^b$  and its induced  $PS_u^a$ .

#### 4. EQUIVALENCE FOR LOTTERIES : A THEOREM

Barberà & Sonnenschein's Theorem is based on the existence of a binary and Paretian  $SSWS$ . In its turn, the  $h$ -Theorem is based on the existence of an anonymous and unanimous  $PS$ . Furthermore, we know the particular  $PS$  which is induced by the initial  $SSWS$  to be always Lucian since based on GCA. We can obtain an equivalence theorem between Barberà & Sonnenschein's Theorem and the  $h$ -Theorem in using both theorems in the sense of an equivalence between two axioms, binarity and the Paretian property on one side and unanimity and anonymity on the other, and the existence of two functions, a power function on one side and a social utility function on the other ; then, we have just to apply GCA in order to link both theorems. In that case, i.e. when considered lotteries (over profiles as over coalitions) are all probabilistic, we write  $h_+$ -Theorem.

EQUIVALENCE (+) THEOREM : *Barberà & Sonnenschein's Theorem is equivalent to the  $h_+$ -Theorem if GCA holds for probabilistic lotteries.*

The proof firstly, consists in showing that binarity and the Paretian property imply anonymity and unanimity if GCA holds and secondly that anonymity and unanimity imply binarity and the Paretian property if GCA holds. The proof is purely logical. This particular version only deals with additive lotteries. From now on, we can consider any induced  $PS$  to be anonymous and unanimous if the initial  $SSWS$  is binary and satisfies the Paretian property. As soon as GCA holds, the induced  $PS$  is Lucian ; hence, under GCA, the induced  $PS$  - if existing - is a Lucian  $PS_u^a$ .

##### 5. FROM PROBABILISTIC LOTTERIES OVER PREFERENCES SOCIAL UTILITY FUNCTIONS

We have just to mix both results of Barberà & Sonnenschein's Theorem, Lucian (+) Lemma and  $h$ -Theorem in order to prove the existence of a social or a coalitional utility based on lotteries over preferences. Lucian (+) Lemma consists in restricting the basic set of lotteries  $L(N)$  into a particular subset  $L^g(N)$  coming from the previous  $SSWS_p^b g_+(\cdot)$ . Thus, in taking the induced power scheme  $PS_u^a h_+^g(\cdot) : 2^N \rightarrow [0,1]$  into  $L^g(N)$ , we satisfy both conditions of the  $h$ -Theorem ever since  $h_+^g(\cdot)$  is Lucian ; i.e. as soon as  $h_+^g(\cdot)$  exists which is guaranteed by Equivalence (+) Theorem.

**COROLLARY (+) :** *Given any  $SSWS_p^b g_+(\cdot) : \mathcal{B}^n \rightarrow L(\mathcal{B})$ ,  $\forall \mathbf{B} \in \mathcal{B}^n$ ,  $\forall x \in X$ , the induced  $PS_u^a h_+^g(\cdot) \in L^g(N)$  is Lucian. Then, there exists a function  $u_h^g(\cdot) : 2^N \rightarrow [0,1]$  such that :  $u_h^g(x(\psi(\mathbf{B}, xBX_x))) = p[g_+(\mathbf{B}), xBX_x]$ .*

*Proof :* Because  $g_+(\cdot)$  is a  $SSWS_p^b$ , there exists a power function  $\mu_g(\cdot)$  such that  $\mu_g(\psi(\mathbf{B}, xBX_x)) = p[g_+(\mathbf{B}), xBX_x]$ . Thanks to Equivalence (+) Theorem, we know the induced  $h_+^g(\cdot)$  to be a Lucian  $PS_u^a$ . Thus, in applying Lucian (+) Lemma, we have  $\mu_g(\psi(\mathbf{B}, xBX_x)) = I_h^g(\psi(\mathbf{B}, xBX_x))$  since  $\mu_g(N) = 1$  by definition and  $h_+^g(\cdot) = I_h^g(\cdot) \in L^g(N)$ . Hence, the  $h$ -Theorem yields,  $u_h^g(x(\psi(\mathbf{B}, xBX_x))) = p[h_+^g(\psi(\mathbf{B}), xBX_x), x]$  which also means  $u_h^g(x(\psi(\mathbf{B}, xBX_x))) = p[g_+(\mathbf{B}), xBX_x]$ .

Q.E.D.

The social utility function  $u_h^g(\cdot)$  we deduce from Corollary (+) is necessarily additive since Barberà & Sonnenschein's Theorem holds for the only probabilistic lotteries. Basically, Corollary (+) allows to set up the utility level of each alternative from lotteries over preferences. In order to generalize Corollary (+) to nonadditive lotteries, we have previously to generalize Barberà & Sonnenschein's Theorem to  $C(\mathcal{B})$ .

## 6. FROM NONADDITIVE LOTTERIES OVER PREFERENCES TO SOCIAL UTILITY FUNCTIONS

The proof of Barberà & Sonnenschein's Theorem is completely independent from the probabilistic nature of considered lotteries. If rewriting both axioms of binarity and that dealing with the Paretian property according to nonadditive lotteries over preferences, we can immediately obtain the two basic generalized results of neutrality and nonperversity. Hence, it is obvious to conclude by :

**BARBERA & SONNENSCHNEIN'S GENERALIZED THEOREM :** *Given any  $SSWS_p^b g(\cdot) : \mathcal{B}^n \rightarrow C(\mathcal{B})$ , there exists a function  $\mu_g(\cdot) : 2^N \rightarrow [0,1]$  such that :  $\forall \mathbf{B} \in \mathcal{B}^n, \forall x, y \in X$ ,*

$$\mu_g(\psi(\mathbf{B}, xBy)) = p[g(\mathbf{B}), xBy],$$

*furthermore we have,  $\mu_g(\emptyset) = 0$  ; whenever  $A \supset B$ ,  $\mu_g(A) \geq \mu_g(B)$ .*

*Remark 9 :* The only property which differs from the original theorem deals with the probabilistic constraint of mass-unity (see Billot 1991b) ; actually, since  $\mu_g$  is nonadditive, the sum of any coalition's power and that of its complement could be different from 1. Hence, the probabilistic constraint of mass-unity disappears. The proof is based on same arguments as Barberà & Sonnenschein 1978 ones (p. 248) even though we do not search here any subadditive function but only a nonadditive function. This last characteristic is directly issued from the definition of the  $SSWS_p^b : \mathcal{B}^n \rightarrow C(\mathcal{B})$  ; i.e. the membership of the designed lottery to  $C(\mathcal{B})$ . Since we have proved the Generalized Luce's Theorem in Billot & Thisse 1991, we can establish the following theorem :

**LUCIAN GENERALIZED LEMMA :** *For any given  $SSWS_p^b g(\cdot) : \mathcal{B}^n \rightarrow C(\mathcal{B})$ , GCA is satisfied if and only if there exists a  $g$ -class of distributions of Choquet capacities  $C^g(N)$  such that,  $\forall A \subseteq N$ , for any nonadditive lottery  $c_A^g(\cdot) \in C^g(N)$  we have :*

$$c_A^g(B) = \frac{\mu_g(B)}{\mu_g(A)} \quad \text{for all } B \subseteq A \subseteq N.$$

The proof is exactly the same as Billot & Thisse's one. Each nonadditive lottery  $c_A^g(\cdot) \in C^g(N)$  corresponds to the induced PS  $h^g(\cdot)$  because of GCA. Thus, by definition,

the induced PS  $h^s(\cdot) = c^s(\cdot)$  is always Lucian. Hence, we can prove the following theorem in mixing the proofs of Barberà & Sonnenschein's Generalized Theorem,  $h$ -Theorem and Lucian Generalized Lemma.

GENERALIZED COROLLARY : *Given any SSWS<sub>p</sub><sup>b</sup>  $g(\cdot) : \mathcal{B}^n \rightarrow C(\mathcal{B})$ ,  $\forall \mathbf{B} \in \mathcal{B}^n$ ,  $\forall x \in X$ , the induced PS<sub>n</sub><sup>a</sup>  $h^s(\cdot) \in C^s(N)$  is Lucian. Then, there exists a function  $u_n^s(\cdot) : 2^N \rightarrow [0,1]$  such that :  $u_n^s(x(\psi(\mathbf{B}, xBX_x))) = p[g(\mathbf{B}), xBX_x]$ .*

This last result is just a generalization of Corollary (+) to  $C(N)$ . The social or coalitional utility that we define according to a nonadditive lottery over agents is nonadditive. If the induced PS<sub>n</sub><sup>a</sup>  $h^s(\cdot)$  is subadditive, if  $h^s(\cdot)$  is a possibility for example, then the social utility  $u_n^s(\cdot)$  is subadditive since possibilistic (see Zadeh 1978 and Billot & Thisse 1991). Actually, all the properties of  $u_n^s(\cdot)$  come from those of  $h^s(\cdot)$  which in their turn come from those of  $g(\cdot)$ .

When rewriting every previous axiom according to nonadditive lotteries, we can easily generalize the Equivalence (+) Theorem to a new one, between the Barberà & Sonnenschein's Generalized Theorem and the  $h$ -Theorem ;

EQUIVALENCE GENERALIZED THEOREM : *Barberà & Sonnenschein's Generalized Theorem is equivalent to the  $h$ -Theorem if GCA holds for nonadditive lotteries.*

This last theorem guarantees the equivalence for nonadditive lotteries and thus the existence of a method for passing from nonadditive lotteries over profiles to nonadditive lotteries over coalitions as soon as GCA holds. It means also the induced PS<sub>n</sub><sup>a</sup>  $h(\cdot)$  to be always Lucian. This way, for example, possibilistic lotteries over preferences could come from a possibilistic, hence subadditive power function (regardless of  $\#X$ ) which implies possibilistic lotteries over coalitions coming from a possibilistic, hence subadditive utility function. In Billot & Thisse 1991, following Lovasz 1983, we proved any submodular (i.e. subadditive) utility function (for a basic finite set of alternatives) to imply the existence of what we called a decreasing *discrete marginal utility function*. Then, we proposed to analyze this property in assimilating subadditivity and discrete concavity. Hence, in order to restore the complete Barberà & Sonnenschein's result in so far as the exhibition of a subadditive power function leading to lotteries over preferences, we could restrict the previous social utility function to the only discrete concave ones. In that very case, which is particularly



intuitive (since corresponding to a decreasing discrete marginal utility function), the power function should be obviously subadditive. In return, lotteries over preferences should be also subadditive.

#### CONCLUSION

This paper deals with the only strict relations of preference. The extension to weak orderings does not change a priori neither the methodology nor the basic sense of results. One of the most interesting comments of Barberà & Sonnenschein's paper consists in interpreting the power function as a distribution of *dictatorial* power. Thus, in dividing this dictatorial power by means of lotteries over preferences, one could diminish the risk of dictatorship. Finally, in our paper, we try to found a method which allows to define this power distribution directly from alternatives by means of a social utility. This last exists whenever we assume a Power Scheme to be anonymous and unanimous. In other words, the social utility function can be interpreted as a distribution of utility upon the set of alternatives. Since we know Barberà & Sonnenschein to be equivalent to ours *h*-Theorem when GCA holds, we know also that dividing the dictatorial power is equivalent to divide the social utility if the Power Scheme is socially consistent, i.e. Lucian. Basically, this conclusion confirms the Arrovian analysis of aggregation.

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