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AXIOMATICS AND CONSTRUCTION OF THE CENTRAL PLACE SYSTEM

Agnès GAHITTE

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AXIOMATICS AND CONSTRUCTION OF THE CENTRAL PLACE SYSTEM

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Abstract: The construction of the løschian landscape is a basic element in the Theory of Economic Regions. It is based on an hexagonal lattice and løschian numbers having properties used by DACEY (10,11,12,13) and MARSHALL (16,17,18,19,20). In fact, the former has not tested his model. In this paper our purpose is to prove that DACEY failed to build mathematically the Central Place System, and to propose a new method of construction of the løschian landscape.

Keywords: Löschian Numbers - Central Place System - Theory of Economic Regions - Shape of Market Areas.

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INTRODUCTION

CHRISTALLER (8) has created the Theory of Economic Regions and LOSCH (15) has developed it before the second World War, second chapter of his reference book: "The Economics of Location".

TARRANT (22) has endeavored to show how it is possible to develop the Löschen landscape on a rhombic lattice using non orthogonal reference axes.

Next to him, two different "schools" (BEAVON and MABIN (4,5,6,7), later on BEAVON (1,2,3) and MARSHALL (16,17,18,19)) began to deal with the same subject. MARSHALL (17) asserted that LOSCH's single good model was too restrictive.

The aim of this paper is not to question the Löschen Theory of Regions but to explain its construction. Of course, we could blame LOSCH (15) for his lack of explanation concerning the structure of his landscape. A number of authors have ignored the problem in putting that part of the theory aside.

MARSHALL (17,18,19,20) has constructed a landscape which is not a Löschen one because its elaboration doesn't respect location and maximisation constraints enforced by LOSCH (15). Then he denies the existence of alternating city rich and city poor sectors but his claim doesn't disturb the Löschen scheme...

In fact DACEY (10,11,12,13) in 1964 has been the forerunner of a strictly mathematical approach of this theory. He has worked out the first postulates which allow us to formulate the Central Place model, and has transformed a concealed structuration into an adequate mathematical language: mathematics revealed to be an effectual implement.

Farms have in this framework analysis equivalent properties to those of points in an hexagonal lattice, basis of the system. The number of farms completely supplied is characterized by a Löschen number provided with properties of Number theory (10,11,16). To fix this number comes to the same thing as to resolve a Diophantine Equation, which uses integers only.

In "Geometry of Central Place Theory" DACEY (12,p 120) has restated the essential elements of LOSCH 's argument and summarized the most important properties of the Central Place System. Our purpose is specially to examine the algorithm T5 (12, p120). DACEY asserts that it must be able to give the primitive function point n(n+z). An attempt will be made here to prove that this method is not efficient to locate some centers of market areas.

The mathematical proof will be given just as examples. These one will show that it is impossible to make a choice between several function points of the form n(n+z) in using T5.

A constructive solution allowing a logical construction of the landscape must be treated by computer.
Figure 1: Hexagonal lattice used in the Central Place System.
1 Mathematical bases

The Geometry of Central Place is based on Lattice Theory and particularly on the study of the hexagonal lattice which with an angle of periodic rotation of $\pi/3$ generates regular hexagons (COXETER (9), HILBERT and COHN VOSSEN (14), PECAUT (21)). The lattice is based on a six fold axis.

11 Triangular Lattice

This study uses a lattice based on an axis which determines integer's coordinates.

111 Lattice characterization

In 1964 DACEY (15, p 63) published his first paper on LOSCH's Central Place System.

Put $P$ a symmetrical lattice with a six fold axis. If we arbitrarily choose a point $O$ as lattice origin, the location of any other lattice point can be defined relatively to $O(0,0)$ by the vector $T$ such as $T = u_1 t_1 + v_2 t_2$, $u \in Z, v \in Z$ in $(0,t_1,t_2)$.

The plane is constructed as a linear lattice having translation period $t_1$ repeated at an interval $t_2$ (with $\| t_1 \| = 1$ et $\| t_2 \| = 1$).

In the lattice we use an angle of periodic rotation of $\pi/3$, $\angle Ot_1 = \pi/3$ (see figure 1).

We get six sextants which cover the plane. If we note $\varphi$ the six sextants we defined the lattice points $P$ by:

$$\varphi = \varphi T = \{ u t_1 + v t_2 \} u \in Z, v \in Z.$$

112 The sextants

If we examine the hexagonal lattice we see that it is simple to use only one sextant $\varphi_1 T$ for example ($\varphi_1 T$ has a vertex $O$):

$$\varphi T = \varphi \{ x t_1 + y t_2 \} \text{ with } x \in N, y \in N^*$$

This sextant does not contain $O(0,0)$ but contains all points such as:

$$(0,1) \ldots (0,n)$$

$$(1,1) \ldots (1,n)$$

$$(m,1) \ldots (m,n)$$

On the axis $Ot_2$ abscissa is noted $x = 0, \ldots , n$ and on $Ot_2$ (y axis) we have $y = 1, \ldots , m$.

By rotation of $\varphi_1 T$ all the lattice points are obtained on the six fold axis:

$$PT = (\varphi_1 T \cup \varphi_2 T \cup \varphi_3 T \cup \varphi_4 T \cup \varphi_5 T \cup \varphi_6 T) + 0$$

$$\varphi_A T \cap \varphi_B T = \emptyset \text{ with } A \neq B, A = 1 \ldots 6$$

$$B = 1 \ldots 6.$$
Figure 2: Coordinates of the typical parallelogram.

Figure 3: Examples of reduced parallelogram: the triangles surrounding the point 0.
For example put $c_{ij} = i t_1 + j t_2$, $c_{ij}$ is a particular point of the sublattice $\varphi_1 T$. The point $A(1,1)$ is noted $c_{11} = 1 t_1 + 1 t_2$.

$\varphi \{ c_{ij} \}$ represents the subset of lattice points obtained by rotation and translation of $c_{ij}$.

By translation of the vector $i t_1 + j t_2$ and by rotation of $\pi/3$ a sublattice is obtained. This collection of points is noted:

$$\{ u t_1 + v j t_2 \} \ u, v \in \mathbb{Z}.$$ 

12 Dirichlet Regions

The lattice is a collection of points having integers coordinates. An arbitrary point $0$ is chosen as the origin. In figure 2 the points $(1,0), (1,1)$ and $(0,1)$ form with $0(0,0)$ a parallelogram: it is the typical parallelogram because translations change the lattice in basic cells filling the plane without gap.

A parallelogram is formed by four points $0, X, XY, Y$. The translation $T = (X, Y)$ changes it into another one having point $T'$ as first vertex (see figure 3). By joining the vertices $X$ and $Y$ of that reduced parallelogram and the two vertices corresponding of each "repliqua" we get six identical triangles whose vertices are the lattice points and have non obtus angle. If we connect the circumcenters of these six equilateral triangles we obtain a Dirichlet region: a polygon that contains a lattice point at its center and every point of the plane is closer to that lattice point than to any other lattice point.

The löschian system is developed on a six fold axis, for this axis the primitive Dirichlet region is always a regular hexagon which contains a lattice point as center.

121 Definition

The fundamental region of a plane is a parallelogram which has four lattice points for its vertices and no other points on its edges or faces. Such a region is defined by the points:

$$i t_1 + j t_2; (i+1) t_1 + j t_2; i t_1 + (j+1) t_2; (i+1) t_1 + (j+1) t_2.$$ 

When this region has the shortest possible sides we call it reduced parallelogram.

Dirichlet region is a regular hexagon whose interior consists of all points in the plane which are closer to particular lattice point than to any other lattice point. The set of Dirichlet regions (each of them surrounding a lattice point) exhausts the plane without overlapping.

The fundamental or primitive Dirichlet region contains only one lattice point, which is the center of each region, and is called a region of degree 0 and order 1.

The concept of Dirichlet region may be extended to polygons containing more than one lattice point, and such a region is called a multiple Dirichlet region:

that region has a lattice point at its center and one or more other points within its interior, on an edge or at a vertex.

Each Dirichlet region is identified by degree. A region is said to have a degree $q$ if the sum of the three following values equals $q$:

i) the number of lattice points interior to the region

ii) one half the number of lattice points on edges of the region

iii) one third the number of lattice points at vertices of the region.

A point that is interior to a Dirichlet region, including the center, is said to belong to that region.
If 0(0,0) and any other point (i,j) are centers of two adjacent multiple Dirichlet regions they define a unique set of regions, because by rotation and translation (i,j) is carried into the lattice forming the sublattice \( \{ u_i t_1 + v j t_2 \} \) and no other point is the center of that Dirichlet region. Because of the symmetry of the lattice, the hexagons have identical area and contain identical numbers of lattice points, when they belong to the same set of regions.

122 Order of a Dirichlet region of degree 1

Consider 0 and (i,j) the centers of adjacent Dirichlet regions. By rotation and translation of vector (i,j), we get \( \{ u_i t_1 + v j t_2 \} \), sublattice which includes the origin 0. We can show by induction that the number of points in each region having as center (i,j) is given by:

\[
k_{ij} = i^2 + ij + j^2
\]

\( k_{ij} \) is called the order of the regions defined by \( \{ u_i t_1 + v j t_2 \} \) and by any Dirichlet region determined by a vector of the form \( i t_1 + j t_2 \) is called a first degree region.

123 Permissible orders of Dirichlet regions of degree 1

The order of a first degree region is by \( k_{ij} \),

\[
k_{ij} = i^2 + ij + j^2, \text{ a positive integer}; \text{ however not all integers values occur as permissible orders for first degree regions. The set of integers } \{ k \} \text{ which are orders of first degree regions is given by }
\]

\[
\{ k \} = \{ i^2 + ij + j^2 \} \text{ for } i \neq j \text{ and } i \neq 1.
\]

Location of a point which is the center of Dirichlet region of order \( k_{ij} \) and adjacent to a region of same order centered in 0(0,0) is obtained in substituing in \( i t_1 + j t_2 \) every (i,j) \((i \in N, j \in N)\) satisfying \( i^2 + ij + j^2 \).

For each (i,j) centers of these regions are determined by rotation and translation of the vector (i,j).

124 Orders of separable first degree regions.

An interesting collection of first degree Dirichlet regions having important economic implications to LOshch's Theory of Central Place are those orders in which all the lattice points are interior to a region, that is no lattice points are located on edges or at vertices of this region. This system is called separable Dirichlet regions. A separable region cannot be located on X axis or on a secondary symmetry axis (i=j). The number of points in a separable region must, to retain symmetry, be an integer multiple of six plus an additional center point. All centers of these regions of degree 1, noted \( \{ u_j t_1 + v j t_2 \} \), are given by rotation and translation of (us, vs) for which the following three conditions hold:

\[
\begin{align*}
klj &= 6z + 1 \quad z \in N^* \\
ij &\neq 1 \\
j &\neq 0
\end{align*}
\]
Figure 4: Superposition of three Dirichlet regions of order 7, 13, 19.
Figure 5: the first market areas, they are lattice L(3), L(4), L(7)

Table I: Elements of Q for the first values of i and j.

<table>
<thead>
<tr>
<th>i</th>
<th>j</th>
<th>$i^2 + j^2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>13</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>19</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>16</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>21</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>28</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>37</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>48</td>
</tr>
<tr>
<td>5</td>
<td>0</td>
<td>25</td>
</tr>
<tr>
<td>...</td>
<td>...</td>
<td>...</td>
</tr>
</tbody>
</table>
In figure 4 we see the superposition of three separable Dirichlet regions because:

i) \( k_{ij} = 6 \times 1 + 1 = 7, k_{ij} = 6 \times 2 + 1, k_{ij} = 6 \times 3 + 1 \)

ii) in all these cases the center is the point \((2,1)\) and \(2 \neq 1\)

iii) \(1 \neq 0\)

125 Degree of hierarchical Dirichlet regions

\[ \rightarrow i t + j t \rightarrow \]

determines a partition of the lattice into first degree Dirichlet regions. A point \((i + f, j + g)\) with \(f \in \mathbb{N}\), \(g \in \mathbb{N}\) is the center of an \(n\) degree Dirichlet region of order \(k_{ij}\) if, and only if:

(i) that region contains exactly \(k_{ij}\) centers of Dirichlet region of order \(k_{ij}\)

(ii) \(O\) is also the center of \(n\) degree Dirichlet region of order \(k_{ij}\)

If a point is the center of \(n\) degree Dirichlet regions then it will be the center of \(n-1, n-2\)... degree regions.

By induction we conclude that the number of lattice points in a \(A\) Dirichlet region of order \(k_{ij}\), called \(k_{ij}^A\), is given by:

\[ k_{ij} = h^A + \left[ \begin{array}{c} A \\ 1 \end{array} \right] + \ldots + \left[ \begin{array}{c} A \\ A-1 \end{array} \right] h + 1 \quad \text{(with } h = k_{ij} - 1) \]

For example put \(i = 1, j = 3\), then \(k_{ij} = 13, h = 12\), if the degree is \(A = 2\) we get \(13^2 = 12^2 + \left[ \begin{array}{c} 2 \\ 1 \end{array} \right] 12 + 1\).

126 Location of centers of the Dirichlet regions of order \(n\)

This location is given by finding a pair of integers \((i,j)\), that satisfies \(k_{ij}^A\). By translation and rotation the location of all \(A\) degree centers is determined. In general \((i,j)\) is not single valued, then the hierarchical system of Dirichlet regions is not uniquely determined.

A separable \(A\) degree Dirichlet region of order \(k_{ij}\) is characterized by

\[ \begin{cases} 
(i) & k_{ij} = 6z + 1, z \in \mathbb{N}^* \\
(ii) & i \neq j \\
(iii) & j \neq 0 
\end{cases} \]

13 General properties

DACEY (17, p 113) stated without proofs these useful properties, these proofs are given by GAHITTE (22 bis, p 33).

Pl: On an hexagonal lattice, the area \(H\) of a Dirichlet region with distance \(t\) between centers of adjacent regions is

\[ H = t \sqrt{3} \]

(DACEY(17, p 113) gives \(H = t^2 = \sqrt{3}/2\). By another method BEAVON and MABIN obtain \(H = 3 b^2 \sqrt{3}/2\) with \(b = t/\sqrt{3}\).
P2: In an hexagonal lattice the number of points \( n \) assigned to a Dirichlet region with area \( h \) and the distance \( t \) between adjacent centers is \( n = 2h \sqrt{3} = t^2 \).

(DACEY (17, p 113) finds \( n = h \sqrt{2/3} \), BEAVON and MABIN (5, p 32) give \( n = t^2 \)).

P3: In an hexagonal lattice the distance from the origin to the point \((i,j)\) is given by \( t_{ij} = \sqrt{i^2 + ij + j^2} \)

P4: \( t_{ij} \) gives the possible degrees of Dirichlet regions on an hexagonal lattice. Let \( \{q\} \) denote the possible degrees of a Dirichlet region, we get \( \{q\} = \{i^2 + ij + j^2\} \) \( i \in \mathbb{N}, j \in \mathbb{N} \).

14 Löschian numbers

In the Löschian model, the size of market area is measured by the number of farms that this area contains. As these areas are hexagonal only a part of the farms buy the goods at the center of supply (center of market area). The total number of farms supplied by the supplied center is an integer characterized by the function \( Q \) such as:

\[ Q = \{i^2 + ij + j^2\}, i \in \mathbb{N}, j \in \mathbb{N} \]

The elements of \( Q \) are noted \( n \), \( n = i^2 + ij + j^2 \) is a Diophantine equation

"Q gives the values of \( t^2 \), (\( t \) is the distance between two suppliers of the same good" MARSHALL (16, p 423). We can also see that \( t = a[n] \) (\( a=1 \) the distance between two lattice points is unitary).

We can say that an integer is a löschian number, if and only if, its factorization satisfies the condition such as the prime number 2 and all the prime numbers of the form \( (6x-1) \) have even power. The proof of this assertion is given by MARSHALL (16, p422) and GAHITTE (22bis).

A löschian number is also an integer as 3 or every prime number of the form \( (6x+1) \) having even or odd power. In fact we have proved that an integer \( n \) such as \( n = i^2 + ij + j^2 \) is löschian if its factorization in prime numbers (as 2 or every prime number of the form \( 6x-1 \)) is such as these numbers have odd power, and 3 or prime numbers of the form \( 6x+1 \) have even or odd power.

Then we can conclude that not only löschian numbers have this kind of factorization, but every integer having this kind of factorization is a löschian number. See table 1.

For example 2692 is a löschian number because 2692 = 2^2 . 673; 2 have an odd power and 672 ≠ 2^A \( \# (6x-1)^B \) with \( A \) even.

But 303 is not a löschian number:

303 = 3^3 . 101

and 101 = (6x-1)^1, 101 have an even power.
(i, j) in $i^2 + ij + j^2$ has a concrete sense: it represents the coordinates of a supply center relatively to metropolis locate at $O(0,0)$.

For solving the Diophantine equations $n = i^2 + ij + j^2$ we only can use iterative process.

The construction of the löschian landscape (the test of the model) needs some postulates and definitions (DACEY(12)). We note $HN$ the Central Place system, it contains a set of Dirichlet regions for each degree lower than $N$.

P1: At least one lattice point in $HN$ is the center of the Dirichlet region with degree $q$, $qN$.

P2: If any lattice point in $HN$ is assigned to a Dirichlet region with degree $q$, then every lattice point in $HN$ is assigned to exactly one Dirichlet region with degree $q$.

P3: There is at least one sector of $\pi/6$ radians with locus at any lattice point $(i, j)$ that contains at least one vertex of each Dirichlet region with center $(i, j)$.

A1: The origin of the lattice is the center of Dirichlet regions with degree $q$, $qN$. This assumption is valid only if it exists a lattice point which is the center of a Dirichlet region in each tessellation occurring in $HN$.

T1: There exist at least one lattice point which is the center of Dirichlet regions with degree $q$, $q \in \{q\}, qN$.

D1: definition of the function of a point: a point which is the center of a Dirichlet region with degree $n$ has a function $n$.

D2: definition of a primitive point: if $(i, j)$ has the function $n$ and no other function $n$ point is closer to the origin point $O$, $(i, j)$ is a primitive function $n$ point.

D3: definition of multiple function: a point that is the center of Dirichlet regions of degrees $n$ and $(n+z)$, $z \in N$, have function $n(n+z)$.

D4: definition of order: A point that has function $n$, $n<n'$, $n', n' \in \{q\}$ have order $n$. If $(i, j)$ has order $n$ and no other order $n$ point is closer to $O(0,0)$, then $(i, j)$ is a primitive order $n$ point.

Some properties of the Central Place system $HN$

L1: The value of $q$ is an element of the set $\{q\} = \{i^2 + ij + j^2\}$ for $i \in N, j \in N$.

L2: If $N$ and $N+z$, $z \in N$, and no integer between $N$ and $N+z$, $N+z$ is an element of $\{q\}$, then all system $Hz$ for which $N\leq z$ are identical.

A2: $N \in \{q\}$. The sublattice $P1$ is partitioned into two sectors of $\pi/6$ radians. These two $30^\circ$ sectors are identified as the upper half of $P1$ and the lower half of $P1$, labeled $Pu$ and $Pl$. Boundaries between $Pu$ and $P1$ are defined by secondary axis $(i = j)$ of $P1$.

If $(i, j) \in P1$ then

(i) $(i, j)$ is interior to $Pu$ if $i > j > 0$
(ii) $(i, j)$ is interior to $P1$ if $j > i > 0$
(iii) $(i, j)$ is on one edge if $i = 0$, $j = 0$ or $i = j$

A3: The $P1$ does not contain any vertex of a Dirichlet region with center at $O$
T2: If a point is interior to Pu, then that point is not a primitive function point. The primitive function point is located at \((i,j)\), \(i\) and \(j\) integers such as \(n = i^2 + ij + j^2\), \((i,j)\) is not unique.

T3: If \((i,j)\) is a primitive function point, the subcollection of function points is \(CN = \{ u_1 t_1 + v_1 t_2 \}, u, v \in Z\).

T4: Suppose that \(n, n+z \in \mathbb{N}\) and that \(n \in \{ q \}\), then the primitive function point \(n(n+z)\) exists. Further the primitive function \((n(n+z)...n+z')\) point exists if and only if, \(n, n+z,...n+z' \in \mathbb{N}\) and \(n, n+z,...n+z'\) are all elements of \(\{ q \}\).

2 Location of the multiple function point.

21 Theoretical review of DACEY's algorithm

During the construction of the löschian Central Place system we must locate what DACEY calls the "primitive function points" Suppose that \((i,j)\) is a primitive function point which is the center of a Dirichlet region \((n = i^2 + ij + j^2)\). We must consider three cases

(i) \(n\) is a prime number
(ii) \(n\) is a perfect square
(iii) \(n\) is a multiple function point neither prime number nor square.

This last case is treated by DACEY (12, p 120) with the algorithm T5.

211 Terms of the algorithm

T5 : Assume \((i,j)\) and \((h,k)\) are respectively primitive function point \(n\) and \(n+z\) points. The primitive function point \(n(n+z)\) is located at \((r,s)\) where \(s \in \mathbb{N}, r \in \mathbb{N}\) given by the equations:

\[
\begin{align*}
r &= [\alpha h + \beta (h+k)] = [\gamma i + \delta (i+j)] \\
\gamma j + \delta i
\end{align*}
\]

\(\alpha\) and \(\gamma\) are positive integers and \(\beta\) and \(\delta\) are integers subject to the conditions:

(a) \(\beta(hk + h^2 + k^2) = \gamma(ik - hj) + \delta(ik + jk + ih)\)

(b) \(\alpha = \gamma j - \beta h - \delta i\)

There is an infinite number of such solutions and the required set of values is the one which minimizes \((\gamma^2 + \gamma \delta + \delta^2)\). (4) may be solved, but DACEY doesn't know an efficient algorithm. Also, the solution is not necessarily unique. When (4) is solved iteratively the following inequalities may be used.

\[
\begin{align*}
(c) & \frac{\alpha h}{h+k} < \frac{\beta}{h} & \text{and } \alpha > 0 \\
(d) & \frac{\gamma i}{i+j} < \frac{\delta}{i} & \text{and } \gamma > 0
\end{align*}
\]
"By repeated applications of T5 to pairs of functions an expression is obtained that gives coordinates of primitive points with multiplicity greater than 2" DACEY (12, p 120).

212 Discussion of the algorithm.

In fact to resolve this system we must change it into an homogeneous one. We must find $\alpha, \beta, \gamma, \delta$, and not $r$ and $s$. $r$ and $s$ are known, we can find them in solving the next equations:

\[
\begin{align*}
 n + z &= h^2 + hk + k^2 \\
 n &= i^2 + ij + j^2
\end{align*}
\]

\[
\begin{align*}
 n(n+z) &= r^2 + rs + s^2
\end{align*}
\]

If $(h,k)$ and $(i,j)$ are known then $n + z$, $n$ and so $n(n+z)$ are known too.

\[
\begin{align*}
 n(n+z) &= (h^2 + hk + k^2) (i^2 + ij + j^2) = r^2 + rs + s^2
\end{align*}
\]

The new system can be written as:

\[
\begin{align*}
| ah + \beta(h+k) - \gamma i - \delta(i+j) &= 0 \\
 ak - \beta h - \gamma j + \delta i &= 0 \\
| \gamma j - \delta i - s &= 0 \\
| \gamma i - \delta(i+j) - r &= 0 \\
| ah + \beta(h+k) - r &= 0 \\
| ak - \beta h - s &= 0
\end{align*}
\] (6)

2121 Equality constraints

Equations (a) and (b) are obtained by linear combination of the above ones.

Proof: isolate from (6) the first two equations, then we get:

\[
\begin{align*}
| ah + \beta(h+k) - \gamma i + \delta(i+j) &= 0 \\
 ak - \beta h - \gamma j + \delta i &= 0
\end{align*}
\] (7)

By multiplication of the first equation (by $k$) and of the second one (by $h$) and by subtraction of them we obtain:

\[
\begin{align*}
 \beta k (h+k) - \gamma ik - \delta k(i+j) + \beta h^2 + \gamma hj - \delta ih = 0
\end{align*}
\]

\[
\begin{align*}
 \beta (kh + h^2 + k^2) = \gamma (ik - hj) + \delta (ki + kj + ih)
\end{align*}
\]

which is exactly equation (a).

$\alpha$ is obtained by transposing $\beta, \gamma, \delta$ in the second equation of (7).

We can see that (a), (b) are redundant equations relatively to the system (6). It was not necessary to introduce them next to (4).

We solve (6) by linear combination of the last two equations, and by addition, we get:

\[
\begin{align*}
| ah + \beta(h + k) - r &= 0 \\
 ak - \beta h - s &= 0
\end{align*}
\] (1)

\[
\begin{align*}
 | -k \\
 | h \\
 | h + k
\end{align*}
\] (2)
(1) $-\beta k (h + k) + rk - \beta h^2 - sh = 0 \iff$

\[
\beta(h^2 + hk + k^2) = sh - rk \iff
\]

\[
\beta = \frac{sh - rk}{hk + h^2 + k^2} = \frac{sh - rk}{n + z} \quad n + z \neq 0
\]

(2) $\alpha h^2 - rh + \alpha k (h+k) - sh - sk = 0 \iff$

\[
\alpha(h^2 + hk + k^2) = rh + sk + sh \iff
\]

\[
\alpha = \frac{rh + sh + sk}{h^2 + hk + k^2} = \frac{rh + sh + sk}{n + z} \quad \text{with } n + z \neq 0
\]

In the same way, by linear combination of the third and fourth equations we obtain:

\[
\begin{vmatrix}
\gamma_j - \delta i - s = 0 \\
\gamma_i + \delta(i+j) - r = 0
\end{vmatrix}
\]

(1) $-\delta i^2 - is - \delta j(i+j) + rj = 0 \iff$

\[
\delta(i^2 + ij + j^2) = rj - is \iff
\]

\[
\delta = \frac{rj - is}{n} \quad n \neq 0
\]

(2) $\gamma j(i+j) - s(i+j) + \gamma i^2 - ri = 0 \iff$

\[
\gamma(i^2 + ij + j^2) = s(i+j) + ri \iff
\]

\[
\gamma = \frac{si + sj + ri}{n} \quad n \neq 0
\]

Moreover DACEY uses a minimization criterion. The required set of values is that which minimizes $(\gamma^2 + \gamma \delta + \delta^2)$. We can demonstrate that this criterion is not able to allow a choice between more than two multiple primitive function points:

\[
\gamma^2 + \gamma \delta + \delta^2 = n + z
\]

Proof:

\[
n^2(\delta^2 + \gamma \delta + \delta^2) = n^2 \left[ \left( \frac{rj - is}{n} \right)^2 + \left( \frac{si + sj + ri}{n} \right)^2 + \left( \frac{rj - is}{n} \right) \left( \frac{si + sj + ri}{n} \right) \right]
\]

\[
= r^2 j^2 + s^2 j^2 - 2ri sj + r^2 i^2 + si + sj + 2ris (i+j)
\]

\[
+ (rj - ri)(ri + sj + s) = (r^2 + rs + s^2)(i^2 + ij + j^2) \iff
\]

\[
n^2 (\delta^2 + \gamma \delta + \gamma^2) = n \cdot n(n+z) \iff
\]

\[
\delta^2 + \gamma \delta + \gamma^2 = n + z \quad \text{with } n \neq 0.
\]
Using values of $\alpha$ and $\beta$ we are going to prove that $\alpha^2 + \alpha\beta + \beta^2 = n$

Proof:

$$(n + z)^2 (\alpha^2 + \alpha\beta + \beta^2) = (n + z)^2 \left[ \frac{(sh-rk)^2}{(n+z)^2} + \frac{(rh+sh+sk)^2}{(n+z)^2} + \frac{(rh+sh+sk)(sh-rk)}{n+z} \right]$$

$$= r^2k^2 + s^2h^2 + r^2r^2 + s^2s^2 + s^2k^2 + 2hsk -$$

$$s^2h^2 + 2srh + rh + rs^2h^2 - rskh - rskh + rs^2h^2 - hks$$

$$= k^2 (r^2 + rs + s^2) + h^2 (r^2 + rs + s^2)$$

$$+ kh (r^2 + rs + s^2)$$

Thus, to use $\gamma^2 + \gamma\delta + \delta^2$ as a minimization criterion is equivalent to use $n + z$. Now as $r$ and $s$ verify $n(n+z) = r^2 + rs + s^2$, in many cases the choice between numerous multiple primitive function points $n(n+z)$ will be impossible because they will have the same $n+z$.

For example:

If $n(n+z) = 273$ two solutions exist $(11,8)$ and $(16,1)$ with $n = 3$ and $n+z = 91$. For those points it is not possible to determine the optimal point. The same argument can be applied to $\alpha^2 + \alpha\beta + \beta^2$ if we choose it as a criterion of maximization or minimization.

2122 Inequality constraints

Concerning inequality constraints (5) we can prove that they are always true whatever the values of $r$, $s$, $i$, $j$, $k$, $h$, $n$, $n+z$.

Proof: the first inequality constraint c) become:

$$- h \frac{(rh+sh+sk)}{h+k} < \frac{rk-sh}{n+z}$$

$$n+z \neq 0$$

$$n \neq 0$$

$$<==>$$

$$- h \frac{(rh+sh+sk)}{h+k} < \frac{rk-sh}{n+z} \leq \frac{rk-sh+sk}{k}$$

but $h+k > 0$ and $n > 0$ so we can write:

$$<==>$$

$$- rh^2 - sh^2 - skh \leq rhk - sh^2 + rh^2 - shk$$

$$rhk - shk \leq rhk + shk + sk^2$$

$$<==>$$

$$- rh^2 - rhk - rk^2 < 0$$

$$- shk - sh^2 - sk^2 \leq 0$$
We can say that the system is always checked. The second constraint d) is equivalent to:

\[ \begin{align*}
\Rightarrow & \quad \frac{i}{i+j} \left( \frac{r_i + s_i + s_j}{n} \right) < \frac{r_j - s_j}{n} \quad \text{with } n > 0 \\
\Rightarrow & \quad \frac{-s_{i+j}}{i+j} < \frac{r_{i+j}}{i+j} \quad \text{with } i+j > 0 \\
\Rightarrow & \quad r(i^2 + ij + j^2) > 0 \\
\Rightarrow & \quad s(i^2 + ij + j^2) > 0 \\
\Rightarrow & \quad \frac{rn}{n} > 0 \\
\Rightarrow & \quad \frac{sn}{n} > 0
\end{align*} \]

with \( r \in N, s \in N, n \in N^* \) we can affirm that this second system is always true.

For example:

(i) \( n(n+z) = 12, n = 3, i = 1, j = 1, n+z = 4, h = 2, k = 0, r = 2 \) 
\( s = 2 \) we get \( \alpha = 2, \beta = -1, \gamma = 2, \delta = 0 \) 
with \ \( -2 < -1 \leq 0 \)

(ii) \( n(n+z) = 27, r = 3, s = 3, n = 3, i = 1, j = 1 n+z = 9, h = 3 \) 
\( k = 0, \alpha = 2, \beta = -1, \gamma = 3, \delta = 0 \) with \ \( -2 < -1 \leq 0 \)

When there is at least two solutions the constraints do not allow us to make a choice. For example: \( n(n+z) = 588, \) two solutions exist \((22,4)\) and \((14,4)\)

(1) \( (14,4), n = 21, i = 4, j = 1, n+z = 28, h = 4, k = 2, \) if we take DACEY's minimization criterion this solution will be preferred to the other because \( n+z = 28 \) we get \( \alpha = -5, \beta = -1, \gamma = 6, \delta = -2 \) with

\[ \begin{align*}
-10 \leq -1 & < -1 \\
\frac{24}{5} \leq -2 & < \frac{3}{2}
\end{align*} \]

de the constraints are always checked.
(2) for (22,4) n = 4, i = 2, j = 0, n+z = 147, h = 11, k = 2, this solution will not be taken into account because it does not minimize n+z, we obtain \( \alpha = 2, \beta = 0, \gamma = 13, \delta = -2 \) with

\[
\begin{align*}
-22 &< 0 \leq 4 \\
3 &< 11 \\
11 &< 0 \\
\end{align*}
\]

Four other solutions for \( \alpha, \beta, \gamma, \delta \) exist but do not minimize \( \gamma^2 + \gamma \delta + \delta^2 \), in any constraints are always verify. We can find cases for which the criterion is respected and the constraints always checked, the choice is then impossible.

If we take \( n(n+z) = 868 \) the solutions are (18,16) and (26,6).

(1) for (18,16) n = 4, i = 2, j = 0, n+z = 217, h = 9, k = 8, \( \alpha = -2, \beta = 0, \gamma = -17, \delta = -8 \) constraints are

\[
\begin{align*}
-18 &< 0 \leq 16 \\
17 &< 9 \\
17 &< -8 < 0 \\
\end{align*}
\]

(2) for (26,6) n = 4, i = 2, j = 0, n+z = 217, h = 13, k = 3, \( \alpha = 2, \beta = 0, \gamma = -16, \delta = -3 \) with

\[
\begin{align*}
-13 &< 0 \leq 2 \\
18 &< 8 \\
16 &< -3 > 0. \\
\end{align*}
\]

We can say that (18,16) and (26,6) are equivalent points, but it is incorrect!

The conclusion can be the following one:

T5 is an inefficient algorithm because

- some of the equations are redundant
- the minimization criterion is incorrect because we can not choose between primitive function points
- inequality constraints are inefficient because they are always true when \( r \in \mathbb{N}, s \in \mathbb{N} \).
Figure 6: Location of centers of Dirichlet regions in the Central Place System.

Figure 7: Superposition of three market areas: (N02, N02, N05)
Discussion of results obtained with the algorithm T5

The algorithm has been tested for $n(n+z) = 12$ till 5000

No constraints on the integers $r$ and $s$

Example 1:

$n(n+z) = 1981$, two solutions exist $(31, 20)$ and $(44, 1)$ with $n = 7$ and $n+z = 283$. $(31, 20)$ and $(44, 1)$ are located in the city rich sector then those points are primitive function points. So we can use the algorithm T5.

1. $(44, 1) n = 7, i = 2, j = 1, n+z = 283, h = 13, k = 6, \delta = -87$  \[ \beta = 251, \gamma = \frac{134}{7}, \] this last value is incorrect then T5 precludes that point.

2. for $(20, 31) n = 7, i = 2, j = 1, n+z = 283, h = 13, k = 6, \delta = -6, \alpha = 2.7$ which is impossible T5 again precludes that point.

On the other hand, T5 gives $(20, 31) (\alpha = 3, \beta = -1, \gamma = 19, \delta = -6)$ which is not a primitive function point. Many other examples can be given. Then we can be sure that this algorithm does not supply the good primitive function point.

$r$ and $s$ are such $r \geq s$

For the first example $n(n+z) = 1981$ there is no solution. We can think that the constraints imposed in $a$ and $y$ were too restrictive. A new test has been made.

$a$ and $p$ are such as $a \in \mathbb{R}, y \in \mathbb{R}$

The above examples have been reexamined:

Example 1:

1. for $(44, 1)$ we get $a = 2.09, \beta = 0.89, \gamma = 13, \delta = 6$ the minimization criterion is inefficient

2. for $(31, 20)$ we get $a = 2.77, \beta = 0.26, \gamma = 13.43, \delta = -1.29$ with $n = 7, n+z = 283$.

In fact in $(44, 1)$ we have $n = 7$ when in $(31, 20)$ there is no center of market area.

Other maximization criterions have been considered:

- maximize the number of integers unknown, which is incorrect for $n(n+z) = 399$ (see table 1)
- maximize $a-\beta$ or $a+\beta$ or $\gamma-\delta$ or $\gamma+\delta$ or minimize it.
- maximize $a/\beta$ or minimize it.
Table 1: new test of T5 for \( n(n+z) = 399\), \( \alpha \in \mathbb{R} \), \( \beta \in \mathbb{R} \), \( \gamma \in \mathbb{R} \), \( \delta \in \mathbb{R} \)

<table>
<thead>
<tr>
<th>z</th>
<th>s</th>
<th>n(n+z)</th>
<th>n</th>
<th>n+z</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \gamma )</th>
<th>( \delta )</th>
<th>( \alpha + \beta )</th>
<th>( \gamma + \delta )</th>
<th>( -\alpha / \beta )</th>
<th>( \gamma / \delta )</th>
<th>( \alpha - \beta )</th>
<th>( \gamma - \delta )</th>
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</thead>
<tbody>
<tr>
<td>13</td>
<td>10</td>
<td>399</td>
<td>3</td>
<td>133</td>
<td>1.86</td>
<td>-0.29</td>
<td>11</td>
<td>1</td>
<td>1.57</td>
<td>12</td>
<td>6.41</td>
<td>11</td>
<td>2.1</td>
<td>10</td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>399</td>
<td>3</td>
<td>133</td>
<td>1.98</td>
<td>-0.73</td>
<td>11</td>
<td>1</td>
<td>1.25</td>
<td>12</td>
<td>2.71</td>
<td>11</td>
<td>2.7</td>
<td>10</td>
</tr>
<tr>
<td>13</td>
<td>10</td>
<td>399</td>
<td>7</td>
<td>57</td>
<td>3</td>
<td>1</td>
<td>8</td>
<td>-1</td>
<td>2</td>
<td>7</td>
<td>3</td>
<td>-8</td>
<td>4</td>
<td>9</td>
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<td>21</td>
<td>4.86</td>
<td>-1.29</td>
<td>4.6</td>
<td>0.2</td>
<td>3.57</td>
<td>4.89</td>
<td>3.76</td>
<td>22</td>
<td>6.1</td>
<td>4</td>
</tr>
<tr>
<td>17</td>
<td>5</td>
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<td>3</td>
<td>133</td>
<td>1.64</td>
<td>0.14</td>
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<td>13</td>
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<td>1.5</td>
<td>5</td>
</tr>
<tr>
<td>17</td>
<td>5</td>
<td>399</td>
<td>3</td>
<td>133</td>
<td>1.86</td>
<td>-0.29</td>
<td>9</td>
<td>4</td>
<td>1.57</td>
<td>13</td>
<td>6.41</td>
<td>2.2</td>
<td>2.1</td>
<td>5</td>
</tr>
<tr>
<td>17</td>
<td>5</td>
<td>399</td>
<td>7</td>
<td>57</td>
<td>2.79</td>
<td>-0.32</td>
<td>7</td>
<td>1</td>
<td>2.47</td>
<td>8</td>
<td>8.71</td>
<td>7</td>
<td>3.1</td>
<td>6</td>
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<tr>
<td>17</td>
<td>5</td>
<td>399</td>
<td>19</td>
<td>21</td>
<td>4.43</td>
<td>-0.14</td>
<td>4</td>
<td>1</td>
<td>4.29</td>
<td>5</td>
<td>31.6</td>
<td>4</td>
<td>4.5</td>
<td>3</td>
</tr>
</tbody>
</table>

If we only use the minimization criterion of DACEY (13,10) on line 4 and (17,5) on line 8 are possible and if we impose criterion 1 the choice proceeds on (17,5) when the solution is (13,10). If we maximize \( \alpha + \beta \) it is insatisfactory because we will choose (17,5). A lot of other examples can be given revealing the same inefficiency. In view of these results we have forsaken the algorithm. Then, it was necessary to propose a new method of construction of the loschian landscape based on concepts set up by DACEY.

3 New method of elaboration of the Central Place System

To throw out the algorithm doesn't mean that all the postulates and definitions of DACEY are incorrect.

In 1965 DACEY(11) has written "when the vector \( c_{ij} \) such as:

\[
c_{ij} = i \mathbf{t}_1 + j \mathbf{t}_2
\]

and the point \( 0 \) \((0,0)\) are located and are the centers of adjacents DIRICHLET regions of degree \( n \), by rotation of \( \pi /3 \) and translation of the vector \((i,j)\) points of the lattice \( \mathcal{R}_1 \{ c_{ij} \} \) are obtained and are the centers of these regions".

The strict application of this rule is unable to determine the landscape constructed by LOSCH.
Explanation of the location of the points

This method must allow us to locate all the solutions \((i,j)\) of the Diophantine equation:

\[ k_{ij} = i^2 + ij + j^2 \]

(1) when there is a single solution, the center of the DIRICHLET region will be located in \((i,j)\). This one will have an order \(k_{ij}\).

To locate centers of DIRICHLET adjacents regions primitive function point is displaced in the lattice with vectorial translation of what we call "primitive" vector \((i,j)\) till a fixed couple of coordinates (in our case \((50,50)\)).

The part obtained after \((k-1)\) translations is noted \(A' = k (i,j)\), this point is again displaced in the lattice by rotation of \(\frac{\pi}{3}\) angle, it means that point \(A''\) image of \(A'\) by that rotation is get as follows:

\[
A'' = A' \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = k (i,j) \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} = (-kj, ki + kj)
\]

\(A''\) is again translated by \((i,j)\) we get point \(A'''\) such as

\[ A''' = (-kj + i, ki + kj + j) \]

Thus we obtain all the centers of adjacent Dirichlet region of function \(n\). The points which are neither perfect squares nor multiple function points and which after this last translation have negative abscissa are rotated of \(-\frac{\pi}{3}\) angle. In this case the matrix is

\[
\begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}
\]

The points obtained are then translated with the primitive vector \((i,j)\).

(2) when the solution is not unique the point is located priority where the maximum of points are already located (whatever the way they have been obtained).

These points are previously the centers of function \(n\) and \(n+z\). If there is the same number of centers we must choose the point which has the higher order.

For example \(n(n+z) = 91\), two points are possible \((6,5)\) and \((9,1)\). In \((6,5)\) there is no center of DIRICHLET region and in \((9,1)\) there is one center of function \(n = 7\). For \(n(n+z) = 273\) we must choose between \((11,8)\) and \((16,1)\). In \((11,8)\) there are three regions of function \(n = 3, 13, 21\) then in \((16,1)\) there is \(n = 3, 7, 13\) therefore we choose \((11,8)\).
32 Two landscapes

Another interesting case exist:

the case where the number of coinciding centers is the same and when they have the same functions (for example $n(n+z) = 169$ located in $(8,7)$ or in $(13,0)$ where is already the point of function $n = 13$).

In fact no criterion can allow us to choose because the works of LOSCH are of no help (15):

"First we lay the nets so that all of them shall at least one center in common. Here a metropolis will arise...Second, we turn the nets about this center so as to get six sectors with many and six sectors with only a few production sites. With this arrangement the greatest number of location coincide..."

These conditions are respected for two points $(8,7)$ and $(13,0)$ for the function $n(n+z) = 169$.

This is not the single one and its location will influence the one of multiple function of 169:

For example $n(n+z) = 7 \times 169 = 1183$, the primitive function point 1183 can be located in $(26,13)$, $(29,9)$ or $(31,9)$. If we locate the center of DIRICHLET region of order 169 in $(8,7)$ the center of order 1183 region will be located in $(26,13)$ as will as those of order 7, 13, 91, in $(29,9)$ there is no center except the one of order 1 located everywhere in the lattice; in $(31,9)$ we get the point which is the center of the region of order 1321. But if we locate the center of the DIRICHLET region of order 169 in $(13,0)$ the region of order 1183 will be centered in $(26,13)$ as the regions of function 7, 13, 91, 169.

In $(29,9)$ there is a point of order 1, in $(31,9)$ is centered a region of order 1321. We can identify two cases:

(i) the first one (case 1) chooses the primitive function point having the smaller abscissa, when at least two points have the same number of coinciding centers with the same function.

(ii) the second one (case 2) chooses the primitive function point having the higher abscissa in the same situation as (i).

then according as the choice which is made the landscape will be modified (table 2).

For $i < 50$ ($i \in \mathbb{N}$) and $j \leq 50$ ($j \in \mathbb{N}$), if we choose the point having the smaller abscissa (case 1) we locate 13 supplementary centers of region in the city rich sector (P1) than if we choose case 2.
Table 2: Differences between case 1 and case 2

<table>
<thead>
<tr>
<th>max (i,j)</th>
<th>nb pts</th>
<th>nb pts</th>
<th>nb pts</th>
<th>nb pts</th>
<th>nb pts</th>
<th>nb pts</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>case 1 (8,7)</td>
<td>case 2 (13,0)</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>(50,50)</td>
<td>3731</td>
<td>412</td>
<td>2870</td>
<td>3709</td>
<td>418</td>
<td>2873</td>
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<tr>
<td>(100,100)</td>
<td>16378</td>
<td>1011</td>
<td>13109</td>
<td>16295</td>
<td>1032</td>
<td>13129</td>
</tr>
<tr>
<td>(150,150)</td>
<td>38542</td>
<td>1674</td>
<td>31414</td>
<td>38378</td>
<td>1716</td>
<td>31456</td>
</tr>
<tr>
<td>(200,200)</td>
<td>70608</td>
<td>2407</td>
<td>58199</td>
<td>70304</td>
<td>2468</td>
<td>58292</td>
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<tr>
<td>(250,250)</td>
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<td>3152</td>
<td>93645</td>
<td>112150</td>
<td>3236</td>
<td>93770</td>
</tr>
</tbody>
</table>

Remarques: I < J city poor sector
I > J city rich sector
I = J city rich and poor sectors boundary
CONCLUSION

This paper has provided a new method of construction of the loschian landscape and has proved the inefficiency of an algorithm elaborated by DACEY. Our system is based on translations and rotations. To settle the mosaic of Dirichlet regions two cases must be distinguished.

(i) if \((i,j)\) is a primitive function point such as \(j = 0\). Then we must translate this point with a "primitive" vector \((i,j)\) till a fixed point \((50,50)\) for example. The point obtained is rotated with an angle of \(\pi/3\), the matrix transformation is

\[
\begin{bmatrix}
0 & 1 \\
-1 & 1
\end{bmatrix}
\]

this point must again be translated with the primitive vector \((i,j)\) till \((50,50)\).

(ii) otherwise, after this second translation the points having negative abscissa rotated with a rotation of angle of \(-\pi/3\) with a matrix transformation of

\[
\begin{bmatrix}
1 & -1 \\
1 & 0
\end{bmatrix}
\]

the points thus obtained are translated with the primitive vector \((i,j)\).

Of course, this test can be enlarged, nevertheless we can assert that the sectorial dichotomy exists, even if MARSHALL \((17,18,19)\) has an opposite view. The loschian Theory is then corroborated.

DACEY's algorithm is in fact useless, incorrect and his landscape description is incomplete because he only makes translations and rotation of angle \(\pi/3\); then a lot of points will be located in incorrect places.

If we respect our method of elaborating the Central place landscape with the help of mathematical concepts, the landscape obtained is in perfect agreement with LOSCH's figure, what we wanted to reach.
Bibliography


(7) __________, City Rich and City Poor Sectors : A Comment on the Construction of L"oschian Landscape, Geographical Analysis, vol. 10 n°1, January 1978, pp 77-82.


(22) TARRANT J, Comments on the LOSCH Central Place System, Geographical Analysis, vol 5, 1973, pp 113-121.