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A finite volume scheme for a seawater intrusion model with cross-diffusion

Ahmed Ait Hammou Oulhaj

Abstract We consider a finite volume scheme for a seawater intrusion model. It is based on a two-point flux approximation with upwind mobilities. The scheme preserves at the discrete level the main features of the continuous problem: the non-negativity of the solutions, the decay of the energy and the control of the entropy and its dissipation. Moreover the scheme converges towards a weak solution to the problem. Numerical results are provided to illustrate the behavior of the model and of the scheme.

Key words: Finite Volume scheme, entropy stability, decay of energy.

MSC (2010): 65M08, 65N08, 35Q30

1 The continuous problem and objectives

Let Ω be a polygonal open bounded and connected subset of \mathbb{R}^2 , and $T > 0$ be a finite time horizon. We consider the following cross-diffusion system of degenerate parabolic equations

$$\begin{cases} \partial_t f - \nabla \cdot (\mu f \nabla (f + g + b)) = 0 & \text{in } \Omega_T := \Omega \times (0, T), \\ \partial_t g - \nabla \cdot (g \nabla (\mu f + g + b)) = 0 & \text{in } \Omega_T, \\ \nabla f \cdot \mathbf{n} = \nabla g \cdot \mathbf{n} = 0, & \text{on } \partial\Omega \times (0, T), \\ f|_{t=0} = f_0 \geq 0, \quad g|_{t=0} = g_0 \geq 0, & \text{in } \Omega. \end{cases} \quad (1)$$

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It models the seawater intrusion in an unconfined aquifer (see [4]). We assume that the impermeable interface between the saltwater and the bedrock is given by $z = b(x)$. The saltwater and the freshwater are assumed to be immiscible. The interfaces between the salt-and freshwater, and between the freshwater and the dry soil is located at height $z = g(x, t) + b(x)$ and $z = f(x, t) + g(x, t) + b(x)$ respectively (cf. Fig. 1). The parameter μ is given by $\mu = \frac{\rho_{\text{fresh}}}{\rho_{\text{salt}}} \in (0, 1)$, with ρ the mass density of the fluid. The initial data f_0 and g_0 are assumed to belong to $L^\infty(\Omega)$.

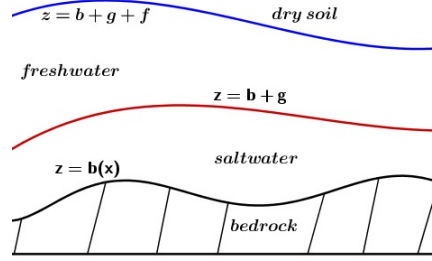


Fig. 1 Description of an unconfined aquifer

Following [2, 6, 5], there exists a weak solution to the problem (1). Moreover

$$f \geq 0, g \geq 0 \quad \text{a.e in } \Omega_T. \quad (2)$$

We recall the definition of entropy (resp. energy) functional introduced in [2, 6]:

$$\mathfrak{H}(f, g) = \int_{\Omega} \left[\Gamma(g) + \frac{1}{\mu} \Gamma(f) \right] dx, \quad \text{where } \Gamma(s) = s \log s - s + 1,$$

$$\mathfrak{E}(f, g) = \int_{\Omega} \left[\frac{\mu}{2} (f + g + b)^2 + \frac{1 - \mu}{2} (g + b)^2 \right] dx.$$

Multiplying (formally) the first equation of (1) by $\frac{1}{\mu} \log f$ (resp. $\mu(f + g + b)$) and the second equation by $\log g$ (resp. $\mu f + g + b$), integrating over Ω and summing both equalities, get the classical entropy/dissipation property:

$$\frac{d}{dt} \mathfrak{H}(f, g) + \frac{1 - \mu}{2} \int_{\Omega} \left[(\nabla f)^2 + (\nabla g)^2 \right] dx \leq \frac{1}{2(\mu + 1)} \int_{\Omega} (\nabla b)^2 dx, \quad (3)$$

and the decay of the energy functional along time:

$$\frac{d}{dt} \mathfrak{E}(f, g) + \int_{\Omega} \left[\mu^2 f (\nabla(f + g + b))^2 + g (\nabla(\mu f + g + b))^2 \right] dx = 0. \quad (4)$$

In this work, We propose a finite volume scheme for the problem (1). This scheme is based on a two-point flux approximation with upwind mobilities. It is designed in order to preserve at the discrete level the main features of the continuous problem: the nonnegativity of the solutions (2), the decay of the energy (4), and the control of the entropy and of its dissipation (3). Based on these estimates, the convergence of the scheme can be proved. We refer to [1] for this purpose.

2 The finite volume scheme

An admissible mesh of Ω is given by a family \mathcal{T} of a control volumes (open and convex polygons), a family \mathcal{E} of edges, and a family of points $(x_K)_{K \in \mathcal{T}}$ which satisfy Definition 9.1 in [3]. This definition implies that the straight line between two neighboring centers of cells (x_K, x_L) is orthogonal to the edge $\sigma = K|L$.

We distinguish the interior edges $\sigma \in \mathcal{E}_{\text{int}}$ and the boundary edges $\sigma \in \mathcal{E}_{\text{ext}}$. The set of edges \mathcal{E} equals the union $\mathcal{E}_{\text{int}} \cup \mathcal{E}_{\text{ext}}$. For a control volume $K \in \mathcal{E}$, we denote by \mathcal{E}_K the set of its edges, by $\mathcal{E}_{\text{int},K}$ the set of its interior edges, and by $\mathcal{E}_{\text{ext},K}$ the set of edges of K included in $\partial\Omega$.

Furthermore, we denote by d the distance in \mathbb{R}^2 and by m the Lebesgue measure in \mathbb{R}^2 or \mathbb{R} . We assume that the mesh satisfies the following regularity requirement: there exists $\zeta > 0$ such that it holds

$$d(x_K, \sigma) \geq \zeta d(x_K, x_L), \quad \forall K \in \mathcal{T}, \forall \sigma \in \mathcal{E}_{\text{int},K}, \text{ with } \sigma = K|L. \quad (5)$$

For all $\sigma \in \mathcal{E}_{\text{int},K}$, with $\sigma = K|L$, we define $d_\sigma = d(x_K, x_L)$, and the transmissibility coefficient $\tau_\sigma = \frac{m(\sigma)}{d_\sigma}$, $\sigma \in \mathcal{E}$. The size of the mesh is $\delta = \max_{K \in \mathcal{T}}(\text{diam}(K))$.

Let N be a positive integer, and $\Delta t = \frac{T}{N}$; then a uniform discretization of $(0, T)$ is given by the family $(t^n)_{n \in \{0, \dots, N\}}$ where $t^n = n\Delta t$.

We denote by \mathcal{D} an admissible space-time discretization of Ω_T composed of an admissible mesh \mathcal{T} of Ω and the values Δt and N . The size of this space-time discretization \mathcal{D} is defined by $\eta = \max(\delta, \Delta t)$.

Remark 1. Voronoï meshes and triangular meshes with uniformly acute angles are a typical examples of admissible meshes satisfying (5).

The initial conditions are discretized by

$$s_{\mathcal{T}}^0 = \sum_{K \in \mathcal{T}} s_K^0 \mathbf{1}_K, \quad \text{where } s_K^0 = \frac{1}{m(K)} \int_K s_0(x) dx, \quad \forall K \in \mathcal{T}, \quad \text{with } s = f \text{ or } g,$$

and $\mathbf{1}_K$ is the characteristic function on K . The discretization of problem (1) is given by the following set of nonlinear equations: for $K \in \mathcal{T}$ and $0 \leq n \leq N-1$

$$m(K) \frac{f_K^{n+1} - f_K^n}{\Delta t} + \sum_{\sigma \in \mathcal{E}_{\text{int},K}} \tau_\sigma f_\sigma^{n+1} \mu (u_K^{n+1} - u_L^{n+1}) = 0, \quad (6)$$

$$m(K) \frac{g_K^{n+1} - g_K^n}{\Delta t} + \sum_{\sigma \in \mathcal{E}_{\text{int},K}} \tau_\sigma g_\sigma^{n+1} (v_K^{n+1} - v_L^{n+1}) = 0, \quad (7)$$

where $u_K^{n+1} = f_K^{n+1} + g_K^{n+1} + b_K$, $v_K^{n+1} = \mu f_K^{n+1} + g_K^{n+1} + b_K$, with $b_K = b(x_K)$,

$$f_\sigma^{n+1} = \begin{cases} (f_K^{n+1})^+ & \text{if } u_K^{n+1} \geq u_L^{n+1}, \\ (f_L^{n+1})^+ & \text{if } u_K^{n+1} < u_L^{n+1}, \end{cases} \quad g_\sigma^{n+1} = \begin{cases} (g_K^{n+1})^+ & \text{if } v_K^{n+1} \geq v_L^{n+1}, \\ (g_L^{n+1})^+ & \text{if } v_K^{n+1} < v_L^{n+1}, \end{cases} \quad (8)$$

with $x^+ = \max(x, 0)$.

Proposition 1. (existence of a discrete solution) For $n \in \{0, \dots, N\}$ there exists (at least) one solution of the scheme (6)-(8). Moreover $f_K^n \geq 0$, $g_K^n \geq 0$, for $K \in \mathcal{T}$.

Proof. Let us to prove the nonnegativity of f_K^n (which is similar for g_K^n). This property clearly holds for $n = 0$. Assume now the nonnegativity holds at time step n , and assume that $f_K^{n+1} < 0$, for some $K \in \mathcal{T}$. In view of the definition (8) of f_σ^{n+1} one has that

$$f_K^{n+1} = f_K^n - \frac{\mu \Delta t}{m(K)} \sum_{\sigma \in \mathcal{E}_{\text{int},K}} \tau_\sigma \underbrace{(f_K^{n+1})^+}_{=0} (u_K^{n+1} - u_L^{n+1})^+ - (f_L^{n+1})^+ (u_K^{n+1} - u_L^{n+1})^- \geq 0,$$

yielding a contradiction, ensuring that $f_K^{n+1} \geq 0$, $\forall K \in \mathcal{T}, \forall n \geq 0$. The proof of existence is detailed in [1]. \square

3 Entropy and energy estimates

We introduce a discrete version of entropy (resp. energy) functional:

$$\mathfrak{H}^n := \mathfrak{H}(f_K^n, g_K^n) = \sum_{K \in \mathcal{T}} m(K) \left(\frac{1}{\mu} \Gamma(f_K^n) + \Gamma(g_K^n) \right),$$

$$\mathfrak{E}^n := \mathfrak{E}(f_K^n, g_K^n) = \sum_{K \in \mathcal{T}} m(K) \left(\frac{\mu}{2} (f_K^n + g_K^n + b_K)^2 + \frac{1-\mu}{2} (g_K^n + b_K)^2 \right).$$

We establish now the discrete counterpart of (3) and (4).

Proposition 2. There exists C depending only on T, Ω and b such that

$$\sup_{n \in \{0, \dots, N-1\}} \mathfrak{H}^{n+1} + \sum_{n=0}^{N-1} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_\sigma \left[(f_K^{n+1} - f_L^{n+1})^2 + (g_K^{n+1} - g_L^{n+1})^2 \right] \leq C.$$

Proof. We multiply (6) (resp. (7)) by $\Delta t \frac{\log f_K^{n+1}}{\mu}$ (resp. $\Delta t \log g_K^{n+1}$) and sum over $K \in \mathcal{T}$, provides that: $A + B + C = 0$, where

$$\begin{aligned} A &= \sum_{K \in \mathcal{T}} m_K \left[\frac{1}{\mu} (f_K^{n+1} - f_K^n) \log f_K^{n+1} + (g_K^{n+1} - g_K^n) \log g_K^{n+1} \right], \\ B &= \Delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int},K}} \tau_\sigma f_\sigma^{n+1} (u_K^{n+1} - u_L^{n+1}) \log f_K^{n+1}, \\ C &= \Delta t \sum_{K \in \mathcal{T}} \sum_{\sigma \in \mathcal{E}_{\text{int},K}} \tau_\sigma g_\sigma^{n+1} (v_K^{n+1} - v_L^{n+1}) \log g_K^{n+1}. \end{aligned}$$

By the convexity of Γ , we find that

$$\mathcal{J}^{n+1} - \mathcal{J}^n = \sum_{K \in \mathcal{T}} m(K) \left[\frac{1}{\mu} (\Gamma(f_K^{n+1}) - \Gamma(f_K^n)) + \Gamma(g_K^{n+1}) - \Gamma(g_K^n) \right] \leq A.$$

We can rewrite B and C as:

$$\begin{aligned} B &= \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} \tau_\sigma f_\sigma^{n+1} (u_K^{n+1} - u_L^{n+1}) (\log f_K^{n+1} - \log f_L^{n+1}), \\ C &= \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} \tau_\sigma g_\sigma^{n+1} (v_K^{n+1} - v_L^{n+1}) (\log g_K^{n+1} - \log g_L^{n+1}). \end{aligned}$$

It follows from the convexity of \exp that

$$a(\log a - \log b) \geq a - b \geq b(\log a - \log b) \quad \forall a, b \in [0, +\infty[,$$

where we have used the convention $\log(0) = -\infty$ and $0 \log(0) = 0$. Hence, in view of the definition (8), one has

$$\begin{aligned} B &\geq \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} \tau_\sigma \left((f_K^{n+1} - f_L^{n+1})^2 + (g_K^{n+1} - g_L^{n+1})(f_K^{n+1} - f_L^{n+1}) \right. \\ &\quad \left. + (b_K - b_L)(f_K^{n+1} - f_L^{n+1}) \right), \end{aligned}$$

$$\begin{aligned} C &\geq \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma=K|L}} \tau_\sigma \left((g_K^{n+1} - g_L^{n+1})^2 + \mu(g_K^{n+1} - g_L^{n+1})(f_K^{n+1} - f_L^{n+1}) \right. \\ &\quad \left. + (b_K - b_L)(g_K^{n+1} - g_L^{n+1}) \right). \end{aligned}$$

Combining these inequalities, one deduces that

$$\begin{aligned}
& \mathfrak{H}^{n+1} - \mathfrak{H}^n + \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (f_K^{n+1} - f_L^{n+1})^2 + \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (g_K^{n+1} - g_L^{n+1})^2 \\
& \quad + (\mu + 1) \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (f_K^{n+1} - f_L^{n+1})(g_K^{n+1} - g_L^{n+1}) \\
& \leq -\Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (b_K - b_L) \left[(f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) \right] := D.
\end{aligned}$$

Using the Young inequality, one has for all $\varepsilon > 0$

$$D \leq \frac{1}{2\varepsilon} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (b_K - b_L)^2 + \frac{\varepsilon}{2} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} \left[(f_K^{n+1} - f_L^{n+1}) + (g_K^{n+1} - g_L^{n+1}) \right]^2.$$

We choose $\varepsilon = 1 + \mu$, we have

$$\begin{aligned}
|D| & \leq \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} \left(\frac{1}{2(\mu + 1)} (b_K - b_L)^2 + (\mu + 1) (f_K^{n+1} - f_L^{n+1})(g_K^{n+1} - g_L^{n+1}) \right) \\
& \quad + \frac{\mu + 1}{2} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} \left((f_K^{n+1} - f_L^{n+1})^2 + (g_K^{n+1} - g_L^{n+1})^2 \right).
\end{aligned}$$

Finally, one has

$$\begin{aligned}
& \mathfrak{H}^{n+1} - \mathfrak{H}^n + \frac{1 - \mu}{2} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (f_K^{n+1} - f_L^{n+1})^2 \\
& \quad + \frac{1 - \mu}{2} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (g_K^{n+1} - g_L^{n+1})^2 \leq \frac{1}{2(\mu + 1)} \Delta t \sum_{\substack{\sigma \in \mathcal{E}_{\text{int}} \\ \sigma = K|L}} \tau_{\sigma} (b_K - b_L)^2.
\end{aligned}$$

Summing over $n = 0, \dots, N - 1$, concludes the proof of Proposition 2. \square

Proposition 3. For $n \in \{0, \dots, N - 1\}$

$$\sup_{n \in \{0, \dots, N - 1\}} \mathfrak{E}^{n+1} \leq \mathfrak{E}^0.$$

Proof. We multiply (6) (resp. (7)) by $\Delta t \mu u_K^{n+1}$ (resp. $\Delta t v_K^{n+1}$) and sum over $K \in \mathcal{T}$. Summing both equalities and reorganizing the sums, we get $A + B = 0$, where

$$\begin{aligned}
A &= \sum_{K \in \mathcal{T}} m(K) \left[\mu \left((f_K^{n+1} + g_K^{n+1} + b_K) - (f_K^n + g_K^n + b_K) \right) \left(f_K^{n+1} + g_K^{n+1} + b_K \right) \right] \\
&\quad + \sum_{K \in \mathcal{T}} m(K) \left[(1 - \mu) \left((g_K^{n+1} + b_K) - (g_K^n + b_K) \right) \left(g_K^{n+1} + b_K \right) \right], \\
B &= \Delta t \sum_{\substack{\sigma \in \delta_{\text{int}} \\ \sigma = K|L}} \tau_\sigma \left(\mu^2 f_\sigma^{n+1} \left(u_K^{n+1} - u_L^{n+1} \right)^2 + g_\sigma^{n+1} \left(v_K^{n+1} - v_L^{n+1} \right)^2 \right).
\end{aligned}$$

We use the following inequality: $(a - b)a \geq \frac{1}{2}(a^2 - b^2)$, $\forall a, b \in \mathbb{R}$, to get

$$\begin{aligned}
A &\geq \sum_{K \in \mathcal{T}} m(K) \left[\frac{\mu}{2} \left((f_K^{n+1} + g_K^{n+1} + b_K)^2 - (f_K^n + g_K^n + b_K)^2 \right) \right] \\
&\quad + \sum_{K \in \mathcal{T}} m(K) \left[\frac{1 - \mu}{2} \left((g_K^{n+1} + b_K)^2 - (g_K^n + b_K)^2 \right) \right] = \mathfrak{E}^{n+1} - \mathfrak{E}^n.
\end{aligned}$$

Summing over $n = 0, \dots, N - 1$, concludes the proof of Proposition 3. \square

Let us remark that Proposition 2 and Proposition 3 give a discrete $L^2(0, T; H^1(\Omega))$ and $L^\infty(0, T; L^2(\Omega))$ bounds on the approximate solutions. These estimates are sufficient to prove the convergence of the scheme when the discretization parameters δ and Δt tend to 0. We refer to [1] for the proof. As a byproduct, this convergence result ensures the existence of a weak solution to the continuous model. The question of uniqueness of the solution is open as far as we know.

4 Numerical results

Let us provide some illustrations of the behaviour of the numerical scheme (6)-(8). The scheme leads to a nonlinear system that we solve thanks to the Newton-Raphson method. The numerical analysis of the scheme was carried out for a uniform time discretization of $(0, T)$ only in order to avoid heavy notations. In order to increase the robustness of the algorithm and to ensure the convergence of the Newton-Raphson iterative procedure, we used an adaptive time step procedure in the practical implementation. More precisely, we associate a maximal time step Δt_{max} for the mesh. If the Newton-Raphson method fails to converge after 30 iterations —we choose that the ℓ^∞ norm of the residual has to be smaller than 10^{-10} as stopping criterion—, the time step is divided by two. If the Newton-Raphson method converges, the time step is multiplied by two and projected on $[0, \Delta t_{\text{max}}]$.

In our test case, the domain is the unit square, i.e., $\Omega = (0, 1)^2$. We consider an admissible triangular mesh made of 14336 triangles. We choose $b(x, y) = \max \left(0, \frac{1}{2} \left(1 - 16(x - 1/2)^2 \right) \left((\cos(\pi y) + 2) \right) \right)$. We set $\mu = 0.9$, and

$$f_0(x,y) = \begin{cases} \frac{1}{2} & \text{if } x \leq \frac{1}{4}, \\ 0 & \text{elsewher,} \end{cases} \quad g_0(x,y) = \begin{cases} b\left(\frac{1}{2},0\right) - b(x,y) - \left(x - \frac{1}{2}\right) & \text{if } x \leq \frac{1}{2}, \\ 0 & \text{elsewhere.} \end{cases}$$

Fig. 2 shows the evolution of $b(x)$ (black), $b(x) + g(x,t)$ (red) and $b(x) + g(x,t) + f(x,t)$ (blue) at different times, and also the evolution of the energy along time. About the model, we observe that there is convergence towards an equilibrium state, with horizontal interfaces as expected.

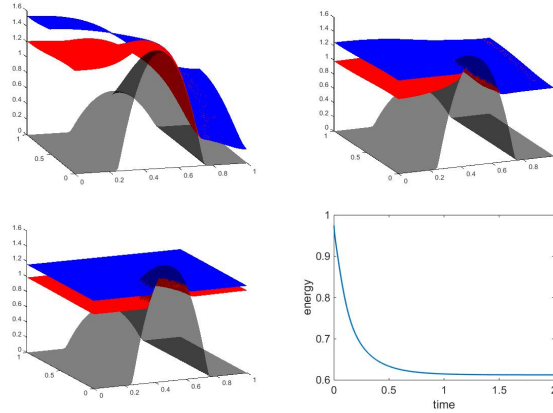


Fig. 2 Behaviour of the model at $t = 0.2$, $t = 0.79$, $t = 12$, and evolution of the energy along time

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