Uniform-in-Time Convergence of Numerical Schemes for a Two-Phase Discrete Fracture Model

Jerome Droniou, Julian Hennicker, Roland Masson

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Abstract Flow and transport in fractured porous media are of paramount importance for many applications such as petroleum exploration and production, geological storage of carbon dioxide, hydrogeology, or geothermal energy. We consider here the two-phase discrete fracture model introduced in [3] which represents explicitly the fractures as codimension one surfaces immersed in the surrounding matrix domain. Then, the two-phase Darcy flow in the matrix is coupled with the two-phase Darcy flow in the fractures using transmission conditions accounting for fractures acting either as drains or barriers. The model takes into account complex networks of fractures, discontinuous capillary pressure curves at the matrix fracture interfaces and can be easily extended to account for gravity including in the width of the fractures. It also includes a layer of damaged rock at the matrix fracture interface with its own mobility and capillary pressure functions. In this work, the convergence analysis carried out in [3] in the framework of gradient discretizations [2] is extended to obtain the uniform-in-time convergence of the discrete solutions to a weak solution of the model.

Key words: Discrete fracture model, two-phase Darcy flow, uniform-in-time convergence, gradient discretization method

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1 Continuous model

We give here a brief overview of the notations, and refer to [3] for more details. Ω is a bounded polytopal domain of $\mathbb{R}^d$ ($d = 2,3$), partitioned into a fracture domain Γ and a matrix domain $\Omega\backslash\Gamma$. The network of fractures is $\Gamma = \bigcup_{i\in I}\Gamma_i$, where each Γ_i is planar and has therefore two faces $\mathfrak{a}^+(i)$ and $\mathfrak{a}^-(i)$. Set $\chi = \{\mathfrak{a}^+(i), \mathfrak{a}^-(i) \mid i \in I\}$ the set all faces and write, for simplicity, $\Gamma_{\mathfrak{a}^+(i)} = \Gamma_{\mathfrak{a}^-(i)} = \Gamma_i$. For $\mathfrak{a} \in \chi$, $\gamma_\mathfrak{a}$ is the one-sided trace operator on $\Gamma_\mathfrak{a}$ and $\mathbf{n}_\mathfrak{a}$ denotes the unit normal vector directed from the face $\mathfrak{a}$ to the matrix domain. The following notations, in which $\overline{u}^\alpha_\mu$ is the phase pressure in the medium μ and phase α, are used throughout the paper.

$$
\begin{aligned}
&M_m = \Omega\,,\ M_f = \Gamma \text{ and } M_\mathfrak{a} = \Gamma_\mathfrak{a}\,;\ s^+ = \max(0,s)\,,\ s^- = (-s)^+\,;\\
&(\overline{p}_m, \overline{p}_f) = (\overline{u}^1_m - \overline{u}^2_m, \overline{u}^1_f - \overline{u}^2_f) \text{ (capillary pressures)};\ [\![\overline{u}^\alpha]\!]_\mathfrak{a} = \gamma_\mathfrak{a}\overline{u}^\alpha_m - \overline{u}^\alpha_f.
\end{aligned}
$$

The assumptions in the rest of this paper are:

- The matrix-valued functions Λ_m and Λ_f, permeability tensors in the matrix and fracture domains, respectively, are uniformly coercive tensors.
- The functions T_f (half-normal transmissibility in the fracture network), ϕ_m and ϕ_f (porosities of the matrix and fracture, respectively), and d_f (fracture width) are bounded measurable and uniformly positive.
- The phase mobilities $k^\alpha_\mu\colon M_\mu \times [0,1] \to \mathbb{R}$ are bounded uniformly positive Caratheodory functions, $h^\alpha_\mu \in L^2((0,T)\times M_\mu)$ and $\eta > 0$.
- The saturation $S^1_\mu : M_\mu \times \mathbb{R} \to [0,1]$ of the non wetting phase is a Caratheodory function; for a.e. $\mathbf{x} \in M_\mu$, $S^1_\mu(\mathbf{x},\cdot)$ is a non-decreasing Lipschitz continuous function on $\mathbb{R}$; $S^1_\mu(\cdot,q)$ is piecewise constant on a finite partition $(M^j_\mu)_{j\in J_\mu}$ of polytopal subsets of M_μ, for all $q \in \mathbb{R}$. Not indicating the phase in the saturation means that $\alpha = 1$, that is, $S_\mu = S^1_\mu$. Of course, $S^2_\mu = 1 - S^1_\mu$ The initial capillary pressures $(\overline{p}_{m,0}, \overline{p}_{f,0})$ belong to $H^1(\Omega\setminus\overline{\Gamma}) \times L^2(\Gamma)$.

For $\varphi_\mu \in L^2((0,T)\times M_\mu)$ and a.e. $(t,\mathbf{x}) \in (0,T)\times M_\mu$, we let

$$
S^\alpha_\mu(\varphi_\mu)(t,\mathbf{x}) = S^\alpha_\mu(\mathbf{x},\varphi_\mu(t,\mathbf{x})) \quad\text{and}\quad [kS]^\alpha_\mu(\varphi_\mu)(t,\mathbf{x}) = k^\alpha_\mu(\mathbf{x}, S^\alpha_\mu(\mathbf{x},\varphi_\mu(t,\mathbf{x}))).
$$

The PDEs model writes: find phase pressures $(\overline{u}^\alpha_m, \overline{u}^\alpha_f)$ and velocities $(\mathbf{q}^\alpha_m, \mathbf{q}^\alpha_f)$ ($\alpha = 1,2$), such that

$$
\left\{
\begin{aligned}
\phi_m\partial_t S^\alpha_m(\overline{p}_m) + \operatorname{div}(\mathbf{q}^\alpha_m) &= h^\alpha_m && \text{on } (0,T)\times\Omega\setminus\overline{\Gamma}\\
\mathbf{q}^\alpha_m &= -[kS]^\alpha_m(\overline{p}_m)\ \Lambda_m\nabla\overline{u}^\alpha_m && \text{on } (0,T)\times\Omega\setminus\overline{\Gamma}\\
\phi_f d_f\partial_t S^\alpha_f(\overline{p}_f) + \operatorname{div}_\tau(\mathbf{q}^\alpha_f) - \sum_{\mathfrak{a}\in\chi} Q^\alpha_{f,\mathfrak{a}} &= d_f h^\alpha_f && \text{on } (0,T)\times\Gamma\\
\mathbf{q}^\alpha_f &= -d_f[kS]^\alpha_f(\overline{p}_f)\ \Lambda_f\nabla_\tau\overline{u}_f && \text{on } (0,T)\times\Gamma\\
(\overline{p}_m,\overline{p}_f)|_{t=0} &= (\overline{p}_{m,0},\overline{p}_{f,0}) && \text{on } (\Omega\setminus\overline{\Gamma})\times\Gamma,
\end{aligned}
\right.
\tag{1a}
$$

coupled with the matrix-fracture transmission conditions for all $\mathfrak{a} \in \chi$

$$\begin{cases} \mathbf{q}_m^\alpha \cdot \mathbf{n}_\mathfrak{a} + Q_{f,\mathfrak{a}}^\alpha = \eta \partial_t S_\mathfrak{a}^\alpha(\gamma_\mathfrak{a} \overline{p}_m) \\ \qquad\qquad Q_{f,\mathfrak{a}}^\alpha = [kS]_f^\alpha(\overline{p}_f) T_f [\![\overline{u}^\alpha]\!]_\mathfrak{a}^- - [kS]_\mathfrak{a}^\alpha(\gamma_\mathfrak{a}\overline{p}_m) T_f [\![\overline{u}^\alpha]\!]_\mathfrak{a}^+. \end{cases} \tag{1b}$$

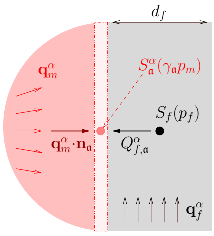

Fig. 1 Illustration of the coupling condition. It can be seen as an upwind two point approximation of $Q_{f,\mathfrak{a}}^\alpha$. The upwinding takes into account the damaged rocktype of porous thickness η at the matrix-fracture interfaces.

To give the weak formulation of this model, set $V^0 = V_m^0 \times V_f^0$ with

$$V_m^0 = \{v \in H^1(\Omega \backslash \overline{\Gamma}) \mid \gamma_{\partial\Omega} v = 0 \text{ on } \partial\Omega\},$$
$$V_f^0 = \{v \in H^1(\Gamma) \mid \gamma_{\partial\Gamma_i} v = 0 \text{ on } \partial\Gamma_i \cap \partial\Omega \text{ for all } i \in I\}.$$

The space $H^1(\Gamma)$ is made of functions whose restriction to each Γ_i belong to $H^1(\Gamma_i)$, and whose traces are continuous at fracture intersections $\partial\Gamma_i \cap \partial\Gamma_j$. Here, $\partial\Gamma_i$ is the boundary of Γ_i respective to the hyperplane containing Γ_i, and γ is the trace operator. We abridge $\sum_{\mu\in\{m,f\}}$, $\sum_{\mathfrak{a}\in\chi}$ and $\sum_{\alpha=1}^2$ into, respectively, $\sum_\mu$, $\sum_\mathfrak{a}$ and $\sum_\alpha$.

Definition 1 (Weak solution of the model). A weak solution of the model is $(\overline{u}_m^\alpha, \overline{u}_f^\alpha)_{\alpha=1,2} \in [L^2(0,T;V_m^0) \times L^2(0,T;V_f^0)]^2$ such that, for any $\alpha = 1,2$ and any $(\overline{\varphi}_m^\alpha, \overline{\varphi}_f^\alpha) \in C_0^\infty([0,T)\times\Omega) \times C_0^\infty([0,T)\times\Gamma)$,

$$\begin{aligned}
&\sum_\mu \Big(-\int_0^T \int_{M_\mu} \phi_\mu S_\mu^\alpha(\overline{p}_\mu) \partial_t \overline{\varphi}_\mu^\alpha \mathrm{d}\tau_\mu \mathrm{d}t + \int_0^T \int_{M_\mu} [kS]_\mu^\alpha(\overline{p}_\mu)\, \Lambda_\mu \nabla \overline{u}_\mu^\alpha \cdot \nabla \overline{\varphi}_\mu^\alpha \mathrm{d}\tau_\mu \mathrm{d}t \\
&\quad - \int_{M_\mu} \phi_\mu S_\mu^\alpha(\overline{p}_{\mu,0}) \overline{\varphi}_\mu^\alpha(0,\cdot) \mathrm{d}\tau_\mu \Big) + \sum_\mathfrak{a} \int_0^T \int_{\Gamma_\mathfrak{a}} \mathscr{F}(\gamma_\mathfrak{a}\overline{p}_m, \overline{p}_f, [\![\overline{u}^\alpha]\!]_\mathfrak{a}) [\![\overline{\varphi}^\alpha]\!]_\mathfrak{a} \mathrm{d}\tau \mathrm{d}t \\
&\quad - \sum_\mathfrak{a} \Big(\int_0^T \int_{\Gamma_\mathfrak{a}} \eta S_\mathfrak{a}^\alpha(\gamma_\mathfrak{a}\overline{p}_m) \partial_t \gamma_\mathfrak{a} \overline{\varphi}_m^\alpha \mathrm{d}\tau \mathrm{d}t + \int_{\Gamma_\mathfrak{a}} \eta S_\mathfrak{a}^\alpha(\gamma_\mathfrak{a}\overline{p}_{m,0}) \gamma_\mathfrak{a} \overline{\varphi}_m^\alpha(0,\cdot) \mathrm{d}\tau \Big) \\
&\qquad = \sum_\mu \int_0^T \int_{M_\mu} h_\mu^\alpha \overline{\varphi}_\mu^\alpha \mathrm{d}\tau_\mu,
\end{aligned} \tag{2}$$

where $\mathscr{F}(s_1,s_2,s_3) = T_f([kS]_\mathfrak{a}^\alpha(s_1)s_3^+ - [kS]_f^\alpha(s_2)s_3^-)$, $\mathrm{d}\tau_m(\mathbf{x}) = \mathrm{d}\mathbf{x}$ and $\mathrm{d}\tau_f(\mathbf{x}) = d_f(\mathbf{x})\mathrm{d}\tau(\mathbf{x})$ ($\mathrm{d}\tau$ being the $(d-1)$-dimensional measure on the fractures).

2 The gradient scheme

Definition 2 (Gradient Discretization (GD)). A spatial gradient discretisation for a DFN is $\mathscr{D}_S = (X^0, (\Pi_{\mathscr{D}_S}^\mu, \nabla_{\mathscr{D}_S}^\mu)_{\mu\in\{m,f\}}, ([\![\cdot]\!]_{\mathfrak{a},\mathscr{D}_S})_{\mathfrak{a}\in\chi}, (\mathbb{T}_{\mathscr{D}_S}^\mathfrak{a})_{\mathfrak{a}\in\chi})$, where

- X^0 is a finite dimensional space of degrees of freedom,
- $\Pi^\mu_{\mathscr{D}_S} : X^0 \to L^2(M_\mu)$ reconstructs a function on M_μ from the DOFs,
- $\nabla^\mu_{\mathscr{D}_S} : X^0 \to L^2(M_\mu)^{\dim M_\mu}$ reconstructs a gradient on M_μ from the DOFs,
- $[\![\cdot]\!]_{\mathfrak{a},\mathscr{D}_S} : X^0 \to L^2(\Gamma_\mathfrak{a})$ reconstructs, from the DOFs, a jump on $\Gamma_\mathfrak{a}$ between the matrix and fracture,
- $\mathbb{T}^\mathfrak{a}_{\mathscr{D}_S} : X^0 \to L^2(\Gamma_\mathfrak{a})$ reconstructs, from the DOFs, a trace on $\Gamma_\mathfrak{a}$ from the matrix.

Here, $\Pi^\mu_{\mathscr{D}_S}$ and $\mathbb{T}^\mathfrak{a}_{\mathscr{D}_S}$ are piecewise constant reconstructions in the sense of [2], which implies that if $g : \mathbb{R} \to \mathbb{R}$ then $\Pi^\mu_{\mathscr{D}_S} g(w) = g(\Pi^\mu_{\mathscr{D}_S} w)$ and $\mathbb{T}^\mathfrak{a}_{\mathscr{D}_S} g(w) = g(\mathbb{T}^\mathfrak{a}_{\mathscr{D}_S} w)$. $\mathscr{D}_S$ is extended into a space-time GD $\mathscr{D} = (\mathscr{D}_S, (\mathrm{I}^\mu_\mathscr{D})_{\mu\in\{m,f\}}, (t_n)_{n=0,\dots,N})$ with

- $0 = t_0 < t_1 < \cdots < t_N = T$ a discretisation of the time interval $[0,T]$,
- $\mathrm{I}^m_\mathscr{D} : H^1(\Omega \setminus \overline{\Gamma}) \to X^0$ and $\mathrm{I}^f_\mathscr{D} : L^2(\Gamma) \to X^0$ are operators designed to interpolate initial conditions.

The spatial operators are extended into space-time operators the following way. If $w = (w_n)_{n=0,\dots,N+1} \in (X^0)^{N+1}$, and $\Psi_{\mathscr{D}_S} = \Pi^\mu_{\mathscr{D}_S}$, $\nabla^\mu_{\mathscr{D}_S}$, $[\![\cdot]\!]_{\mathfrak{a},\mathscr{D}_S}$ or $\mathbb{T}^\mathfrak{a}_{\mathscr{D}_S}$, then $\Psi_\mathscr{D} w$ is defined on $[0,T] \times M_\mu$ or $[0,T] \times \Gamma_\mathfrak{a}$ by

$$\Psi_\mathscr{D} w(0,\cdot) = \Psi_{\mathscr{D}_S} u_0 \text{ and, } \forall n \in \{0,\dots,N-1\}\,,\ \forall t \in (t_n, t_{n+1}]\ \Psi_\mathscr{D} w(t,\cdot) = \Psi_{D_S} w_{n+1}.$$

We also define the discrete time derivative $\delta_t w : (0,T] \to X^0$ by, for the same n and t as above, $\delta_t w(t) = \frac{w_{n+1}-w_n}{t_{n+1}-t_n}$.

The gradient scheme for (2) is: find $(u^\alpha)_{\alpha=1,2} \in [(X^0)^{N+1}]^2$ such that, setting $p = u^1 - u^2$, we have $p_0 = (\mathrm{I}^m_\mathscr{D} \overline{p}_{m,0}, \mathrm{I}^f_\mathscr{D} \overline{p}_{f,0})$ and, for $\alpha = 1,2$ and $v^\alpha \in (X^0)^{N+1}$,

$$\begin{aligned}
&\sum_\mu \int_0^T \int_{M_\mu} \Big(\phi_\mu \Pi^\mu_\mathscr{D} \Big(\delta_t S^\alpha_\mu(p) \Big) \Pi^\mu_\mathscr{D} v^\alpha + [kS]^\alpha_\mu (\Pi^\mu_\mathscr{D} p)\ \Lambda_\mu \nabla^\mu_\mathscr{D} u^\alpha \cdot \nabla^\mu_\mathscr{D} v^\alpha \Big) \mathrm{d}\tau_\mu \mathrm{d}t \\
&+ \sum_\mathfrak{a} \Big(\int_0^T \int_{\Gamma_\mathfrak{a}} \mathscr{F}(\mathbb{T}^\mathfrak{a}_\mathscr{D} p, \Pi^f_\mathscr{D} p, [\![u^\alpha]\!]_{\mathfrak{a},\mathscr{D}}) [\![v^\alpha]\!]_{\mathfrak{a},\mathscr{D}} \mathrm{d}\tau \mathrm{d}t \\
&\quad + \int_0^T \int_{\Gamma_\mathfrak{a}} \eta \mathbb{T}^\mathfrak{a}_\mathscr{D} \Big(\delta_t S^\alpha_\mathfrak{a}(p) \Big) \mathbb{T}^\mathfrak{a}_\mathscr{D} v^\alpha \mathrm{d}\tau \mathrm{d}t \Big) = \sum_\mu \int_0^T \int_{M_\mu} h^\alpha_\mu \Pi^\mu_\mathscr{D} v^\alpha \mathrm{d}\tau_\mu \mathrm{d}t. \qquad (3)
\end{aligned}$$

3 Main result

Theorem 1. *Under the assumptions of Section 1, let* $(\mathscr{D}^l)_{l\in\mathbb{N}}$ *be a* coercive, consistent, limit-conforming *and* compact *sequence of space-time GD (see [3]), and let* $(u^{\alpha,l})_{l\in\mathbb{N}}$ *be such that* $u^{\alpha,l} \in (X^0_l)^{N_l+1}$ *is a sequence of solutions of* (3) *with* $\mathscr{D} = \mathscr{D}_l$. *Then, there exists a weak solution* $(\overline{u}^\alpha_m, \overline{u}^\alpha_f)_{\alpha=1,2}$ *of the model such that, for all* $\mu \in \{m,f\}$ *and* $\mathfrak{a} \in \chi$, $S_\mu(\overline{p}_\mu) : [0,T] \to L^2(M_\mu)$ *and* $S_\mathfrak{a}(\gamma_\mathfrak{a} \overline{p}_m) : [0,T] \to L^2(\Gamma_\mathfrak{a})$ *are continuous and, up to a subsequence as* $l \to \infty$, *with* $\overline{p} = \overline{u}^1 - \overline{u}^2$,

$$\Pi^{\mu}_{\mathscr{D}^l} S_\mu(p^l) \longrightarrow S_\mu(\overline{p}_\mu) \textit{ in } L^\infty(0,T;L^2(M_\mu)),$$
$$\mathbb{T}^{\mathfrak{a}}_{\mathscr{D}^l} S_{\mathfrak{a}}(p^l) \longrightarrow S_{\mathfrak{a}}(\gamma_{\mathfrak{a}}\overline{p}_m) \textit{ in } L^\infty(0,T;L^2(\Gamma_{\mathfrak{a}})).$$

Notations and preliminary results. Before proving this theorem, we recall some convergence results established in [3], under the assumptions of Theorem 1. Here, if $(w^l)_{l\in\mathbb{N}}$ is a sequence of functions in $L^2((0,T)\times M)$ for some measured space M, "$w^l \to w$ in L^2" means that the convergence holds in $L^2((0,T)\times M)$.

There exists a weak solution $\overline{u} = (\overline{u}_m, \overline{u}_f)$ such that, up to a subsequence as $l\to\infty$, for all $\mu \in \{m,f\}$ and $\mathfrak{a}\in\chi$, with $p = u^1 - u^2$ and $\overline{p}_\mu = \overline{u}^1_\mu - \overline{u}^2_\mu$,

$$\Pi^{\mu}_{\mathscr{D}^l} u^{\alpha,l} \rightharpoonup \overline{u}^\alpha_\mu\,,\ \nabla^{\mu}_{\mathscr{D}^l} u^{\alpha,l} \rightharpoonup \nabla\overline{u}^\alpha_\mu \text{ and } [\![u^{\alpha,l}]\!]_{\mathfrak{a},\mathscr{D}^l} \rightharpoonup [\![\overline{u}^\alpha]\!]_{\mathfrak{a}} \text{ weakly in } L^2, \tag{4}$$
$$\Pi^{\mu}_{\mathscr{D}^l} S_\mu(p^l) \to S_\mu(\overline{p}_\mu) \text{ and } \mathbb{T}^{\mathfrak{a}}_{\mathscr{D}^l} S_{\mathfrak{a}}(p^l) \to S_{\mathfrak{a}}(\gamma_{\mathfrak{a}}\overline{p}_m) \text{ strongly in } L^2. \tag{5}$$

The functions $S_\mu(\overline{p}_\mu) : [0,T] \to L^2(M_\mu)$ and $S_{\mathfrak{a}}(\gamma_{\mathfrak{a}}\overline{p}_m) : [0,T] \to L^2(\Gamma_{\mathfrak{a}})$ are continuous for the weak topologies of $L^2(M_\mu)$ and $L^2(\Gamma_{\mathfrak{a}})$, respectively. Moreover, for any $(T^l)_{l\in\mathbb{N}} \subset [0,T]$ that converges to some T^∞,

$$\begin{aligned}&\Pi^{\mu}_{\mathscr{D}^l} S_\mu(p^l)(T^l) \to S_\mu(\overline{p}_\mu)(T^\infty) \text{ weakly in } L^2(M_\mu), \text{ and}\\ &\mathbb{T}^{\mathfrak{a}}_{\mathscr{D}^l} S_{\mathfrak{a}}(p^l)(T^l) \to S_{\mathfrak{a}}(\gamma_{\mathfrak{a}}\overline{p}_m)(T^\infty) \text{ weakly in } L^2(\Gamma_{\mathfrak{a}}).\end{aligned} \tag{6}$$

There exists $\rho_{\mathfrak{a}} \in L^2((0,T)\times\Gamma_{\mathfrak{a}})$ such that

$$\mathscr{F}(\mathbb{T}^{\mathfrak{a}}_{\mathscr{D}^l} p^l, \Pi^{f}_{\mathscr{D}^l} p^l, [\![u^{\alpha,l}]\!]_{\mathfrak{a},\mathscr{D}^l}) \to \rho_{\mathfrak{a}} \text{ weakly in } L^2, \tag{7}$$

and, for all $\varphi \in [L^2(0,T;V^0_m)\times L^2(0,T;V^0_f)]^2$,

$$\sum_{\alpha,\mathfrak{a}} \int_0^T \int_{\Gamma_{\mathfrak{a}}} \rho_{\mathfrak{a}} [\![\overline{\varphi}^\alpha]\!]_{\mathfrak{a}} \mathrm{d}\tau\mathrm{d}t = \sum_{\alpha,\mathfrak{a}} \int_0^T \int_{\Gamma_{\mathfrak{a}}} \mathscr{F}(\gamma_{\mathfrak{a}}\overline{p}_m, \overline{p}_f, [\![\overline{u}^\alpha]\!]_{\mathfrak{a}}) [\![\overline{\varphi}^\alpha]\!]_{\mathfrak{a}} \mathrm{d}\tau\mathrm{d}t. \tag{8}$$

For $\rho = \mu \in \{m,f\}$ or $\rho = \mathfrak{a}\in\chi$, let $R_{S_\rho(\mathbf{x},\cdot)}$ be the range of $S_\rho(\mathbf{x},\cdot)$ and $[S_\rho(\mathbf{x},\cdot)]^i : R_{S_\rho(\mathbf{x},\cdot)} \to \mathbb{R}$ be its pseudo-inverse defined by

$$[S_\rho(\mathbf{x},\cdot)]^i(q) = \begin{cases} \inf\{z\in\mathbb{R}\,|\,S_\rho(\mathbf{x},z) = q\} & \text{if } q > S_\rho(\mathbf{x},0)\,,\\ 0 & \text{if } q = S_\rho(\mathbf{x},0)\,,\\ \sup\{z\in\mathbb{R}\,|\,S_\rho(\mathbf{x},z) = q\} & \text{if } q < S_\rho(\mathbf{x},0). \end{cases}$$

Let $B_\rho(\mathbf{x},\cdot) : \mathbb{R} \to [0,\infty]$ be given by $B_\rho(\mathbf{x},q) = \int_{S_\rho(\mathbf{x},0)}^q [S_\rho(\mathbf{x},\cdot)]^i(\tau)\mathrm{d}\tau$ if $q \in R_{S_\rho(\mathbf{x},\cdot)}$, $B_\rho(\mathbf{x},q) = +\infty$ otherwise. $B_\rho(\mathbf{x},\cdot)$ is convex l.s.c. and $B_\rho(\mathbf{x},S_\rho(\mathbf{x},\cdot))$ has a sub-quadratic growth: $B_\rho(\mathbf{x},S_\rho(\mathbf{x},r)) \le Kr^2$ for some K not depending on $\mathbf{x}$ or r.

The following continuous (based on [1, Lemma 3.6]) and discrete energy relations hold. For all $T_0 \in [0,T]$,

$$
\begin{aligned}
&\sum_\mu \int_{M_\mu} \phi_\mu \Big[B_\mu(S_\mu(\overline{p}_\mu)(T_0))\mathrm{d}\tau_\mu - \int_{M_\mu} \phi_\mu B_\mu(S_\mu(\overline{p}_\mu)(0))\Big]\mathrm{d}\tau_\mu \\
&+\sum_{\mathfrak{a}} \int_{\Gamma_{\mathfrak{a}}} \Big[\eta B_{\mathfrak{a}}(S_{\mathfrak{a}}(\gamma_{\mathfrak{a}}\overline{p}_m)(T_0))\mathrm{d}\tau_\mu - \int_{\Gamma_{\mathfrak{a}}} \eta B_{\mathfrak{a}}(S_{\mathfrak{a}}(\gamma_{\mathfrak{a}}\overline{p}_m)(0))\Big]\mathrm{d}\tau \\
&+\sum_{\alpha,\mu} \int_0^{T_0}\int_{M_\mu} [kS]_\mu^\alpha(\overline{p}_\mu)\Lambda_\mu \nabla \overline{u}_\mu^\alpha \cdot \nabla \overline{u}_\mu^\alpha \mathrm{d}\tau_\mu \mathrm{d}t \\
&+\sum_{\alpha,\mathfrak{a}} \int_0^{T_0}\int_{\Gamma_{\mathfrak{a}}} \mathscr{F}(\gamma_{\mathfrak{a}}\overline{p}_m, \overline{p}_f, [\![\overline{u}^\alpha]\!]_{\mathfrak{a}})[\![\overline{u}^\alpha]\!]_{\mathfrak{a}} \mathrm{d}\tau \mathrm{d}t = \sum_{\alpha,\mu}\int_0^{T_0}\int_{M_\mu} h_\mu^\alpha \overline{u}_\mu^\alpha \mathrm{d}\tau_\mu \mathrm{d}t \qquad (9)
\end{aligned}
$$

and, if k is chosen such that $T_0 \in (t_k, t_{k+1}]$,

$$
\begin{aligned}
&\sum_\mu \int_{M_\mu} \phi_\mu \Big[B_\mu(S_\mu(\Pi^\mu_{\mathscr{D}_S^l} p^l)(T_0)) - B_\mu(S_\mu(\Pi^\mu_{\mathscr{D}_S^l} p_0))\Big]\mathrm{d}\tau_\mu \\
&+\sum_{\mathfrak{a}} \int_{\Gamma_{\mathfrak{a}}} \eta \Big[B_{\mathfrak{a}}(S_{\mathfrak{a}}(\mathbb{T}^{\mathfrak{a}}_{\mathscr{D}_S^l} p^l)(T_0)) - B_{\mathfrak{a}}(S_{\mathfrak{a}}(\mathbb{T}^{\mathfrak{a}}_{\mathscr{D}_S^l} p_0))\Big]\mathrm{d}\tau \\
&+\sum_{\alpha,\mu} \int_0^{T_0}\int_{M_\mu} [kS]_\mu^\alpha(\Pi^\mu_{\mathscr{D}^l} p^l)\Lambda_\mu \nabla^\mu_{\mathscr{D}^l} u^{\alpha,l} \cdot \nabla^\mu_{\mathscr{D}^l} u^{\alpha,l} \mathrm{d}\tau_\mu \mathrm{d}t \\
&+\sum_{\alpha,\mathfrak{a}} \int_0^{T_0}\int_{\Gamma_{\mathfrak{a}}} \mathscr{F}(\mathbb{T}^{\mathfrak{a}}_{\mathscr{D}^l} p^l, \Pi^f_{\mathscr{D}^l} p^l, [\![u^{\alpha,l}]\!]_{\mathfrak{a},\mathscr{D}^l})[\![u^{\alpha,l}]\!]_{\mathfrak{a},\mathscr{D}^l} \mathrm{d}\tau \mathrm{d}t \\
&\qquad \le \sum_{\alpha,\mu}\int_0^{t_{k+1}}\int_{M_\mu} h_\mu^\alpha \Pi^\mu_{\mathscr{D}^l} u^{\alpha,l} \mathrm{d}\tau_\mu \mathrm{d}t. \qquad (10)
\end{aligned}
$$

Proof of Theorem 1. The proof follows the ideas initially introduced in [1]. By the characterisation [2, Lemma 4.8] of uniform-in-time convergence, it suffices to prove that, for any sequence $(T^l)_{l\in\mathbb{N}} \subset [0,T]$ converging to some T^∞,

$$
\begin{aligned}
&\Pi^\mu_{\mathscr{D}^l} S_\mu(p^l)(T^l) \to S_\mu(\overline{p}_\mu)(T^\infty) \text{ in } L^2(M_\mu), \\
&\mathbb{T}^{\mathfrak{a}}_{\mathscr{D}^l} S_{\mathfrak{a}}(p^l)(T^l) \to S_{\mathfrak{a}}(\gamma_{\mathfrak{a}}\overline{p}_m)(T^\infty) \text{ in } L^2(\Gamma_{\mathfrak{a}}).
\end{aligned} \qquad (11)
$$

Applying the discrete energy relation (10) to $T_0 = T^l$ yields

$$
\begin{aligned}
&\sum_\mu \int_{M_\mu} \phi_\mu B_\mu(S_\mu(\Pi^\mu_{\mathscr{D}_S^l} p^l)(T^l))\mathrm{d}\tau_\mu + \sum_{\mathfrak{a}}\int_{\Gamma_{\mathfrak{a}}} \eta B_{\mathfrak{a}}(S_{\mathfrak{a}}(\mathbb{T}^{\mathfrak{a}}_{\mathscr{D}_S^l} p^l)(T^l))\mathrm{d}\tau \\
&\le \int_{M_\mu} \phi_\mu B_\mu(S_\mu(\Pi^\mu_{\mathscr{D}_S^l} p_0))\mathrm{d}\tau_\mu + \sum_{\mathfrak{a}}\int_{\Gamma_{\mathfrak{a}}} \eta B_{\mathfrak{a}}(S_{\mathfrak{a}}(\mathbb{T}^{\mathfrak{a}}_{\mathscr{D}_S^l} p_0))\mathrm{d}\tau \\
&-\sum_{\alpha,\mu} \int_0^{T^l}\int_{M_\mu} [kS]_\mu^\alpha(\Pi^\mu_{\mathscr{D}^l} p^l)\Lambda_\mu \nabla^\mu_{\mathscr{D}^l} u^{\alpha,l} \cdot \nabla^\mu_{\mathscr{D}^l} u^{\alpha,l} \mathrm{d}\tau_\mu \mathrm{d}t \\
&-\sum_{\alpha,\mathfrak{a}} \int_0^{T^l}\int_{\Gamma_{\mathfrak{a}}} \mathscr{F}(\mathbb{T}^{\mathfrak{a}}_{\mathscr{D}^l} p^l, \Pi^f_{\mathscr{D}^l} p^l, [\![u^{\alpha,l}]\!]_{\mathfrak{a},\mathscr{D}^l})[\![u^{\alpha,l}]\!]_{\mathfrak{a},\mathscr{D}^l} \mathrm{d}\tau \mathrm{d}t
\end{aligned}
$$

$$+\sum_{\alpha,\mu}\int_0^{t_{k(l)+1}}\int_{M_\mu} h_\mu^\alpha \Pi_{\mathscr{D}^l}^\mu u^{\alpha,l}\mathrm{d}\tau_\mu \mathrm{d}t = \mathscr{A}_1+\mathscr{A}_2-\mathscr{A}_3-\mathscr{A}_4+\mathscr{A}_5. \qquad (12)$$

where $k(l)$ is such that $T^l \in (t_{k(l)}, t_{k(l)+1}]$. The consistency of $(\mathscr{D}^l)_{l\in\mathbb{N}}$ shows that $\Pi_{\mathscr{D}_S^l}^\mu p_0 = \Pi_{\mathscr{D}_S^l}^\mu \mathrm{I}_{\mathscr{D}^l}^\mu \overline{p}_{\mu,0} \to \overline{p}_\mu(0)$ in $L^2(M_\mu)$, $\mathbb{T}_{\mathscr{D}_S^l}^{\mathfrak{a}} p_0 = \mathbb{T}_{\mathscr{D}_S^l}^{\mathfrak{a}} \mathrm{I}_{\mathscr{D}^l}^m \overline{p}_{m,0} \to \gamma_{\mathfrak{a}}\overline{p}_m(0)$ in $L^2(\Gamma_{\mathfrak{a}})$. Since $B_\rho \circ S_\rho$ is sub-quadratic, we infer

$$\mathscr{A}_1+\mathscr{A}_2 \to \int_{M_\mu}\phi_\mu B_\mu(S_\mu(\overline{p}_\mu(0)))\mathrm{d}\tau_\mu + \sum_{\mathfrak{a}}\int_{\Gamma_{\mathfrak{a}}}\eta B_{\mathfrak{a}}(S_{\mathfrak{a}}(\gamma_{\mathfrak{a}}\overline{p}_m(0)))\mathrm{d}\tau. \qquad (13)$$

The convergence of $\mathscr{A}_5$ is trivial from the weak convergence of $\Pi_{\mathscr{D}}^\mu u^{\alpha,l}$ and the fact that $t_{k(l)+1} \to T^\infty$:

$$\mathscr{A}_5 \to \sum_{\alpha,\mu}\int_0^{T^\infty}\int_{M_\mu} h_\mu^\alpha \overline{u}_\mu^\alpha \mathrm{d}\tau_\mu \mathrm{d}t. \qquad (14)$$

Consider Lemma 1 applied to $F^l((t,\mathbf{x}),\xi) = \mathbf{1}_{(0,T^l)}(t)[kS]_\mu^\alpha(\Pi_{\mathscr{D}^l}^\mu p^l)(t,\mathbf{x})\Lambda_\mu(\mathbf{x})\xi$ and $W^l = \nabla_{\mathscr{D}^l}^\mu u^{\alpha,l}$. By (4) and (5), $W^l \to W := \nabla\overline{u}_\mu^\alpha$ weakly in $L^2((0,T)\times M_\mu)$ and, up to a subsequence, $\mathbf{1}_{(0,T^l)}\Pi_{\mathscr{D}^l}^\mu S_\mu(p^l)\Lambda_\mu \to \mathbf{1}_{(0,T^\infty)}[kS]_\mu^\alpha(\overline{p}_\mu)\Lambda_\mu$ a.e. on $M_\mu\times(0,T)$ while remaining bounded. Since F^l is monotonic with respect to its second argument, the assumptions of Lemma 1 are satisfied with $\rho = \mathbf{1}_{(0,T^\infty)}[kS]_\mu^\alpha(\overline{p}_\mu)\Lambda_\mu\nabla\overline{u}_\mu^\alpha$, and therefore

$$\liminf_{l\to\infty}\mathscr{A}_3 \geq \sum_{\alpha,\mu}\int_0^{T^\infty}\int_{M_\mu}[kS]_\mu^\alpha(\overline{p}_\mu)\Lambda_\mu\nabla\overline{u}_\mu^\alpha\cdot\nabla\overline{u}_\mu^\alpha\mathrm{d}\tau_\mu\mathrm{d}t. \qquad (15)$$

To study the limit of $\mathscr{A}_4$, we apply again Lemma 1, this time with $F^l((t,\mathbf{x}),\xi) = \mathscr{F}(\mathbb{T}_{\mathscr{D}^l}^{\mathfrak{a}} p^l(t,\mathbf{x}), \Pi_{\mathscr{D}^l}^f p^l(t,\mathbf{x}),\xi)$ and $W^l = [\![u^{\alpha,l}]\!]_{\mathfrak{a},\mathscr{D}^l}$. From the definition of $\mathscr{F}$ it can be readily checked that F^l is monotonic with respect to its first argument. Using therefore the strong convergences (5) of $S_{\mathfrak{a}}^\alpha(\mathbb{T}_{\mathscr{D}^l}^{\mathfrak{a}} p^l)$ and $S_\mu(\Pi_{\mathscr{D}^l}^f p^l)$, the weak convergence (4) of $[\![u^{\alpha,l}]\!]_{\mathfrak{a},\mathscr{D}^l}$ and the convergence property (7)–(8) of $F^l(\cdot,W^l) = \mathscr{F}(\mathbb{T}_{\mathscr{D}^l}^{\mathfrak{a}} p^l, \Pi_{\mathscr{D}^l}^f p^l, [\![u^{\alpha,l}]\!]_{\mathfrak{a},\mathscr{D}^l})$, the assumptions of Lemma 1 are satisfied and

$$\liminf_{l\to\infty}\mathscr{A}_4 \geq \sum_{\alpha,\mathfrak{a}}\int_0^T\int_{\Gamma_{\mathfrak{a}}}\mathscr{F}(\gamma_{\mathfrak{a}}\overline{p}_m,\overline{p}_f,[\![\overline{u}^\alpha]\!]_{\mathfrak{a}})[\![\overline{u}^\alpha]\!]_{\mathfrak{a}}\mathrm{d}\tau\mathrm{d}t. \qquad (16)$$

Gathering (13), (14), (15) and (16) into (12) and using the energy equality (9) yields

$$\begin{aligned}\limsup_{l\to\infty}&\Big(\sum_\mu\int_{M_\mu}\phi_\mu B_\mu(S_\mu(\Pi_{\mathscr{D}_S^l}^\mu p^l)(T^l))\mathrm{d}\tau_\mu + \sum_{\mathfrak{a}}\int_{\Gamma_{\mathfrak{a}}}\eta B_{\mathfrak{a}}(S_{\mathfrak{a}}(\mathbb{T}_{\mathscr{D}_S^l}^{\mathfrak{a}} p^l)(T^l))\mathrm{d}\tau\Big)\\ &\leq \sum_\mu\int_{M_\mu}\phi_\mu B_\mu(S_\mu(\overline{p}_\mu)(T^\infty))\mathrm{d}\tau_\mu + \sum_{\mathfrak{a}}\int_{\Gamma_{\mathfrak{a}}}\eta B_{\mathfrak{a}}(S_{\mathfrak{a}}(\overline{p}_f)(T^\infty))\mathrm{d}\tau. \qquad (17)\end{aligned}$$

On the other hand, the weak L^2 convergences (6) and the fact that the functions B_ρ are convex lower semi-continuous give, by [1, Lemma 3.4],

$$\sum_\mu \int_{M_\mu} \phi_\mu B_\mu(S_\mu(\overline{p}_\mu)(T^\infty))\mathrm{d}\tau_\mu \leq \liminf_{l\to\infty} \sum_\mu \int_{M_\mu} \phi_\mu B_\mu(S_\mu(\Pi^\mu_{\mathscr{D}^l_S} p^l)(T^l))\mathrm{d}\tau_\mu \quad (18)$$

$$\sum_\mathfrak{a} \int_{\Gamma_\mathfrak{a}} \eta B_\mathfrak{a}(S_\mathfrak{a}(\overline{p}_f)(T^\infty))\mathrm{d}\tau \leq \liminf_{l\to\infty} \sum_\mathfrak{a} \int_{\Gamma_\mathfrak{a}} \eta B_\mathfrak{a}(S_\mathfrak{a}(\mathbb{T}^\mathfrak{a}_{\mathscr{D}^l_S} p^l)(T^l))\mathrm{d}\tau. \quad (19)$$

Combining (17), (18) and (19) yields, by [2, Lemma 4.33],

$$\sum_\mu \int_{M_\mu} \phi_\mu B_\mu(S_\mu(\overline{p}_\mu)(T^\infty))\mathrm{d}\tau_\mu = \lim_{l\to\infty} \sum_\mu \int_{M_\mu} \phi_\mu B_\mu(S_\mu(\Pi^\mu_{\mathscr{D}^l_S} p^l)(T^l))\mathrm{d}\tau_\mu$$

$$\sum_\mathfrak{a} \int_{\Gamma_\mathfrak{a}} \eta B_\mathfrak{a}(S_\mathfrak{a}(\overline{p}_f)(T^\infty))\mathrm{d}\tau = \lim_{l\to\infty} \sum_\mathfrak{a} \int_{\Gamma_\mathfrak{a}} \eta B_\mathfrak{a}(S_\mathfrak{a}(\mathbb{T}^\mathfrak{a}_{\mathscr{D}^l_S} p^l)(T^l))\mathrm{d}\tau.$$

The proof of (11), and thus of Theorem 1, is then completed using the exact same reasoning as in [1, Section 4.3]. □

Lemma 1 (Weak Fatou by monotonicity). *Let $k \geq 1$, M be a measured space, and let $(F^l)_{l\in\mathbb{N}}$ be Caratheodory functions $M \times \mathbb{R}^k \to \mathbb{R}^k$ such that, for a.e. $\mathbf{z} \in M$ and all $\xi, \eta \in \mathbb{R}^k$, $[F^l(\mathbf{z},\xi) - F^l(\mathbf{z},\eta)] \cdot [\xi - \eta] \geq 0$. Let $(W^l)_{l\in\mathbb{N}}$ such that, as $l \to \infty$, $W^l \rightharpoonup W$ weakly in $L^2(M)^k$, $(F^l(\cdot,W))_{l\in\mathbb{N}}$ converges strongly in $L^2(M)^k$, and $F^l(\cdot,W^l) \rightharpoonup \rho$ weakly in $L^2(M)^k$. Then $\int_M \rho(\mathbf{z}) \cdot W(\mathbf{z})\mathrm{d}\mathbf{z} \leq \liminf_{l\to\infty} \int_M F^l(\mathbf{z},W^l(\mathbf{z})) \cdot W^l(\mathbf{z})\mathrm{d}\mathbf{z}$.*

Proof. We have $[F^l(\mathbf{z},W^l) - F^l(\mathbf{z},W)] \cdot [W^l - W] \geq 0$. Integrate and develop:

$$0 \leq \int_M F^l(\mathbf{z},W^l) \cdot W^l \mathrm{d}\mathbf{z} - \int_M F^l(\mathbf{z},W^l) \cdot W \mathrm{d}\mathbf{z} + \int_M F^l(\mathbf{z},W) \cdot \Big[W^l - W\Big]\mathrm{d}\mathbf{z}. \quad (20)$$

The last term goes to 0 by strong convergence of $F^l(\cdot,W)$ and weak convergence of W^l. By weak convergence of $F^l(\cdot,W^l)$, the second term goes to $\int_M \rho \cdot W$. The proof is concluded by taking the inferior limit of (20).

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