Multirate Symbolic Models for Incrementally Stable Switched Systems

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Abstract: Methods for computing approximately bisimilar symbolic models for incrementally stable switched systems are usually based on discretization of time and space, where the value of time and space sampling parameters must be carefully chosen in order to achieve a desired precision. This often results in symbolic models that have a very large number of transitions, especially when the time sampling, and thus the space sampling parameters are small. In this paper, we present an approach to the computation of symbolic models for switched systems using multirate time sampling, where the period of symbolic transitions is a multiple of the control (i.e. switching) period. We show that multirate symbolic models are approximately bisimilar to the original incrementally stable switched system. The main contribution of the paper is the explicit determination of the optimal sampling ratio between transition and control periods, which minimizes the number of transitions in the symbolic model. Interestingly, this optimal sampling ratio is mainly determined by the state space dimension and the number of modes of the switched system. Finally, an illustration of the proposed approach is shown for the boost DC-DC converter, which shows the benefit of multirate symbolic models.

Keywords: Approximate bisimulation, switched systems, symbolic control, multirate sampling, incremental stability.

1. INTRODUCTION

Hybrid systems are dynamical systems combining both discrete and continuous behaviors and for which verification and control problems cannot be solved by the classical tools of continuous control. For this reason, over recent years, several studies focused on the use of discrete abstractions of hybrid systems, also called symbolic models (see Tabuada (2009) and the refences therein). The main advantage of these approaches is that they enable the use of existing control techniques developed in the areas of supervisory control of discrete event systems (Cassandras and Lafortune (2009)) and algorithmic game theory (Bloem et al. (2012)).

The symbolic models and the original system are often related by some approximate equivalence relationships such as approximate bisimulation (Girard and Pappas (2007)) or alternating approximate bisimulation (Pola and Tabuada (2009)). In particular, it was shown that approximately bisimilar discrete abstractions are computable for several classes of incrementally stable systems including nonlinear systems with or without disturbances (Pola et al. (2008), Pola and Tabuada (2009)) or switched systems with or without dwell-time assumption (Girard et al. (2010)).

In this paper, we deal with incrementally stable switched systems, for which symbolic models can be computed by discretizing time and space. It was shown in Girard et al. (2010), that discrete abstractions of arbitrary precision can be obtained by carefully choosing time and space sampling parameters. However, for a given precision, the choice of a small time sampling parameter imposes to choose a small space sampling parameter resulting in symbolic models with a prohibitively large number of transitions. This constitutes a limiting factor of the approach because the size of the symbolic models is a crucial factor for computational efficiency of symbolic controller synthesis.

The main idea of this paper is to reduce the size of the symbolic model by using a multirate sampling. Multirate sampling has been used in the area of discrete-time systems to face some of the sampling processes disadvantages (see e.g. Monaco and Normand-Cyrot (1991)). In this paper, we present an approach to the computation of multirate symbolic models for switched systems, where the period of symbolic transitions is a multiple of the control (i.e. switching) period. We show that multirate symbolic models are approximately bisimilar to the original incrementally stable switched system. A similar approach has been explored in the symbolic control literature in the context of nonlinear digital control systems (Majumdar and Zamani (2012)). The main contribution of the current paper is the explicit determination of the optimal sampling ratio between transition and control periods which minimizes the number of transitions in the symbolic model. Interestingly, this optimal sampling ratio is mainly determined by the state space dimension and the number of modes of the switched system.

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This paper is organized as follows. In Section 2, we introduce the notions of transitions systems, approximate bisimulation and incrementally stable switched systems. In Section 3, we present the construction of symbolic models for incrementally stable switched systems using multirate sampling. In Section 4, we establish the optimal sampling ratio between control and transition periods which minimizes the number of transitions in the symbolic model. Finally, in Section 5, we illustrate our approach using the boost DC-DC converter, which shows the benefits of the proposed multirate symbolic models.

Notations: $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{N}^+$ denote the sets of integers of non-negative integers and of positive integers, respectively. $\mathbb{R}, \mathbb{R}_0^+$ and $\mathbb{R}^+$ denote the sets of real, of non-negative real and of positive real numbers, respectively. For $s \in \mathbb{R}_0^+$, $\lfloor s \rfloor$ denote its integer part, i.e. the largest nonnegative integer $r \in \mathbb{N}$ such that $r \leq s$. For $x \in \mathbb{R}^n$, $x[i]$ denotes its i-th coordinate, $i = 1, \ldots, n$; $\|x\|$ denotes the Euclidean norm of x. A continuous function $\gamma : \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\gamma(0) = 0$; γ is said to belong to class $\mathcal{K}_\infty$ if γ is $\mathcal{K}$ and $\gamma(r) \to \infty$ as $r \to \infty$. A continuous function $\beta : \mathbb{R}_0^+ \times \mathbb{R}_0^+ \to \mathbb{R}_0^+$ is said to belong to class $\mathcal{KL}$ if, for each fixed s, the map $\beta(\cdot, s)$ belongs to class $\mathcal{K}$, and for each fixed nonzero r, the map $\beta(r, \cdot)$ is strictly decreasing and $\beta(r, s) \to 0$ as $s \to \infty$.

2. PRELIMINARIES

2.1 Transition systems

We present the notion of transition systems, which allows us to describe in a unified framework the switched system and its symbolic model.

Definition 1. A *transition system* is a tuple $T = (X, U, Y, \Delta, X^0)$ consisting of:

- a set of states X;
- a set of inputs U;
- a set of outputs Y;
- a transition relation $\Delta \subseteq X \times U \times X \times Y$;
- a set of initial states $X^0 \subseteq X$.

T is said to be *metric* if the set of outputs Y is equipped with a metric d, *symbolic* if X and U are finite or countable sets.

The transition $(x, u, x', y) \in \Delta$ will be denoted $(x', y) \in \Delta(x, u)$ and means that the system can evolve from state x to state x' under the action of input u, while producing output y. An input $u \in U$ belongs to the set of *enabled* inputs at state $x \in X$, denoted $Enab(x)$, if $\Delta(x, u) \neq \emptyset$. T is said to be *deterministic* if for all $x \in X$ and for all $u \in Enab(x)$, $\Delta(x, u)$ consists of a unique element. State $x \in X$ is said to be *blocking* if $Enab(x) = \emptyset$, otherwise it is said to be *non-blocking.* A *trajectory* of the transition system is a finite or infinite sequence of transitions $\sigma = (x^0, u^0, y^0)(x^1, u^1, y^1)\,(x^2, u^2, y^2)\ldots$ where $(x^{i+1}, y^i) \in \Delta(x^i, u^i)$, for $i \geq 0$. It is *initialized* if $x^0 \in X^0$. A state $x \in X$ is reachable if there exists an initialized trajectory such that $x^i = x$, for some $i \geq 0$. The transition system is said to be *non-blocking* if all reachable states are non-blocking. The *output behavior* associated to the trajectory σ is the sequence of outputs $y^0 y^1 y^2 \ldots$

In this paper, we consider the approximation relationship for transition systems based on the notion of approximate bisimulation (Girard and Pappas (2007)), which requires that the distance between the output behaviors of two transition systems remains bounded by some specified precision. The following definition is taken from Girard et al. (2016) and generalizes that of Girard and Pappas (2007) to accommodate the encoding of the output map within the transition relation.

Definition 2. Let $T_1 = (X_1, U, Y, \Delta_1, X_1^0)$ and $T_2 = (X_2, U, Y, \Delta_2, X_2^0)$ be two metric transition systems with the same input set U and the same output set Y equipped with a metric d. Let $\varepsilon \geq 0$ be a given precision. A relation $R \subseteq X_1 \times X_2$ is said to be an *ε-approximate bisimulation relation* between T_1 and T_2 if for all $(x_1, x_2) \in R$, $u \in U$:

$$\begin{aligned}\forall (x_1', y_1) \in \Delta_1(x_1, u), &\exists (x_2', y_2) \in \Delta_2(x_2, u),\\ &d(y_1, y_2) \leq \varepsilon \text{ and } (x_1', x_2') \in R;\\ \forall (x_2', y_2) \in \Delta_2(x_2, u), &\exists (x_1', y_1) \in \Delta_1(x_1, u),\\ &d(y_1, y_2) \leq \varepsilon \text{ and } (x_1', x_2') \in R.\end{aligned}$$

The transition systems T_1 and T_2 are said to be *ε-approximately bisimilar*, denoted $T_1 \sim_\varepsilon T_2$, if:

- $\forall x_1 \in X_1^0$, $\exists x_2 \in X_2^0$, such that $(x_1, x_2) \in R$;
- $\forall x_2 \in X_2^0$, $\exists x_1 \in X_1^0$, such that $(x_1, x_2) \in R$.

The approximate bisimulation relation guarantees that for each output behavior of T_1 (respectively of T_2), there exists an output behavior of T_2 (respectively of T_1) such that the distance between these output behaviors is uniformly bounded by ε.

2.2 Incrementally stable switched systems

We introduce the class of systems that we consider in this paper:

Definition 3. A *switched system* is a quadruple $\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F)$, where:

- $\mathbb{R}^n$ is the state space;
- $P = \{1, \ldots, m\}$ is the finite set of modes;
- $\mathcal{P}$ is a subset of $\mathcal{S}(\mathbb{R}_0^+, P)$ which denotes the set of piecewise constant functions from $\mathbb{R}_0^+$ to P, continuous from the right and with a finite number of discontinuities on every bounded interval of $\mathbb{R}_0^+$ (which guarantees the absence of Zeno behaviors);
- $F = \{f_1, \ldots, f_m\}$ is a collection of vector fields indexed by P.

For all $p \in P$, we denote by Σ_p the continuous *subsystem* of Σ defined by the differential equation:

$$\dot{\mathbf{x}}(t) = f_p(\mathbf{x}(t)). \tag{1}$$

We make the assumption that the vector field f_p is locally Lipschtiz and forward complete, so that solutions of (1) are uniquely defined for all $t \in \mathbb{R}_0^+$. Necessary and sufficient conditions for a system to be forward complete can be found in Angeli and Sontag (1999).

A *switching signal* of Σ is a function $\mathbf{p} \in \mathcal{P}$, the discontinuities of $\mathbf{p}$ are called *switching times.* A piecewise $\mathcal{C}^1$ function $\mathbf{x} : \mathbb{R}_0^+ \to \mathbb{R}^n$ is said to be a *trajectory* of Σ if it is continuous and there exists a switching signal $\mathbf{p} \in \mathcal{P}$ such that, at each $t \in \mathbb{R}_0^+$ where the function $\mathbf{p}$ is continuous, $\mathbf{x}$ is continuously differentiable and satisfies:

$$\dot{\mathbf{x}}(t) = f_{\mathbf{p}(t)}(\mathbf{x}(t)).$$

We will denote $\mathbf{x}(t, x, \mathbf{p})$ the point reached at time $t \in \mathbb{R}_0^+$ from the initial condition x under the switching signal $\mathbf{p}$. We will denote $\mathbf{x}(t, x, p)$ the point reached by Σ at time $t \in \mathbb{R}_0^+$ from the initial condition x under the constant switching signal $\mathbf{p}(t) = p$, for all $t \in \mathbb{R}_0^+$. For all p, we have $\mathbf{x}(t, x, p) = \phi_t^p(x)$ where ϕ_t^p is the flow associated to the vector field f_p.

The construction of the approximately bisimilar symbolic models of switched systems are generally based on the notion of incremental stability. In Angeli (2002), incremental global asymptotic stability (δ-GAS) is characterized using Lyapunov functions for nonlinear systems. An extension of this result to the class of switched systems was presented in Girard et al. (2010).

Definition 4. A switched system Σ is *incrementally globally uniformly asymptotically stable* (δ-GUAS) if there exists a $\mathcal{KL}$ function β such that for all $t \in \mathbb{R}_0^+$, for all $x, y \in \mathbb{R}^n$ and for all switching signal $\mathbf{p} \in \mathcal{P}$, the following condition is satisfied:

$$\|\mathbf{x}(t, x, \mathbf{p}) - \mathbf{x}(t, y, \mathbf{p})\| \leq \beta(\|x - y\|, t).$$

Intuitively, incremental stability means that all trajectories associated to the same switching signal converge to the same trajectory, independently of their initial conditions. The notion of incremental stability for switched systems can also be proved using dissipation inequalities as follows:

Definition 5. : A smooth function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_0^+$ is a *common δ-GUAS Lyapunov function* for Σ if there exist $\mathcal{K}_\infty$ functions $\underline{\alpha}$, $\overline{\alpha}$ and $\kappa \in \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^n$, for all $p \in P$,

$$\underline{\alpha}(\|x - y\|) \leq V(x, y) \leq \overline{\alpha}(\|x - y\|); \quad (2)$$

$$\frac{\partial V}{\partial x}(x, y) f_p(x) + \frac{\partial V}{\partial y}(x, y) f_p(y) \leq -\kappa V(x, y). \quad (3)$$

Theorem 1. (Girard et al. (2010)). Consider a switched system $\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F)$ with a common δ-GUAS Lyapunov function, then Σ is δ-GUAS.

In the following, and in order to construct symbolic models for the switched systems, we shall make the sumplementary assumption that there exists a $\mathcal{K}_\infty$ function γ such that:

$$\forall x, y, z \in \mathbb{R}^n, \ |V(x, y) - V(x, z)| \leq \gamma(\|y - z\|). \quad (4)$$

Remark 1. In Girard et al. (2010), it is shown that this assumption is verified, if we are interested in the dynamics of the switched system on a compact set $C \subseteq \mathbb{R}^n$ and V is $\mathcal{C}^1$ on C. Then, we have for all $x, y, z \in C$,

$$|V(x, y) - V(x, z)| \leq c_\gamma \|y - z\| \text{ with } c_\gamma = \max_{x,y \in C} \left\| \frac{\partial V}{\partial y}(x, y) \right\|.$$

Then, (4) holds for the linear $\mathcal{K}_\infty$ function given by $\gamma(r) = c_\gamma r$, regardless of the linear or nonlinear nature of the system dynamics.

Remark 2. For all $x \in \mathbb{R}^n$ we have $V(x, x) = 0$, then for all $x, y \in \mathbb{R}^n$ we have:

$$V(x, y) \leq |V(x, y) - V(x, x)| \leq \gamma(\|x - y\|)$$

Then, there is no loss of generality in assuming that the second inequality in (2) holds with $\overline{\alpha} = \gamma$.

3. MULTIRATE SYMBOLIC MODELS

3.1 Multirate sampling of switched systems

In this paper we focus on switched systems $\Sigma = (\mathbb{R}^n, P, \mathcal{P}, F)$ for which the switching is periodically controlled with control period $\tau \in \mathbb{R}^+$ (i.e. $\mathcal{P}$ is the set of switching signals whose switching times occur at multiples of the period τ). Given a switched system Σ and a control period τ, we define the transition system $T_\tau(\Sigma) = (X, U, Y, \Delta_\tau, X^0)$ as follows:

- The set of states is $X = \mathbb{R}^n$;
- The set of inputs is $U = P$;
- The set of outputs $Y = \mathbb{R}^n$;
- The transition relation is given for $x, x' \in X$, $u \in U$, $y \in Y$, by $(x', y) \in \Delta_\tau(x, u)$ if and only if
$$x' = \phi_\tau^u(x) \text{ and } y = x;$$
- the set of initial states $X^0 = \mathbb{R}^n$.

$T_\tau(\Sigma)$ is deterministic, non-blocking and metric when the set of outputs Y is equipped with Euclidean metric given by $d(y, y') = \|y - y'\|$.

In the previous transition system, the period of transitions coincides with the control period τ. In this paper, we deal with more general multirate sampling where the period of transitions is a multiple $r\tau$ of the control period τ where the *sampling ratio* $r \in \mathbb{N}^+$. We thus define the multirate transition system $T_{\tau,r}(\Sigma) = (X, U_r, Y_r, \Delta_{\tau,r}, X^0)$ where:

- The set of states is $X = \mathbb{R}^n$;
- The set of inputs is $U_r = P^r$;
- The set of outputs $Y_r = \mathbb{R}^{n \times r}$;
- The transition relation is given for $x, x' \in X$, $u \in U_r$ with $u = (p_1, \ldots, p_r)$, $y \in Y_r$, by $(x', y) \in \Delta_{\tau,r}(x, u)$ if and only if
$$\begin{aligned} x' &= \phi_\tau^{p_r} \circ \phi_\tau^{p_{r-1}} \circ \ldots \circ \phi_\tau^{p_1}(x) \text{ and} \\ y &= \left(x, \phi_\tau^{p_1}(x), \ldots, \phi_\tau^{p_{r-1}} \circ \ldots \circ \phi_\tau^{p_1}(x)\right); \end{aligned}$$
- the set of initial states $X^0 = \mathbb{R}^n$.

$T_{\tau,r}(\Sigma)$ is deterministic, non-blocking and metric when the set of outputs Y_r is equipped with the following metric d_r:

$$\forall y = (y_1, \ldots, y_r), y' = (y'_1, \ldots, y'_r) \in Y_r, \quad d_r(y, y') = \max_{i \in \{1,\ldots,r\}} \|y_i - y'_i\|. \quad (5)$$

Let us remark that for $r = 1$, $T_{\tau,r}(\Sigma)$ coincides with $T_\tau(\Sigma)$. When $r \neq 1$, the following result shows that $T_\tau(\Sigma)$ and $T_{\tau,r}(\Sigma)$ produce equivalent output behaviors.

Proposition 1. For any output behavior $(y^0, y^1, y^2, \ldots)$ of $T_\tau(\Sigma)$, there exists an output behavior $(y_r^0, y_r^1, y_r^2, \ldots)$ of $T_{\tau,r}(\Sigma)$ with $y_r^i = (z_{r,1}^i, \ldots, z_{r,r}^i)$ for $i \geq 0$ such that

$$\forall i \geq 0, j = 1, \ldots, r, \ z_{r,j}^i = y^{ir+j-1}. \quad (6)$$

Conversely, for any output behavior $(y_r^0, y_r^1, y_r^2, \ldots)$ of $T_{\tau,r}(\Sigma)$ with $y_r^i = (z_{r,1}^i, \ldots, z_{r,r}^i)$ for $i \geq 0$, there exists an output behavior $(y^0, y^1, y^2, \ldots)$ of $T_\tau(\Sigma)$ such that (6) holds.

Proof. Consider a trajectory $\sigma = (x^0, u^0, y^0)(x^1, u^1, y^1)$ $(x^2, u^2, y^2)\dots$ of $T_\tau(\Sigma)$ and let us consider the trajectory $\sigma_r = (x_r^0, u_r^0, y_r^0)(x_r^1, u_r^1, y_r^1)$ $(x_r^2, u_r^2, y_r^2)\dots$ of $T_{\tau,r}(\Sigma)$ with $x_r^0 = x^0$ and $u_r^i = (u^{ir}, \dots, u^{ir+r-1})$ for $i \geq 0$. Then by construction of $T_\tau(\Sigma)$ and $T_{\tau,r}(\Sigma)$, we have that (6) holds. The proof of the converse result comes similarly. □

Remark 3. Using $T_\tau(\Sigma)$ or $T_{\tau,r}(\Sigma)$ for the purpose of synthesis provides identical guarantees on the sampled behavior (with period τ) of the switched system, since the output behaviors of both transition systems are equivalent. However, it leads to different implementations of switching controllers. For controllers synthesized using $T_{\tau,r}(\Sigma)$, the sensing and actuation period are equal to τ; while for controllers synthesized using $T_{\tau,r}(\Sigma)$, the actuation period remains equal to τ when the sensing period is equal to $r\tau$. In the latter case, at sensing instants, the controller selects a sequence of r modes, each of which is actuated for a duration τ.

3.2 Construction of symbolic models

For an incrementally stable switched system Σ, a construction of symbolic models that are approximately bisimilar to $T_\tau(\Sigma)$ has been proposed in Girard et al. (2010), based on a discretization of the state-space $\mathbb{R}^n$. Theorem 4.1 in that paper, shows that symbolic models of arbitrary precision can be computed by using a sufficiently fine discretization of the state-space. However, this usually results in symbolic models that have a very large number of transitions, especially when the control period τ is small.

In this section, we establish a similar result for the multirate transition system $T_{\tau,r}(\Sigma)$. This idea is inspired by the work presented in Majumdar and Zamani (2012), in which symbolic models are computed for digital control systems using multirate sampling. Our results can be seen as an extension to the class of switched systems. In addition, in the following section, we will provide a theoretical analysis allowing us to choose the optimal sampling ratio r, minimizing the number of transitions in the symbolic model, which is not available in Majumdar and Zamani (2012).

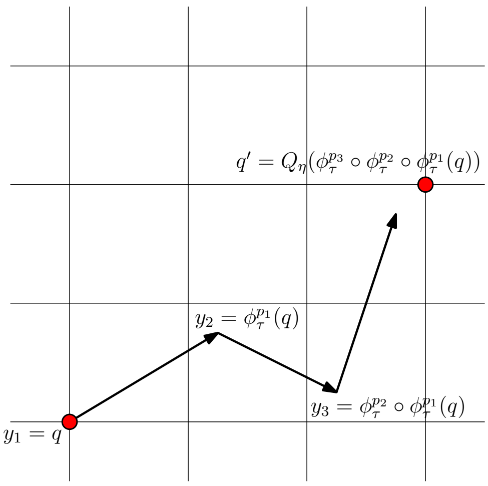

Fig. 1. A transition $(q', y) \in \Delta_{\tau,r}^\eta(q, u)$ of the multirate symbolic model $T_{\tau,r}^\eta(\Sigma)$ with $r = 3$, $u = (p_1, p_2, p_3)$ and $y = (y_1, y_2, y_3)$.

Let $\eta \in \mathbb{R}^+$ be a space sampling parameter, the set of states $\mathbb{R}^n$ is approximated by the lattice:

$$[\mathbb{R}^n]_\eta = \frac{2\eta}{\sqrt{n}}\mathbb{Z}^n.$$

We associate a quantizer $Q_\eta : \mathbb{R}^n \longrightarrow [\mathbb{R}^n]_\eta$ given by $Q_\eta(x) = q$ if and only if

$$\forall i = 1, \dots, n,\ q[i] - \frac{\eta}{\sqrt{n}} \leq x[i] < q[i] + \frac{\eta}{\sqrt{n}}.$$

We can easily show that for all $x \in \mathbb{R}^n$, $\|Q_\eta(x) - x\| \leq \eta$.

Let us then define the transition system $T_{\tau,r}^\eta(\Sigma) = (X^\eta, U_r, Y_r, \Delta_{\tau,r}^\eta, X^{\eta,0})$ as follows:

- The set of states is $X^\eta = [\mathbb{R}^n]_\eta$;
- The set of inputs is $U_r = P^r$;
- The set of outputs $Y_r = \mathbb{R}^{n\times r}$;
- The transition relation is given for $q, q' \in X^\eta$, $u \in U_r$ with $u = (p_1, \dots, p_r)$, $y \in Y_r$, by $(q', y) \in \Delta_{\tau,r}^\eta(q, u)$ if and only if
$$\begin{aligned} q' &= Q_\eta\big(\phi_\tau^{p_r} \circ \phi_\tau^{p_{r-1}} \circ \dots \circ \phi_\tau^{p_1}(q)\big) \text{ and} \\ y &= \big(q, \phi_\tau^{p_1}(q), \dots, \phi_\tau^{p_{r-1}} \circ \dots \circ \phi_\tau^{p_1}(q)\big); \end{aligned}$$
- the set of initial states $X^{\eta,0} = [\mathbb{R}^n]_\eta$.

$T_{\tau,r}^\eta(\Sigma)$ is symbolic, deterministic, non-blocking and metric when the set of outputs Y_r is equipped with the metric d_r given by (5). The construction of the symbolic transition relation is illustrated in Figure 1. We can now state the following approximation result:

Theorem 2. Consider a switched system Σ, and let us assume that there exists a common δ-GUAS Lyapunov function V for Σ such that (4) holds for some $\mathcal{K}_\infty$ function γ. Let time and space sampling parameters $\tau, \eta \in \mathbb{R}^+$, sampling ratio $r \in \mathbb{N}^+$ and precision $\varepsilon \in \mathbb{R}^+$ satisfy:

$$\eta \leq \gamma^{-1}\left((1 - e^{-r\kappa\tau})\underline{\alpha}(\varepsilon)\right) \quad (7)$$

then, the transition systems $T_{\tau,r}(\Sigma)$ and $T_{\tau,r}^\eta(\Sigma)$ are ε-approximately bisimilar.

Proof. We start by proving that the relation R defined by:

$$R = \{(x, q) \in X \times X^\eta |\ V(x, q) \leq \underline{\alpha}(\varepsilon)\}$$

is an ε-approximate bisimulation relation between $T_{\tau,r}(\Sigma)$ and $T_{\tau,r}^\eta(\Sigma)$.

Let $(x, q) \in R$, $u \in U_r$ with $u = (p_1, \dots, p_r)$, and $(x', y) \in \Delta_{\tau,r}(x, u)$, then $x' = \phi_\tau^{p_r} \circ \dots \circ \phi_\tau^{p_1}(x)$. Let $(q', z) \in \Delta_{\tau,r}^\eta(q, u)$, then $\|\phi_\tau^{p_r} \circ \dots \circ \phi_\tau^{p_1}(q) - q'\| \leq \eta$. It follows from equation (4) that

$$|V(x', q') - V(x', \phi_\tau^{p_r} \circ \dots \circ \phi_\tau^{p_1}(q))| \leq \gamma(\eta).$$

It follows that

$$\begin{aligned} V(x', q') &\leq V(x', \phi_\tau^{p_r} \circ \dots \circ \phi_\tau^{p_1}(q)) + \gamma(\eta) \\ &\leq V(\phi_\tau^{p_r} \circ \dots \circ \phi_\tau^{p_1}(x), \phi_\tau^{p_r} \circ \dots \circ \phi_\tau^{p_1}(q)) + \gamma(\eta) \\ &\leq e^{-r\kappa\tau}V(x, q) + \gamma(\eta) \\ &\leq e^{-r\kappa\tau}\underline{\alpha}(\varepsilon) + \gamma(\eta) \\ &\leq \underline{\alpha}(\varepsilon) \end{aligned}$$

where the third inequality comes from (3), the fourth inequality comes from the fact that $(x, q) \in R$ and the fifth inequality comes from (7). Thus, $(x', q') \in R$.

In addition, we have by definition of the transition relations that

$$y = \big(x, \phi_\tau^{p_1}(x), \ldots, \phi_\tau^{p_{r-1}} \circ \ldots \circ \phi_\tau^{p_1}(x)\big),$$
$$z = \big(q, \phi_\tau^{p_1}(q), \ldots, \phi_\tau^{p_{r-1}} \circ \ldots \circ \phi_\tau^{p_1}(q)\big).$$

Then, by (2) and since $(x, q) \in R$, we have

$$\|x - q\| \leq \underline{\alpha}^{-1}\big(V(x, q)\big) \leq \varepsilon.$$

Moreover, for $i = 1, \ldots, r-1$, by (2), (3) and since $(x, q) \in R$, we have

$$\begin{aligned}&\|\phi_\tau^{p_i} \circ \ldots \circ \phi_\tau^{p_1}(x) - \phi_\tau^{p_i} \circ \ldots \circ \phi_\tau^{p_1}(q)\| \\ &\leq \underline{\alpha}^{-1}\big(V(\phi_\tau^{p_i} \circ \ldots \circ \phi_\tau^{p_1}(x), \phi_\tau^{p_i} \circ \ldots \circ \phi_\tau^{p_1}(q))\big) \\ &\leq \underline{\alpha}^{-1}\big(V(x, q)\big) \leq \varepsilon.\end{aligned}$$

It follows that $d_r(y, z) \leq \varepsilon$ and the first condition of the definition (2) holds.

In a similar way, we can prove that for all $(q', z) \in \Delta^\eta_{\tau,r}(q, u)$ there exists $(x', y) \in \Delta_{\tau,r}(x, u)$ such that $(x', q') \in R$ and $d_r(y, z) \leq \varepsilon$. Therefore, R is an ε-approximate bisimulation relation between $T_{\tau,r}(\Sigma)$ and $T^\eta_{\tau,r}(\Sigma)$.

Now, let $x \in X^0 = \mathbb{R}^n$, and $q \in X^{\eta,0} = [\mathbb{R}^n]_\eta$, given by $q = Q_\eta(x)$, then $\|x - q\| \leq \eta$. Following Remark 2, we assume the second inequality of (2) holds with $\overline{\alpha} = \gamma$. It follows that

$$V(x, q) \leq \gamma(\|x - q\|) \leq \gamma(\eta) \leq \underline{\alpha}(\varepsilon)$$

where the last inequality comes from (7). Hence $(x, q) \in R$. Conversely, for all $q \in X^{\eta,0} = [\mathbb{R}^n]_\eta$, let $x \in X^0 = \mathbb{R}^n$, given by $x = q$, then $V(x, q) = 0$ and $(x, q) \in R$. Therefore, $T_{\tau,r}(\Sigma)$ and $T^\eta_{\tau,r}(\Sigma)$ are ε-approximately bisimilar. □

Remark 4. From the previous Theorem and using the fact that $\overline{\alpha} = \gamma$, we can recover when $r = 1$, the original approximation result given in Theorem 4.1 of Girard et al. (2010).

Some remarks regarding the size of the symbolic models are in order. It appears from (7) that using larger sampling ratio $r \in \mathbb{N}^+$ allows us to use larger values of $\eta \in \mathbb{R}^+$ and thus coarser discretizations of the state space. This results in symbolic models with fewer symbolic states. However, the number of transitions initiating from a symbolic state is m^r and thus grows exponentially with the sampling ratio. Hence, the advantage of using multirate symbolic models in terms of number of transitions in the symbolic model is yet unclear. This issue is addressed in the following section, where we determine the optimal value of the sampling ratio.

4. OPTIMAL SAMPLING RATIO

In the following, we consider multirate symbolic models $T^\eta_{\tau,r}(\Sigma)$, where we restrict the set of states to some compact set $C \subseteq \mathbb{R}^n$ with nonempty interior. The number of symbolic states in $X^\eta \cap C$ is then accurately estimated by $\frac{v_C}{\eta^n}$, where $v_C \in \mathbb{R}^+$ is a positive constant proportional to the volume of C. Then, the total number of symbolic transitions initiating from states in $X^\eta \cap C$ is $v_C \frac{m^r}{\eta^n}$. We assume in the following that the number of modes $m \geq 2$.

4.1 Problem formulation

In this section, given a desired precision $\varepsilon \in \mathbb{R}^+$, and a control period $\tau \in \mathbb{R}^+$, we establish the optimal values $r^* \in \mathbb{N}^+$ and $\eta^* \in \mathbb{R}^+$, which minimizes the number of symbolic transitions while satisfying (7). Since, C is a compact set, following Remark 1, we assume that (4) holds for a linear $\mathcal{K}_\infty$ function γ given by $\gamma(r) = c_\gamma r$ where $c_\gamma \in \mathbb{R}^+$. Thus, we aim at solving the following mixed integer nonlinear program:

$$\begin{array}{ll}\text{Minimize} & v_C \dfrac{m^r}{\eta^n} \\ \text{over} & r \in \mathbb{N}^+,\ \eta \in \mathbb{R}^+ \\ \text{under} & \eta \leq (1 - e^{-r\kappa\tau}) \dfrac{\underline{\alpha}(\varepsilon)}{c_\gamma}\end{array} \tag{8}$$

Let us first remark that for a given $r \in \mathbb{N}^+$, the optimal value $\eta \in \mathbb{R}^+$ is obviously obtained as $\eta = (1 - e^{-r\kappa\tau}) \frac{\underline{\alpha}(\varepsilon)}{c_\gamma}$. It follows that (8) is equivalent to the following integer program:

$$\begin{array}{ll}\text{Minimize} & v_C \dfrac{c_\gamma^n}{(\underline{\alpha}(\varepsilon))^n} \dfrac{m^r}{(1 - e^{-r\kappa\tau})^n} \\ \text{over} & r \in \mathbb{N}^+\end{array} \tag{9}$$

The value $v_C \frac{c_\gamma^n}{(\underline{\alpha}(\varepsilon))^n} \in \mathbb{R}^+$ does not depend on r and thus does not affect the solution of (9), which can finally be equivalently formulated as:

$$\begin{array}{ll}\text{Minimize} & g(r) = \dfrac{m^r}{(1 - e^{-r\kappa\tau})^n} \\ \text{over} & r \in \mathbb{N}^+\end{array} \tag{10}$$

A first interesting information that comes from (10) is that the optimal sampling ratio only depends on the control period $\tau \in \mathbb{R}^+$, the dimension of the state-space $n \in \mathbb{N}^+$, the number of modes $m \in \mathbb{N}^+$ and the decay rate $\kappa \in \mathbb{R}^+$ of the δ-GUAS Lyapunov function. In particular, it is noteworthy that it is independent of the desired precision $\varepsilon \in \mathbb{R}^+$ and of the compact set C.

4.2 Explicit solution

In this section, we show that the previous optimization problems can be solved explicitly. We first consider the relaxation of the integer program (10) over the positive real numbers:

Lemma 1. Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be given as in (10). Then, g has a unique minimizer $\tilde{r}^* \in \mathbb{R}^+$ given by

$$\tilde{r}^* = \frac{1}{\kappa\tau} \ln\left(1 + \frac{n\kappa\tau}{\ln(m)}\right). \tag{11}$$

Moreover, g is strictly decreasing on $(0, \tilde{r}^*]$ and strictly increasing on $[\tilde{r}^*, +\infty)$.

Proof. Let us compute the first order derivative of g:

$$\begin{aligned}g'(r) &= \frac{1}{(1 - e^{-r\kappa\tau})^{2n}} \Big(\ln(m) m^r (1 - e^{-r\kappa\tau})^n \\ &\qquad - m^r n\kappa\tau e^{-r\kappa\tau} (1 - e^{-r\kappa\tau})^{n-1}\Big) \\ &= \frac{m^r}{(1 - e^{-r\kappa\tau})^{n+1}} \Big(\ln(m)(1 - e^{-r\kappa\tau}) - n\kappa\tau e^{-r\kappa\tau}\Big) \\ &= \frac{\ln(m) m^r}{(1 - e^{-r\kappa\tau})^{n+1}} \left(1 - e^{-r\kappa\tau}\left(1 + \frac{n\kappa\tau}{\ln(m)}\right)\right).\end{aligned}$$

By remarking that $\frac{\ln(m) m^r}{(1 - e^{-r\kappa\tau})^{n+1}} > 0$ for all $r \in \mathbb{R}^+$, it is easy to see that $1 - e^{-r\kappa\tau}\big(1 + \frac{n\kappa\tau}{\ln(m)}\big)$ and thus $g'(r)$ is negative on $(0, \tilde{r}^*)$, zero at $\tilde{r}^*$ and positive on $(\tilde{r}^*, +\infty)$. The result stated in Lemma 1 follows immediately. □

We can now state our main result:

Theorem 3. For any desired precision $\varepsilon \in \mathbb{R}^+$, and any control period $\tau \in \mathbb{R}^+$, the optimal parameters $r^* \in \mathbb{N}^+$ and $\eta^* \in \mathbb{R}^+$, solutions of (8), which minimize the number of symbolic transitions of $T^{\eta}_{\tau,r}(\Sigma)$, initiating from states in $X^{\eta} \cap C$, while satisfying (7), are given by

$$r^* = \lfloor \tilde{r}^* \rfloor \text{ or } r^* = \lfloor \tilde{r}^* \rfloor + 1 \tag{12}$$

$$\text{and } \eta^* = (1 - e^{-r^*\kappa\tau})\frac{\underline{\alpha}(\varepsilon)}{c_\gamma} \tag{13}$$

where $\tilde{r}^*$ is given by (11).

Proof. From Lemma 1, it follows that

$$\forall r \in \mathbb{N}^+, \text{ with } r < \lfloor \tilde{r}^* \rfloor, \; g(r) > g(\lfloor \tilde{r}^* \rfloor)$$

and

$$\forall r \in \mathbb{N}^+, \text{ with } r > \lfloor \tilde{r}^* \rfloor + 1, \; g(r) > g(\lfloor \tilde{r}^* \rfloor + 1).$$

Then, it folllows that the minimal value of g over $\mathbb{N}^+$ is obtained for $r^* = \lfloor \tilde{r}^* \rfloor$ or $r^* = \lfloor \tilde{r}^* \rfloor + 1$. Then, from the discussions in Section 4.1, it follows that the solution of (8) is given by r^* and $\eta^* = (1 - e^{-r^*\kappa\tau})\frac{\underline{\alpha}(\varepsilon)}{c_\gamma}$. □

In practice, we compute the optimal parameters of the multirate symbolic models by evaluating the function g at $\lfloor \tilde{r}^* \rfloor$ and $\lfloor \tilde{r}^* \rfloor + 1$. We then pick the one out of two values r^*, which minimizes g and compute η^* using (9).

We would like to point out that the previous result can be applied to either linear or nonlinear switched systems. The only requirement is that we restrict the analysis to a compact subset of $\mathbb{R}^n$. Finally, it is interesting to remark that for small values of the control period $\tau \in \mathbb{R}^+$, the optimal sampling ratio r^* is mainly determined by the state space dimension and the number of modes.

Corollary 1. There exists $\overline{\tau} \in \mathbb{R}^+$, such that for any desired precision $\varepsilon \in \mathbb{R}^+$, and any control period $\tau \in (0, \overline{\tau}]$, the optimal parameters $r^* \in \mathbb{N}^+$ and $\eta^* \in \mathbb{R}^+$, solutions of (8), which minimize the number of symbolic transitions of $T^{\eta}_{\tau,r}(\Sigma)$, initiating from states in $X^{\eta} \cap C$, while satisfying (7), are given by

$$r^* = \left\lfloor \frac{n}{\ln(m)} \right\rfloor \text{ or } r^* = \left\lfloor \frac{n}{\ln(m)} \right\rfloor + 1$$

$$\text{and } \eta^* = (1 - e^{-r^*\kappa\tau})\frac{\underline{\alpha}(\varepsilon)}{c_\gamma}.$$

Proof. Let $\overline{\tau}$ be given by

$$\overline{\tau} = \frac{2\ln(m)}{n\kappa}\left(1 - \frac{\left\lfloor \frac{n}{\ln(m)} \right\rfloor}{\frac{n}{\ln(m)}}\right). \tag{14}$$

From Theorem 2.2 in Baker (1990), we have that for all $n, m \in \mathbb{N}^+$ with $m \geq 2$, $\frac{n}{\ln(m)} \in \mathbb{R}^+ \setminus \mathbb{N}^+$. Then, it follows that $\lfloor \frac{n}{\ln(m)} \rfloor < \frac{n}{\ln(m)}$ and that $\overline{\tau} > 0$.

Now, let us remark that for all $\theta \in \mathbb{R}^+$, we have that $\theta(1 - \frac{\theta}{2}) \leq \ln(1 + \theta) \leq \theta$. Let $\tilde{r}^*$ be given by (11), then it follows from the previous inequalities that for all $\tau \in \mathbb{R}^+$.

$$\frac{n}{\ln(m)}\left(1 - \frac{n\kappa\tau}{2\ln(m)}\right) \leq \tilde{r}^* \leq \frac{n}{\ln(m)}.$$

Then, using (14), it follows that for all $\tau \in (0, \overline{\tau}]$,

$$\left\lfloor \frac{n}{\ln(m)} \right\rfloor \leq \tilde{r}^* \leq \frac{n}{\ln(m)}$$

which implies that $\lfloor \tilde{r}^* \rfloor = \lfloor \frac{n}{\ln(m)} \rfloor$. The stated result is then a consequence of Theorem 3. □

5. ILLUSTRATING EXAMPLE

As an illustration, we consider a boost DC-DC converter described by a two-dimensional switched affine system with two modes (i.e. $n = 2$, $m = 2$) and given by

$$\dot{\mathbf{x}}(t) = A_{\mathbf{p}(t)}\mathbf{x}(t) + b$$

with $x(t) = [i_l(t)\; v_c(t)]^T$, $b = [\frac{v_s}{x_l}\; 0]^T$, and

$$A_1 = \begin{bmatrix} -\frac{r_l}{x_l} & 0 \\ 0 & -\frac{1}{x_c}\frac{1}{r_0 + r_c} \end{bmatrix}, \; A_2 = \begin{bmatrix} -\frac{1}{x_l}(r_l + \frac{r_0 r_c}{r_0 + r_c}) & -\frac{1}{x_l}\frac{r_0}{r_0 + r_c} \\ \frac{1}{x_c}\frac{r_0}{r_0 + r_c} & -\frac{1}{x_c}\frac{1}{r_0 + r_c} \end{bmatrix}.$$

In the following, we use the numerical values from Beccuti et al. (2005), expressed in the per-unit system: $x_c = 70$, $x_l = 3$, $r_c = 0.005$, $r_l = 0.05$, $r_0 = 1$ and $v_s = 1$. For a better numerical conditioning, we rescaled the second variable of the system, the new state becomes $x(t) = [i_l(t)\; 5v_c(t)]^T$; (the matrices A_1, A_2 and vector b are modified accordingly). It has been shown in Girard et al. (2010) that this switched systems admits a common δ-GUAS Lyapunov function of the form $V(x, y) = \sqrt{((x - y)^T P(x - y))}$ with

$$P = \left[\begin{smallmatrix} 1.0224 & 0.0084 \\ 0.0084 & 1.0031 \end{smallmatrix}\right].$$

Then, equations (2), (3) and (4) hold with $\underline{\alpha}(s) = s$, $\overline{\alpha}(s) = 1.0127s$, $\kappa = 0.014$ and $\gamma(s) = 1.0127s$.

We compute multirate symbolic models using the approach described in Section 3. We set the control period $\tau = 0.5$ and the desired precision $\varepsilon = 0.025$. We restrict the dynamics to a compact subset of $\mathbb{R}^2$ given by $C = [1.3, 1.5] \times [5.65, 5.75]$. We compute the symbolic models for several sampling ratios $r = 1, \ldots, 10$, the space sampling parameter is then chosen as $\eta = (1 - e^{-r\kappa\tau})\frac{\underline{\alpha}(\varepsilon)}{c_\gamma}$. Figure 2 shows the number of symbolic transitions as a function of r and we can see that this number is minimal for $r = 3$.

Using (14), we compute $\overline{\tau} = 15.19$. Thus, $\tau \in (0, \overline{\tau}]$ and the assumptions of Corollary 1 hold. In particular, since

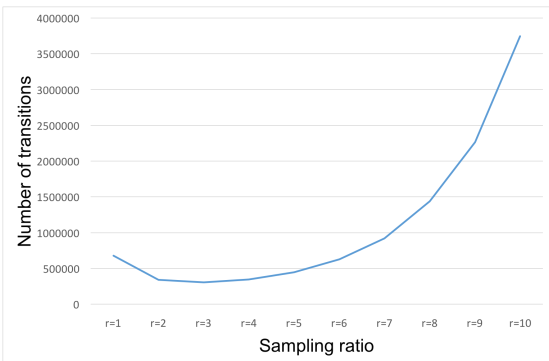

Fig. 2. Number of symbolic transitions in the multirate symbolic models for different values of the sampling ratio r.

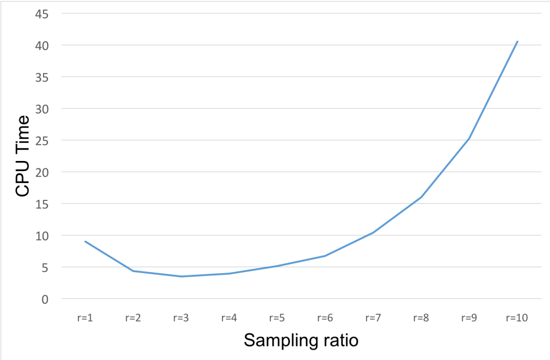

Fig. 3. Computation times in seconds for generating the multirate symbolic models and synthesizing controllers for different values of the sampling ratio r (Implementation in MATLAB, Processor 2.7 GHz Intel Core i5, Memory 8 GB 1867 MHz DDR3).

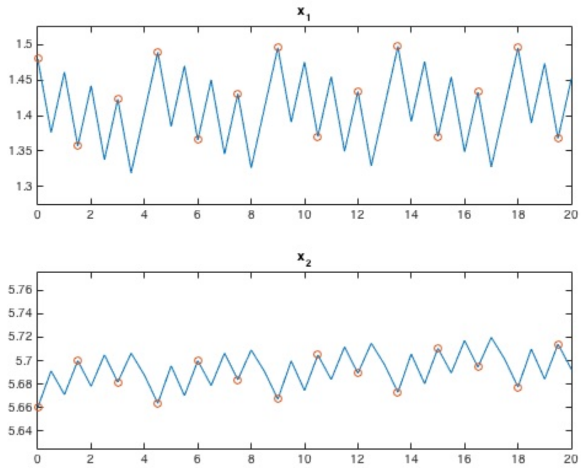

Fig. 4. Trajectory of the switched system controlled with the symbolic controller for the initial state $x^0 = [1.48\ 5.66]^T$. The control period is $\tau = 0.5$ while the transition period is $3\tau = 1.5$ (instants of transitions are indicated with circles).

$\frac{n}{\ln(m)} = 2.89$, the optimal sampling ratio is either 2 or 3. We can then check numerically that $g(3) < g(2)$ where g is given by (10). This provides us with the optimal sampling ratio $r^* = 3$, which is consistent with the experimental data.

We now synthesize safety controllers (see e.g. Tabuada (2009)), which keep the output of the symbolic models inside the compact region C. Figure 3 reports the computation times for generating symbolic models and synthesizing controllers for $r = 1, \ldots, 10$. We can check that using the optimal sampling ratio $r = 3$ allows us to reduce the computation times by more than 50% in comparison to the classical approach corresponding to $r = 1$. For $r = 3$, Figure 4 shows a trajectory of the switched system controlled with the symbolic controller for the initial state $x^0 = [1.48\ 5.66]^T$.

6. CONCLUSION

In this paper, we have proposed the use of multirate sampling for the computation of symbolic abstractions for incrementally stable switched systems, and show that our technique allows us to produce symbolic models with fewer transitions than the existing approaches. The optimal sampling ratio has been determined theoretically and validated experimentally on the boost DC-DC converter example, which shows that multirate symbolic models enables controller synthesis at a reduced computational cost. In a future work, we will extend this approach to the class of switched systems with multiple Lyapunov functions and a dwell time.

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