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Self-Triggered Control for Sampled-data Systems using Reachability Analysis ⋆

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Abstract: In this work, we design the sampling policy in sampled-data systems. It is known that implementing such systems using variable sampling periods, instead of a constant period, is more efficient in terms of performance and resource utilization. Thus, after rewriting the system in the framework of impulsive linear systems, a self-triggered control strategy obtained using reachability analysis is proposed in order to define the sampling period as a function of the state.

Keywords: Reachability; Invariance; Impulsive systems; Self triggered control; Sampled-data systems;

1. INTRODUCTION

Technologies based on integrating digital controllers within physical systems are becoming more pervasive (intelligent buildings and cars, advanced manufacturing plants, smart medical devices, etc.). The interaction between the two corresponding cyber and physical worlds defines the scope of this work. More precisely, we analyze and design the behavior of a sampler in such cyber-physical systems where the instants at which sampling occurs strongly influence the stability and performance of the overall system. Given the dynamics of the system and the control law, the simplest strategy for a sampler to work is to sample periodically with a fixed sampling period (time-triggered sampling). Alternatively, this period could vary so that sampling occurs only when needed. In fact, implementing sampled-data systems using variable sampling periods is proved to be more efficient in terms of performance and resource utilization [Tabuada (2007); Donkers and Heemels (2012); Fiter et al. (2012)]. In literature two frameworks define the latter strategy: Event-triggered [Tabuada (2007)] and Self-triggered [Mazo et al. (2009); Fiter et al. (2012)]. The first control strategy requires dedicated hardware to continuously monitor the state of the plant and calls for sampling whenever it is necessary. On the other hand, the second strategy emulates the first one but requires to know the state just at the sampling instants and thus results in less intensive on-line computations.

This work proposes a self-triggered control strategy, obtained using reachability analysis, in order to define the sampling period as a function of the state. In other words, we define, using off-line computations, a fixed set of sampling periods as well as their associated regions of the state space. Then in real-time and at each sampling instant, the next sampling period is chosen from the fixed pool depending on which predefined region the state of the system lies in.

Our contribution is mainly based on a previous work [Al Khatib et al. (2016)]. Therein, we rely on reachability tools [Le Guernic (2009)] to compute contracting sets [Blanchini (1991, 1999)] for sampled-data systems in order to verify stability for all sampling periods defined within a lower and upper-bound. In this paper, we make use of these contracting sets and design a map from the state-space to a set of sampling periods in order to enlarge the upper-bound found earlier while guaranteeing stability and satisfying, in terms of performance, a specific decay rate.

The rest of the paper is organized as follows. First, some preliminary notations are defined before formulating the self-triggered control problem in Section 2. The main results are discussed in Section 3. Applications on sampled-data control systems and comparisons with existing results are then discussed, before concluding our work. An appendix is added at the end to present the reachability-based over-approximation scheme from [Al Khatib et al. (2016)] and the proofs of all the stated results.

Notations Let $\mathbb{R}$, $\mathbb{R}_0^+$, $\mathbb{R}^+$, $\mathbb{N}$, $\mathbb{N}^+$ denote the sets of reals, nonnegative reals, positive reals, nonnegative integers and positive integers, respectively. For $I \subseteq \mathbb{R}_0^+$, let $\mathbb{N}_I = \mathbb{N} \cap I$. Let $|.|$ be a norm on $\mathbb{R}^n$, and let $\mathcal{B}$ denote the associated unit ball. Given a real matrix $A \in \mathbb{R}^{n\times n}$, $|A|$ is the norm of A induced by the norm $|.|$. Given $\mathcal{S} \subseteq \mathbb{R}^n$ and a real matrix $A \in \mathbb{R}^{n\times n}$, the set $A\mathcal{S} = \{x \in \mathbb{R}^n : (\exists y \in \mathcal{S} : x = Ay)\}$; for $a \in \mathbb{R}$, $a\mathcal{S} = (aI_n)\mathcal{S}$ where I_n is the $n \times n$ identity matrix. The convex hull of $\mathcal{S}$ is denoted by $\mathrm{ch}(\mathcal{S})$. The interior of $\mathcal{S}$ is denoted $\mathrm{int}(\mathcal{S})$. We denote the set of all subsets of $\mathbb{R}^n$ by $2^{\mathbb{R}^n}$. We denote by $\mathcal{B}_0(\mathbb{R}^n)$ the set of bounded subsets of $\mathbb{R}^n$ containing 0 in their interior.

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For any $\mathcal{S} \in \mathcal{B}_0(\mathbb{R}^n)$, there exist $\underline{c}, \overline{c} \in \mathbb{R}^+$ such that $\underline{c}\mathcal{B} \subseteq \mathcal{S} \subseteq \overline{c}\mathcal{B}$. A polytope $\mathcal{P}$ is a subset of $\mathbb{R}^n$ which can be defined as the intersection of a finite number of closed half-spaces, that is $\mathcal{P} = \{x \in \mathbb{R}^n : Hx \leq 1\}$ where $H \in \mathbb{R}^{r\times n}$ and the vector of inequalities is interpreted component-wise.

2. PROBLEM FORMULATION

2.1 Sampled-data systems

We consider sampled-data systems given in the form:

$$\dot{z}(t) = Az(t) + Bu(t) \tag{1}$$

$$u(t) = Kz(t_k) \quad t_k < t \leq t_{k+1}, \tag{2}$$

where $z \in \mathbb{R}^p$ is the state of the system, $u \in \mathbb{R}^m$ is the control input, $A \in \mathbb{R}^{p\times p}$, $B \in \mathbb{R}^{p\times m}$, and the static feedback gain is $K \in \mathbb{R}^{m\times p}$. The sequence of sampling instants $(t_k)_{k\in\mathbb{N}}$ is increasing and divergent with $t_0 = 0$. The initial state is given as $z(0) = z_0$. We assume that K is designed such that the matrix $A + BK$ is Hurwitz.

2.2 Impulsive system reformulation

In order to develop stability conditions based on reachability analysis, it is more convenient to write (1-2) in the form of impulsive systems. Now let $e(t) = z(\alpha(t)) - z(t)$ with $\alpha(t) = t_k$ for $t_k < t \leq t_{k+1}$, then we have:

$$\begin{aligned} \dot{x}(t) &= A_c x(t) \quad t \neq t_k \\ x(t_k^+) &= A_r x(t_k), \end{aligned} \tag{3}$$

where

$$A_c = \begin{pmatrix} A+BK & BK \\ -A-BK & -BK \end{pmatrix}, A_r = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, x(t) = \begin{pmatrix} z(t) \\ e(t) \end{pmatrix}.$$

We then have $A_c, A_r \in \mathbb{R}^{n\times n}$ and the state of the impulsive system (3) $x \in \mathbb{R}^n$, with $n = 2p$ and $x(0) = x_0$.

Letting $\Delta_k = t_{k+1} - t_k$ for all $k \in N$, we define:

$$\Delta_k \in [\underline{h}, \theta(x(t_k))], \ \forall k \in \mathbb{N}, \tag{4}$$

with $\theta : \mathbb{R}^n \to \mathbb{R}^+$ and $\underline{h} \in \mathbb{R}^+$ a given lower bound on Δ to avoid Zeno phenomena.

Stability of (3-4) is given in the following sense:

Definition 1. (β^*-stability). Let $\beta \in \mathbb{R}^+$, system (3-4) is β^*-stable if there exist $C \in \mathbb{R}^+$ and $\varepsilon^* \in \mathbb{R}^+$ such that:

$$|x(t)| \leq Ce^{-(\beta+\varepsilon^*)t}|x_0|, \ \forall t \in \mathbb{R}^+. \tag{5}$$

This paper provides a solution to the following problem:

Problem 1. (Self-triggered control). Given $A_c, A_r \in \mathbb{R}^{n\times n}$, $\underline{h} \in \mathbb{R}^+$, and $\beta \in \mathbb{R}^+$ as a performance measure, define a strategy (4) that renders (3) β^*-stable while enlarging $\theta(x(t_k))$, for all $k \in \mathbb{N}$.

3. SELF-TRIGGERED CONTROL SYNTHESIS

In this section we propose an approach to solve Problem 1. Our approach is divided into two distinct parts. Primarily, we fix the value of $\theta(x)$, for all $x \in \mathbb{R}^n$, to a given value $\overline{h} > \underline{h}$ and compute a contracting polytope that ensures β^*-stability of (3-4) with $\theta(x(t_k)) = \overline{h}$, for all $k \in \mathbb{N}$. Next, we use the computed set to enlarge $\theta(x)$, in (4), based on the position of x in the state space.

3.1 Finding the Contracting set

Let us define the map $\Phi : 2^{\mathbb{R}^n} \to 2^{\mathbb{R}^n}$, given for all $\mathcal{S} \subseteq \mathbb{R}^n$ and $h, h' \in \mathbb{R}^+$ with $h \leq h'$ by

$$\Phi_{[h,h']}(\mathcal{S}) = \bigcup_{\tau\in[h,h']} e^{\tau(A_c+\beta I)} A_r \mathcal{S}. \tag{6}$$

We define the iterates of Φ as $\Phi^0_{[h,h']}(\mathcal{S}) = \mathcal{S}$ for all $\mathcal{S} \subseteq \mathbb{R}^n$, and $\Phi^{k+1} = \Phi \circ \Phi^k$ for all $k \in \mathbb{N}$. The next proposition establishes some properties of Φ.

Proposition 2. Let $h, h' \in \mathbb{R}^+$, $a \in \mathbb{R}$, and $\mathcal{S} \subset \mathbb{R}^n$. The map Φ satisfies the following properties:

(a) $\Phi_{[h,h']}(a\mathcal{S}) = a\Phi_{[h,h']}(\mathcal{S})$.
(b) $\Phi_{[h,h']}(\mathcal{S}) = \bigcup_{x\in\mathcal{S}} \Phi_{[h,h']}(\{x\})$.
(c) If $\mathcal{S}$ is bounded, so is $\Phi_{[h,h']}(\mathcal{S})$.
(d) $\Phi_{[h,h']}(\mathrm{ch}(\mathcal{S})) \subseteq \mathrm{ch}(\Phi_{[h,h']}(\mathcal{S}))$.

The idea to synthesize a contracting polytope

$$\mathcal{P} = \{x \in \mathbb{R}^n : Hx \leq 1\}, H \in \mathbb{R}^{r\times n}, \tag{7}$$

for (3-4), with $\theta(x(t_k)) = \overline{h}$ for all $k \in \mathbb{N}$, is inspired from the following lemma which defines in theory an explicit form of $\mathcal{P}$ whenever the system is β^*-stable. Note that this result is based on Proposition 2 and on Corollary 5 in [Al Khatib et al. (2016)].

Lemma 3. Let $\mathcal{S} \in \mathcal{B}_0(\mathbb{R}^n)$, $\beta \in \mathbb{R}^+$, $\underline{h} \in \mathbb{R}^+$, and $\overline{h} \in \mathbb{R}^+$. System (3-4) is β^*-stable, with $\theta(x(t_k)) = \overline{h}$ for all $k \in \mathbb{N}$, if and only if there exist $l \in \mathbb{N}^+$ and $\epsilon \in (0, 1)$ such that

$$\Phi_{[\underline{h},\overline{h}]}(\mathcal{P}) \subseteq \epsilon\mathcal{P}, \tag{8}$$

where $\mathcal{P} = \mathrm{ch}\left(\bigcup_{j=0}^{l-1} \epsilon^{-j}\Phi^j_{[\underline{h},\overline{h}]}(\mathcal{S})\right)$.

It is often impossible to exactly compute Φ. Thus we use in this work an over-approximation $\overline{\Phi} : \mathcal{B}_0(\mathbb{R}^n) \to \mathcal{B}_0(\mathbb{R}^n)$ satisfying the following assumption:

Assumption 4. For all $\mathcal{S} \in \mathcal{B}_0(\mathbb{R}^n)$, $\Phi(\mathcal{S}) \subseteq \overline{\Phi}(\mathcal{S})$.

The iterates of the map $\overline{\Phi}$ are defined similarly to those of Φ. In addition, we rely on an effective computation of the over-approximation scheme in Appendix 6.1. Therein the set $\overline{\Phi}(\mathcal{S})$ is indeed a polytope for any $\mathcal{S} \in \mathcal{B}_0(\mathbb{R}^n)$.

In the following result, we synthesize the contracting set based on the map $\overline{\Phi}$.

Corollary 5. Let $\beta \in \mathbb{R}^+$, $\underline{h} \in \mathbb{R}^+$, and $\overline{h} \in \mathbb{R}^+$. Under Assumption 4, if there exist $\mathcal{S} \in \mathcal{B}_0(\mathbb{R}^n)$ and $(k, i, \rho) \in \mathbb{N}^+ \times \mathbb{N}_{[0,k-1]} \times (0, 1)$ such that

$$\overline{\Phi}^k(\mathcal{S}) \subseteq \rho\overline{\Phi}^i(\mathcal{S}), \tag{9}$$

then

(a) System (3) is β^*-stable, with $\theta(x(t_k)) = \overline{h}$ for all $k \in \mathbb{N}$.
(b) There exists $\epsilon \in (0, 1)$ such that $\Phi_{[\underline{h},\overline{h}]}(\mathcal{P}) \subseteq \epsilon\mathcal{P}$, where

$$\mathcal{P} = \mathrm{ch}\left(\bigcup_{j=0}^{k-1} \epsilon^{-j}\overline{\Phi}^j_{[\underline{h},\overline{h}]}(\mathcal{S})\right). \tag{10}$$

Practically, we compute the contracting set (7) as the following: we start iterating forward from a chosen set $\mathcal{S}$ by computing, at each iteration k, $\overline{\Phi}^k_{[\underline{h},\overline{h}]}(\mathcal{S})$ until condition

(9) is satisfied. Consequently, Corollary 5 allows us to set $\mathcal{P}$ as given by (10).

3.2 Sampling strategy design

Suppose that for a given $\overline{h} \in \mathbb{R}^+$ we have a contracting set $\mathcal{P}$ for (3), with $\theta(x(t_k)) = \overline{h}$ for all $k \in \mathbb{N}$. We intend further to increase the upper-bound on sampling, i.e. $\overline{h}$, for some regions in the state space while conserving β^*-stability.

We consider a polytopic covering of $q \in \mathbb{N}$ polytopes $\{\mathcal{P}_s : s \in \mathbb{N}_{[1,q]}\}$, such that

$$\mathcal{P} = \bigcup_{s=1}^{q} \mathcal{P}_s, \tag{11}$$

and a set of sampling periods $\{h_s \geq \overline{h} : s \in \mathbb{N}_{[1,q]}\}$, such that

$$\overline{\Phi}_{[\underline{h},h_s]}(\mathcal{P}_s) \subseteq \text{int}(\mathcal{P}). \tag{12}$$

Two coverings are suggested in Section 3.3: the first relies on the facets of the contracting polytope $\mathcal{P}$ and the second on the discrete-time behavior of the system. In fact the latter is inspired by [Fiter et al. (2012)]; therein conic coverings are computed instead of polytopic ones. Now we define a sampling strategy as (4) with

$$\theta(x) = \max\{h_s \in \{h_1, \ldots, h_q\} : x \in \gamma(x)\mathcal{P}_s\}, \tag{13}$$

where

$$\gamma(x) = \min\{\gamma \in \mathbb{R}^+ : x \in \gamma\mathcal{P}\}. \tag{14}$$

Eventually, the following instumental result solves Problem 1.

Theorem 6. Given a contracting set $\mathcal{P}$ by (7), a set of polytopic coverings $\{\mathcal{P}_s : s \in \mathbb{N}_{[1,q]}\}$ satisfying (11), and a set of sampling periods $\{h_s \geq \overline{h} : s \in \mathbb{N}_{[1,q]}\}$ satisfying (12), then under Assumption 4 (3-4) is β^*-stable with θ given by (13).

Note that Theorem 6 guarantees robustness of the sampling strategy in the sense that at any $t_k \in \mathbb{R}^+$, the next sampling instant t_{k+1} can take any value within $[t_k+\underline{h}, t_k+\theta(x(t_k))]$, while guaranteeing β^*-stability. Furthermore, a consequence of Theorem 6 is that the map γ, given by (14), is a set-induced Lyapunov function [Blanchini and Miani (2007)] for (3-4) that obviously decreases at the sampling times t_k for all $k \in \mathbb{N}$.

3.3 Polytopic covering

We propose two different methods to compute a polytopic covering $\{\mathcal{P}_s : s \in \mathbb{N}_{[1,q]}\}$ satisfying (11).

3.4 Method 1: Using the facets of the contracting polytope

Let the contracting set $\mathcal{P}$ be given in the form (7), where $H \in \mathbb{R}^{r\times n}$, then $\mathcal{P}_s$ are defined for all $s \in \mathbb{N}_{[1,q]}$ by

$$\mathcal{P}_s = \{x \in \mathbb{R}^n : H_s x \leq 1, (H_i - H_s)x \leq 0 \ \ \forall i \neq s\}, \tag{15}$$

with $q = r$ as the number of facets of $\mathcal{P}$ and H_s as the sth row of H.

Note that with this method no additional offline computations are required after we compute $\mathcal{P}$. As for the online computations, given the state at a sampling instant, i.e. $x(t_k)$, the latest next sampling is defined as

$$t_{k+1} = t_k + \mathsf{max}\{h_k : k = \mathsf{argmax}_{s\in\mathbb{N}_{[1,q]}} H_s x(t_k)\},$$

which requires only q multiplications of $n-$dimensional vectors and one argmax operation.

3.5 Method 2: Using the discrete-time behavior of the system

Given a scalar $\sigma > \overline{h}$, $q \in \mathbb{N}$, and $H \in \mathbb{R}^{r\times n}$ as the matrix defining the contracting set $\mathcal{P}$ in (7), we define a set of sampling times $\{T_s = \overline{h} + (s-1)\frac{\sigma-\overline{h}}{q-1} : s \in \mathbb{N}_{[1,q]}\}$. Then $\mathcal{P}_s$ are defined for all $s \in \mathbb{N}_{[1,q]}$ by

$$\mathcal{P}_s = \{x \in \mathbb{R}^n : \begin{pmatrix} H \\ He^{T_1(A_c+\beta I)}A_s \\ \vdots \\ He^{T_s(A_c+\beta I)}A_s \end{pmatrix} x \leq 1\}. \tag{16}$$

Note that the maximum σ is given such that the intersection of the boundary of $\mathcal{P}$ and $\mathcal{P}_s$ that corresponds to $T_s = \sigma$ is not empty and the larger the constant q is chosen the better sampling times are obtained but resulting in more complex on-line computation. Concerning the complexity, in this case the additional off-line computations required, after finding $\mathcal{P}$, are those that correspond for computing (16) for all $s \in \mathbb{N}_{[1,q]}$. Also this method is more complex than Method 1 for on-line computations since at each state $x(t_k)$ the latest next sampling instant is given by

$$t_{k+1} = t_k + t_k + \mathsf{max}\{h_s \in \{1, \ldots, q\} : x(t_k) \in \mathcal{P}_s\},$$

which requires at most one max operation, $\sum_{s=1}^{q}(rs)$ multiplications of $n-$dimensional vectors, and the same latter number of inequality checks.

We remark that in fact the on-line computations are reduced by half in both methods since the contracting set $\mathcal{P}$ is practically projected on the first $p = \frac{n}{2}$ dimensions and hence all vectors' dimensions will be reduced by half. This results from the fact that the deleted dimensions correspond to the error e which is null for (3) at all sampling instants t_k^+.

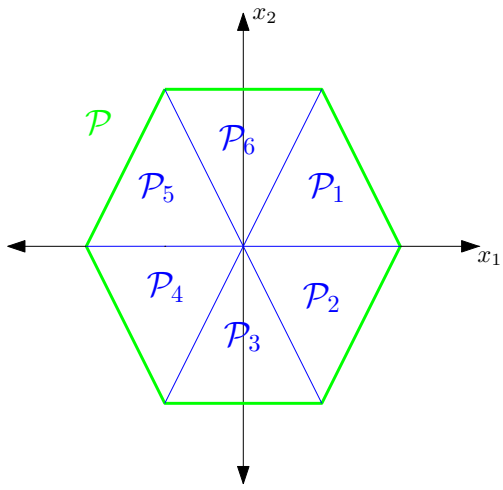

Fig. 1. Covering the contracting polytope $\mathcal{P}$ of dimension 2 with $q = 6$ polytopic regions $\mathcal{P}_s$ using Method 1.

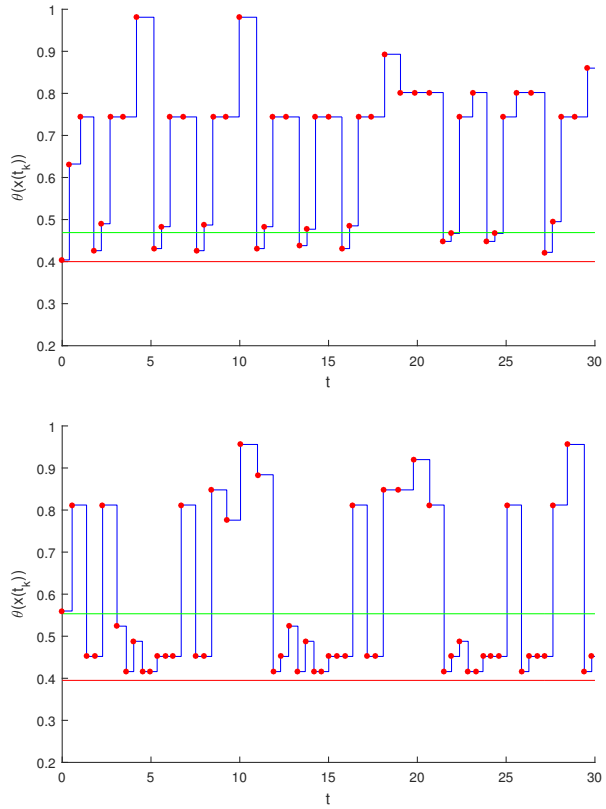

Fig. 2. Inter-execution times $\theta(x(t_k))$, in Example 7, for a decay rate (up) $\beta = 0$ using Method 1 for covering $\mathcal{P}$ and (down) $\beta = 0.05$ using Method 2.

4. NUMERICAL RESULTS

We conduct several experiments to validate the efficiency of our proposed self-triggered control approach. In the sequel, we also compare our results with existing approaches in literature. Our implementation relies on the matlab Mpt toolbox [Herceg et al. (2013)] where all reported experiments are realized on a desktop with i7 4790 processor of frequency 3.6 GHz and a 8 GB RAM.

Example 7. Consider the following system from [Fiter et al. (2012)] given by (1-2) with:

$$A = \begin{pmatrix} -0.5 & 0 \\ 0 & 3.5 \end{pmatrix}, B = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, K = \begin{pmatrix} 1.02 & -5.62 \end{pmatrix}.$$

Reformulating the problem into (3-4), the contracting set $\mathcal{P}$ is computed with $\Delta_k \in [0.01, 0.4]$. This implies that the maximum value of Δ in (4) is at least $\overline{h} = 0.4s$. After covering $\mathcal{P}$ with $q = 272$ polytopes using Method 1 in Section 3.3, sampling intervals $h_1, \ldots, h_{272}$ are defined such that (12) holds for $\beta = 0$. For a constant sampling greater than $T_{max} = 0.469s$ the system becomes unstable. Whereas, we can go with our approach beyond the limit T_{max} for some regions of the state space (up to $0.981s$).

We fix $\beta = 0$ and run simulations from 1000 different initial positions, uniformly distributed on the unit circle, with a duration corresponding to 30 resets for each one. If we sample each time with $\Delta_k = \theta(x(t_k)), \forall k \in \mathbb{N}$, the resulting average inter-sampling time is $T_{av} = 0.676s > T_{max}$. Considering Method 2, we cover $\mathcal{P}$ with $q = 16$ polytopes after setting $\sigma = 0.96$. Then we get an average sampling interval of $T_{av} = 0.77$ for the same previous experiment.

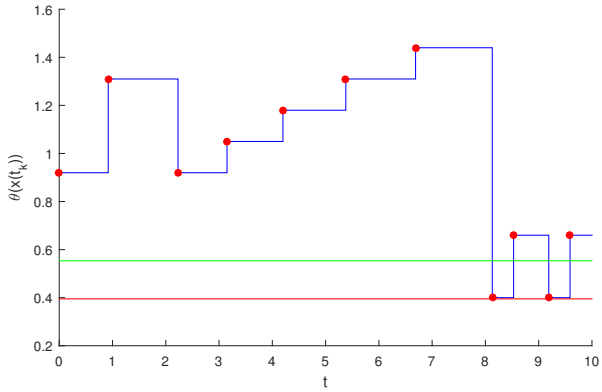

Fig. 3. Inter-execution times $\theta(x(t_k))$, in Example 8, for a decay rate $\beta = 0$.

Now we set $\beta = 0.05$ and compute a contracting set $\mathcal{P}$ with $\Delta_k \in [0.01, 0.38]$. After covering $\mathcal{P}$ with $q = 16$ polytopes, using Method 2 with $\sigma = 0.96$, we rerun the simulations from 1000 different initial positions as done previously to get an average inter-sampling interval of $T_{av} = 0.5913s > T_{max}$. For the two cases of $\beta = 0$ and $\beta = 0.05$, Figure 2 shows, for a random initial state, the sampling intervals (blue/piecewise constant curve), with the lower-bound of the off-line computed state dependent sampling function (red/lowerhorizontal line), and the limit T_{max} of the periodic case (green/upper horizontal line). The sampling times are represented by the red dots assuming that we are always sampling with $\Delta_k = \theta(x(t_k)), \forall k \in \mathbb{N}$.

Example 8. We cite another example with higher dimension ($p = 4$) from [Fiter et al. (2012)] given by (1-2) with:

$$A = \begin{pmatrix} 1.38 & -0.2 & 6.71 & -5.67 \\ -0.58 & -4.29 & 0 & 0.67 \\ 1.06 & 4.27 & -6.65 & 5.89 \\ 0.04 & 4.27 & 1.34 & -2.1 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 5.67 & 0 \\ 1.13 & -3.14 \\ 1.13 & 0 \end{pmatrix},$$

$$K = \begin{pmatrix} -0.1006 & 0.2469 & 0.0952 & 0.2447 \\ -1.4099 & 0.1966 & -0.0139 & -0.0823 \end{pmatrix}.$$

Reformulating the problem into an impulsive linear system (3-4), we compute the contracting set $\mathcal{P}$ for $\Delta_k \in [0.05, 0.4]$.

We cover $\mathcal{P}$ with $q = 13$ polytopes using Method 2 in Section 3.3 after setting $\sigma = 2.1$. Correspondingly we have 13 different sampling intervals given by $\{h_1, \ldots, h_{13}\} = \{0.4, \ldots, 2.1\}$. We check then that (12) holds for $\beta = 0$ and run a simulation for $10s$ to validate our results. Albeit for a constant sampling greater than $T_{max} = 0.553s$ the system becomes unstable we can go with our approach up to 2.1s for some regions of the state space and sample in average by $T_{av} = 0.746s$. These results are comparable with those in literature where for the first 10s, [Mazo et al. (2009)] actuated 32 times in the best mentioned case, [Fiter et al. (2012)] sampled 17 times, and as Figure 3 shows only 11 samplings were required using our method.

Now, we take $\beta = 0.06$ and an arbitrary initial position. Using a similar covering as in the previous case, Figure 4 shows the set-induced Lyapunov function $\gamma(x)$ which

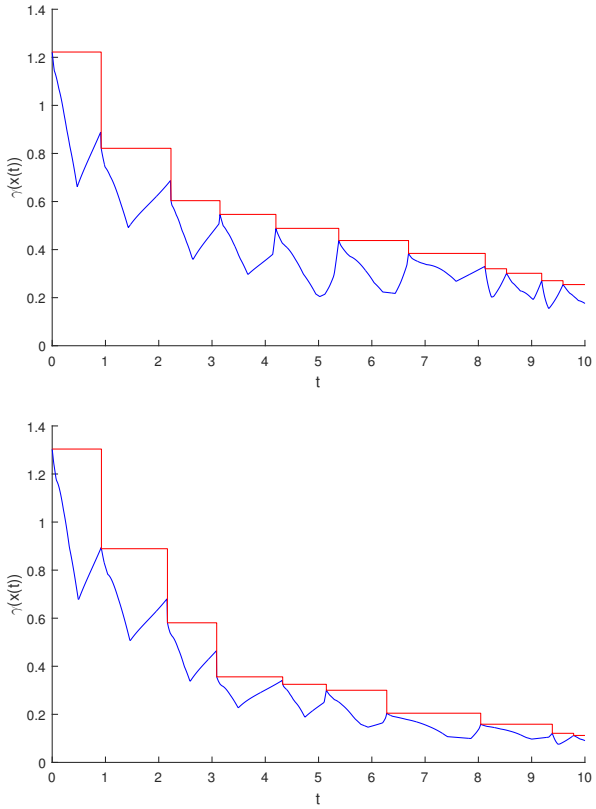

Fig. 4. The Lyapunov function $\gamma(x(t))$, in Example 8, for (up) $\beta = 0$ and (down) $\beta = 0.06$.

obviously decreases at the sampling instants t_k ensuring β^*-stability for the two cases $\beta = 0$ and $\beta = 0.06$.

5. CONCLUSION

We suggested a self-triggered sampling strategy to define the instants at which the sampler in a sampled-data system should sample while guaranteeing β^*-stability. This strategy is shown to be competitive with existing methods in literature and better, in the discussed cases, than the case of sampling with a constant sampling period.

Further work is needed to optimize the sampling strategy and the choice of the contracting set, which is designed in Section 3.1. Another issue rises from the fact that the predefined performance constrain only the decay rate β. Consequently, we may need other guarantees on performance like the over-shoot, rise time, or any other temporal logic specification.

6. APPENDIX

6.1 Over-approximation scheme

We define the set valued map $\overline{\Phi}$, satisfying Assumption 4 with Φ given for all $\mathcal{S} \subseteq \mathbb{R}^n$ by (6). For the computation of $\overline{\Phi}$, we use efficient and accurate algorithms presented in [Le Guernic (2009)], for over-approximating the reachable set of a linear system. For that purpose, let us introduce some notations.

Given a real matrix $A \in \mathbb{R}^{n\times n}$, $\langle A\rangle$ is the matrix whose elements are the absolute values of the elements of A. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, the interval hull of $\mathcal{S}$ is the smallest n-dimensional interval containing the set $\mathcal{S}$ and is denoted by $\square(\mathcal{S})$. The symmetric interval hull of $\mathcal{S}$ is the smallest symmetric (with respect to 0) n-dimensional interval containing $\mathcal{S}$ and is denoted by $\boxdot(\mathcal{S})$. Given $\mathcal{S}, \mathcal{S}' \subseteq \mathbb{R}^n$, the Minkowski sum of $\mathcal{S}$ and $\mathcal{S}'$ is $\mathcal{S} \oplus \mathcal{S}' = \{x + x' : x \in \mathcal{S}, x' \in \mathcal{S}'\}$. Also for $[t, t'] \subset \mathbb{R}^+$ we denote $\mathcal{R}^A_{[t,t']}(\mathcal{S}) = \bigcup_{\tau\in[t,t']} e^{\tau A}\mathcal{S}$. Next, given a matrix $H \in \mathbb{R}^{r\times n}$, let H_i, $i \in \mathbb{N}_{[1,r]}$ denote the row vectors of H. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, let us define the polytope $\Gamma_H(\mathcal{S}) = \{x \in \mathbb{R}^n : Hx \leq b\}$ where $b_i = \sup_{x\in\mathcal{S}} H_i x$, $i \in \mathbb{N}_{[1,r]}$. In other words, $\Gamma_H(\mathcal{S})$ is the smallest polytope whose facets directions are given by H and containing $\mathcal{S}$. Let us remark that if $\mathcal{S}$ is bounded and if 0 is in the interior of $\mathrm{ch}(\{H_1, \ldots, H_r\})$, then $\Gamma_H(\mathcal{S})$ is bounded. In addition, if $\mathcal{S}$ is convex, then it can be approximated arbitrarily close by $\Gamma_H(\mathcal{S})$ by taking a sufficient number of facets directions $H_1, \ldots, H_m$.

Theorem 9. (Le Guernic (2009)). For $\delta \in \mathbb{R}^+$, $A \in \mathbb{R}^{n\times n}$ and a set $\mathcal{S} \subseteq \mathbb{R}^n$, let

$$\overline{\mathcal{R}}^A_{[0,\delta]}(\mathcal{S}) = \bigcup_{i=1}^{N} \overline{\mathcal{R}}^A_{[(i-1)h,ih]}(\mathcal{S}) \tag{17}$$

where $N \in \mathbb{N}^+$, $h = \delta/N$ is the time step, and $\overline{\mathcal{R}}^A_{[(i-1)h,ih]}(\mathcal{S})$ is defined by the recurrence equation:

$$\overline{\mathcal{R}}^A_{[0,h]}(\mathcal{S}) = \mathrm{ch}(\mathcal{S}, e^{hA}\,\mathcal{S}) \;\oplus\; 1/4\;\epsilon_h(\mathcal{S}),$$
$$\overline{\mathcal{R}}^A_{[ih,(i+1)h]}(\mathcal{S}) = e^{hA}\,\overline{\mathcal{R}}^A_{[(i-1)h,ih]}(\mathcal{S}),\; i \in \mathbb{N}_{[1,N-1]},$$

with

$$\epsilon_h(\mathcal{S}) = \square(\langle A\rangle^{-1}(e^{h\langle A\rangle} - I) \boxdot (A(I - e^{hA})\mathcal{S})) \oplus$$
$$\square(\langle A\rangle^{-2}(e^{h\langle A\rangle} - I - h\langle A\rangle) \boxdot (A^2 e^{hA}\mathcal{S})).$$

Then, $\mathcal{R}^A_{[0,\delta]}(\mathcal{S}) \subseteq \overline{\mathcal{R}}^A_{[0,\delta]}(\mathcal{S})$.

The over-approximation of Φ is then as follows:

Corollary 10. Let the matrix $H \in \mathbb{R}^{r\times n}$, such that 0 is in the interior of $\mathrm{ch}(\{H_1, \ldots, H_r\})$. Let $\overline{\Phi}$ be given by

$$\overline{\Phi}(\mathcal{S}) = \Gamma_H\left(\mathrm{ch}(\overline{\mathcal{R}}^{(A_c+\beta I)}_{[0,h'-h]}(e^{h(A_c+\beta I)}A_r\mathcal{S}))\right), \tag{18}$$

where $\overline{\mathcal{R}}^{(A_c+\beta I)}_{[0,h'-h]}(e^{h(A_c+\beta I)}A_r\mathcal{S})$ is computed as in Theorem 9. Then, $\overline{\Phi}$ satisfies Assumption 4.

Let us remark that practically the computation of $\overline{\Phi}(\mathcal{S})$ is fairly simple using an implementation based on support functions [Le Guernic and Girard (2010)]. Indeed, if $\mathcal{S}$ is a polytope, then using the properties of support functions, the computation of $\overline{\Phi}(\mathcal{S})$ reduces to solving a set of linear programs.

We refer the reader to Section 3.3.1 in [Al Khatib et al. (2016)] for an efficient computation of the initial set $\mathcal{S}$.

6.2 Proofs

Proof of Proposition 2: In order to simplify the notations, we denote $\hat{\Phi} := \Phi_{[h,h']}$. $\hat{\Phi}$ satisfies the following:

(a) It is clear that for $a \in \mathbb{R}$, we have $\hat{\Phi}(a\mathcal{S}) = a\hat{\Phi}(\mathcal{S})$.
(b) Using the definition of Φ:

$$\hat{\Phi}(\mathcal{S}) = \bigcup_{\tau\in[h,h']} e^{\tau(A_c+\beta I)}A_r\mathcal{S} = \bigcup_{\tau\in[h,h']}\bigcup_{x\in\mathcal{S}} e^{\tau(A_c+\beta I)}A_r\{x\}$$
$$= \bigcup_{x\in\mathcal{S}}\bigcup_{\tau\in[h,h']} e^{\tau(A_c+\beta I)}A_r\{x\} = \bigcup_{x\in\mathcal{S}} \hat{\Phi}(\{x\}).$$

(c) Let $x' \in \hat{\Phi}(\mathcal{S})$. Then there exists $x \in \mathcal{S}$ and $\tau \in [h, h']$ such that $x' = e^{\tau(A_c+\beta I)}A_r x$. Then,

$$|x'| \leq e^{\overline{h}|(A_c+\beta I)|}|A_r||x|.$$

Hence, if $\mathcal{S}$ is bounded, so is $\hat{\Phi}(\mathcal{S})$.

(d) Let $x' \in \hat{\Phi}(\mathrm{ch}(\mathcal{S}))$, then there exist $x \in \mathrm{ch}(\mathcal{S})$ and $\tau \in [h, h']$ such that $x' = e^{\tau(A_c+\beta I)}A_r x$. Since $x \in \mathrm{ch}(\mathcal{S})$, there exist $z, y \in \mathcal{S}$ and $\lambda \in [0, 1]$ such that $x = \lambda z + (1-\lambda)y$. Then, by linearity

$$x' = \lambda e^{\tau(A_c+\beta I)}A_r z + (1-\lambda)e^{\tau(A_c+\beta I)}A_r y.$$

By remarking that $e^{\tau(A_c+\beta I)}A_r z \in \hat{\Phi}(\mathcal{S})$ and $e^{\tau(A_c+\beta I)}A_r y \in \hat{\Phi}(\mathcal{S})$, it follows that $x' \in \mathrm{ch}(\hat{\Phi}(\mathcal{S}))$. Thus, $\hat{\Phi}(\mathrm{ch}(\mathcal{S})) \subseteq \mathrm{ch}(\hat{\Phi}(\mathcal{S}))$.

□

Proof of Lemma 3: The lemma is a consequence of Corollary 5 in [Al Khatib et al. (2016)] and due to space limitation the proof is omitted. □

Proof of Corollary 5: $\mathcal{S}' = \bigcup_{j=0}^{k-1} \epsilon^{-j}\overline{\Phi}^j(\mathcal{S})$ with $\epsilon = \rho^{\frac{1}{k}}$ then $\mathcal{P}$ in (7) is equal to $ch(\mathcal{S}')$. Using Proposition 2 and Assumption 4 we can follow the same lines as in the proof of Corollary 8 in [Al Khatib et al. (2016)] to prove that

(a) for all trajectories $\hat{x}$ of

$$\hat{x}_{k+1} \in \Phi_{[\underline{h},\overline{h}]}(\{\hat{x}_k\}),\ k \in \mathbb{N} \tag{19}$$

we have

$$|\hat{x}_k| \leq C\epsilon^k\,|\hat{x}_0|\,,\ \forall k \in \mathbb{N}. \tag{20}$$

Then following the lines in the proof of Lemma 3 we deduce that System (3) is β^*-stable with $\theta(x(t_k)) = \overline{h}$, for all $k \in \mathbb{N}$.

(b)

$$\Phi_{[\underline{h},\overline{h}]}(\mathcal{S}') \subseteq \epsilon\mathcal{S}'. \tag{21}$$

Now using (21) and item (d) of Proposition 2 we have

$$\begin{aligned}\Phi_{[\underline{h},\overline{h}]}(\mathcal{P}) &\subseteq \Phi_{[\underline{h},\overline{h}]}(ch(\mathcal{S}')) \subseteq ch(\Phi_{[\underline{h},\overline{h}]}(\mathcal{S}')) \\ &\subseteq ch(\epsilon\mathcal{S}') = \epsilon\mathcal{P}.\end{aligned} \tag{22}$$

□

Proof of Theorem 6: The state of (3-4) at any sampling instant t_{k+1}, $k \in \mathbb{N}$ is given by

$$\begin{aligned}x_{k+1} &= e^{A_c\Delta_k}A_r x_k \\ &= e^{-\Delta_k\beta}e^{(A_c+\beta I)\Delta_k}A_r x_k \quad \forall \Delta_k \in [\underline{h}, \theta(x_k)].\end{aligned}$$

Then there exist $h_s = \theta(x_k)$ and $\mathcal{P}_s$ such that

$$x_{k+1} \in e^{-\Delta_k\beta}\Phi_{[\underline{h},h_s]}(\gamma(x_k)\mathcal{P}_s) \quad \forall \Delta_k \in [\underline{h}, \theta(x_k)]. \tag{23}$$

Using (12), properties of Φ in Proposition 2, and Assumption 4 we get

$$\begin{aligned}x_{k+1} &\in e^{-\Delta_k\beta}\gamma(x_k)\Phi_{[\underline{h},h_s]}(\mathcal{P}_s) \subseteq e^{-\Delta_k\beta}\gamma(x_k)\overline{\Phi}_{[\underline{h},h_s]}(\mathcal{P}_s) \\ &\subseteq \mathrm{int}(e^{-\Delta_k\beta}\gamma(x_k)\mathcal{P}) \quad \forall \Delta_k \in [\underline{h}, \theta(x_k)].\end{aligned}$$

In other words there exists $\epsilon \in (0, 1)$ such that

$$x_{k+1} \in \epsilon e^{-\Delta_k\beta}\gamma(x_k)\mathcal{P} \quad \forall \Delta_k \in [\underline{h}, \theta(x_k)]. \tag{24}$$

The definition of γ gives

$$x_{k+1} \in \gamma(x_{k+1})\mathcal{P} \subseteq \epsilon e^{-\Delta_k\beta}\gamma(x_k)\mathcal{P} \quad \forall \Delta_k \in [\underline{h}, \theta(x_k)]. \tag{25}$$

This implies that for all $\Delta_i \in [\underline{h}, \theta(x_i)]$

$$\gamma(x_{k+1}) \leq \epsilon e^{-\Delta_k\beta}\gamma(x_k) \leq \cdots \leq \epsilon^{k+1}e^{-\sum_{i=0}^{k}\Delta_i\beta}\gamma(x_0). \tag{26}$$

Since $\mathcal{P} \in \mathcal{B}_0(\mathbb{R}^n)$, then there exist $\underline{c} \in \mathbb{R}^+$, $\overline{c} \in \mathbb{R}^+$ such that $\underline{c}\mathcal{B} \subseteq \mathcal{P} \subseteq \overline{c}\mathcal{B}$. Thus for any $x \in \mathbb{R}^n$:

$$\frac{|x|}{\overline{c}} \leq \gamma(x) \leq \frac{|x|}{\underline{c}}. \tag{27}$$

Using (25), (26), (27), and $t_k = \sum_{i=0}^{k-1}\Delta_i$ for all $\Delta_i \in [\underline{h}, \theta(x_i)]$ yields

$$|x_k| \leq \epsilon^i e^{-t_k\beta}\frac{\overline{c}}{\underline{c}}|x_0|. \tag{28}$$

Now, let $t \in \mathbb{R}^+$ and $k \in \mathbb{N}$ be such that $t \in (t_k, t_{k+1}]$, then $t - t_k \leq h_{max}$ and $k \geq t/\overline{h} - 1$, with $h_{max} = \max_{i\in\{1,\ldots,q\}} h_i$. Moreover,

$$\begin{aligned}|x(t)| &\leq \epsilon^i e^{|A_c|h_{max}}|A_r|e^{-t_k\beta}\frac{\overline{c}}{\underline{c}}|x_0| \\ &\leq \frac{e^{(|A_c|+\beta)h_{max}}|A_r|\frac{\overline{c}}{\underline{c}}}{\epsilon}e^{-t(\beta-\frac{ln\epsilon}{h_{max}})}|x_0|,\end{aligned}$$

which finishes the proof since $\epsilon \in (0, 1)$. □

REFERENCES

Al Khatib, M., Girard, A., and Dang, T. (2016). Stability verification and timing contract synthesis for linear impulsive systems using reachability analysis. *Nonlinear Analysis: Hybrid Systems*. doi: http://dx.doi.org/10.1016/j.nahs.2016.08.007.

Blanchini, F. (1991). Ultimate boundedness control for uncertain discrete-time systems via set-induced lyapunov functions. In *Conference on Decision and Control*, 1755–1760.

Blanchini, F. (1999). Survey paper: Set invariance in control. *Automatica*, 35(11), 1747–1767.

Blanchini, F. and Miani, S. (2007). *Set-theoretic methods in control*. Springer.

Donkers, M. and Heemels, W. (2012). Output-based event-triggered control with guaranteed-gain and improved and decentralized event-triggering. *Automatic Control, IEEE Transactions on*, 57(6), 1362–1376.

Fiter, C., Hetel, L., Perruquetti, W., and Richard, J.P. (2012). A state dependent sampling for linear state feedback. *Automatica*, 48(8), 1860–1867.

Herceg, M., Kvasnica, M., Jones, C., and Morari, M. (2013). Multi-Parametric Toolbox 3.0. In *European Control Conference*, 502–510. `http://control.ee.ethz.ch/ mpt`.

Le Guernic, C. (2009). *Reachability analysis of hybrid systems with linear continuous dynamics*. Ph.D. thesis, Université Joseph-Fourier-Grenoble I.

Le Guernic, C. and Girard, A. (2010). Reachability analysis of linear systems using support functions. *Nonlinear Analysis: Hybrid Systems*, 4(2), 250–262.

Mazo, M., Anta, A., and Tabuada, P. (2009). On self-triggered control for linear systems: Guarantees and complexity. In *Control Conference (ECC), 2009 European*, 3767–3772. IEEE.

Tabuada, P. (2007). Event-triggered real-time scheduling of stabilizing control tasks. *Automatic Control, IEEE Transactions on*, 52(9), 1680–1685.