R-boundedness Approach to linear third differential equations in a UMD Space

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ABSTRACT

The aim of this work is to study the existence of a periodic solutions of third order differential equations $z'''(t) = Az(t) + f(t)$ with the periodic condition $x(0) = x(2\pi), x'(0) = x'(2\pi)$ and $x''(0) = x''(2\pi)$. Our approach is based on the R-boundedness and L^p-multiplier of linear operators.
Keywords: differential equations, L^p-multipliers.

Contents

1 Introduction

Motivated by the fact that functional differential equations arise in many areas of applied mathematics, this type of equations has received much attention in recent years. In particular, the problem of existence of periodic solutions, has been considered by several authors. We refer the readers to papers [[1], [6], [9], [14]] and the references listed therein for informations on this subject.

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In this work, we study the existence of periodic solutions for the following differential equations

$$\begin{cases} z'''(t) = Az(t) + f(t) \\ x(0) = x(2\pi), x'(0) = x'(2\pi) \text{ and } x''(0) = x''(2\pi). \end{cases} \tag{1.1}$$

where $A : D(A) \subseteq X \to X$ is a linear closed operator on Banach space $(X, \|.\|)$ and α can be any real number and $f \in L^p(\mathbb{T}, X)$ for all $p \geq 1$.
Hale [18] and Webb [23] firstly studied the first order delay equation:

$$u'(t) = Au(t) + F(u_t) + f(t), \tag{1.2}$$

Bátkai et al. [5] obtained results on the hyperbolicity of delay equations using the theory of operatorvalued Fourier multipliers. Bu [8] has studied C^α-maximal regularity for the problem (1.2) on R. Recently, Lizama [14] obtained necessary and sufficient conditions for the first order delay equation (1.2) to have L^p-maximal regularity using multiplier theorems on L^p-(𝕋;X), and C^α-maximal regularity of the corresponding equation on the real line has been studied by Lizama and Poblete [15].
Arendt [1] gave necessary and sufficient conditions for the existence of periodic solutions of the following evolution equation.

$$\frac{d}{dt}x(t) = Ax(t) + f(t) \text{ for } t \in \mathbb{R},$$

where A is a closed linear operator on an UMD-space Y.
Hernan et al [9], studied the existence of periodic solutions for the class of linear abstract neutral functional differential equation described in the following form:

$$\frac{d}{dt}[x(t) - Bx(t-r)] = Ax(t) + G(x_t) + f(t) \quad \text{for } t \in \mathbb{R}$$

where $A : D(A) \to X$ and $B : D(B) \to X$ are closed linear operator such that $D(A) \subset D(B)$ and $G \in B(L^p([-2\pi, \ 0], \ X); \ X)$.
Bahaj et al [3] studied the existence of periodic solution of second degenerate differential equation described in the following form:

$$(Mx)''(t) + Ax(t) + G(x_t) = f(t) \quad \text{for } t \in \mathbb{R}$$

where $A : D(A) \to X$ and $M : D(M) \to X$ are closed linear operator such that $D(A) \subset D(M)$ and $G \in B(L^p([-2\pi, \ 0], \ X); \ X)$.
The organization of this work is as follows: In section 2, collects definitions and basic properties of R-bounded, UMD space and Fourier multipliers, In section 3, we study the sufficient Conditions For the Periodic solutions of Eq. (1.1), In section 4, we establish the periodic solution for the equation (1.1) of this work solely in terms of a property of R-boundedness for the sequence of operators $-ik^3(-ik^3 + (\alpha - 1)A)^{-1}$. We optain that the following assertion are equivalent in UMD space :

1) $D_A = \dfrac{d^3}{dt^3} - A : H^{3,p}(\mathbb{T}, X) \cap L^p(D(A), X) \to L^p(\mathbb{T}, X)$ is an isomorphism.

2) $\sigma_{\mathbb{Z}}(\Delta) = \phi$, $\left\{-ik^3\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is R-bounded.

3) $\forall f \in L^p(\mathbb{T}, X)$ there exists a unique 2π-periodic strong L^p-solution of $Eq.(1.1)$.

In section 5, we propose an application. In section 6, we give the conclusion.

2 Vector-valued space and preliminaries

Let X be a Banach Space. Firstly, we denote By $\mathbb{T}$ the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. There is an identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$. We consider the interval $[0, 2\pi)$ as a model for $\mathbb{T}$.
Given $1 \leq p < \infty$, we denote by $L^p(\mathbb{T}; X)$ the space of 2π-periodic locally p-integrable functions from $\mathbb{R}$ into X, with the norm:

$$\|f\|_p := \left(\int_0^{2\pi} \|f(t)\|^p \, dt \right)^{1/p}$$

For $f \in L^p(\mathbb{T}; X)$, we denote by $\hat{f}(k)$, $k \in \mathbb{Z}$ the k-th Fourier coefficient of f that is defined by:

$$\hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt \quad \text{for} \quad k \in \mathbb{Z} \quad \text{and} \quad t \in \mathbb{R}.$$

For $1 \leq p < \infty$, the periodic vector-valued space is defined by

$$H^{1,p}(\mathbb{T}; X) = \{u \in L^p(\mathbb{T}, X) : \exists v \in L^p(\mathbb{T}, X), \hat{v}(k) = ik\hat{u}(k) \text{ for all } \ k \in \mathbb{Z}\} \tag{2.1}$$

2.1 UMD Space

Definition 2.1. Let $\varepsilon \in]0, 1[$ and $1 < p < \infty$. Define the operator H_ε by:
for all $f \in L^p(\mathbb{R}; X)$

$$(H_\varepsilon f)(t) := \frac{1}{\pi} \int_{\varepsilon < |s| < \frac{1}{\epsilon}} \frac{f(t-s)}{s} ds$$

if $\lim\limits_{\varepsilon \to 0} H_\varepsilon f := Hf$ exists in $L^p(\mathbb{R}; X)$ Then Hf is called the Hilbert transform of f on $L^p(\mathbb{R}, X)$.

Definition 2.2. [1]
A Banach space X is said to be UMD space if the Hilbert transform is bounded on $L^p(\mathbb{R}; \ X)$ for all $1 < p < \infty$.

2.2 R-bounded and L^p-multiplier

Let X and Y be Banach spaces. Then $B(X, Y)$ denotes, the space of bounded linear operators from X to Y.

Definition 2.3. [1]
A family of operators $T = (T_j)_{j \in \mathbb{N}^*} \subset B(X, Y)$ is called R-bounded (**Rademacher bounded or randomized bounded**), if there is a constant $C > 0$ and $p \in [1, \infty)$ such that

for each $n \in N, T_j \in \mathrm{T}, x_j \in X$ and for all independent, symmetric, $\{-1,1\}$-valued random variables r_j on a probability space (Ω, M, μ) the inequality

$$\left\| \sum_{j=1}^{n} r_j T_j x_j \right\|_{L^p(0,1;Y)} \leq C \left\| \sum_{j=1}^{n} r_j x_j \right\|_{L^p(0,1;X)}$$

is valid. The smallest C is called R-bounded of $(T_j)_{j\in\mathbb{N}^*}$ and it is denoted by $R_p(T)$.

Definition 2.4. [1]
For $1 \leq p < \infty$, a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathbf{B}(X,Y)$ is said to be an L^p-multiplier if for each $f \in L^p(\mathbb{T}, X)$, there exists $u \in L^p(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

Proposition 2.1. [[1], *Proposition* 1.11]
Let X be a Banach space and $\{M_k\}_{k\in\mathbb{Z}}$ be an L^p-multiplier, where $1 \leq p < \infty$. Then the set $\{M_k\}_{k\in\mathbb{Z}}$ is R-bounded.

Theorem 2.1. ***(Marcinkiewicz operator-valud multiplier Theorem).***
Let X, Y be UMD spaces and $\{M_k\}_{k\in\mathbb{Z}} \subset B(X,Y)$. If the sets $\{M_k\}_{k\in\mathbb{Z}}$ and $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ are
R-bounded, then $\{M_k\}_{k\in\mathbb{Z}}$ is an L^p-multiplier for $1 < p < \infty$.

Theorem 2.2. *($Fejer's$ theorem)*
Let $f \in L^p(\mathbb{T}, X)$. Then

$$f = \lim_{n\to\infty} \sigma_n(f)$$

in $L^p(\mathbb{T}, X)$ where

$$\sigma_n(f) := \frac{1}{n+1} \sum_{m=0}^{n} \sum_{k=-m}^{m} e_k \hat{f}(k)$$

with $e_k(t) := e^{ikt}$.

Lemma 2.3. [1]. *Let $f, g \in L^p(\mathbb{T}; X)$. If $\hat{f}(k) \in D(A)$ and $A\hat{f}(k) = \hat{g}(k)$ for all $k \in \mathbb{Z}$ Then*

$$f(t) \in D(A) \ \text{ and } \ Af(t) = g(t) \ \text{ for all } \ t \in [0, 2\pi].$$

3 Sufficient Conditions For the Periodic solutions of Eq. (1.1)

In this section, we will give conditions which guarantee the periodic solution of the some second differential equation. We denote by
$H^{1,p}(\mathbb{T}; X) = \{u \in L^p(\mathbb{T}, X) : \exists v \in L^p(\mathbb{T}, X), \hat{v}(k) = ik\hat{u}(k) for \ \ all \ \ k \in \mathbb{Z}\}$
$H^{2,p}(\mathbb{T}; X) = \left\{u \in L^p(\mathbb{T}, X) : \exists v \in L^p(\mathbb{T}, X), \hat{v}(k) = -k^2\hat{u}(k) for \ \ all \ \ k \in \mathbb{Z}\right\}$
$H^{3,p}(\mathbb{T}; X) = \left\{u \in L^p(\mathbb{T}, X) : \exists v \in L^p(\mathbb{T}, X), \hat{v}(k) = -ik^3\hat{u}(k) for \ \ all \ \ k \in \mathbb{Z}\right\}$

Definition 3.1. [14]
For $1 \leq p < \infty$, we say that a sequence $\{M_k\}_{k\in\mathbb{Z}} \subset \mathbf{B}(X,Y)$ is an $(L^p, H^{3,p})$-multiplier, if for each $f \in L^p(\mathbb{T},X)$ there exists $u \in H^{3,p}(\mathbb{T},Y)$ such that $\hat{u}(k) = M_k\hat{f}(k) \quad$ for all $\; k \in \mathbb{Z}$.

Lemma 3.1. [1]
Let $1 \leq p < \infty$ and $(M_k)_{k\in\mathbb{Z}} \subset \mathbf{B}(X)$ ($\mathbf{B}(X)$ is the set of all bounded linear operators from X to X). Then the following assertions are equivalent:
(i) $(M_k)_{k\in\mathbb{Z}}$ is an $(L^p, H^{1,p})$-multiplier.
(ii) $(ikM_k)_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

We define

$$D_A := \frac{d^3}{dt^3} - A$$

$$\Delta_k = (-ik^3 I + A) \;\; and \;\; \sigma_{\mathbb{Z}}(\Delta) = \{k \in \mathbb{Z} : \Delta_k \; is \; not \; bijective\}$$

We begin by establishing our concept of strong solution for Eq. (1.1)

Definition 3.2. Let $f \in L^p(\mathbb{T};X)$. A function $x \in H^{3,p}(\mathbb{T};X)$ is said to be a 2π-periodic strong L^p-solution of Eq.(1.1) if $x(t) \in D(A)$ for all $t \geq 0$ and Eq. (1.1) holds almost every where.

Proposition 3.1. *Let A be a closed linear operator defined on an UMD space X. Suppose that*
$\sigma_{\mathbb{Z}}(\Delta) = \phi$ *.Then the following assertions are equivalent :*

(i) $\left(-ik^3(-ik^3 I + A)^{-1}\right)_{k\in\mathbb{Z}}$ *is an L^p-multiplier for $1 < p < \infty$*

(ii) $\left(-ik^3(-ik^3 I + A)^{-1}\right)_{k\in\mathbb{Z}}$ *is R-bounded.*

Proof. (i) $\Rightarrow$ (ii) As a consequence of Proposition **(2.1)**
(ii) $\Rightarrow$ (i) Define $M_k = -ik^3 N_k$ where $N_k = (-ik^3 I + A)^{-1}$. By Marcinkiewcz Theorem it is sufficient to prove that the set $\{k(M_{k+1} - M_k)\}_{k\in\mathbb{Z}}$ is R-bounded. Since

$$\begin{aligned}
k\,[M_{k+1} - M_k] &= k[-i(k+1)^3(-i(k+1)^3 I + A)^{-1} + ik^3(-ik^3 I + A)^{-1}] \\
&= k[-i(k+1)^2 N_{k+1} + ik^2 N_k] \\
&= kN_{k+1}[-i(k+1)^3((-k^3 I + A))) + ik^3((-i(k+1)^3 I + A)]N_k \\
&= kN_{k+1}[-i(k+1)^3 A + ik^3 A)]N_k \\
&= ikN_{k+1}[(k^3 - (k+1)^3)A]N_k \\
&= ikN_{k+1}[-(3k^2 + 3k + 1)A]N_k \\
&= ik^3 N_{k+1}[-(3 + 3\frac{1}{k} + \frac{1}{k^2})](I + ik^3 N_k) \\
&= \frac{k^3}{(k+1)^3} M_{k+1}[-(3 + 3\frac{1}{k} + \frac{1}{k^2})](I - M_k)
\end{aligned}$$

Since products and sums of R-bounded sequences is R-bounded [[**14**]. Remark 2.2]. Then the proof is complete. □

Lemma 3.2. *Let $1 \leq p < \infty$. Suppose that $\sigma_{\mathbb{Z}}(\Delta) = \phi$ and D_A is surjective. Then D_A is bijective.*

Proof. We have D_A is surjective the $H^{3,p}(\mathbb{T}, X)$ to $L^p(\mathbb{T}, X)$. Then

$$\forall f \in L^p(\mathbb{T}, X)\ \exists z \in H^{3,p}(\mathbb{T}, X) \text{ such that } \ D_A z = f$$

Suppose that there exists z_1 and z_2 such that $D_A z_1 = f$ and $D_A z_2 = f$. then for $z = z_1 - z_2$ we have $D_A z = 0$. Taking Fourier transform, we obtain that

$$(-ik^3 + A)\hat{z}(k) = 0, k \in \mathbb{Z}.$$

i.e

$$\Delta_k \hat{z}(k) = 0$$

It follows that $\hat{z}(k) = 0$ for every $k \in \mathbb{Z}$ and therefore $z = 0$. Then $z_1 = z_2$ and D_A is bijective. □

Theorem 3.3. *Let X be a Banach space. Suppose that the operator $D_A := \frac{d^3}{dt^3} - A$ is an isomorphism of $H^{3,p}(\mathbb{T}, X)$ onto $L^p(\mathbb{T}, X)$ for $1 \leq p < \infty$. Then*

1. *for every $k \in \mathbb{Z}$ the operator $\Delta_k = (-ik^3 I + A)$ has bijective,*
2. *$\left\{-ik^3 \Delta_k^{-1}\right\}_{k \in \mathbb{Z}}$ is R-bounded.*

Before to give the proof of Theorem **(3.3)**, we need the following Lemma.

Lemma 3.4. *if $x \in Ker\Delta_k$, then $e^{ikt}x \in KerD_A$*

Proof. $x \in Ker\Delta_k \Rightarrow -ik^3 x = Ax$.
Put $z(t) = e^{ikt}x$, then

$$\begin{aligned} z'''(t) &= (ike^{ikt}x)'' \\ &= (-k^2 e^{ikt}x)' \\ &= -ik^3 e^{ikt}x \\ &= e^{ikt}Ax \\ &= Az(t) \\ &\Rightarrow D_A z(t) = 0 \\ &\Rightarrow e^{ikt}x \in KerD_A. \end{aligned}$$

Proof of Theorem (3.3): 1) Let $k \in \mathbb{Z}$ and $y \in X$. Then for $f(t) = e^{ikt}y$, there exists $z \in H^{3,p}(\mathbb{T}; X)$ such that:

$$D_A z(t) = f(t)$$

Taking Fourier transform, we deduce that:
$(-ik^3 + A)\hat{z}(k) = \hat{f}(k) = y \Rightarrow \Delta_k = (-ik^3 + A)$ is surjective.
Let $u \in Ker\Delta_k$. By Lemma **3.4**, we have $e^{ikt}u \in KerD_A$, then $u = 0$ and $(-ik^3 + A)$ is injective.

2) Let $f \in L^p(\mathbb{T};X)$. By hypothesis, there exists a unique $z \in H^{3,p}(\mathbb{T},X)$ such that $D_A z = f$. Taking Fourier transforms, we deduce that

$$\hat{z}(k) = (-ik^3 + A)^{-1}\hat{f}(k) \quad \text{for all } \ k \in \mathbb{Z}.$$

Hence

$$-ik^3\hat{z}(k) = -ik^3(-ik^3 + A)^{-1}\hat{f}(k) \ \text{ for all } \ k \in \mathbb{Z}$$

Since $z \in H^{3,p}(\mathbb{T};X)$, then there exists $v \in L^p(\mathbb{T};X)$ such that

$$\hat{v}(k) = -ik^3\hat{z}(k) = -ik^3(-k^3 + A)^{-1}\hat{f}(k).$$

Then $\left\{-ik^3\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is an L^p-multiplier and $\left\{-ik^3\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is R-bounded. □

4 Main result

Our main result in this section is to establish that the converse of Theorem **3.3**, are true, provided X is an UMD space.

Lemma 4.1. *Let $1 \le p < \infty$ and $(M_k)_{k\in\mathbb{Z}} \subset \mathbf{B}(X)$. Then the following assertions are equivalent:*
(i) $(M_k)_{k\in\mathbb{Z}}$ is an $(L^p, H^{2,p})$-multiplier.
(ii) $(-k^2M_k)_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

Proof. We have

$$\begin{aligned}
&(-k^2M_k)_{k\in\mathbb{Z}} \ \text{ is an } (L^p,L^p) - multiplier\\
&\Leftrightarrow ik(ikM_k)_{k\in\mathbb{Z}} \text{ is an } (L^p,L^p) - multiplier\\
&\Leftrightarrow (ikM_k)_{k\in\mathbb{Z}} \text{ is an } (L^p,H^{1,p}) - multiplier \text{ (by Lemma3.1)}
\end{aligned}$$

then the proof ase soon as Lemma (3.1). □

Lemma 4.2. *Let $1 \le p < \infty$ and $(M_k)_{k\in\mathbb{Z}} \subset \mathbf{B}(X)$. Then the following assertions are equivalent:*
(i) $(M_k)_{k\in\mathbb{Z}}$ is an $(L^p, H^{3,p})$-multiplier.
(ii) $(-ik^3M_k)_{k\in\mathbb{Z}}$ is an (L^p, L^p)-multiplier.

Theorem 4.3. *Let X be an UMD space and $A : D(A) \subset X \to X$ be an closed linear operator. Then the following assertions are equivalent for $1 < p < \infty$.*

(1) *The operator $D_A := \frac{d^3}{dt^3} - A$ is an isomorphism of $H^{3,p}(\mathbb{T},X) \cap L^p(D(A),X)$ onto $L^p(\mathbb{T},X)$.*

(2) *$\sigma_{\mathbb{Z}}(\Delta) = \phi$ and $\left\{-ik^3\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is R-bounded.*

Proof. 1) ⇒ 2) see Theorem **(3.3)**
1) ⇐ 2) Let $f \in L^p(\mathbb{T};X)$. Define $\Delta_k = (-ik^3I + A)$,
By Lemma **3.1**, the family $\left\{-ik^3\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is an L^p-multiplier it is equivalent to

the family $\left\{\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is an L^p-multiplier that maps $L^p(\mathbb{T};X)$ into $H^{3,p}(\mathbb{T};X)$ (Lemma 4.2), namely there exists $z\in H^{3,p}(\mathbb{T},X)$ such that

$$\hat{z}(k)=\Delta_k^{-1}\hat{f}(k)=(-ik^3I+A)^{-1}\hat{f}(k) \tag{4.1}$$

In particular, $z\in L^p(\mathbb{T};X)$ and there exists $v\in L^p(\mathbb{T};X)$ such that $\hat{v}(k)=-ik^3\hat{z}(k)$ By Theorem **2.2**, we have

$$z(t)=\lim_{n\to+\infty}\frac{1}{n+1}\sum_{m=0}^{n}\sum_{k=-m}^{m}e^{ikt}\hat{z}(k)$$

Using now (4.1) we have:

$$(-ik^3I-A)\hat{z}(k)=\hat{f}(k)\quad\text{for all}\ \ k\in\mathbb{Z}.$$

i.e

$$\widehat{(D_Az)}(k)=\hat{f}(k)\quad\text{for all}\ \ k\in\mathbb{Z}.$$

Since A is closed, then $z(t)\in D(A)$ and $D_Az(t)=f(t)$ [Lemma **2.3**].
Uniqueness, suppose that $\exists z_1,z_2: D_Az_1(t)=f(t)$ and $D_Az_2(t)=f(t)$.
Then

$$z=z_1-z_2\in ker(D_A)\ \text{i.e}\ \frac{d^3}{dt^3}x(t)=Ax(t).$$

Taking Fourier transform, we deduce that:
$\Delta_k\hat{z}(k)=0\Rightarrow\hat{z}(k)=0\ \forall k\in\mathbb{Z}\Rightarrow z=0$. i.e $z_1=z_2$. Or D_A is linear operator then D_A is isomorphism. □

Corollary 4.4. *Let X be an UMD space and $A: D(A)\subset X\to X$ be an closed linear operator. Then the following assertions are equivalent for $1<p<\infty$.*

(1) *for every $f\in L^p(\mathbb{T};X)$ there exists a unique 2π-periodic strong L^p-solution of Eq. (1.1).*

(2) *$\sigma_{\mathbb{Z}}(\Delta)=\phi$ and $\left\{-ik^3\Delta_k^{-1}\right\}_{k\in\mathbb{Z}}$ is R-bounded.*

Proof. By theorem 4.3, we have

(2) $\Leftrightarrow D_A: H^{3,p}(\mathbb{T},X)\cap L^p(D(A),X)\to L^p(\mathbb{T},X)$ is an isomorphism.
$\Leftrightarrow\forall f\in L^p(\mathbb{T},X)$ there exits a unique $z\in H^{3,p}(\mathbb{T},X)\cap L^p(D(A),X): D_Az=f$
$\Leftrightarrow\forall f\in L^p(\mathbb{T},X)$ there exits a unique $z\in H^{3,p}(\mathbb{T},X)\cap L^p(D(A),X): z=D_A^{-1}f$
$\Leftrightarrow\forall f\in L^p(\mathbb{T},X)$there exists a unique 2π-periodic strong L^p-solution of *Eq.*(1.1).

□

5 Application

To apply the provious result, we propose the following partial functional differential equation

$$\begin{cases} \frac{\partial^3}{\partial t^3}w(t,x) = \frac{\partial^2}{\partial t^2}w(t,x) + g(t,x), x \in [0,\pi], t \geq 0, \\ \qquad w(0,t) = w(\pi,t) = 0, t \geq 0. \\ \qquad w(x,t) = w_0(x,t), x \in [0,\pi], -r \leq t \leq 0. \end{cases} \tag{5.1}$$

Let $X = C_0[0,\pi] = \{u \in C([0,\pi],\mathbb{R}) : u(0) = u(\pi) = 0\}$. we define the linear operator $A : D(A) \subset X \to X$ by

$$\begin{cases} Ay = y'' \\ D(A) = \{y \in C^2([0,\pi],\mathbb{R}) : y(0) = y(\pi) = 0\}. \end{cases}$$

Put

$$y(t)(x) = w(t,x) \text{ and } f(t)(x) = g(t,x)$$

Thus, Eq. (5.1) takes the following abstract form

$$\frac{d^3}{dt^3}y(t) = Ay(t) + f(t) \tag{5.2}$$

It is well known that A is linear operator. Then by [[9], Section 3.7] (see also references therein), there exists a constant $c > 0$ such that

$$||(-ik^3 - A)|| \leq \frac{c}{1+|k^3|}$$

Then

$$\sup_{k\in\mathbb{Z}} ||k^3(-ik^3 - A)|| < +\infty$$

We deduce from Corollary (4.4) that the above periodic problem has L^p-strong solution.

6 Conclusion

we are obtained necessary and sufficient conditions to guarantee existence and uniqueness of periodic solutions to the equation $z'''(t) = Az(t) + f(t)$ in terms of either the R-boundedness of the modified resolvent operator determined by the equation. Our results are obtained in the UMD spaces.

References

[1] W.Arend and S.Bu, *The operator-valued Marcinkiewicz multiplier theorem and maximal regularity, Math.Z. 240, (2002), 311-343.*

[2] W. Arendt, S. Bu, Operator-valued Fourier multipliers on periodic Besov spaces and applications, Proc. Edinburgh Math. Soc. 47, (1), (2004), 15-33.

[3] M.Bahaj, R.Bahloul et O.Sidki, Periodic solutions of degenerate equations with finite delay in UMD space, journal of Advances in Dynamical Systems and Application. ISSN 0973-5321, Volume 10, Number 1, (2015) pp. 23-31.

[4] R.Bahloul, *Periodic solutions of differential equations with two variable in vector-valued function space, Asian Journal of Mathematics and Computer Research*,**12**(1): 44-53, 2016 ISSN: 2395-4205 (P), ISSN: 2395-4213 (O).

[5] R.Bahloul, K.Ezzinbi et O.Sidki, Periodic Solutions in UMD spaces for some neutral partial function differential equations, Advances in Pure Mathematics, (2016), 6, 713-726, http: dx.doi.org/10.4236/apm.2016.610058

[6] S.Bu and G.Cai, Periodic solutions of third-order integrodifferential equations in vector-valued functional spaces, Journal of Evolution Equations. May 2016.

[8] Bu S. Maximal regularity of second order delay equations in Banach spaces. Acta Math Sin (Engl Ser), 2009, 25: 2128

[9] A. Favini and A. Yagi, Degenerate Differential Equations in Banach Spaces, Pure and Appl. Math., 215, Dekker, New York, Basel, Hong Kong, 1999.

[10] S.Bu and Fang Yi, Maximal regularity of second order delay equationsin Banach spaces, Science China Mathematics, January 2010 Vol. 53 No.1: 5162.

[11] V. Keyantuo and C. Lizama, Fourier multipliers and integro-differential equations in Banach spaces, J. London Math. Soc. 69 (2004), 737750.

[12] V. Keyantuo and C. Lizama, Maximal regularity for a class of integro-differential equations with infinite delay in Banach spaces, Studia Math.168 (2005), 2550.

[13] J. Lindenstrauss and L. Tzafriri, Classical Banach Spaces II. Spinger, Berlin, 1996.

[14] C. Lizama, Fourier multipliers and periodic solutions of delay equations in Banach spaces,J. Math. Anal. Appl. 324 (2006), 921933.

[15] C. Lizama and V. Poblete, Periodic solutions of fractional differential equations with delay,J. Evol. Equ. 11 (2011), no. 1, 5770.

[16] C. Lizama and V. Poblete, Maximal regularity of delay equations in Banach spaces, Studia Math. 175 (2006), 91102.

[17] Y.Hino, T.Naito,N. Van Minh and J.S.Shin, Almost periodic solution of Differential Equations in Banach Spaces, Taylor and Francis, London,2002.

[18] Hale J K. Theory of Functional Differential Equations. New York: Springer-Verlag, 1977

[19] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, 1983.

[20] V.Poblete, Maximal regularity of second-order equations with delay, J. Differential Equations 246 (2009) 261276.

[21] J.Wu, Theory and Applications of Partial Differential Equations, Appl, Math .Sci. 119, Springer-verlag, 1969.

[22] L. Weis, Operator-valued Fourier multiplier theorems and maximal Lpregularity.

[23] Webb G. Functional differential equations and nonlinear semigroups in Lp-spaces. J Differential Equations, 1976, 29: 7189

[24] Sylvain Koumla, Khalil Ezzinbi and Rachid Bahloul, mild solutions for some partial functional integrodifferential equations with finite delay in Frechet spaces. SeMA (2016), P 1-13. DOI 10.1007/s40324-016-0096-7

[25] X. Zhang. Some results of linear fractional order time-delay system. Appl. Math. Comput. 197 (2008),407411.