



HAL
open science

Periodic solutions of integro-differential equations in Banach space having Fourier type

Bahloul Rachid

► **To cite this version:**

Bahloul Rachid. Periodic solutions of integro-differential equations in Banach space having Fourier type. 2017. hal-01540174

HAL Id: hal-01540174

<https://hal.science/hal-01540174>

Preprint submitted on 15 Jun 2017

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

Periodic solutions of integro-differential equations in Banach space having Fourier type

Bahloul Rachid¹

¹ *Department of Mathematics, Faculty of Sciences and Technology, Fez, Morocco.*

ABSTRACT

The aim of this work is to study the existence of a periodic solutions of integro-differential equations $\frac{d}{dt}[x(t) - L(x_t)] = A[x(t) - L(x_t)] + G(x_t) + \int_{-\infty}^t a(t-s)x(s)ds + f(t)$, ($0 \leq t \leq 2\pi$) with the periodic condition $x(0) = x(2\pi)$, where $a \in L^1(\mathbb{R}_+)$. Our approach is based on the M-boundedness of linear operators, Fourier type, $B_{p,q}^s$ -multipliers and Besov spaces.

Keywords: integro-differential equations, Fourier type, $B_{p,q}^s$ -multipliers.

1 Introduction

The aim of this paper is to study the existence and of solutions for some neutral functional integro-differential equations with delay by using methods of maximal regularity in Besov spaces. Motivated by the fact that neutral functional integro-differential equations with finite delay arise in many areas of applied mathematics, this type of equations has received much attention in recent years. In particular, the problem of existence of periodic solutions, has been considered by several authors. We refer the readers to papers [[1], [7], [9], [14]] and the references listed therein for information on this subject. One of the most important tools to prove maximal regularity is the theory of Fourier multipliers. They play an important role in the analysis of parabolic problems. In recent years it has become apparent that one needs not only the classical theorems but also vector-valued extensions with operator-valued multiplier functions or symbols. These extensions allow to treat certain problems for evolution equations with partial differential operators in an elegant and efficient manner in analogy to ordinary differential equations. For some recent papers on the sujet, we refer to Arendt-Bu [1], Bu et al [11] and Hernan et al [12].

We characterize the existence of periodic solutions for the following integro-differential equations in vector-valued spaces and Besov. In the case of vector-valued space, our results involve UMD spaces, the concept of R-boundedness and a condition on the resolvent operator. We remark that many of the most powerful modern theorems are valid in UMD spaces, i.e., Banach spaces in which martingale are unconditional differences. The probabilistic definition of UMD spaces turns out to be equivalent to the L^p -boundedness of the Hilbert transform, a transformation which is, in a sense, the typical representative example of a multiplier

1 : *E-mail address : bahloul33r@hotmail.com*

operator. On the other hand the notion of M-boundedness has played an important role in the functional approach to partial differential equations.

In the case of Besov spaces, our results involve only boundedness of the resolvent.

In this work, we study the existence of periodic solutions for the following integro-differential equations

$$\frac{d}{dt}[x(t) - L(x_t)] = A[x(t) - L(x_t)] + G(x_t) + \int_{-\infty}^t a(t-s)x(s)ds + f(t) \quad (1.1)$$

where $A : D(A) \subseteq X \rightarrow X$ is a linear closed operator on Banach space $(X, \|\cdot\|)$ and $f \in L^p(\mathbb{T}, X)$ for all $p \geq 1$. For $r_{2\pi} := 2\pi N$ (some $N \in \mathbb{N}$) L and G are in $B(L^p([-r_{2\pi}, 0], X); X)$ is the space of all bounded linear operators and x_t is an element of $L^p([-r_{2\pi}, 0], X)$ which is defined as follows

$$x_t(\theta) = x(t + \theta) \text{ for } \theta \in [-r_{2\pi}, 0].$$

In [1], Arendt gave necessary and sufficient conditions for the existence of periodic solutions of the following evolution equation.

$$\frac{d}{dt}x(t) = Ax(t) + f(t) \text{ for } t \in \mathbb{R},$$

where A is a closed linear operator on an UMD-space Y .

In [12], Hernan et al, studied the existence of periodic solution for the class of linear abstract neutral functional differential equation described in the following form:

$$\frac{d}{dt}[x(t) - Bx(t-r)] = Ax(t) + G(x_t) + f(t) \text{ for } t \in \mathbb{R}$$

where $A : D(A) \rightarrow X$ and $B : D(B) \rightarrow X$ are closed linear operator such that $D(A) \subset D(B)$ and $G \in B(L^p([-2\pi, 0], X); X)$.

In [8], Sylvain Koumla, Khalil Ezzinbi, Rachid Bahloul established mild solutions for some partial functional integrodifferential equations with finite delay in Frchet spaces

$$\frac{d}{dt}x(t) = Ax(t) + \int_0^t B(t-s)x(s)ds + f(t, x_t) + h(t, x_t)$$

This work is organized as follows : After preliminaries in the second section, we give a main result and the conclusion.

2 Preliminaries

Let X be a Banach Space. Firstly, we denote By \mathbb{T} the group defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$. There is an identification between functions on \mathbb{T} and 2π -periodic functions on \mathbb{R} . We consider the interval $[0, 2\pi)$ as a model for \mathbb{T} .

Given $1 \leq p < \infty$, we denote by $L^p(\mathbb{T}; X)$ the space of 2π -periodic locally p -integrable functions from \mathbb{R} into X , with the norm:

$$\|f\|_p := \left(\int_0^{2\pi} \|f(t)\|^p dt \right)^{1/p}$$

For $f \in L^p(\mathbb{T}; X)$, we denote by $\hat{f}(k)$, $k \in \mathbb{Z}$ the k -th Fourier coefficient of f that is defined by:

$$\mathcal{F}(f)(k) = \hat{f}(k) = \frac{1}{2\pi} \int_0^{2\pi} e^{-ikt} f(t) dt \text{ for } k \in \mathbb{Z} \text{ and } t \in \mathbb{R}.$$

For $1 \leq p < \infty$, the periodic vector-valued space is defined by.

Let S be the Schwartz space on \mathbb{R} and let S' be the space of all tempered distributions on \mathbb{R} . Let $\phi(\mathbb{R})$ be the set of all systems $\phi = \{\phi_j\}_{j \geq 0} \subset S$ satisfying

$$\begin{aligned} \text{supp}(\phi_0) &\subset [-2, 2] \\ \text{supp}(\phi_j) &\subset [-2^{j+1}, -2^{j-1}] \cup [2^{j-1}, 2^{j+1}], \quad j \geq 1 \\ \sum_{j \geq 0} \phi_j(t) &= 1, \quad t \in \mathbb{R} \end{aligned}$$

Let $1 \leq p, q \leq \infty, s \in \mathbb{R}$ and $\phi = (\phi_j)_{j \geq 0} \in \phi(\mathbb{R})$. We define the X -valued periodic Besov spaces by

$$B_{p,q}^{s,\phi}(\mathbb{T}; X) = \{f \in D'(\mathbb{T}, X) : \|f\|_{B_{p,q}^{s,\phi}} = (\sum_{j \geq 0} 2^{sjq})^{1/q} \|\sum_{k \in \mathbb{Z}} e_k \phi_j(k) \hat{f}(k)\|_p^q\}$$

Proposition 2.1. [14]

- 1) $B_{p,q}^s((0, 2\pi); X)$ is a Banach space;
- 2) Let $s > 0$. Then $f \in B_{p,q}^{s+1}((0, 2\pi); X)$ if and only if f is differentiable and $f' \in B_{p,q}^s((0, 2\pi); X)$

Definition 2.1. [14]

For $1 \leq p < \infty$, a sequence $\{M_k\}_{k \in \mathbb{Z}} \subset B(X, Y)$ is a $B_{p,q}^s$ -multiplier if for each $f \in B_{p,q}^s(\mathbb{T}, X)$, there exists $u \in B_{p,q}^s(\mathbb{T}, Y)$ such that $\hat{u}(k) = M_k \hat{f}(k)$ for all $k \in \mathbb{Z}$.

Definition 2.2. [1]

The Banach space X has Fourier type $r \in [1, 2]$ if there exists $C_r > 0$ such that

$$\|\mathcal{F}(f)\|_{r'} \leq C_r \|f\|_r, \quad f \in L^r(\mathbb{R}, X)$$

where $\frac{1}{r'} + \frac{1}{r} = 1$.

Definition 2.3. [14]

Let $\{M_k\}_{k \in \mathbb{Z}} \subseteq B(X, Y)$ be a sequence of operators. $\{M_k\}_{k \in \mathbb{Z}}$ is M -bounded of order 1 (or M -bounded) if

$$\sup_k \|M_k\| < \infty \text{ and } \sup_k \|k(M_{k+1} - M_k)\| < \infty \quad (2.1)$$

Theorem 2.1. [1]

Let X and Y be Banach spaces having Fourier type $r \in [1, 2]$ and let $\{M_k\}_{k \in \mathbb{Z}} \subseteq B(X, Y)$ be a sequence satisfying (2.1). Then for $1 \leq p, q < \infty, s \in \mathbb{R}$, $\{M_k\}_{k \in \mathbb{Z}}$ is an $B_{p,q}^s$ -multiplier.

Lemma 2.2. [7]

Let $L : L^p(\mathbb{T}, X) \rightarrow X$ be a bounded linear operator. Then

$$\widehat{L(u)}(k) = L(e_k \hat{u}(k)) := L_k \hat{u}(k) \text{ for all } k \in \mathbb{Z}$$

and $\{L_k\}_{k \in \mathbb{Z}}$ is r -bounded such that

$$R_p((L_k)_{k \in \mathbb{Z}}) \leq (2r_{2\pi})^{1/p} \|L\|.$$

Next we give some preliminaries. Given $a \in L^1(\mathbb{R}^+)$ and $u : [0, 2\pi] \rightarrow X$ (extended by periodicity to \mathbb{R}), we define

$$F(t) = \int_{-\infty}^t a(t-s)u(s)ds.$$

Let $\tilde{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t)dt$ be the Laplace transform of a . An easy computation shows that:

$$\hat{F}(k) = \tilde{a}(ik)\hat{u}(k), \text{ for all } k \in \mathbb{Z} \quad (2.2)$$

2.1 Main result

For convenience, we introduce the following notations:

$B_k = kA(L_{k+1} - L_k)$, $P_k = k(\tilde{a}(i(k+1)) - \tilde{a}(ik))$, $Q_k = k(L_{k+1} - L_k)$, $R_k = k(G_{k+1} - G_k)$. In order to give our result, the following hypotheses are fundamental.

H_1 : $\{L_k\}_{k \in \mathbb{Z}}$, $\{G_k\}_{k \in \mathbb{Z}}$ and $\{\tilde{a}(ik)\}_{k \in \mathbb{Z}}$ are M-bounded (i.e $\{P_k\}_{k \in \mathbb{Z}}$, $\{Q_k\}_{k \in \mathbb{Z}}$ and $\{R_k\}_{k \in \mathbb{Z}}$ are bounded)

H_2 : $\sup_{k \in \mathbb{Z}} \|B_k\| < \infty$

Definition 2.4. : Let $1 \leq p, q < \infty$ and $s > 0$. We say that a function $x \in B_{p,q}^s(\mathbb{T}; X)$ is a strong $B_{p,q}^s$ -solution of (1.1) if $(x(t) - L(x_t)) \in D(A)$, $Dx_t \in B_{p,q}^{s+1}(\mathbb{T}; X)$ and equation (1.1) holds for a.e $t \in \mathbb{T}$.

We prove the following result.

Lemma 2.3. : Let X be a Banach space and A be a linear closed operator. Suppose that $(ikD_k - AD_k - G_k - \tilde{a}(ik))$ is bounded invertible and $ik(ikD_k - AD_k - G_k - \tilde{a}(ik))^{-1}$ is bounded. Then $\{(ikD_k - AD_k - G_k - \tilde{a}(ik))^{-1}\}_{k \in \mathbb{Z}}$ and $\{ik(ikD_k - AD_k - G_k - \tilde{a}(ik))^{-1}\}_{k \in \mathbb{Z}}$ are M-bounded.

Proof. Let $S_k = ikN_k$, $N_k = (C_k - AD_k)^{-1}$ and $C_k = ikD_k - G_k - \tilde{a}(ik)$.

For convenience, we introduce the following result

$$\begin{aligned} C_k - C_{k+1} &= [ikD_k - G_k - \tilde{a}(ik)] - [i(k+1)D_{k+1} - G_{k+1} - \tilde{a}(i(k+1))] \\ &= ikI - ikL_k - G_k - \tilde{a}(ik) - (i+k)(I - L_{k+1}) + G_{k+1} + \tilde{a}(i(k+1)) \\ &= -iI + iL_{k+1} + ik(L_{k+1} - L_k) + (G_{k+1} - G_k) + (\tilde{a}(i(k+1)) - \tilde{a}(ik)) \end{aligned}$$

Then we have

$$\begin{aligned} k(C_k - C_{k+1}) &= -ikI + ikL_{k+1} + ik(k(L_{k+1} - L_k)) + k(G_{k+1} - G_k) + k(\tilde{a}(i(k+1)) - \tilde{a}(ik)) \\ &= -ikI + ikL_{k+1} + ikQ_k + R_k + P_k \end{aligned}$$

Now, we are going to show that

$$\begin{cases} \sup_k \|k(N_{k+1} - N_k)\| < \infty, \\ \sup_k \|k(S_{k+1} - S_k)\| < \infty \end{cases} \quad (2.3)$$

By hypothesis we have, $\{N_k\}_{k \in \mathbb{Z}}$, $\{S_k\}_{k \in \mathbb{Z}}$, $\{T_k\}_{k \in \mathbb{Z}}$ and $\{F_k\}_{k \in \mathbb{Z}}$ are bounded. Then We have

$$\begin{aligned}
\sup_{k \in \mathbb{Z}} \|k(N_{k+1} - N_k)\| &= \sup_{k \in \mathbb{Z}} \left\| (k[(C_{k+1} - ADk + 1)^{-1} - (C_k - ADk)^{-1}]) \right\| \\
&= \sup_{k \in \mathbb{Z}} \|kN_{k+1}[C_k - ADk - C_{k+1} + AD_{k+1}]N_k\| \\
&= \sup_{k \in \mathbb{Z}} \|kN_{k+1}[(C_k - C_{k+1}) - A(L_{k+1} - L_k)]N_k\| \\
&= \sup_{k \in \mathbb{Z}} \|kN_{k+1}[k(C_k - C_{k+1}) - kA(L_{k+1} - L_k)]N_k\| \\
&= \sup_{k \in \mathbb{Z}} \|N_{k+1}[-ikI + ikL_{k+1} + ikQ_k + R_k + P_k - B_k]N_k\| \\
&= \sup_{k \in \mathbb{Z}} \|[-N_{k+1} + N_{k+1}L_{k+1} + N_{k+1}Q_k]S_k + N_{k+1}R_kN_k + N_{k+1}P_kN_k - N_{k+1}B_kN_k\|.
\end{aligned}$$

We obtain:

$$\sup_{k \in \mathbb{Z}} \|k(N_{k+1} - N_k)\| < \infty \quad (2.4)$$

On the other hand, we have

$$\begin{aligned}
\sup_{k \in \mathbb{Z}} \|k(S_{k+1} - S_k)\| &= \sup_{k \in \mathbb{Z}} \left\| k[i(k+1)(C_{k+1} - AD_{k+1})^{-1} - ik(C_k - AD_k)^{-1}] \right\| \\
&= \sup_{k \in \mathbb{Z}} \|kN_{k+1}[i(k+1)(C_k - A(I - L_k)) - ik(C_{k+1} - A(I - L_{k+1}))]N_k\| \\
&= \sup_{k \in \mathbb{Z}} \|kN_{k+1}[ik(C_k - C_{k+1}) + i(C_k - A(I - L_k)) - ikA(L_{k+1} - L_k)]N_k\| \\
&= \sup_{k \in \mathbb{Z}} \|N_{k+1}[k(C_k - C_{k+1})]ikN_k + ikN_{k+1} - N_{k+1}kA(L_{k+1} - L_k)ikN_k\| \\
&= \sup_{k \in \mathbb{Z}} \|N_{k+1}[k(C_k - C_{k+1})]ikN_k + ikN_{k+1} - N_{k+1}B_kikN_k\| \\
&= \sup_{k \in \mathbb{Z}} \left\| N_{k+1}[-ikI + ikL_{k+1} + ikQ_k + R_k + P_k]S_k + \frac{k}{k+1}S_{k+1} - N_{k+1}B_kS_k \right\| \\
&= \sup_{k \in \mathbb{Z}} \left\| \frac{k}{k+1}S_{k+1}[-I + L_{k+1} + Q_k]S_k + N_{k+1}[R_k + P_k]S_k + \frac{k}{k+1}S_{k+1} - N_{k+1}B_kS_k \right\|
\end{aligned}$$

Then

$$\sup_{k \in \mathbb{Z}} \|k(S_{k+1} - S_k)\| < \infty$$

so, $(N_k)_{k \in \mathbb{Z}}$ and $(S_k)_{k \in \mathbb{Z}}$ are M-bounded. \square

Lemma 2.4. : *Let X be a Banach space and A be a linear closed operator. Suppose that $(ikD_k - AD_k - G_k - \tilde{a}(ik))$ is bounded invertible and $ik(ikD_k - AD_k - G_k - \tilde{a}(ik))^{-1}$ is bounded. Then $\{G_kN_k\}_{k \in \mathbb{Z}}$ and $\{\tilde{a}(ik)N_k\}_{k \in \mathbb{Z}}$ are M-bounded.*

Proof. Let $T_k = G_kN_k$ and $F_k = \tilde{a}(ik)N_k$.

we have

$$\begin{aligned}
\sup_{k \in \mathbb{Z}} \|k(T_{k+1} - T_k)\| &= \sup_{k \in \mathbb{Z}} \|(k[G_{k+1}N_{k+1} - G_kN_k])\| \\
&= \sup_{k \in \mathbb{Z}} \|k(G_{k+1} - G_k)N_{k+1} + G_kk(N_{k+1} - N_k)\| \\
&\leq \sup_{k \in \mathbb{Z}} \|k(G_{k+1} - G_k)N_{k+1}\| + \sup_{k \in \mathbb{Z}} \|G_kk(N_{k+1} - N_k)\| \\
&\leq \sup_{k \in \mathbb{Z}} \|R_kN_{k+1}\| + \sup_{k \in \mathbb{Z}} \|G_k\| \sup_{k \in \mathbb{Z}} \|k(N_{k+1} - N_k)\|
\end{aligned}$$

Then by (2.4) we have

$$\sup_{k \in \mathbb{Z}} \|k(T_{k+1} - T_k)\| < \infty$$

and

$$\begin{aligned}
\sup_{k \in \mathbb{Z}} \|k(F_{k+1} - F_k)\| &= \sup_{k \in \mathbb{Z}} \|(k[\tilde{a}(i(k+1))N_{k+1} - \tilde{a}(ik)N_k])\| \\
&= \sup_{k \in \mathbb{Z}} \|k(N_{k+1} - N_k)\tilde{a}(i(k+1)) + k(\tilde{a}(i(k+1)) - \tilde{a}(ik))N_k\| \\
&= \sup_{k \in \mathbb{Z}} \|k(N_{k+1} - N_k)\tilde{a}(i(k+1)) + F_kN_k\|
\end{aligned}$$

Then by (2.4) we have

$$\sup_{k \in \mathbb{Z}} \|k(F_{k+1} - F_k)\| < \infty$$

So, $(T_k)_{k \in \mathbb{Z}}$ and $(F_k)_{k \in \mathbb{Z}}$ are M-bounded. \square

Theorem 2.5. *Let $1 \leq p, q < \infty$ and $s > 0$. Let X be a Banach space having Fourier type $r \in]1, 2]$ and A be a linear operator. If $(ikD_k - AD_k - G_k - \tilde{a}(ik))$ is bounded invertible and $ik(ikD_k - AD_k - G_k - \tilde{a}(ik))^{-1}$ is bounded. Then for every $f \in B_{p,q}^s(\mathbb{T}, X)$ there exist a unique strong $B_{p,q}^s$ -solution of (1.1).*

Proof. Define $S_k = ikN_k$, $N_k = (ikD_k - AD_k - G_k - \tilde{a}(ik))^{-1}$, $F_k = \tilde{a}(ik)N_k$ and $T_k = G_kN_k$ for $k \in \mathbb{Z}$. Since by Lemma(2.3) and Lemma(2.4), $(S_k)_{k \in \mathbb{Z}}$, $(N_k)_{k \in \mathbb{Z}}$, $(F_k)_{k \in \mathbb{Z}}$ and $(T_k)_{k \in \mathbb{Z}}$ are M-bounded, we have by Theorem 2.1 that $(S_k)_{k \in \mathbb{Z}}$, $(N_k)_{k \in \mathbb{Z}}$, $(F_k)_{k \in \mathbb{Z}}$ and $(T_k)_{k \in \mathbb{Z}}$ are an $B_{p,q}^s$ -multipliers. Since $D_kS_k - AD_kN_k - T_k - \tilde{a}(ik)N_k = I$ (because $(ikD_k - AD_k - G_k - F_k)N_k = I$), we deduce AD_kN_k and D_kS_k are also an $B_{p,q}^s$ -multiplicateur.

Now let $f \in B_{p,q}^s(\mathbb{T}, X)$. Then there exist $u, v, w, q, x \in B_{p,q}^s(\mathbb{T}, X)$, such that

$\hat{u}(k) = N_k\hat{f}(k)$, $\hat{v}(k) = D_kS_k\hat{f}(k)$, $\hat{w}(k) = T_k\hat{f}(k)$, $\hat{x}(k) = F_k\hat{f}(k)$ and $\hat{q}(k) = AD_kN_k\hat{f}(k)$ for all $k \in \mathbb{Z}$. So, We have $(\hat{u}(k) - L_k\hat{u}(k)) \in D(A)$ and $A(\hat{u}(k) - L_k\hat{u}(k)) = \hat{q}(k)$ for all $k \in \mathbb{Z}$, we deduce that $(u(t) - L(u_t)) \in D(A)$. On the other hand $\exists v \in B_{p,q}^s(\mathbb{T}, X)$ such that $\hat{v}(k) = D_kS_k\hat{f}(k) = ikD_kN_k\hat{f}(k) = ikD_k\hat{u}(k)$. Then we obtain $(Du_t)' = v(t)$ a.e. Since $Du_t \in B_{p,q}^{s+1}(\mathbb{T}, X)$.

We have $\widehat{(Du_t)'}(k) = ikD_k\hat{u}(k)$, $(A(u(\cdot) - L(u_\cdot)))^\wedge(k) = AD_k\hat{u}(k)$, $\widehat{Gu}_\cdot(k) = G_k\hat{u}(k)$ and $\int_{-\infty}^t a(t-s)u(s)ds(k) = \tilde{a}(ik)\hat{u}(k)$ for all $k \in \mathbb{Z}$, It follows from the identity

$$ikD_kN_k - AD_kN_k - G_kN_k - \tilde{a}(ik)N_k = I$$

that

$$(u(t) - L(u_t))' = A(u(t) - L(u_t)) + G(u_t) + \int_{-\infty}^t a(t-s)u(s)ds + f(t)$$

For the uniqueness we suppose two solutions u_1 and u_2 , then $u = u_1 - u_2$ is strong L^p -solution of equation (1.1) corresponding to the function $f = 0$, taking Fourier transform, we get $(ikD_k - AD_k - G_k - \tilde{a}(ik))\hat{u}(k) = 0$, which implies that $\hat{u}(k) = 0$ for all $k \in \mathbb{Z}$ and $u(t) = 0$. Then $u_1 = u_2$. The proof is completed. \square

3 Conclusion

We are obtained necessary and sufficient conditions to guarantee existence and uniqueness of periodic solutions to the equation $\frac{d}{dt}[\mathbf{x}(t) - \mathbf{L}(\mathbf{x}_t)] = \mathbf{A}[\mathbf{x}(t) - \mathbf{L}(\mathbf{x}_t)] + \mathbf{G}(\mathbf{x}_t) + \int_{-\infty}^t \mathbf{a}(t-s)\mathbf{x}(s)ds + \mathbf{f}(t)$ in terms of either the M-boundedness of the modified resolvent operator determined by the equation. Our results are obtained in the Besov space.

References

- [1] W. Arendt, S. Bu, *Operator-valued Fourier multipliers on periodic Besov spaces and applications*, Proc. Edinburgh Math. Soc. **47**, (1), 15-33 (2004).
- [2] S. Bu, J. Kim, *Operator-valued Fourier multipliers on periodic Triebel spaces*, Acta Math. Sin. (Engl. Ser.) **21** (2005) 1049-1056.
- [3] A. Bátkai, R. Shvidkoy. *Hyperbolicity of delay equations via Fourier multipliers*. Acta Sci. Math. (Szeged) **69** (2003), 131-145.
- [4] R.Bahloul, M.Bahaj and O.Sidki, *Periodic solutions of degenerate equations with finite delay in UMD space*, journal of Advances in Dynamical Systems and Application. ISSN 0973-5321, Volume 10, Number 1, (2015) pp. 23-31.
- [5] R.Bahloul, *Periodic solutions of differential equations with two variable in vector-valued function space*, Asian Journal of Mathematics and Computer Research, 12(1): 44-53, 2016 ISSN: 2395-4205 (P), ISSN: 2395-4213 (O).
- [6] R.Bahloul, M.Bahaj and O.Sidki, *Existence and uniqueness of periodic solutions integro-differential equations with two variable*, Asian Journal of Mathematics and Computer Research, ISSN No. : 2395-4205 (Print), 2395-4213 (Online), Vol.: 17, Issue.: 3
- [7] K. Ezzinbi, R.Bahloul et O.Sidki, *Periodic Solutions in UMD spaces for some neutral partial function differential equations*, Advances in Pure Mathematics, (2016), 6, 713-726, <http://dx.doi.org/10.4236/apm.2016.610058>.
- [8] Sylvain Koumla, Khalil Ezzinbi and Rachid Bahloul, *mild solutions for some partial functional integrodifferential equations with finite delay in Frechet spaces*. SeMA (2016), P 1-13. DOI 10.1007/s40324-016-0096-7
- [9] S.Bu, “*L^p-Maximal regularity of degenerate delay equations with Periodic Conditions.*” , Banach J.Math.Anal. **8**(2014), no. 2, 49-59;
- [10] A. Bátkai, S. Piazzera. *Semigroups and linear partial differential equations with delay*. J. Math. Anal. Appl. **264** (2001), 1-20.

-
- [11] S.Bu and G.Cai, Mild well-posedness of second order differential equations on the real line, taiwanese journal of mathematics, Vol. 17, No. 1, pp. 143-159, February 2013
- [12] Hernan R.Henriquez, Michelle Pierri and Andrea Prokopczyk Periodic Solutions of abstract neutral functional differential equations, J. Math. Ana. Appl. 385, (2012), 608 - 621
- [13] A. Favini and A. Yagi, Degenerate Differential Equations in Banach Spaces, Pure and Appl.Math., 215, Dekker, New York, Basel, Hong Kong, 1999.
- [14] V. Keyantuo, C. Lizama and V.Poblete, Periodic solutions of integro-differential equations in vector-valued function spaces. J. Differential Equations 246 (2009) 10071037
- [20] Y.Hino, T.Naito, N. Van Minh and J.S.Shin, Almost periodic solution of Differential Equations in Banach Spaces, Taylor and Francis, London, 2002.
- [21] Hale J K. Theory of Functional Differential Equations. New York: Springer-Verlag, 1977
- [22] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, 1983.