Fano-Mukai fourfolds of genus 10 as compactifications of C^4

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FANO-MUKAI FOURFOLDS OF GENUS 10 AS COMPACTIFICATIONS OF $\mathbb{C}^4$

YURI PROKHOROV AND MIKHAIL ZAIDENBERG

Abstract. It is known that the moduli space of smooth Fano-Mukai fourfolds V_{18} of genus 10 has dimension one. We show that any such fourfold is a completion of $\mathbb{C}^4$ in two different ways. Up to isomorphism, there is a unique fourfold V_{18}^{s} acted upon by $\mathrm{SL}_2(\mathbb{C})$. The group $\mathrm{Aut}(V_{18}^{\mathrm{s}})$ is a semidirect product $\mathrm{GL}_2(\mathbb{C})\rtimes(\mathbb{Z}/2\mathbb{Z})$. Furthermore, V_{18}^{s} is a $\mathrm{GL}_2(\mathbb{C})$-equivariant completion of $\mathbb{C}^4$, and as well of $\mathrm{GL}_2(\mathbb{C})$. The restriction of the $\mathrm{GL}_2(\mathbb{C})$-action on V_{18}^{s} to $\mathbb{C}^4 \hookrightarrow V_{18}^{\mathrm{s}}$ yields a faithful representation with an open orbit. There is also a unique, up to isomorphism, fourfold V_{18}^{a} such that the group $\mathrm{Aut}(V_{18}^{\mathrm{a}})$ is a semidirect product $(\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}) \rtimes (\mathbb{Z}/2\mathbb{Z})$. For a Fano-Mukai fourfold V_{18} neither isomorphic to V_{18}^{s}, nor to V_{18}^{a}, the group $\mathrm{Aut}(V_{18})$ is a semidirect product of $(\mathbb{G}_{\mathrm{m}})^2$ and a finite cyclic group whose order is a factor of 6. Besides, we establish that the affine cone over any polarized Fano-Mukai variety V_{18} is flexible in codimension one, and flexible if $V_{18} = V_{18}^{\mathrm{s}}$.

Contents

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1. Introduction

Let V be a compact complex manifold, and let A be a closed analytic subset in V. The pair (V, A) is called a *compactification of* $\mathbb{C}^n$ if $V \setminus A$ is biholomorphically equivalent to $\mathbb{C}^n$. A compactification (V, A) of $\mathbb{C}^n$ is said to be *projective* if V is a smooth projective variety. A celebrated Hirzebruch problem ([Hir54, Problem 27]) asks to describe all possible compactifications of $\mathbb{C}^n$ with $\mathrm{b}_2(V) = 1$. For any projective compactification (V, A) of $\mathbb{C}^n$ the Kodaira dimension of V is negative (see [Kod72, Theorem 3]). It is unknown, however, whether the complement $V \setminus A$ must be automatically biregularly isomorphic to $\mathbb{C}^n$. There is a related open problem ([Zai93]) on existence of an affine algebraic variety X non-isomorphic to $\mathbb{C}^n$ biregularly, but analytically isomorphic to $\mathbb{C}^n$.

In this paper we deal with projective compactifications (V, A) of $\mathbb{C}^n$ with $\mathrm{b}_2(V) = 1$ and with $V \setminus A$ biregularly isomorphic to $\mathbb{C}^n$. Then V is a Fano manifold with Picard number one. In the first nontrivial case $n = 3$ the classification was completed in a series of papers [PS88], [Fur90], [Pro91], [Fur93], see also the references therein. Such a threefold V is one of the following:

- $\mathbb{P}^3$;
- a smooth quadric $Q \subset \mathbb{P}^4$;
- a del Pezzo quintic threefold $V_5 \subset \mathbb{P}^6$;
- a Fano threefolds $V_{22} \subset \mathbb{P}^{13}$ varying in a proper subset of codimension 2 in the moduli space.

All the possibilities for the divisor A are also described.

The four-dimensional case is much more complicated. Indeed, already the very classification of smooth Fano fourfolds is still lacking.

Recall that the *Fano index* $\iota(V)$ of a Fano manifold V is the largest integer ι such that the canonical class K_V is divisible by ι in the Picard group. It is well known that $1 \le \iota(V) \le \dim V + 1$. Furthermore, $\iota(V) = \dim V + 1$ if and only if $V \cong \mathbb{P}^n$ and $\iota(V) = \dim V$ if and only if V is a quadric $Q^n \subset \mathbb{P}^{n+1}$. This implies immediately that $(\mathbb{P}^n, \mathbb{P}^{n-1})$ and (Q^n, Q'), where Q' is a singular hyperplane section of the quadric Q^n, are the only projective compactifications of $\mathbb{C}^n$ with $\mathrm{b}_2(V) = 1$ and $\iota(V) \ge n$.

A Fano manifold V with $\operatorname{rk}\operatorname{Pic}(V) = 1$ is called a *del Pezzo manifold* if $\iota(V) = \dim V - 1$ and a *Fano-Mukai manifold* if $\iota(V) = \dim V - 2$. Compactifications (V, A) of $\mathbb{C}^4$ with $\mathrm{b}_2(V) = 1$ and $\iota(V) = 3$ were classified in [Pro94]. Such a del Pezzo fourfold $V = W_5 \subset \mathbb{P}^7$ is unique up to isomorphism, and there are exactly four possible choices for a divisor $A \subset V$ with $V \setminus A \cong \mathbb{C}^4$. All such divisors A are singular. The singular locus of A is either a plane, or a special line of one of two possible types, or finally a unique ordinary double point.

In [PZ15, § 6.3] we asked the following question:

Which Fano-Mukai fourfolds can serve as compactifications of $\mathbb{C}^4$*?*

We give the answer for the Fano-Mukai fourfolds $V_{18} \subset \mathbb{P}^{12}$ of genus 10 and Fano index 2. According to [Muk89] there is a homogeneous space $\mathrm{G}_2/P \subset \mathbb{P}^{13}$ of the simple algebraic group of type G_2 such that any variety V_{18} is isomorphic to a hyperplane section of G_2/P. Up to isomorphism, such varieties V_{18} form a one-parameter family, see Remark 13.4. It occurs that all of them are compactifications of $\mathbb{C}^4$, see Theorem 1.1. There are two distinguished members of the family. One of them, denoted by V_{18}^{s}, is quasihomogeneous with respect to a $\mathrm{GL}_2(\mathbb{C})$-action and yields a $\mathrm{GL}_2(\mathbb{C})$-equivariant compactification of $\mathbb{C}^4$

and of $\mathrm{GL}_2(\mathbb{C})$. Another one, denoted by V_{18}^{a}, is acted upon by the product $\mathbb{G}_{\mathrm{a}}\times\mathbb{G}_{\mathrm{m}}$. This group is the identity component of $\mathrm{Aut}(V_{18}^{\mathrm{a}})$ and has index two in the full automorphism group. Any other smooth member V_{18} of the family is acted upon by the torus of rank two, see Theorem 1.3 below. More formally, our main results are the following three theorems.

1.1. **Theorem.** *Any smooth Fano-Mukai fourfold $V = V_{18} \subset \mathbb{P}^{12}$ of genus 10 contains at least two distinct $\mathrm{Aut}^0(V)$-invariant cones over rational twisted cubic curves. If $S \subset V$ is such a cone, then there is a unique $\mathrm{Aut}^0(V)$-invariant hyperplane section $A = A_S$ of V with $\mathrm{Sing}(A) = S$ such that (V, A) is an $\mathrm{Aut}^0(V)$-equivariant compactification of $\mathbb{C}^4$.*

1.2. **Theorem.** *Given a Fano-Mukai fourfold $V = V_{18} \subset \mathbb{P}^{12}$ of genus 10 consider a Mukai realization of V as a hyperplane section of $\mathrm{G}_2/P \hookrightarrow \mathbb{P}^{13}$, see Section 7. Then the $\mathrm{Aut}^0(V)$-action on V extends to an $\mathrm{Aut}^0(V)$-action on G_2/P induced by the natural G_2-action.*

1.3. **Theorem.** (i) *There exists a smooth Fano-Mukai fourfold V_{18}^{s} of genus 10 with*

$$\mathrm{Aut}(V_{18}^{\mathrm{s}}) \cong \mathrm{GL}_2(\mathbb{C}) \rtimes (\mathbb{Z}/2\mathbb{Z}), \tag{1.3.1}$$

where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{GL}_2(\mathbb{C})$ via $M \mapsto (M^t)^{-1}$. Furthermore, $\mathrm{Aut}^0(V_{18}^{\mathrm{s}}) \cong \mathrm{GL}_2(\mathbb{C})$ has a principal dense open orbit in V_{18}^{s} and exactly two fixed points. Any Fano-Mukai fourfold V_{18} of genus 10 whose automorphism group has non-abelian identity component is isomorphic to V_{18}^{s}.

(ii) *There exists a smooth Fano-Mukai fourfold V_{18}^{a} of genus 10 with*

$$\mathrm{Aut}(V_{18}^{\mathrm{a}}) \cong (\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}) \rtimes (\mathbb{Z}/2\mathbb{Z}), \tag{1.3.2}$$

where the generator of $\mathbb{Z}/2\mathbb{Z}$ acts by the inversion $g \mapsto g^{-1}$ on $\mathbb{G}_{\mathrm{a}}\times\mathbb{G}_{\mathrm{m}}$. Such a fourfold is unique up to isomorphism.

(iii) *For any smooth Fano-Mukai fourfold V_{18} of genus 10 non-isomorphic to one of the V_{18}^{s} and V_{18}^{a} one has*

$$(\mathbb{G}_{\mathrm{m}})^2 \subset \mathrm{Aut}(V_{18}) \subset (\mathbb{G}_{\mathrm{m}})^2 \rtimes (\mathbb{Z}/6\mathbb{Z}). \tag{1.3.3}$$

See also Theorems 12.1 and 13.5 for some additional information. In particular, it occurs that the fourfold V_{18}^{s} contains two one-parameter families of cones over twisted cubics and a unique pair of $\mathrm{Aut}^0(V)$-invariant such cubic cones. The number of cubic cones in V_{18}^{a} equals 4, and equals 6 in $V_{18} \ncong V_{18}^{\mathrm{s}}, V_{18}^{\mathrm{a}}$. Any such cone S defines an $\mathrm{Aut}^0(V)$-invariant compactification (V_{18}, A_S) of $\mathbb{C}^4$ as in Theorem 1.1.

The proofs of Theorems 1.1–1.3 are done in Section 13. They use a construction of V_{18} starting with $\mathbb{P}^4$ via a sequence of two Sarkisov links, see Sections 2–4 for details. The first Sarkisov link gives the quintic del Pezzo fourfold $W_5 \subset \mathbb{P}^7$. In Sections 5–6 we study the automorphism group of W_5 (cf. [PdV99]), its action on W_5, and the stabilizers of certain rational normal quintic scrolls in W_5. Such a scroll F serves as the center of blowup for the second Sarkisov link. For a properly chosen F the stabilizer of F in $\mathrm{Aut}^0(W_5)$ is isomorphic to the automorphism group $\mathrm{Aut}^0(V_{18})$ of the resulting Fano-Mukai fourfold V_{18}. On the other hand, the construction of S. Mukai ([Muk88]) embeds any Fano-Mukai fourfold V_{18} onto a hyperplane section of $\mathrm{G}_2/P \subset \mathbb{P}^{13}$. Using results of M. Kapustka and K. Ranestad ([KR13]) we compute in Section 7 the stabilizers of these hyperplane sections as subgroups of $\mathrm{Aut}^0(V_{18})$. It remains to show that such a stabilizer coincides actually with the whole group $\mathrm{Aut}^0(V_{18})$, cf. Theorem 1.2. To this end, following again ([KR13]) we study in detail the Hilbert schemes of lines and of rational normal cubic scrolls on V_{18} (see Sections 8–9), and the subschemes of rational cubic cones (see Section 10). The latter cones occur to be in one-to-one correspondence with the lines on certain singular del Pezzo sextic surfaces. This geometry allows us to describe the automorphism groups of the fourfolds V_{18}, see Sections 11-12. This leads finally in Section 13 to our main results. In Section 14 (see Theorem 14.3) we deduce the flexibility in codimension one of the affine cones over the Fano-Mukai fourfolds V_{18} and the flexibility if $V_{18} = V_{18}^{\mathrm{s}}$, cf. [AFK$^+$13]. The concluding Section 15 contains some remarks and open problems.

2. Linking the del Pezzo fourfold W_5 to $\mathbb{P}^4$

2.1. In this and the next sections $W = W_5 \subset \mathbb{P}^7$ stands for a del Pezzo quintic fourfold realized as a smooth section of the Grassmannian $\mathrm{Gr}(2,5)$ under its Plücker embedding in $\mathbb{P}^9$ by a general linear subspace of codimension 2 in $\mathbb{P}^9$. In fact, a del Pezzo quintic fourfold is unique up to isomorphism ([Fuj81]). We use the following description of the planes in W (see the classical paper [Tod30] for more details and [PdV99, § 6] for a modern treatment).

2.1.1. We regard the Grassmannian $\mathrm{Gr}(2,5)$ as the variety of lines in $\mathbb{P}^4 = \mathbb{P}(\mathbb{C}^5)$. Recall that any plane in $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$ is a Schubert variety of one of the following two types :

- $\sigma_{2,2}$, that is, the Schubert variety of lines in a fixed plane $\mathbb{P}^2 \subset \mathbb{P}^4$;
- $\sigma_{3,1}$, that is, the Schubert variety of lines passing through a fixed point and contained in a fixed 3-space $\mathbb{P}^3 \subset \mathbb{P}^4$.

2.1.2. Let $\mathscr{P}$ be a pencil of hyperplane sections which cut out a del Pezzo quintic fourfold W in $\mathrm{Gr}(2,5) \subset \mathbb{P}^9$. It can be treated as a pencil of skew-symmetric bilinear forms $\lambda_1 q_1 + \lambda_2 q_2 \in (\wedge^2 \mathbb{C}^5)^\vee$. Since W is smooth, each form $\lambda_1 q_1 + \lambda_2 q_2 \in \mathscr{P}$ is of rank 4. Consider the map $\upsilon : \mathscr{P} \to \mathbb{P}(\mathbb{C}^5)$ that sends a form $\lambda_1 q_1 + \lambda_2 q_2$ to the projectivization of its kernel $\ker(\lambda_1 q_1 + \lambda_2 q_2) \subset \mathbb{C}^5$. This map is given by the Pfaffians of the corresponding matrix. Hence the image $\upsilon(\mathscr{P})$ is a conic in $\mathbb{P}^4 = \mathbb{P}(\mathbb{C}^5)$ ([PdV99, Prop. 6.3]). The linear span $\Theta = \langle \upsilon(\mathscr{P}) \rangle$ is a plane in $\mathbb{P}(\mathbb{C}^5)$, which is a maximal common isotropic subspace for the forms $\lambda_1 q_1 + \lambda_2 q_2 \in \mathscr{P}$. Such a subspace is unique. This defines a unique $\sigma_{2,2}$-plane $\Xi \subset W$ parameterizing the lines in $\Theta \cong \mathbb{P}^2$. On the other hand, there is a one-parameter family of $\sigma_{3,1}$-planes Π_γ parameterizing the lines passing through a point $P_\gamma \in \upsilon(\mathscr{P})$ and contained in the three-dimensional subspace in $\mathbb{P}^4$ orthogonal to P_γ with respect to any form $\lambda_1 q_1 + \lambda_2 q_2 \in \mathscr{P}$. Let $\Upsilon \subset \Xi$ be the dual conic of $\upsilon(\mathscr{P}) \subset \Theta$. Each plane Π_γ meets Ξ along a tangent line l_γ to Υ at a point $\gamma \in \Upsilon$. Any two distinct planes Π_γ and $\Pi_{\gamma'}$ meet at a unique point $l_\gamma \cap l_{\gamma'}$ on $\Xi \setminus \Upsilon$.

The planes $\{\Pi_\gamma\}_{\gamma \in \Upsilon}$ and Ξ are the only planes contained in W. The union $R = \bigcup_{\gamma \in \Upsilon} \Pi_\gamma$ is a hyperplane section of W. The threefold R contains also Ξ and is singular along Ξ (see, e.g., [Tod30], [PZ16, Prop. 3.4] and the references therein). The triple $W \supset R \supset \Xi$ plays an important role in what follows.

Consider the following Sarkisov link (for the proofs, see [Fuj81], [Pro94], [PZ16, Prop. 4.9] and the references therein).

2.2. **Proposition.** *In the notation as before, the following hold.*

(i) *There is a commutative diagram*

$$
\begin{array}{ccccccccccc}
 & \widehat{E} & \subset & & \widehat{W} & & & \supset & \widehat{R} & & \\
 & \swarrow & & \rho\swarrow & & \searrow\varphi & & & & \searrow & \\
\Xi & \subset & R & \subset & W & \overset{\phi}{\dashrightarrow} & \mathbb{P}^4 & \supset & E = \langle\Gamma\rangle & \supset & \Gamma
\end{array}
\tag{2.2.1}
$$

where ρ is the blowup of the plane Ξ in W and φ is the blowup of a rational twisted cubic curve $\Gamma \subset \mathbb{P}^4$.

(ii) *The ρ-exceptional divisor $\widehat{E} = \rho^{-1}(\Xi)$ is the proper transform in $\widehat{W}$ of the linear span $\langle\Gamma\rangle \cong \mathbb{P}^3$ of Γ in $\mathbb{P}^4$. The φ-exceptional divisor $\widehat{R}$ is the proper transform of R in $\widehat{W}$.*

(iii) *The morphism φ (ρ, respectively) is defined by the linear system $|H^* - \widehat{E}|$ ($|2L^* - \widehat{R}|$, respectively) on $\widehat{W}$, where H (L, respectively) is the class of hyperplane section on W (on $\mathbb{P}^4$, respectively), and $H^* = \rho^*(H)$, $L^* = \varphi^*(L)$. The birational map $\phi : W \subset \mathbb{P}^7 \dashrightarrow \mathbb{P}^4$ is the linear projection with center Ξ.*

2.2.2. **Corollary** ([Pro94])**.** *(W, R) is a compactification of $\mathbb{C}^4$.*

Proof. Indeed, due to (2.2.1) we have isomorphisms

$$W \setminus R \cong \widehat{W} \setminus (\widehat{R} \cup \widehat{E}) \cong \mathbb{P}^4 \setminus \langle\Gamma\rangle \cong \mathbb{C}^4. \tag{2.2.3}$$

□

2.2.4. **Corollary.** [1] *There is an isomorphism* $\mathrm{Aut}(W) \cong \mathrm{Aut}(\mathbb{P}^4, \Gamma)$.

Proof. Indeed, the plane $\Xi \subset W$ is $\mathrm{Aut}(W)$-invariant. Due to diagram (2.2.1) one has $\mathrm{Aut}(W) = \mathrm{Aut}(W, \Xi) \cong \mathrm{Aut}(\mathbb{P}^4, \Gamma)$. □

2.3. **Lemma.** *In the notation as before, the following hold.*

(a) *$\varphi|_{\widehat{R}} : \widehat{R} \to \Gamma$ is a $\mathbb{P}^2$-bundle.*

(b) *For any $\gamma \in \Gamma$, the fiber $\widehat{\Pi}_\gamma := (\varphi|_{\widehat{R}})^{-1}(\gamma) \subset \widehat{R}$ is sent by ρ isomorphically onto a plane $\Pi_\gamma \subset R$.*

(c) *$\rho|_{\widehat{R}} : \widehat{R} \to R$ is the normalization morphism.*

(d) *Let $\widehat{\Xi} := \widehat{E} \cap \widehat{R} = (\rho|_{\widehat{R}})^{-1}(\Xi)$. Then $\widehat{\Xi} \cong \mathbb{P}^1 \times \mathbb{P}^1$ and $\varphi|_{\widehat{\Xi}} : \widehat{\Xi} \to \Xi$ is a double cover branched along the conic $\Upsilon \subset \Xi$ (see (2.1.2)).*

Proof. (a) follows since $\widehat{R} = \mathbb{P}(\mathscr{N}^\vee_{\Gamma/\mathbb{P}^4})$ is the exceptional divisor of the blowup of Γ in $\mathbb{P}^4$.

(b) Let L be a hyperplane in $\mathbb{P}^4$. On $\widehat{W}$ one has ([PZ16, Prop. 4.9]):

$$L^* \sim H^* - \widehat{E} \quad \text{and} \quad \widehat{R} \sim H^* - 2\widehat{E}.$$

Hence φ is defined by the linear system $|H^* - \widehat{E}|$ on $\widehat{W}$, and

$$H^* \sim 2L^* - \widehat{R} \quad \text{and} \quad \widehat{E} \sim L^* - \widehat{R}. \tag{2.3.1}$$

So, ρ is the morphism given by the base point free linear system $|2L^* - \widehat{R}|$ on $\widehat{W}$, and the map ϕ^{-1} in (2.2.1) is defined by the family of quadrics in $\mathbb{P}^4$ passing through Γ.

We have: $\mathscr{O}_{\widehat{R}}(\widehat{R}) = \mathscr{O}_{\mathbb{P}(\mathscr{N}^\vee_{\Gamma/\mathbb{P}^4})}(-1)$. Since $\mathscr{O}_{\widehat{\Pi}_\gamma}(L^*) = \mathscr{O}_{\widehat{\Pi}_\gamma}$, we obtain

[1] Cf. Lemma 5.4.

$$\mathscr{O}_{\widehat{\Pi}_\gamma}(H^*) = \mathscr{O}_{\widehat{\Pi}_\gamma}(2L^* - \widehat{R}) \cong \mathscr{O}_{\mathbb{P}^2}(1).$$

It follows that

$$H^2 \cdot \Pi_\gamma = (H^*)^2 \cdot \widehat{\Pi}_\gamma = 1.$$

Therefore, $\{\Pi_\gamma\}$ is a family of planes in $\mathbb{P}^7$ that covers R. However, there is just one such family, see 2.1.2. This proves (b).

(c) $\widehat{R}$ is smooth being an exceptional divisor of the blowup φ with a smooth center. Hence to establish (c) it suffices to show that $\rho|_{\widehat{R}} : \widehat{R} \to R$ is a finite birational morphism. Suppose that $\varphi|_{\widehat{R}}$ contracts a curve $J \subset \widehat{R}$. Clearly, J is not contained in a fiber $\cong \mathbb{P}^2$ of $\rho|_{\widehat{R}}$. Therefore, J meets any fiber of $\rho|_{\widehat{R}}$. But then the planes Π_γ should pass all through a common point $\rho(J)$, which is not the case (see 2.1.2). Since ρ is a birational morphism of smooth varieties, $\rho|_{\widehat{R}}$ is birational.

(d) By construction, $\widehat{\Xi} = \widehat{E} \cap \widehat{R}$ is the exceptional divisor of the blowup $\varphi|_{\widehat{E}} : \widehat{E} \to \langle\Gamma\rangle \cong \mathbb{P}^3$ with center Γ. Since $\mathscr{N}_{\Gamma/\mathbb{P}^3} \cong \mathscr{O}_{\mathbb{P}^1}(5) \oplus \mathscr{O}_{\mathbb{P}^1}(5)$ (see e.g. [EvdV81]), we have $\widehat{E} \cap \widehat{R} \cong \mathbb{P}\big(\mathscr{N}^\vee_{\Gamma/\mathbb{P}^3}\big) \cong \mathbb{P}^1 \times \mathbb{P}^1$. On $\widehat{W}$ one has $L^{*2} \cdot \widehat{R}^2 = 0 = L^{*3} \cdot \widehat{R}$. By (2.3.1) and [PZ16, Lem. 2.3, Lem. 4.6] the degree of the morphism $\rho|_{\widehat{\Xi}} : \widehat{\Xi} \to \Xi$ equals

$$H^{*2} \cdot \widehat{E} \cdot \widehat{R} = (2L^* - \widehat{R}) \cdot (L^* - \widehat{R}) \cdot \widehat{R} = 5\widehat{R}^3 \cdot L^* - \widehat{R}^4 = 2.$$

It follows that $\rho|_{\widehat{\Xi}} : \widehat{\Xi} \to \Xi$ is equivalent to a covering $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^2$ of degree 2 branched along a smooth conic. Furthermore, $\varphi|_{\widehat{\Xi}} : \widehat{\Xi} \to \Gamma$ is equivalent to a canonical projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$, with fibers being the lines $\widehat{\Pi}_\gamma \cap \widehat{\Xi}$. These lines are sent by ρ to the tangent lines $\Pi_\gamma \cap \Xi$ of Υ. Hence Υ is the branching divisor of $\widehat{\Xi} \to \Xi$ and the image of the ramification divisor of type $(1,1)$ on $\widehat{\Xi} \cong \mathbb{P}^1 \times \mathbb{P}^1$. Now (d) follows. ☐

2.4. **Lemma.** *There exists a smooth rational curve $\widehat{\Gamma} \subset \widehat{R}$ such that*

- *the restrictions $\varphi|_{\widehat{\Gamma}} : \widehat{\Gamma} \to \Gamma$ and $\rho|_{\widehat{\Gamma}} : \widehat{\Gamma} \to \rho(\widehat{\Gamma})$ are isomorphisms;*
- *$\rho(\widehat{\Gamma}) \subset R$ is a twisted cubic;*
- *$\widehat{E} \cap \widehat{\Gamma} = \varnothing$ and $\big\langle \rho(\widehat{\Gamma}) \big\rangle \cap \Xi = \varnothing$.*

Proof. Fix a point $P \in \mathbb{P}^4 \setminus \langle\Gamma\rangle$. Let $N \subset \mathbb{P}^4$ be the cone over Γ with vertex P, and let $\widehat{N}$ be the proper transform of N in $\widehat{W}$. Consider the curve $\widehat{\Gamma} = \widehat{N} \cap \widehat{R}$. We claim that $\widehat{\Gamma}$ satisfies the conditions of the lemma. Indeed, since $\Gamma \subset N$ is a Cartier divisor, $\varphi|_{\widehat{N}} : \widehat{N} \to N$ is an isomorphism, and $\varphi(\widehat{\Gamma}) = \Gamma$. Hence $\varphi|_{\widehat{\Gamma}} : \widehat{\Gamma} \to \Gamma$ is an isomorphism as well.

Furthermore, one has $\Gamma = N \cap \langle\Gamma\rangle$, and N meets $\langle\Gamma\rangle$ transversely along Γ. It follows that $\widehat{E}$ and $\widehat{\Gamma}$ are disjoint. Since $\widehat{E}$ is the exceptional divisor of ρ and $\rho(\widehat{E}) = \Xi$, we conclude that $\rho|_{\widehat{\Gamma}} : \widehat{\Gamma} \to \rho(\widehat{\Gamma})$ is an isomorphism, and $\rho(\widehat{\Gamma}) \cap \Xi = \varnothing$. Moreover, since $\rho(\widehat{\Gamma})$ does not meet the center Ξ of the birational projection $\phi : W \dashrightarrow \mathbb{P}^4$, the restriction $\phi|_{\rho(\widehat{\Gamma})} : \rho(\widehat{\Gamma}) \to \Gamma$ is an isomorphism preserving the degree. Therefore, $\rho(\widehat{\Gamma})$ is a twisted cubic, since $\Gamma \subset \mathbb{P}^4$ is.

Finally, if Ξ meets $\langle\rho(\widehat{\Gamma})\rangle \cong \mathbb{P}^3$, then $\rho(\widehat{\Gamma})$ has a 2-secant line meeting Ξ. However, in the latter case $\phi|_{\rho(\widehat{\Gamma})} : \rho(\widehat{\Gamma}) \to \Gamma$ cannot be an isomorphism, a contradiction. ☐

3. Linking the Fano-Mukai fourfolds V_{18} to W_5

Any smooth Fano-Mukai fourfold V_{18} of genus 10 can be obtained starting with W_5 via a Sarkisov link. This link can be described as follows.

3.1. **Proposition.** *Let $V = V_{18} \subset \mathbb{P}^{12}$ be a smooth Fano-Mukai fourfold, and let $S \subset V$ be either a smooth two-dimensional cubic scroll, or a cone over a rational twisted cubic curve*[2]. *Then $V \cap \langle S \rangle = S$ as a scheme, and there is a Sarkisov link*

(3.1.1)
$$
\begin{array}{ccccccccc}
 & \widetilde{B} & \subset & & \widetilde{W} & & \supset & \widetilde{A} & \\
 \swarrow & & & \xi \swarrow & & \searrow \eta & & & \searrow \\
S = \langle S \rangle \cap V \subset A & & \subset & V & \overset{\theta}{\dashrightarrow} & W & \supset & B = W \cap \langle F \rangle & \supset F
\end{array}
$$

where

- *ξ is the blowup of S in V with exceptional divisor $\widetilde{B}$;*
- *η is the blowup of a smooth rational quintic scroll $F \subset W$ with exceptional divisor $\widetilde{A}$;*
- *ξ sends $\widetilde{A}$ to a hyperplane section A of V with $\operatorname{Sing}(A) = S$, and η sends $\widetilde{B}$ to the hyperplane section $B = W \cap \langle F \rangle$;*
- *the map $\theta : V \dashrightarrow W = W_5$ comes from the linear projection $\mathbb{P}^{12} \dashrightarrow \mathbb{P}^7$ with center $\langle S \rangle \cong \mathbb{P}^4$.*

Proof. The proof proceeds in several steps.

3.1.2. **Claim.** *We have $V \cap \langle S \rangle = S$ as a scheme.*

Proof. It is well known that any linearly non-degenerate surface of degree 3 in $\mathbb{P}^4$ is an intersection of three quadrics. On the other hand, $V = V_{18} \subset \mathbb{P}^{12}$ is an intersection of quadrics too (see [Isk77, Lem. 2.10]). Let $\mathscr{Q} \subset H^0(\mathbb{P}^{12}, \mathscr{O}_{\mathbb{P}^{12}}(2))$ be the linear system of quadrics passing through V, and let $\mathscr{Q}'$ be the restriction of $\mathscr{Q}$ to $\langle S \rangle$. Then $\mathscr{Q}'$ cuts out on $\langle S \rangle$ a scheme $V \cap \langle S \rangle$ of dimension 2 containing S. Hence $2 \le \dim \mathscr{Q}' \le 3$. Suppose that $\dim \mathscr{Q}' = 2$, that is, $\mathscr{Q}'$ is generated by two linearly independent quadrics $Q_1, Q_2 \in \mathscr{Q}'$. Then $Q_1 \cap Q_2 = S \cup \Pi$, where $\Pi \subset V$ is the residual plane. However, a Fano-Mukai fourfold of genus 10 does not contain any plane ([KR13, Lem. 3]). Hence $\dim \mathscr{Q}' = 3$. So, any quadric in $\langle S \rangle$ containing S is a member of $\mathscr{Q}'$. Since these quadrics cut out in $\mathbb{P}^4$ the surface S, it follows that $S = V \cap \langle S \rangle$ (as a scheme). □

3.1.3. **Claim.** *There exists a line l on V meeting S.*

Proof. Let $L = V \cap \mathbb{P}^{11}$ be a general hyperplane section of V. Then L meets S along a twisted cubic curve, say, Γ'. By the adjunction formula and the Lefschetz hyperplane section theorem, $L \subset \mathbb{P}^{11}$ is an anticanonically embedded Fano threefold with $\operatorname{Pic} L = \mathbb{Z} \cdot K_L$. It is known ([IP99, Prop. 4.2.2]) that $L \subset \mathbb{P}^{11}$ carries a one-parameter family of lines. Since $\operatorname{Pic} L \cong \mathbb{Z}$, the surface $T \subset L$ swept up by these lines meets Γ'. Now the claim follows. □

The next claim concludes the proof of Proposition 3.1.

[2]Such cones are the only cubic cones in V, see 4.5. Hence in the sequel we call them simply cubic cones. We will show that any smooth variety V_{18} contains a cubis scroll or a cubic cone.

3.1.4. **Claim.** *Consider the linear projection* $\theta : \mathbb{P}^{12} \supset V \dashrightarrow W := \theta(V) \subset \mathbb{P}^7$ *with center* $\langle S \rangle \cong \mathbb{P}^4$. *Consider also the blowup* $\xi : \widetilde{W} \to V$ *with center* S. *Then* $W \subset \mathbb{P}^7$ *is a smooth del Pezzo quintic fourfold, and there is a morphism* $\eta : \widetilde{W} \to W$ *such that these objects and morphisms fit in diagram* (3.1.1).

Proof. Let $\widetilde{B}$ be the exceptional divisor of the blowup $\xi : \widetilde{W} \to V$ with center S. One can check that $\widetilde{W}$ is smooth. Let L be a hyperplane section of V. Consider the proper transform $|L^* - \widetilde{B}|$ on $\widetilde{W}$ of the linear system of hyperplanes in $\mathbb{P}^{12}$ passing through S. It is nef and base point free. The morphism $\eta = \Phi_{|L^*-\widetilde{B}|} : \widetilde{W} \to W \subset \mathbb{P}^7$ resolves indeterminacies of θ. Using [PZ16, Lem. 2.3] one computes $(L^* - \widetilde{B})^4 = 5$ (cf. (3.1.9) and the subsequent paragraph). Thus, $\deg W = 5$, and so, η is birational. The divisor $-K_{\widetilde{W}} = (L^* - \widetilde{B}) + L^*$ is ample being the sum of two non-proportional nef divisors. Let $\eta : \widetilde{W} \xrightarrow{\eta'} W' \longrightarrow W$ be the Stein factorization. Then η' is either an isomorphism, or an extremal Mori contraction. Let, as before, H be a hyperplane section of $W \subset \mathbb{P}^7$, and let $H^* = \eta^* H$. Thus, $H^* \sim L^* - \widetilde{B}$ on $\widetilde{W}$.

Let $\widetilde{l} \subset \widetilde{W}$ be the proper transform of a line $l \subset V$ which meets S properly, see 3.1.3. Then $\widetilde{B} \cdot \widetilde{l} \geq 1$ and $L^* \cdot \widetilde{l} = 1$. Hence

$$H^* \cdot \widetilde{l} = (L^* - \widetilde{B}) \cdot \widetilde{l} \leq 0.$$

Since the linear system $|H^*|$ is base point free, we have

$$H^* \cdot \widetilde{l} = 0, \qquad \widetilde{B} \cdot \widetilde{l} = 1, \qquad -K_{\widetilde{W}} \cdot \widetilde{l} = 1. \tag{3.1.5}$$

It follows that a non-ample, nef divisor H^* supports an extremal ray $\widetilde{l}$ contracted by η. In particular, η' is not an isomorphism.

Suppose that η' has a two-dimensional fiber, say, $\widetilde{M}$. By the main theorem in [AW98], $\widetilde{M}$ is isomorphic either to $\mathbb{P}^2$, or to a quadric $Q \subset \mathbb{P}^3$. Moreover, $\mathscr{O}_{\widetilde{M}}(-K_{\widetilde{W}}) \cong \mathscr{O}_{\mathbb{P}^2}(1)$, or, respectively, $\mathscr{O}_{\widetilde{M}}(-K_{\widetilde{W}}) \cong \mathscr{O}_Q(1)$ ([AW98, Prop. 4.11]). On the other hand, $\mathscr{O}_{\widetilde{M}}(-K_{\widetilde{W}}) \cong \mathscr{O}_{\widetilde{M}}(L^*)$ (cf. (4.4.3)). Therefore, $\xi(\widetilde{M})$ is either a plane, or a quadric surface in V. However, V contains neither a plane, nor a quadric surface, see [KR13, Lem. 3 and Cor. 3]. Thus, η' has no two-dimensional fiber, and so, it is not flipping (see e.g. [AW98, Main Theorem]). Thus, η' contracts a divisor, say $\widetilde{A} \subset \widetilde{W}$, to a surface, say, $F' = \eta'(\widetilde{A}) \subset W'$. Again by the main theorem in [AW98], W' and $\widetilde{A}$ are smooth, and $\eta' : \widetilde{W} \to W'$ is the blowup of a smooth surface $F' \subset W'$.

We have $-K_{\widetilde{W}} = 2H^* + \widetilde{B}$. Set $H' = \eta'_* H^*$ and $B' = \eta'_* \widetilde{B}$. Then $-K_{W'} = 2H' + B'$. Since $\operatorname{rk} \operatorname{Pic} W' = 1$, the Fano index of W' is at least 3. Since $H'^4 = H^{*4} = 5$, H' is not divisible. It follows from the classification that W' is a del Pezzo fourfold of degree 5 and $H' \sim B'$ ([Fuj81]). Moreover, $H' = -\frac{1}{3} K_{W'}$ is very ample. Hence $W' \to W$ is an isomorphism. Thus we have $-K_W \sim 3H$ and

$$-K_{\widetilde{W}} \sim 3H^* - \widetilde{A} \sim 2H^* + \widetilde{B}, \quad \widetilde{B} \sim H^* - \widetilde{A}. \tag{3.1.6}$$

It remains to show that the smooth surface $F \subset W$ is a rational quintic scroll. On $\widetilde{W}$ we have $\widetilde{B} \sim H^* - \widetilde{A}$ due to (3.1.6). Since V is smooth and $S \subset V$ has at worst isolated singularities, the intersection numbers $(L^*)^i \cdot \widetilde{B}^{4-i}$ for $i > 0$ can be computed in a usual way (see, e.g., [PZ16, Lem. 2.3]). This gives:

$$(L^*)^4 = 18, \quad (L^*)^3 \cdot \widetilde{B} = 0, \quad (L^*)^2 \cdot \widetilde{B}^2 = -3, \quad L^* \cdot \widetilde{B}^3 = -1. \tag{3.1.7}$$

Since $H^* \sim L^* - \widetilde{B}$ and, by (3.1.6),

$$\widetilde{A} \sim L^* - 2\widetilde{B}, \tag{3.1.8}$$

then the equality $(H^*)^3 \cdot \widetilde{A} = 0$ reads

$$(L^* - \widetilde{B})^3 \cdot (L^* - 2\widetilde{B}) = 0. \tag{3.1.9}$$

Expressing $\widetilde{B}^4$ via the intersection numbers $(L^*)^i \cdot \widetilde{B}^{4-i}$, $i = 1, \ldots, 4$, from (3.1.7) and (3.1.9) one gets $\widetilde{B}^4 = 1$. Furthermore,

$$\deg F = -(H^*)^2 \cdot \widetilde{A}^2 = -(L^* - \widetilde{B})^2 \cdot (L^* - 2\widetilde{B})^2 = 5.$$

Since $\widetilde{B} \sim H^* - \widetilde{A}$, the divisor $B := \eta(\widetilde{B})$ is a hyperplane section of W passing through F. For any such hyperplane section, say, D, the proper transform $\widetilde{D}$ of D in $\widetilde{W}$ belongs to the linear system $|B|$. However, this linear system contains just a single member $\widetilde{B}$. Indeed, the ξ-exceptional divisor $\widetilde{B}$ is covered by lines with negative intersection with $\widetilde{B}$, and so, is not movable. Therefore, there is just one hyperplane section B in W through F, that is, $\langle F \rangle \cap W = B$. Finally, $F \subset \langle F \rangle \cong \mathbb{P}^6$ is a linearly nondegenerate surface of minimal degree, hence a rational quintic scroll, see [EH87, Thm. 1]. □

This ends the proof of Proposition 3.1. □

4. Linking W_5 to V_{18}

4.1. **Notation.** Consider again a smooth del Pezzo quintic fourfold $W = W_5 \subset \mathbb{P}^7$ with a distinguished hyperplane section R of W swept up by the planes contained in W, with a distinguished plane $\Xi = \operatorname{Sing} R \cong \mathbb{P}^2$ and a distinguished smooth conic $\Upsilon \subset \Xi$. Let F be a smooth rational quintic scroll in W, not necessarily contained in R. Let $\eta : \widetilde{W} \to W$ be the blowup with center F and exceptional divisor $\widetilde{A}$. Consider the hyperplane section $B = W \cap \langle F \rangle$ of W, and let $\widetilde{B}$ be the proper transform of B in $\widetilde{W}$. Clearly, B is smooth in codimension 1, and so, $\widetilde{B} \sim H^* - \widetilde{A}$ on $\widetilde{W}$, where H is the class of a hyperplane section of W.

4.1.1. **Remark.** Recall that a rational normal quintic scroll F spans $\mathbb{P}^6$. There exist such scrolls of two different kinds, namely,

(i) $F \cong \mathbb{F}_1$, where the embedding $\mathbb{F}_1 \hookrightarrow \mathbb{P}^6$ is given by the linear system $|s_0 + 3f|$;
(ii) $F \cong \mathbb{F}_3$, where the embedding $\mathbb{F}_3 \hookrightarrow \mathbb{P}^6$ is given by the linear system $|s_0 + 4f|$,

with the exceptional section s_0 and a fiber f.

The following proposition is an analog of Theorem 2.1.a in [PZ15]. The latter theorem is proven in [PZ15] under an additional assumption that the given rational quintic scroll $F \subset W$ does not meet any plane in W along a conic. We do not keep any longer this assumption. Nonetheless, a part of the proof of Theorem 2.1.a in [PZ15] goes through in our setup as well.

4.2. **Proposition.** *For any smooth rational quintic scroll $F \subset \mathbb{P}^7$ contained in W, the data $(W, F, \widetilde{W}, \widetilde{A}, \widetilde{B})$ as in* 4.1 *fits in a Sarkisov link* (3.1.1), *where*

(a) $V = V_{18} \subset \mathbb{P}^{12}$ *is a Fano-Mukai fourfold of genus* 10 *with at worst ordinary double points as singularities;*
(b) ξ *is a birational Mori contraction that contracts the divisor* $\widetilde{B}$ *to a surface* $S = V \cap \langle S \rangle$, *and sends* $\widetilde{A}$ *to a hyperplane section* A *of* V *with* $\mathrm{Sing}(A) = S$*;*
(c) S *is either a smooth rational cubic scroll, or a cone over a rational normal twisted cubic curve;*
(d) ξ *has at most a finite number of two-dimensional fibers. If* $\widetilde{Y}_1, \ldots, \widetilde{Y}_k$ *are the two-dimensional fibers of* ξ, *then both* V *and* S *are smooth outside the finite set* $\xi(\bigcup_i \widetilde{Y}_i)$, *and* $\xi|_{\widetilde{W} \setminus \bigcup_i \widetilde{Y}_i} : \widetilde{W} \setminus \bigcup_i \widetilde{Y}_i \to V \setminus \{\xi(\bigcup_i \widetilde{Y}_i)\}$ *is the blowup of* $S \setminus \{\xi(\bigcup_i \widetilde{Y}_i)\}$. *In particular, if* ξ *has no two-dimensional fiber, then both* V *and* S *are smooth, and* $\widetilde{W} \to V$ *is the blowup of* S.

Proof of Proposition 4.2. The following two facts are borrowed in [PZ15].

4.2.1. **Claim** ([PZ15, Lem. 5.5]). *On* $\widetilde{W}$ *the following equalities hold:*

$$(H^*)^4 = 5, \quad (H^*)^3 \cdot \widetilde{A} = 0, \quad (H^*)^2 \cdot \widetilde{A}^2 = -5, \quad H^* \cdot \widetilde{A}^3 = -8, \quad \widetilde{A}^4 = -6. \tag{4.2.2}$$

4.2.3. **Claim** ([PZ15, Lem. 5.2]). *The linear system* $|2H^* - \widetilde{A}|$ *is base point free and defines a morphism* $\Phi_{|2H^* - \widetilde{A}|} : \widetilde{W} \to \mathbb{P}^{12}$.

Letting $V = \Phi_{|2H^* - \widetilde{A}|}(\widetilde{W}) \subset \mathbb{P}^{12}$, consider the Stein factorization

$$\Phi_{|2H^* - \widetilde{A}|} : \widetilde{W} \xrightarrow{\xi} U \xrightarrow{\psi} V \subset \mathbb{P}^{12}. \tag{4.2.4}$$

An easy computation using (4.2.2) gives

$$(2H^* - \widetilde{A})^4 = 18 \quad \text{and} \quad (2H^* - \widetilde{A})^3 \cdot \widetilde{B} = (2H^* - \widetilde{A})^3 \cdot (H^* - \widetilde{A}) = 0.$$

Therefore, ξ in (4.2.4) is birational and contracts $\widetilde{B}$. Furthermore,

$$\deg V \cdot \deg \psi = 18. \tag{4.2.5}$$

We have $\mathrm{rk}\,\mathrm{Pic}(\widetilde{W}) = 2$. Hence the Mori cone of $\widetilde{W}$ is generated by two extremal rays. These rays are generated by the nef divisors $2H^* - \widetilde{A}$ and H^*. Their sum is an ample anticanonical divisor $-K_{\widetilde{W}}$. So, the morphism ξ in (4.2.4) is a Mori contraction, and $\mathrm{Pic}(U) \cong \mathbb{Z} \cdot L$, where $\xi^* L = 2H^* - \widetilde{A}$. One has

$$L^2 \cdot S = -(2H^* - \widetilde{A})^2 \cdot \widetilde{B}^2 = -(2H^* - \widetilde{A})^2 \cdot (H^* - \widetilde{A})^2 = 3. \tag{4.2.6}$$

Thus, $S = \xi(\widetilde{A})$ is a surface in $\mathbb{P}^{12}$ of degree 3. By the main theorem of [AW98], U has at worst ordinary double points as singularities. Since $-K_{\widetilde{W}} = 2(2H^* - \widetilde{A}) - \widetilde{B}$, we have $-K_U = 2L$, that is, U is a Fano-Mukai fourfold (possibly with ordinary double points). Moreover, outside an at most finite union of two-dimensional fibers of ξ, the morphism ξ is a blowup of a smooth surface in a smooth fourfold. Consequently, the surface S has at worst isolated singularities.

4.2.7. **Claim.** *The morphism* ψ *in* (4.2.4) *is birational.*

Proof. Suppose that $\deg \psi > 1$. Then by (4.2.5), $V \subset \mathbb{P}^{12}$ is a fourfold of degree ≤ 9. By the del Pezzo-Bertini Theorem (see, e.g., [EH87, Thm. 1]) we have $\deg V \geq \mathrm{codim}_{\mathbb{P}^{12}} V + 1 = 9$. Thus, $\deg V = 9$, $\deg \psi = 2$, and V is either a smooth scroll, or a cone over a smooth scroll, or finally a cone with vertex a line over a rational normal curve of degree

9 in $\mathbb{P}^9$, see *loc. cit.* In the first two cases, $\operatorname{rk}\operatorname{Cl}(V) = 2$. On the other hand, there is a natural injection $\psi^* : \operatorname{Cl}(V) \hookrightarrow \operatorname{Cl}(U) \cong \mathbb{Z} \cdot L$. This leads to a contradiction. In the latter case $\operatorname{Cl}(U) = \mathbb{Z} \cdot P$, where P is the class of a plane. As before, this gives a contradiction. □

The following assertion is well known, even in a more general form. However, due to the lack of a reference, we provide an argument.

4.2.8. **Claim.** *The morphism ψ is an isomorphism.*

Proof. It suffices to show that the graded algebra

$$\mathrm{R}(U, L) = \bigoplus_{n \geq 0} H^0(U, \mathscr{O}_U(nL))$$

is generated by its component of degree 1. By our construction, the linear system $|L|$ is base point free. By Bertini's theorem, a general member $X \in |L|$ is smooth. Then X is a smooth Fano threefold, and the anticanonical linear system $|-K_X| = |\mathscr{O}_X(L)|$ is base point free.

By the Kawamata-Viehweg vanishing theorem, we have $H^1(U, \mathscr{O}_U) = 0$. Applying Lemma 2.9 in [Isk77], it is enough to show that the graded algebra

$$\mathrm{R}(X, L) = \bigoplus_{n \geq 0} H^0(X, \mathscr{O}_X(nL))$$

is generated by its component of degree 1. Continuing in the same manner, one arrives at a smooth K3 surface $Z \in |-K_X|$ and a smooth curve $C \in |L|_Z|$, where $K_C = L|_C$. Since the map given by $|K_C| = |L|_C|$ is birational, C is not hyperelliptic. By a theorem of Max Noether ([GH94, Ch. 2, § 3]), the algebra

$$\mathrm{R}(C, K_C) = \bigoplus_{n \geq 0} H^0(C, \mathscr{O}_C(nK_C))$$

is generated by its component $H^0(C, \mathscr{O}_C(K_C))$. Now the claim follows. □

Thus, we may suppose in the sequel that $V = U$ is a Fano-Mukai fourfold of degree 18 (so, of genus 10) with at worst ordinary double points as singularities, as claimed in (a), and $\xi = \Phi_{|2H^* - \widetilde{A}|}$.

Since $L^* \sim 2H^* - \widetilde{A}$ and $\widetilde{B} \sim H^* - \widetilde{A}$, one has $\widetilde{A} \sim L^* - 2\widetilde{B}$ on $\widetilde{W}$. Note that $\widetilde{A} \subset \widetilde{W}$ in (3.1.1) is smooth being a $\mathbb{P}^1$-bundle over a smooth surface F. Letting $A = \xi(\widetilde{A}) \subset V$, one deduces that A is a hyperplane section of V with desingularization $\widetilde{A}$, and $S = \operatorname{Sing}(A)$. This concludes the proof of (b).

To show (c), note that $\deg S = 3$ by (4.2.6), and S has at worst isolated singularities. We claim that $\dim\langle S\rangle > 3$. Indeed, using (4.2.2) one can compute

$$L^* \cdot \widetilde{B}^3 = (2H^* - \widetilde{A}) \cdot (H^* - \widetilde{A})^3 = -1.$$

On the other hand, using [PZ16, Lem. 2.3(ii)] we obtain

$$L^* \cdot \widetilde{B}^3 = -L|_S \cdot K_S + K_V \cdot L \cdot S$$

(the corresponding formula in *loc. cit.* works since S has at worst isolated singularities, and L is movable). So,

$$L|_S \cdot K_S = -L^* \cdot \widetilde{B}^3 - 2L^2 \cdot S = 1 - 6 = -5.$$

If $\dim\langle S\rangle < 4$, then S is a cubic surface in $\mathbb{P}^3$. In this case $L|_S \cdot K_S = -K_S^2 = -3$, a contradiction. Therefore, $\dim\langle S\rangle = 4$, and so, $S \subset \mathbb{P}^4$ is a linearly nondegenerate cubic surface, that is, a surface of minimal degree in $\mathbb{P}^4$. Now the assertion follows by the del Pezzo theorem ([EH87, Thm. 1]). This proves (c). Finally, (d) follows from [AW98, Main Theorem]. This ends the proof of Proposition 4.2. □

Next we examine the alternative case, where ξ in (3.1.1) has a two-dimensional fiber (cf. 4.2.(d)). We need such a simple observation.

4.3. **Remark** (cf. [KPS16, Rem. 5.2.14]). A smooth conic $J \subset \Xi$ touches Υ with even multiplicities if and only if one of the following holds (see Figure 4.3.1):

(i) $J = \Upsilon$;
(ii) J is tangent to Υ in a single point with multiplicity 4;
(iii) J is tangent to Υ in two points.

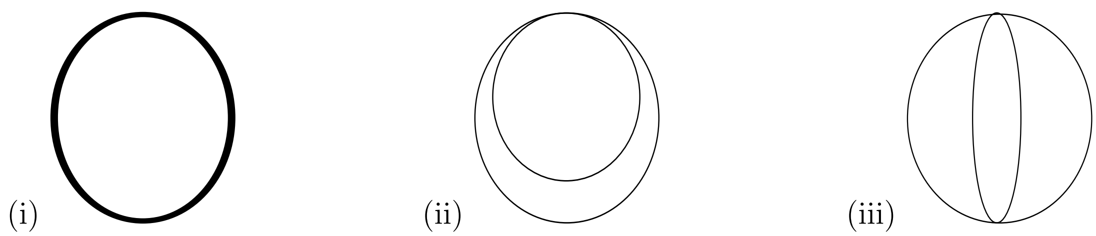


FIGURE 4.3.1.

4.4. **Proposition.** *With the assumptions and notation of* 4.1 *and* 4.2, *suppose that* ξ *has a two-dimensional fiber* $\widetilde{Y}$. *Then the following hold.*

(a) $Y := \eta(\widetilde{Y}) \subset W$ *is a plane, and* $Y \cap F$ *is a reduced (but possibly reducible) conic. Conversely, the proper transform* $\widetilde{Y} \subset \widetilde{W}$ *of any plane* $Y \subset W$ *such that* $Y \cap F$ *is a conic, is a two-dimensional fiber of* ξ.
(b) *If* $F \cong \mathbb{F}_1$, *then* $\widetilde{Y}$ *is a unique two-dimensional fiber of* ξ, *and* $Y \cap F$ *is a smooth conic, the exceptional section of* $F \cong \mathbb{F}_1$.
(c) *One of the following holds.*
  (i) B *is singular at a general point of* $Y := \eta(\widetilde{Y})$, $\xi(\widetilde{Y})$ *is a smooth point of* V *and a (unique) singular point of the surface* S. *This point* $\xi(\widetilde{Y})$ *is of type* $\frac{1}{3}(1,1)$ *in* S. *Furthermore,* $B = R$ *and* $Y = \Xi$.
  (ii) B *is smooth at a general point of* Y, $\xi(\widetilde{Y}) \in V$ *is an ordinary double point, and the surface* S *is smooth at* $\xi(\widetilde{Y})$. *Furthermore,* Y *is a plane of type* $\sigma_{3,1}$ *(that is, a* Π*-plane).*
(d) *If* V *is smooth, then* $F \cong \mathbb{F}_1$, $\widetilde{Y}$ *is the only two-dimensional fiber of* ξ, $B = R$, $Y = \Xi$, S *is a cone over a twisted cubic curve, and the exceptional section of the ruling* $F \to \mathbb{P}^1$ *is a smooth conic* $J := \Xi \cap F$ *of one of types* (i)–(iii) *in Remark* 4.3, *see Figure* 4.3.1.

Proof. By Proposition 4.2(c), S is normal and possesses at most one singular point. It follows from the main theorem and Proposition 4.11 in [AW98] that $Y \cong \mathbb{P}^2$ and

$$\mathscr{O}_{\widetilde{Y}}(3H^* - \widetilde{A}) \cong \mathscr{O}_{\widetilde{Y}}(-K_{\widetilde{W}}) \cong \mathscr{O}_{\mathbb{P}^2}(1). \tag{4.4.1}$$

Since $\widetilde{Y}$ is contracted to a point under $\xi = \Phi_{|2H^* - \widetilde{A}|}$, one has

$$\mathscr{O}_{\widetilde{Y}}(2H^* - \widetilde{A}) = \mathscr{O}_{\widetilde{Y}}. \tag{4.4.2}$$

Then (4.4.1) and (4.4.2) imply

$$\mathscr{O}_{\widetilde{Y}}(H^*) \cong \mathscr{O}_{\mathbb{P}^2}(1) \quad \text{and} \quad \mathscr{O}_{\widetilde{Y}}(\widetilde{A}) \cong \mathscr{O}_{\mathbb{P}^2}(2). \tag{4.4.3}$$

Thus, $Y \subset W$ is a plane, and $Y \cap F$ is a conic. The converse statement is left to the reader.

Assume that $Y \cap F = 2\Lambda$ is a double line. If $F \cong \mathbb{F}_3$, then there exists another line $\Lambda' \subset F$ meeting Λ. The plane Y coincides with the tangent plane to F at the point $\Lambda \cap \Lambda'$. Hence, $Y \cap F = 2\Lambda + \Lambda' \neq 2\Lambda$.

Thus, we may assume that $F \cong \mathbb{F}_1$ and Λ is a ruling of F. Let J be the negative section of $F = \mathbb{F}_1 \to \mathbb{P}^1$, and let J' be a section disjoint with J. Then J is a conic, J' is a twisted cubic in F, and the linear spans $\langle J \rangle$ and $\langle J' \rangle$ are disjoint. The ruling Λ meets J in a point, say, P_0, and J' in P_1. By our assumption, the plane Y is tangent to F along Λ. Hence Y contains the tangent lines $T_{P_0}J$ and $T_{P_1}J'$. It follows that these lines intersect, and then also the linear spans $\langle J \rangle$ and $\langle J' \rangle$ do, a contradiction This proves (a).

(b) In this case the only conic on F is the negative section of $F \cong \mathbb{F}_1$.

To show (c) we note that η provides a local analytic isomorphism of pairs $(\widetilde{B}, \widetilde{Y})$ and (B, Y) at the corresponding generic points of $\widetilde{Y}$ and Y, respectively. The statements of (c) follow now directly from the main theorem in [AW98] and the proof of Proposition 6.3 in *loc. cit.*, except for the equalities $B = R$ and $Y = \Xi$ in (c)(i). Let us show the latter equalities. Since the variety B in (c)(i) is a hyperplane section of W singular along a plane Y (see Definition 4.1), B contains any line in W meeting $\mathrm{Sing}(B) \supset Y$. Any two planes in W intersect. In particular, any plane in W meets Y. Hence B contains any plane in W. Consequently, B coincides with the union $R = \bigcup_{\gamma \in \Upsilon} \Pi_\gamma$, see 2.1.2. It follows finally that $Y = \mathrm{Sing}(R) = \Xi$ and $R = B \supset F$.

(d) Since V is smooth, (c)(i) holds, and so, $B = R$, $Y = \Xi$, and S is the cone over a twisted cubic curve. Since $Y = \Xi$, then according to (a), $F \cap \Xi$ is a conic. This conic contains the exceptional section, say, J of $F \cong \mathbb{F}_n$, $n \in \{1, 3\}$. Thus, $J \subset \Xi$. For $P \in J$, let $\Lambda_P \subset F$ be the ruling through P.

Assume first that $F \cong \mathbb{F}_3$. Then $F \cap \Xi = J + \Lambda_0$, where Λ_0 is the ruling through the point $P_0 = J \cap \Lambda_0$. For a point $P \in J \setminus \{P_0\}$ one has $\Lambda_P \neq \Lambda_0$, and so, $\Lambda_P \not\subset \Xi$. Hence Λ_P is contained in a (unique) plane Π_γ. This defines a (regular) map

$$\psi : J \setminus \{P_0\} \longrightarrow \Upsilon, \qquad P \longmapsto \gamma.$$

Since each plane Π_γ contains a unique ruling Λ_P, this map extends to a bijection $J \to \Upsilon$. The intersection $l_\gamma := \Pi_\gamma \cap \Xi$ is the tangent line to Υ at γ. Then ψ^{-1} is defined as follows

$$\psi^{-1} : \Upsilon \longrightarrow J, \qquad \gamma \longmapsto P = J \cap l_\gamma.$$

This map is bijective if and only if J is a tangent line to Υ. Thus $J = l_{\gamma_0}$ for some $\gamma_0 \in \Upsilon$. If $\Lambda_{\gamma_0} \not\subset \Xi$, then $\Pi_{\gamma_0} \cap F = J + \Lambda_{\gamma_0}$. By (c)(ii) V is singular, a contradiction. Hence $\Lambda_{\gamma_0} \subset \Xi$ and $F \cap \Xi = J + \Lambda_{\gamma_0}$. In this case, $\Lambda_{\gamma_0} \neq J = l_{\gamma_0}$, and so, $\Lambda_{\gamma_0} \cap \Upsilon = \{\gamma_0, \gamma_1\}$, where $\gamma_1 \neq \gamma_0$. On the other hand, Λ_{γ_0} is contained in some plane Π_γ, because this is true for a general Λ_γ. Hence $\Lambda_{\gamma_0} = \Xi \cap \Pi_\gamma$ is tangent to Υ, a contradiction. Thus, $F \cong \mathbb{F}_1$, as stated.

Assume now that $J \neq \Upsilon$. Define a correspondence between Υ and J via

$$Z = \{(P, \gamma) \in J \times \Upsilon \mid P \in l_\gamma\},$$

where l_γ is the line on Ξ tangent to Υ at γ. Clearly, Z is a curve of bidegree $(2, 2)$ on $J \times \Upsilon \cong \mathbb{P}^1 \times \mathbb{P}^1$.

The curve Z admits a natural interpretation as a subvariety of the variety $\mathrm{Fl}(\mathbb{P}^2) \subset \mathbb{P}^2 \times \mathbb{P}^{2\vee}$ of full flags on $\mathbb{P}^2$. Under this interpretation one has $p_1(Z) = J$ and $p_2(Z) = \Upsilon^\vee$, where $\Upsilon^\vee \subset \mathbb{P}^{2\vee}$ is the dual conic and p_1, p_2 are the canonical projections of the product $\mathbb{P}^2 \times \mathbb{P}^{2\vee}$ to the factors. Moreover, $Z = p_1^{-1}(J) \cap p_2^{-1}(\Upsilon^\vee)$. Note that $p_2^{-1}(\Upsilon^\vee) \cong \mathbb{P}^1 \times \mathbb{P}^1$. The restriction $\pi := p_1|_{p_2^{-1}(\Upsilon^\vee)} : p_2^{-1}(\Upsilon^\vee) \to \mathbb{P}^2$ is a double cover branched along Υ.

Consider a generically one-to-one map $\delta : J \to \Upsilon$,

$$P \in J \longmapsto \Lambda_P \longmapsto \Pi_P \longmapsto l_P = \Xi \cap \Pi_P \longmapsto \gamma = l_P \cap \Upsilon \in \Upsilon,$$

where $\Pi_P \subset R$ is the plane containing the ruling $\Lambda_P \subset F$. The graph of δ in $J \times \Upsilon$ is a component of Z. Therefore, Z splits into two components Z_1 and Z_2, which are curves of bidegree $(1, 1)$ on $\mathbb{P}^1 \times \mathbb{P}^1$. Also the preimage $\pi^{-1}(J) = p_1^{-1}(J) \cap p_2^{-1}(\Upsilon^\vee)$ splits into two irreducible components Z_1 and Z_2. This is possible only if J touches the branch locus Υ with even multiplicities, hence only in the cases (ii) and (iii) in 4.3. □

4.5**.** Following [KR13] we call 'cubic scrolls' both smooth cubic scrolls and cones over rational twisted cubic curves. The latter cones in V_{18} will be called 'cubic cones' for short. This does not lead to a confusion. Indeed, since V is an intersection of quadrics ([Isk77, Lem. 2.10]), V does not contain any cubic surface F with $\langle F \rangle \cong \mathbb{P}^3$.

From Propositions 3.1, 4.2, and 4.4 we deduce the following corollaries.

4.5.1. **Corollary.** *The cubic scroll $S \subset V$ in diagram* (3.1.1) *is a cubic cone if and only if V is smooth, $F \subset R$, $F \cong \mathbb{F}_1$, and the exceptional section $J = F \cap \Xi$ of $F \to \mathbb{P}^1$ is a smooth conic touching Υ with even multiplicities. There are three types* 4.3 (i)*,* 4.3(ii)*, and* 4.3(iii) *of such pairs (F, J).*

The next corollary will be used in the proof of Theorem 1.1; cf. also Lemma 9.2.

4.5.2. **Corollary.** *For any cubic cone $S \subset V$ there is a unique hyperplane section A of V with* $\mathrm{Sing}(A) = S$ *such that $V \setminus A \cong \mathbb{C}^4$.*

Proof. By Proposition 3.1, V and S can be included in diagram (3.1.1), where A is a hyperplane section of V with $\mathrm{Sing}(A) = S$. Such a divisor A is unique being the image of the η-exceptional divisor $\widetilde{A} \sim L^* - 2\widetilde{B}$, see (3.1.8). Since V is smooth and S is singular, the cases (d) and (c)(ii) in Propositions 4.2 and 4.4, respectively, are excluded. So, there is a unique two-dimensional fiber $\widetilde{Y}$ of ξ with Sing $S = \xi(\widetilde{Y})$, and we are in case (c)(i) of Proposition 4.4. Due to this proposition, we have

$$\eta(\widetilde{Y}) = \Xi, \quad B = R, \quad \text{and so,} \quad F \subset R.$$

As follows from diagram (3.1.1), there are isomorphisms

$$V \setminus A \cong \widetilde{W} \setminus (\widetilde{A} \cup \widetilde{R}) \cong W \setminus R \cong \mathbb{C}^4,$$

see Corollary 2.2.2. Thus, the pair (V, A) yields a compactification of $\mathbb{C}^4$ into a smooth Fano-Mukai fourfold V of genus 10. $\square$

Finally, we introduce the following notion.

4.6. **Definition.** We say that the pairs (W, F) and (V, S) are *linked* if they fit in diagram (3.1.1) and verify the conditions of one of Propositions 4.2 and 4.4.

In the sequel we deal only with linked pairs that verify the conditions of Corollary 4.5.1.

5. Automorphisms of W_5

Let us introduce the following notation.

5.1. **Notation.** Consider the linear space $M_3 = S^3\mathbb{C}^{2\vee}$ of binary cubic forms $f_3(x, y)$ and the projectivization $\mathbb{P}(M_3 \oplus \mathbb{C}) \cong \mathbb{P}^4$. A point of $\mathbb{P}^4$ can be viewed as a class $[(f_3, c)]$, where $f_3 \in M_3$, $c \in \mathbb{C}$, and $(f_3, c) \neq (0, 0)$. Up to automorphisms of $\mathbb{P}^4$, a twisted cubic curve $\Gamma \subset \mathbb{P}^4$ can be represented as the projectivization

$$\Gamma = \mathbb{P}\left(\left\{(f_3, 0) \in M_3 \oplus \mathbb{C} \setminus \{(0, 0)\} \mid f_3 = (\alpha x + \beta y)^3\right\}\right) \subset \mathbb{P}(M_3 \oplus \{0\}) \subset \mathbb{P}(M_3 \oplus \mathbb{C}).$$

So, the linear span $\langle\Gamma\rangle$ is identified with the linear subspace $\{c = 0\}$ in $\mathbb{P}^4$, and the point $P = \mathbb{P}(\{0\} \oplus \mathbb{C}) \in \mathbb{P}^4 \setminus \langle\Gamma\rangle$ with the class $[(0, 1)]$.

The standard representation of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{C}^2$ induces an irreducible representation of $\mathrm{GL}_2(\mathbb{C})$ on M_3, $(g, f_3) \longmapsto f_3 \circ g^{-1}$. Adding the trivial one-dimensional representation yields a representation of $\mathrm{GL}_2(\mathbb{C})$ on $M_3 \oplus \mathbb{C}$ and, in turn, a $\mathrm{GL}_2(\mathbb{C})$-action on $\mathbb{P}^4$, which fixes the point P and stabilizes the twisted cubic Γ.

Let further $\Delta \subset \mathbb{P}(M_3 \oplus \mathbb{C})$ be the closure of the locus of points $[(f_3, c)] \in \mathbb{P}(M_3 \oplus \mathbb{C})$, whose binary cubic form f_3 has a multiple factor.

5.2. **Remarks.** 1. Clearly, Δ is the cone over $\Delta_0 := \Delta \cap \mathbb{P}(M_3)$ with vertex P. The surface Δ_0 in $\mathbb{P}^3 = \mathbb{P}(M_3)$ is the zero divisor of the discriminant $\mathrm{discr}(f_3)$. This quartic surface has cuspidal singularities along Γ and is smooth outside Γ. In fact, Δ_0 is the tangent developable surface of Γ, that is, the closure of the union of the tangent lines to Γ. The cone N over Γ with vertex P is the singular locus of Δ.

2. The $\mathrm{GL}_2(\mathbb{C})$-action on $\mathbb{P}(M_3 \oplus \mathbb{C}) = \mathbb{P}^4$ has exactly 7 orbits, namely,

$$\{P\},\ \Gamma,\ \Delta_0 \setminus \Gamma,\ \mathbb{P}(M_3) \setminus \Delta_0,\ N \setminus (\Gamma \cup \{P\}),\ \Delta \setminus (N \cup \Delta_0),\ \mathbb{P}(M_3 \oplus \mathbb{C}) \setminus (\mathbb{P}(M_3) \cup \Delta).$$

Thus, P is the unique fixed point of the $\mathrm{GL}_2(\mathbb{C})$-action on $\mathbb{P}^4$, while the last orbit in this list is the open orbit.

3. The center Γ of the blowup φ in (2.2.1) is invariant under the $\mathrm{GL}_2(\mathbb{C})$-action on $\mathbb{P}^4$ defined in 5.1. Hence this action lifts to $\widehat{W}$ stabilizing the exceptional divisors $\widehat{R}$ and $\widehat{E}$ of φ and ρ, respectively, see Proposition 2.2. The $\mathrm{GL}_2(\mathbb{C})$-action on $\widehat{W}$ induces an effective $\mathrm{GL}_2(\mathbb{C})$-action on W via the contraction $\rho : \widehat{W} \to W$ of the $\mathrm{GL}_2(\mathbb{C})$-invariant divisor $\widehat{E}$. Then the birational linear projection $\phi : W \dashrightarrow \mathbb{P}^4$ with center Ξ along with diagram (2.2.1) are $\mathrm{GL}_2(\mathbb{C})$-equivariant. The $\mathrm{GL}_2(\mathbb{C})$-action on W extends to a linear $\mathrm{GL}_2(\mathbb{C})$-action on $\mathbb{P}^7 \supset W$. The latter action has W as an orbit closure, stabilizes R, Ξ, and Υ, and induces a linear $\mathrm{GL}_2(\mathbb{C})$-action on $\mathbb{C}^4 = \mathbb{P}^4 \setminus \mathbb{P}^3 \cong W \setminus R$, see (2.2.3), and a standard $\mathrm{PGL}_2(\mathbb{C})$-action on Ξ with a unique closed orbit Υ.

Using 5.1 – 5.2 we describe in 5.3 – 5.6 the automorphism group $\mathrm{Aut}(W)$ and its action on W.

5.3. **Lemma.** *Let* $\Gamma \subset \mathbb{P}^4$ *be a rational twisted cubic curve, and* $P \in \mathbb{P}^4 \setminus \langle\Gamma\rangle$ *be a point. Then there are isomorphisms*

$$\mathrm{Aut}(\mathbb{P}^4, \Gamma, P) \cong \mathrm{GL}_2(\mathbb{C})/\boldsymbol{\mu}_3 \cong \mathrm{GL}_2(\mathbb{C}), \tag{5.3.1}$$

where the cyclic group $\boldsymbol{\mu}_3$ *of order* 3 *is realized as the subgroup of scalar matrices* $\{\zeta \cdot \mathrm{id}_{\mathbb{C}^2}\}$ *with* $\zeta^3 = 1$.*The stabilizer in* $\mathrm{Aut}(\mathbb{P}^4, \Gamma, P)$ *of a general point* $Q \in \mathbb{P}^4$ *is trivial.*

Proof. We leave to the reader to check the fact that the endomorphism $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{GL}_2(\mathbb{C})$, $A \longmapsto (\det A) \cdot A$, yields the second isomorphism in (5.3.1). Up to an automorphism of $\mathbb{P}^4$ one may suppose that the triple $(\mathbb{P}^4, \Gamma, P)$ is chosen as in 5.1. The $\mathrm{GL}_2(\mathbb{C})$-action on $\mathbb{P}^4$ introduced in 5.1 fixes P and stabilizes Γ. The scalar matrices $\zeta \cdot \mathrm{id} \in \mathrm{GL}_2(\mathbb{C})$ with $\zeta^3 = 1$, and only these, act identically on $\mathbb{P}^4$. This gives an embedding $\mathrm{GL}_2(\mathbb{C})/\boldsymbol{\mu}_3 \hookrightarrow \mathrm{Aut}(\mathbb{P}^4, \Gamma, P)$. In fact, this embedding is an isomorphism. The latter follows by comparing the exact sequences

$$1 \longrightarrow \mathbb{G}_{\mathrm{m}} \longrightarrow \mathrm{Aut}(\mathbb{P}^4, \Gamma, P) \longrightarrow \mathrm{Aut}(\Gamma) = \mathrm{PGL}_2(\mathbb{C}) \longrightarrow 1$$

and

$$1 \longrightarrow \mathbb{G}_{\mathrm{m}} = \mathrm{z}(\mathrm{GL}_2(\mathbb{C})/\boldsymbol{\mu}_3) \longrightarrow \mathrm{Aut}(\mathbb{P}^4, \Gamma, P) \longrightarrow \mathrm{Aut}(\Gamma) = \mathrm{PGL}_2(\mathbb{C}) \longrightarrow 1\,,$$

where $\mathrm{z}(G)$ stands for the center of a group G. The last assertion is immediate. ☐

We use below the following fact.

5.4. **Lemma.** (Piontkowski-Van-de-Ven [PdV99, Thm. 6.6]) *There is an exact sequence*

$$1 \longrightarrow (\mathbb{G}_{\mathrm{a}})^4 \rtimes \mathbb{G}_{\mathrm{m}} \longrightarrow \mathrm{Aut}(W) \xrightarrow{\varrho} \mathrm{Aut}(\Upsilon) = \mathrm{PGL}_2(\mathbb{C}) \longrightarrow 1\,. \tag{5.4.1}$$

Therefore, $\mathrm{Aut}(W)$ *is a connected algebraic group.*

In the next proposition we describe the algebraic Levi decomposition of $\mathrm{Aut}(W)$.

5.5. **Proposition.** (a) *Let* $\mathrm{R_u} = \mathrm{R_u}(\mathrm{Aut}(W))$ *be the unipotent radical of* $\mathrm{Aut}(W)$ *and* L *its reductive Levi subgroup. Then* $\mathrm{R_u} \cong (\mathbb{G}_{\mathrm{a}})^4$ *and* $\mathrm{L} \cong \mathrm{GL}_2(\mathbb{C})$. *Therefore,*

$$\mathrm{Aut}(W) = \mathrm{R_u} \rtimes \mathrm{L} \cong (\mathbb{G}_{\mathrm{a}})^4 \rtimes \mathrm{GL}_2(\mathbb{C}). \tag{5.5.1}$$

Furthermore, a $\mathrm{GL}_2(\mathbb{C})$*-subgroup of* $\mathrm{Aut}(W)$ *is unique up to conjugation.*

(b) *An isomorphism* $\mathrm{Aut}(W) \xrightarrow{\cong} \mathrm{Aut}(\mathbb{P}^4, \Gamma)$ *as in Corollary* 2.2.4 *sends* $\mathrm{R_u}$ *onto the vector group* $\mathrm{Transl}(\mathbb{C}^4) \cong (\mathbb{G}_{\mathrm{a}})^4$ *of the vector space* $\mathbb{C}^4 = \mathbb{P}^4 \setminus \langle\Gamma\rangle$*, and the reductive Levi subgroup* L *onto the stabilizer* $\mathrm{Aut}(\mathbb{P}^4, \Gamma, P) \cong \mathrm{GL}_2(\mathbb{C})$ *of a point* $P \in \mathbb{P}^4 \setminus \langle\Gamma\rangle$.

(c) *The set of reductive Levi subgroups of* $\mathrm{Aut}(W)$ *coincides with the set of stabilizers of points in* $W \setminus R$. *Any Levi subgroup* L *of* $\mathrm{Aut}(W)$ *has a unique fixed point in* $W \setminus R$*, which is the unique fixed point in* $W \setminus R$ *of its center* $\mathrm{z}(\mathrm{L})$*, and acts on* $W \setminus R$ *with a principal open orbit.*

Proof. (a) It follows from (5.4.1) that $\mathrm{R_u}$ coincides with the $(\mathbb{G}_{\mathrm{a}})^4$-subgroup of $\mathrm{Aut}(W)$. The effective $\mathrm{GL}_2(\mathbb{C})$-action on W as in Remark 5.2.3 defines a $\mathrm{GL}_2(\mathbb{C})$-subgroup, say, $\mathrm{L}_0 \subset \mathrm{Aut}(W)$ with $\mathrm{L}_0 \cap \mathrm{R_u} = \{e\}$. Thus, $\mathrm{L}_0 \cong \mathrm{GL}_2(\mathbb{C})$ surjects onto $\mathrm{PGL}_2(\mathbb{C})$ with kernel being the center $\mathrm{z}(\mathrm{L}_0) \cong \mathbb{G}_{\mathrm{m}}$. Letting $\mathrm{L} = \mathrm{L}_0$ we obtain (5.5.1). The last assertion of (a) follows from the Levi-Maltsev-Mostow Theorem, see, e.g., [Mo56] or [Hoh71].

(b) By (a), the isomorphism $\mathrm{Aut}(W) \cong \mathrm{Aut}(\mathbb{P}^4, \Gamma)$ of Corollary 2.2.4 induces an isomorphism of Levi decompositions

$$\mathrm{R_u} \rtimes \mathrm{L} \cong (\mathbb{G}_\mathrm{a})^4 \rtimes \mathrm{GL}_2(\mathbb{C}) \cong \mathrm{Transl}(\mathbb{C}^4) \rtimes \mathrm{Aut}(\mathbb{P}^4, \Gamma, P).$$

Now (b) is straightforward.

(c) By virtue of (b), $W \setminus R$ is the open orbit of $\mathrm{R_u}$ and of $\mathrm{Aut}(W)$. Furthermore, the unipotent radical $\mathrm{R_u} \cong (\mathbb{G}_\mathrm{a})^4$ acts freely (by translations) on $W \setminus R \cong \mathbb{P}^4 \setminus \langle\Gamma\rangle \cong \mathbb{C}^4$. The Levi subgroup $\mathrm{L} \cong \mathrm{GL}_2(\mathbb{C})$ goes onto the stabilizer $\mathrm{Aut}(\mathbb{P}^4, \Gamma, P)$ of a point $P \in \mathbb{C}^4 = \mathbb{P}^4 \setminus \langle\Gamma\rangle$, hence coincides with the stabilizer of the corresponding point in $W \setminus R$. All such stabilizers are conjugated via the action of $\mathrm{R_u} \cong \mathrm{Transl}(\mathbb{C}^4)$ on $W \setminus R \cong \mathbb{C}^4$. The last assertion is straightforward by virtue of Lemma 5.3. □

5.5.2. **Remarks.** 1. Under the identification of the triple $(\mathbb{P}^4, \Gamma, P)$ with $(\mathbb{P}(M_3 \oplus \mathbb{C}), \Gamma, P)$, see 5.1, the $\mathrm{GL}_2(\mathbb{C})$-subgroup $\mathrm{Aut}(\mathbb{P}^4, \Gamma, P) \subset \mathrm{Aut}(\mathbb{P}^4, \Gamma)$ is defined as in 5.1 and 5.3. The $\mathrm{GL}_2(\mathbb{C})$-action by conjugation on $\mathrm{R_u} = \mathrm{Transl}(\mathbb{C}^4) \cong (\mathbb{G}_\mathrm{a})^4$ is then given via the standard irreducible representation of $\mathrm{GL}_2(\mathbb{C})$ on the vector space $M_3 \cong \mathbb{C}^4$ of binary cubic forms. This determines the structure of $\mathrm{Aut}(W)$ as a semidirect product. Identifying $\mathrm{R_u} = \mathrm{Transl}(\mathbb{C}^4)$ with the additive group $(M_3, +)$, the action of $\mathrm{Aut}(W) \cong \mathrm{Aut}(\mathbb{P}^4, \Gamma)$ on $\mathbb{P}^4 = \mathbb{P}(M_3 \oplus \mathbb{C})$ lifts to the action on $M_3 \oplus \mathbb{C}$ via

$$(M_3, +) \rtimes \mathrm{GL}_2(\mathbb{C}) \ni (h, g) : (f, z) \longmapsto (f \circ g^{-1} + zh, z) \in M_3 \oplus \mathbb{C}, \tag{5.5.3}$$

where g is defined modulo multiplication by a cubic root of unity.

2. A torus T in a connected algebraic group G is called *regular* if the centralizer $\mathcal{C}_G(T)$ is solvable, and *singular* otherwise. A torus $T \subset G$ is regular if and only if it is contained just in a finite set of Borel subgroups of G. The maximal tori are regular; see, e.g., [Hum75, Ch. IX, § 24].

By the preceding remark we have $\mathcal{C}_{\mathrm{Aut}(W)}(\mathrm{z}(\mathrm{L})) = \mathrm{L}$ and $\mathcal{C}_{\mathrm{Aut}(W)}(\mathrm{R_u}) = \mathrm{R_u}$. Hence the center $\mathrm{z}(\mathrm{L})$ of a Levi subgroup $\mathrm{L} \subset \mathrm{Aut}(W)$ uniquely determines L and is a singular torus. Using (5.5.1) it is easily seen that, conversely, any singular torus in $\mathrm{Aut}(W)$ is the center $\mathrm{z}(\mathrm{L})$ of a Levi subgroup $\mathrm{L} \subset \mathrm{Aut}(W)$. Any two singular tori in $\mathrm{Aut}(W)$ are conjugated.

3. The orbits of the $\mathrm{Aut}(\mathbb{P}^4, \Gamma)$-action on $\mathbb{P}^4$ are (see Remark 5.2.2)

$$\Gamma, \qquad \Delta_0 \setminus \Gamma, \qquad \langle\Gamma\rangle \setminus \Delta_0, \qquad \mathbb{P}^4 \setminus \langle\Gamma\rangle. \tag{5.5.4}$$

Due to [PdV99, Thm. 6.9], the $\mathrm{Aut}(W)$-action on W has as well exactly four orbits of dimensions 1,2,3, and 4, respectively. These orbits are:

$$\Upsilon, \qquad \Xi \setminus \Upsilon, \qquad R \setminus \Xi, \qquad W \setminus R. \tag{5.5.5}$$

Next we examine the $\mathrm{Aut}(W)$-action on W.

5.6. **Proposition.** *In the notation as before the following hold.*

(a) *The triple* (R, Ξ, Υ) *is* $\mathrm{Aut}(W)$*-invariant.*
(b) $\mathrm{Aut}(W)$ *acts effectively on* R.
(c) *The unipotent radical* $\mathrm{R_u}$ *of* $\mathrm{Aut}(W)$ *acts freely on* $W \setminus R$ *and fixes* Ξ *pointwise.*
(d) *Let* $T = \mathrm{z}(\mathrm{L})$ *be a singular torus in* $\mathrm{Aut}(W)$*, where* L *is a reductive Levi subgroup. Then the fixed point locus* W^T *is* L*-invariant and is a disjoint union*

$$W^T = \{Q\} \cup \Psi \cup \Xi,$$

where $Q \in W \setminus R$ is an isolated point and $\Psi \subset R$ is a twisted cubic curve such that $\langle\Psi\rangle \cap \Xi = \varnothing$ and Ψ meets any plane Π_γ, $\gamma \in \Upsilon$, in a single point $Q_\gamma \in \Pi_\gamma$.

Proof. Statement (a) is immediate from 2.1.2. To show (b) let K be the kernel of the restriction homomorphism $\mathrm{Aut}(W) \to \mathrm{Aut}(R)$, $\alpha \mapsto \alpha|_R$. The action of $\mathrm{Aut}(\mathbb{P}^4, \Gamma)$ induces a representation of this group by automorphisms of the normal bundle $\mathscr{N}_{\Gamma/\mathbb{P}^4}$ and $\widehat{R} \cong \mathbb{P}(\mathscr{N}^\vee_{\Gamma/\mathbb{P}^4})$. The unipotent radical $\mathrm{Transl}(\mathbb{C}^4)$ of $\mathrm{Aut}(\mathbb{P}^4, \Gamma)$ acts trivially on the base Γ. Its action on the total space $\widehat{R}$ induces the $\mathrm{R_u}$-action on R, see Lemma 2.3(a)-(b). The latter action is effective if and only if the former is.

5.6.1. **Claim.** *The induced representation of* $\mathrm{Transl}(\mathbb{C}^4) \cong (\mathbb{G}_\mathrm{a})^4$ *on the normal bundle* $\mathscr{N}_{\Gamma/\mathbb{P}^4}$ *is faithful.*

Proof of Claim 5.6.1. We use Notation 5.1. Let $\pi : (M_3 \oplus \mathbb{C}) \setminus \{0\} \to \mathbb{P}(M_3 \oplus \mathbb{C})$ be the canonical projection, and let $\mathrm{Cone}(\Gamma) = \pi^{-1}(\Gamma)$ be the affine cone over Γ in $(M_3 \oplus \mathbb{C}) \setminus \{0\} \cong \mathbb{C}^5 \setminus \{0\}$. The normal bundle of $\mathrm{Cone}(\Gamma)$ in $M_3 \oplus \mathbb{C}$ is the pullback $\pi^*(\mathscr{N}_{\Gamma/\mathbb{P}^4})$. The induced representation of $\mathrm{Transl}(\mathbb{C}^4) \cong (\mathbb{G}_\mathrm{a})^4$ on the normal bundle $\mathscr{N}_{\Gamma/\mathbb{P}^4}$ is faithful if and only if $(M_3, +) \cong \mathrm{Transl}(\mathbb{C}^4)$ acts effectively on $\mathscr{N}_{\mathrm{Cone}(\Gamma)/(M_3 \oplus \mathbb{C})}$.

An element $g \in (M_3, +)$ acts on $M_3 \oplus \mathbb{C}$ linearly via

$$g.(f, z) = (f + zg, z) \quad \forall (f, z) \in M_3 \oplus \mathbb{C}. \tag{5.6.2}$$

It fixes $\mathrm{Cone}(\Gamma)$ pointwise and acts trivially on the tangent space $T_{(f,0)} \mathrm{Cone}(\Gamma)$. The induced action on the tangent space $T_{(f,0)}(M_3 \oplus \mathbb{C}) \cong M_3 \oplus \mathbb{C}$ is given by the same formula

$$dg.(u, v) = (u + vg, v) \quad \forall (u, v) \in M_3 \oplus \mathbb{C}. \tag{5.6.3}$$

Fix a point $(f, 0) = ((\alpha x + \beta y)^3, 0) \in \mathrm{Cone}(\Gamma)$. The g-action on the normal space

$$\mathscr{N}_{\mathrm{Cone}(\Gamma)/(M_3 \oplus \mathbb{C}),(f,0)} = T_{(f,0)}(M_3 \oplus \mathbb{C})/T_{(f,0)} \mathrm{Cone}(\Gamma)$$

is trivial if and only if $g \in T_{(f,0)} \mathrm{Cone}(\Gamma)$. However, one has

$$\bigcap_{(f,0) \in \mathrm{Cone}(\Gamma)} T_{(f,0)} \mathrm{Cone}(\Gamma) \subset T_{(x^3,0)} \mathrm{Cone}(\Gamma) \cap T_{(y^3,0)} \mathrm{Cone}(\Gamma) = \{0\}.$$

It follows that the only element of $(M_3, +)$ acting trivially on $\mathscr{N}_{\mathrm{Cone}(\Gamma)/(M_3 \oplus \mathbb{C})}$ is $g = 0$. $\square$

5.6.4. **Corollary.** $K \cap \mathrm{R_u} = \{1\}$.

Therefore, (5.4.1) gives an embedding of K in $\mathrm{GL}_2(\mathbb{C}) = \mathrm{Aut}(W)/\mathrm{R_u}$. Since K is normal, K is contained in $\mathrm{z}(\mathrm{GL}_2(\mathbb{C})) \cong \mathbb{G}_\mathrm{m}$. The latter singular torus acts on $\Upsilon \cong \mathbb{P}^1$ with a fixed point, say, Q. The tangent representation of the reductive group $\mathbb{G}_\mathrm{m}$ on $T_Q W$ is faithful. On the other hand, $Q \in \Xi = \mathrm{Sing}\, R$, hence $T_Q W$ is the Zariski tangent space of R at Q. Thus, K acts identically on $T_Q W$, and so, $K = \{1\}$. This proves (b).

The first assertion of (c) follows from the correspondence of Proposition 5.5(b). The second follows from the construction of exact sequence (5.4.1) in [PdV99, Thm. 6.6].

To prove (d) we chose coordinates in $\mathbb{P}^4$ so that the induced action of $\mathrm{z}(\mathrm{L})$ on $\mathbb{P}^4$ is given by the diagonal matrix $\mathrm{diag}(1, 1, 1, 1, \lambda)$, $\lambda \in \mathbb{C}^*$; cf. the proof of (b). This action is identical on $\langle\Gamma\rangle \cong \mathbb{P}^3$, hence also on the normal bundle $\mathscr{N}_{\Gamma/\mathbb{P}^3}$. However, it is effective on the normal bundle $\mathscr{N}_{\Gamma/\mathbb{P}^4}$. This allows to decompose $\mathscr{N}_{\Gamma/\mathbb{P}^4}$ into a direct sum of proper subbundles of ranks 2 and 1, respectively. The rank two subbundle projects in

$\widehat{R} \cong \mathbb{P}(\mathscr{N}^{\vee}_{\Gamma/\mathbb{P}^4})$ to a $\mathbb{P}^1$-subbundle, and the rank one projects to a disjoint section, say, $\widehat{\Psi}$. In each plane $\widehat{\Pi}_\gamma$, $\gamma \in \Gamma$, this gives a line $\widehat{l_\gamma}$ of fixed points of $\mathrm{z(L)}$ and an isolated fixed point $\widehat{\Psi} \cap \widehat{\Pi}_\gamma$. The image of $\widehat{l_\gamma}$ in Π_γ is the line $l_\gamma = \Pi_\gamma \cap \Xi$. The image $\Psi \subset R$ of the curve $\widehat{\Psi} \subset \widehat{R}$ is a second irreducible component of the fixed point locus of $\mathrm{z(L)}|_R$. This curve Ψ is disjoint with Ξ and meets each plane Π_γ in a point. The linear projection $\phi : \mathbb{P}^7 \dashrightarrow \mathbb{P}^4$ with center Ξ as in (2.2.1) sends Ψ isomorphically onto the rational twisted cubic curve Γ. The linear span $\langle \Psi \rangle$ is sent by ϕ onto $\langle \Gamma \rangle \cong \mathbb{P}^3$. It follows that Ψ is a rational twisted cubic curve, and $\langle \Psi \rangle \cap \Xi = \varnothing$. □

In the sequel we need the following simple lemma.

5.7. **Lemma.** *Let $\mathfrak{H}$ be a subgroup of* $\mathrm{Aut}(W)$ *isomorphic either to* $\mathrm{GL}_2(\mathbb{C})$*, or to the* 2*-torus* $(\mathbb{G}_\mathrm{m})^2$*. Then there is a unique Levi subgroup* L *of* $\mathrm{Aut}(W)$ *such that* $\mathrm{z(L)} \subset \mathfrak{H} \subset \mathrm{L}$.

Proof. In the case $\mathfrak{H} \cong \mathrm{GL}_2(\mathbb{C})$ the assertion follows from Proposition 5.5(a). If $\mathfrak{H} \cong (\mathbb{G}_\mathrm{m})^2$, then $\mathfrak{H}$ is a maximal torus in $\mathrm{Aut}(W)$. Being reductive, $\mathfrak{H}$ is contained in a Levi subgroup L of $\mathrm{Aut}(W)$ and contains its center $\mathrm{z(L)}$. Up to conjugation, one may suppose that L is the standard $\mathrm{GL}_2(\mathbb{C})$-subgroup of $\mathrm{Aut}(W) \hookrightarrow \mathrm{Aff}(\mathbb{C}^4)$ as in 5.1, and $\mathfrak{H}$ is the diagonal torus. That is, $W \setminus R \cong \mathbb{P}^4 \setminus \mathbb{P}^3 = \mathbb{C}^4$ is realized as the vector space M_3 of binary cubic forms, where $\mathrm{R_u}$ acts by translations and $\mathrm{GL}_2(\mathbb{C})$ acts via $h.f = f \circ h^{-1}$ for any $h \in \mathrm{GL}_2(\mathbb{C})$ and $f \in M_3$. In particular, $\mathfrak{H}$ acts via

$$(\lambda, \mu).(a_0, a_1, a_2, a_3) = (\lambda^{-3}a_0, \lambda^{-2}\mu^{-1}a_1, \lambda^{-1}\mu^{-2}a_2, \mu^{-3}a_3).$$

Hence $\mathfrak{H}$ has a unique fixed point, say, P, which is the unique fixed point of L in $W \setminus R$. This point determines L uniquely, see Proposition 5.5(c). □

6. Quintic scrolls in W_5 and their stabilizers

In the next proposition we construct a special quintic scroll $F \cong \mathbb{F}_1$ in W contained in R and acted upon by $\mathrm{GL}_2(\mathbb{C})$.

6.1. **Proposition.** *With the notation as in* 5.1 *the following hold.*

(a) *Let $\widehat{\Delta}$ be the proper transform of Δ in $\widehat{W}$. Then the image $\rho(\widehat{\Delta}) \subset W$ is cut out on W by a quadric.*
(b) *The reduced intersection $F = (\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}} \subset \mathbb{P}^6$ is a smooth quintic scroll isomorphic to $\mathbb{F}_1$ and invariant under an effective* $\mathrm{GL}_2(\mathbb{C})$*-action on W.*
(c) *$F \cap \Xi = \Upsilon$ and Ξ is the only plane in W meeting $F \cong \mathbb{F}_1$ along a conic.*

Proof. (a) Since Δ is singular along Γ, we have $\widehat{\Delta} \sim 4L^* - 2\widehat{R} \sim 2H^*$ on $\widehat{W}$, see (2.3.1). Hence $\rho(\widehat{\Delta})$ is cut out on W by a quadric.

(b) First we construct a special quintic scroll $F \subset R$, and then we show that it coincides with $(\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}}$. Let $N = \mathrm{Cone}(\Gamma, P) \subset \mathbb{P}^4$ be as in Remark 5.2.1, and let $\widehat{\Gamma} = \widehat{N} \cap \widehat{R}$ be as in Lemma 2.4. By Lemma 2.3 the normalization morphism $\rho|_{\widehat{R}} : \widehat{R} \to R$ sends each fiber $\widehat{\Pi}_\gamma$ of the $\mathbb{P}^2$-bundle $\varphi|_{\widehat{R}} : \widehat{R} \to \Gamma$ isomorphically onto a plane $\Pi_\gamma \subset R = \bigcup_{\gamma' \in \Gamma} \Pi_{\gamma'}$. The curve $\widehat{\Gamma} \subset \widehat{R}$ is a section of $\varphi|_{\widehat{R}}$ meeting any plane $\widehat{\Pi}_\gamma$ in a single point, say, $\widehat{p}_\gamma$. Set $\Psi := \rho(\widehat{\Gamma})$. By Lemma 2.4 one has $\langle \Psi \rangle \cap \Xi = \varnothing$ and Ψ is a twisted cubic curve. Hence $p_\gamma := \rho(\widehat{p}_\gamma) \in \Pi_\gamma \setminus \Xi$. Letting $\{q_\gamma\} = \Pi_\gamma \cap \Upsilon$, consider the one-parameter family of lines

$\Lambda_\gamma \subset \Pi_\gamma \subset R$ joining p_γ and q_γ. Thus, any Λ_γ meets the conic $\Upsilon \subset \Xi$ and the twisted cubic $\Psi \subset R \setminus \Xi$. Since $\langle\Psi\rangle \cap \langle\Upsilon\rangle = \varnothing$, the join

$$F = \bigcup_{\gamma\in\Upsilon} \Lambda_\gamma \tag{6.1.1}$$

of corresponding points of Υ and Ψ is a rational normal quintic scroll in R ([GH94, Ch. 4, § 3]).

Let us show that $F \subset (\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}}$ for F as in (6.1.1). Due to our choice of $\widehat{\Gamma}$ we have $\widehat{\Delta} \cap \widehat{R} \supset \widehat{N} \cap \widehat{R} = \widehat{\Gamma}$. So, $(\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}} \supset \Psi$. On the other hand, $(\rho(\widehat{\Delta}) \cap \Xi)_{\mathrm{red}}$ is a $\mathrm{GL}_2(\mathbb{C})$-invariant curve. Since Υ is the only $\mathrm{GL}_2(\mathbb{C})$-invariant curve in Ξ, one has $\Upsilon = (\rho(\widehat{\Delta}) \cap \Xi)_{\mathrm{red}} \subset (\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}}$.

For any fiber $\widehat{\Pi}_\gamma$ of $\varphi|_{\widehat{R}} : \widehat{R} \to \Gamma$, where $\gamma \in \Gamma$, the intersection $\widehat{\Pi}_\gamma \cap \widehat{\Delta}$ is the projectivization of the tangent cone to $\Delta \cap \mathbb{P}^3$ at γ, where $\mathbb{P}^3 \subset \mathbb{P}^4$ is a general hyperplane passing through γ. Since the singularity of $\Delta \cap \mathbb{P}^3$ at γ looks locally like a product of a simple cusp singularity by a smooth curve germ, this projectivization is a line $\widehat{\Lambda}_\gamma$ in $\widehat{\Pi}_\gamma \cong \mathbb{P}^2$. The image $\Lambda_\gamma = \rho(\widehat{\Lambda}_\gamma)$ is a line in W passing through the points $p_\gamma \in \Psi \cap \Pi_\gamma$ and $q_\gamma \in \Pi_\gamma \cap \Upsilon$. Thus, $F \subset (\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}}$. On the other hand, $\deg(\rho(\widehat{\Delta}) \cap R) = 10$ and $\deg F = 5$. Furthermore, Δ has cuspidal singularities along $\Gamma = \varphi(\widehat{R})$, hence $\widehat{R}$ is tangent to $\widehat{\Delta}$ along $(\widehat{\Delta} \cap \widehat{R})_{\mathrm{red}}$. Therefore, one has $F = (\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}}$.

The hyperplane section R is invariant under the $\mathrm{Aut}(W)$-action, and $\rho(\widehat{\Delta}) \subset W$ is invariant under the $\mathrm{GL}_2(\mathbb{C})$-action on W as in Remark 5.2.3. Hence the quintic scroll $F = (\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}} \subset W$ is $\mathrm{GL}_2(\mathbb{C})$-invariant as well.

(c) Indeed, by construction,$F \cap \Xi = \Upsilon$ and any plane Π_γ meets F along a ruling Λ_γ and is not tangent to F along Λ_γ. □

6.2. **Lemma.** *Let $J \subset \Xi$, $J \neq \Upsilon$, be a smooth conic meeting Υ with even multiplicities. Then the following hold.*

(a) *The group $\mathrm{Aut}(\Xi, \Upsilon, J)$ of automorphisms of Ξ which leave Υ and J invariant is given by the following table:*

J	$\mathrm{Aut}(\Xi, \Upsilon, J)$
4.3(i)	$\mathrm{Aut}(\Upsilon) \cong \mathrm{PGL}_2(\mathbb{C})$
4.3(ii)	$\mathbb{G}_{\mathrm{a}} \rtimes (\mathbb{Z}/2\mathbb{Z})$
4.3(iii)	$\mathbb{G}_{\mathrm{m}} \rtimes (\mathbb{Z}/2\mathbb{Z})$

(b) *Up to the natural action of $\mathrm{Aut}(\Xi)$, there is a unique configuration (Υ, J) of type 4.3(ii), while the configurations of type 4.3(iii) form a one-parameter family.*

Proof. (a) We leave aside the trivial case $J = \Upsilon$. If $J \neq \Upsilon$, then the action of $\mathrm{Aut}^0(\Xi, \Upsilon, J)$ leaves invariant Υ, J, and the degenerate members of the pencil of conics $\mathscr{P}$ generated by Υ and J. Hence $\mathrm{Aut}^0(\Xi, \Upsilon, J)$ acts identically on the base $\mathbb{P}^1$ of $\mathscr{P}$. So, $\mathrm{Aut}^0(\Xi, \Upsilon, J)$ leaves invariant each member of $\mathscr{P}$.

For J of type 4.3(iii), in appropriate homogeneous coordinates $(x : y : z)$ in $\Xi \cong \mathbb{P}^2$ the general member $J_{(a:b)}$ of $\mathscr{P}$ can be given by equation $ax^2 + byz = 0$. The projective transformations fixing the common points of J and Υ are of the form $(x : y : z) \longmapsto (x : \lambda y : \lambda^{-1} z)$, where $\lambda \in \mathbb{C}^*$. Thus, $\mathrm{Aut}^0(\Xi, \Upsilon, J) \cong \mathbb{G}_{\mathrm{m}}$. The group $\mathrm{Aut}(\Xi, \Upsilon, J)$ is generated by $\mathrm{Aut}^0(\Xi, \Upsilon, J)$ and the involution

$$\kappa : (x : y : z) \longmapsto (x : z : y)$$

that interchanges the two points of the intersection $\Upsilon \cap J$.

For J of type 4.3(ii), the general member $J_{(a:b)}$ of $\mathscr{P}$ can be given by the equation $a(x^2 + yz) + bz^2 = 0$. Then

$$\mathrm{Aut}^0(\Xi, \Upsilon, J) = \{(x : y : z) \longmapsto (x + ez : y + 2ex + e^2 z : z) \mid e \in \mathbb{C}\} \cong \mathbb{G}_{\mathrm{a}},$$

while $\mathrm{Aut}(\Xi, \Upsilon, J)$ is generated by $\mathrm{Aut}^0(\Xi, \Upsilon, J)$ and the involution

$$\kappa : (x : y : z) \longmapsto (-x : y : z).$$

The proof of (b) results from elementary computations. We leave this to the reader. □

6.3. **Proposition.** *Fix a nondegenerate conic $J \subset \Xi$ which touches Υ with even multiplicities, see Figure* 4.3.1. *Fix also a reductive Levi subgroup $\mathrm{L} \subset \mathrm{Aut}(W)$. Let an* L*-invariant twisted cubic curve Ψ be the set of fixed points of* $\mathrm{z}(\mathrm{L})$ *in $R \setminus \Xi$, see Proposition* 5.6(d).

(a) *There exists a* $\mathrm{z}(\mathrm{L})$*-invariant rational normal quintic scroll $F \subset R$, $F \cong \mathbb{F}_1$, with exceptional section $J = F \cap \Xi$.*

(b) *If $J = \Upsilon$, then F as in* (a) *is unique and can be transformed into the scroll $(\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}}$ as in Proposition* 6.1 *by an automorphism from* $\mathrm{R_u}$. *If $J \neq \Upsilon$, then there are exactly two different scrolls F_1 and F_2 as in* (a). *Any such scroll contains Ψ.*

(c) *Furthermore, any F as in* (a) *is invariant under the action on W of the identity component $G = G(\mathrm{L}, J)$ of the subgroup $\mathrm{L} \cap \varrho^{-1}(\mathrm{Aut}(\Xi, \Upsilon, J))$, where $\varrho : \mathrm{Aut}(W) \to \mathrm{Aut}(\Upsilon) = \mathrm{Aut}(\Xi, \Upsilon)$ is the restriction homomorphism, see* (5.4.1).

(d) *For $J \neq \Upsilon$, consider the involution $\kappa \in \mathrm{Aut}(\Xi, \Upsilon, J) \setminus \mathrm{Aut}^0(\Xi, \Upsilon, J)$ (see the proof of Lemma* 6.2(a)). *Let $\widetilde{\kappa} \in \mathrm{L} \cap \varrho^{-1}(\mathrm{Aut}(\Xi, \Upsilon, J)) \subset \mathrm{Aut}(W)$ be such that $\varrho(\widetilde{\kappa}) = \kappa$. Then $\widetilde{\kappa}$ interchanges F_1 and F_2. In particular, $(W, F_1) \cong (W, F_2)$.*

Proof. (a) Assume first that $J = \Upsilon$. By Proposition 5.6 (d) any plane Π_γ, $\gamma \in \Upsilon$, meets Ψ in a unique point, say, p_γ, and meets Υ tangentially in a unique point q_γ. Indeed, $l_\gamma := \Pi_\gamma \cap \Xi$ is the tangent line to Υ at q_γ. The lines (p_γ, q_γ) sweep up a normal rational quintic scroll $F = F(\Psi)$. This scroll F is L-invariant and meets the plane Ξ along Υ. In particular, F is $\mathrm{z}(\mathrm{L})$-invariant. If $\mathrm{L} = \mathrm{L}_0$ comes from the standard $\mathrm{GL}_2(\mathbb{C})$-action on $\mathbb{P}(M_3 \oplus \mathbb{C})$ as in 5.1, then F coincides with $(\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}}$, see the proof of Proposition 6.1. Otherwise, $\mathrm{L} = g\mathrm{L}_0 g^{-1}$ for some $g \in \mathrm{R_u}$ such that $g(F) = (\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}}$.

Let further $J \neq \Upsilon$. Consider the $2:1$ morphism

$$\delta : J \longrightarrow \Upsilon, \quad \Pi_\gamma \cap J \longmapsto \Pi_\gamma \cap \Upsilon = \{q_\gamma\}.$$

By our assumption, the ramification indices of δ are even. Hence δ admits two distinct sections, say, $\sigma_1, \sigma_2 : \Upsilon \to J$, see the proof of Proposition 4.4(d). Letting $t_{\gamma,i} = \sigma_i(q_\gamma)$, consider the smooth quintic scroll $F_i = F_i(J, \mathrm{L}) \cong \mathbb{F}_1$ formed by the lines $(p_\gamma, t_{\gamma,i})$, $\gamma \in \Upsilon$. The conic $J \supset \Xi$ is a common exceptional section of $F_i \to \mathbb{P}^1$, $i = 1, 2$, and Ψ is a common section with $\Psi^2 = 1$. Besides, the scrolls F_1 and F_2 have common rulings passing through the points of $J \cap \Upsilon$. Since J and Ψ are pointwise fixed under the $\mathrm{z}(\mathrm{L})$-action (see Proposition 5.6(d)), each ruling of F_i is $\mathrm{z}(\mathrm{L})$-invariant and represents an orbit closure of $\mathrm{z}(\mathrm{L})$. Hence F_i is $\mathrm{z}(\mathrm{L})$-invariant for $i = 1, 2$.

(b) Let now $F \cong \mathbb{F}_1$ be a smooth rational $\mathrm{z(L)}$-invariant quintic scroll in R with exceptional section $J \subset \Xi$. Since $\mathrm{z(L)}$ acts identically on Ξ, see again Proposition 5.6(d), the $\mathrm{z(L)}$-action on F leaves invariant each ruling Λ_γ, where Λ_γ represents a $\mathrm{z(L)}$-orbit closure. Hence, there are exactly two fixed points of $\mathrm{z(L)}$ on Λ_γ. One of these points runs over J when γ runs over Υ, and the other one runs over Ψ. In particular, $\Psi \subset F$.

The line Λ_γ is contained in a unique plane Π_γ through the point $p_\gamma \in \Psi \cap \Lambda_\gamma$. It passes through the point p_γ and one of the intersection points $t_{\gamma,1}$, $t_{\gamma,2} \in \Pi_\gamma \cap J$. It follows that F coincides with one of the scrolls F_1 and F_2 constructed in (a), where $F_1 = F_2 = F$ in the case $J = \Upsilon$.

To complete the proof of (b) in the case $J = \Upsilon$, consider the standard quintic scroll $F^{\mathrm{s}} := (\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}}$ with $F^{\mathrm{s}} \cap \Xi = \Upsilon$. It is invariant under the action on W of a special Levi subgroup $\mathrm{L}^{\mathrm{s}} \cong \mathrm{GL}_2(\mathbb{C})$ as in Proposition 6.1(b). The Levi subgroups L and L^{s} are conjugated via an element $g \in \mathrm{R_u}$. Clearly, g conjugates their singular tori $\mathrm{z(L)}$ and $\mathrm{z(L^s)}$, and sends the twisted cubic curve $\Psi \subset F$ to the corresponding $\mathrm{z(L^s)}$-fixed curve $\Psi^{\mathrm{s}} \subset F^{\mathrm{s}}$. By virtue of the preceding uniqueness result, it follows that $g(F) = F^{\mathrm{s}}$.

Now (c) is immediate. Notice that G is a connected subgroup of L which leaves J invariant. We claim that G sends the rulings of F into rulings. Indeed, these rulings are $\mathrm{z(L)}$-orbit closures meeting J. Since $\mathrm{L} = \mathcal{C}_L(\mathrm{z(L)})$, L acts on the set of $\mathrm{z(L)}$-orbits. Hence G sends the rulings of F into $\mathrm{z(L)}$-orbit closures meeting J. By connectedness of G, it sends the rulings of F into rulings. Thus, the scroll F is invariant under G.

(d) Notice that $\widetilde{\kappa} \in \mathrm{L}$, Ψ is L-invariant, and J is invariant under $\kappa = \widetilde{\kappa}|_\Xi$. Furthermore, $\widetilde{\kappa} \in \mathrm{L}$ commutes with $\mathrm{z(L)}$, hence sends the $\mathrm{z(L)}$-orbits into $\mathrm{z(L)}$-orbits. It follows from the construction of F_1 and F_2 in (b) that $F_1 \cup F_2$ is invariant under $\widetilde{\kappa}$. It can be readily seen that $\widetilde{\kappa}(F_i) = F_j$ for $i \neq j$. □

From Lemma 6.2(a) and Proposition 6.3(c) we deduce such a corollary.

6.3.1. **Corollary.** *Consider a* $\mathrm{z(L)}$*-invariant quintic scroll* $F \subset R$ *as in Proposition* 6.3(a) *along with the subgroup* $G = G(\mathrm{L}, J) \subset \mathrm{L}$ *as in* 6.3(c). *Then the following hold.*

(a) $\mathrm{Aut}^0(W, F) = (\mathrm{R_u} \cap \mathrm{Aut}^0(W, F)) \rtimes G$, *and*

(b) *G is isomorphic either to* $\mathrm{GL}_2(\mathbb{C})$, *or to* $\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$, *or to* $(\mathbb{G}_{\mathrm{m}})^2$ *provided the conic* $J = F \cap \Xi$ *is of type* 4.3(i), 4.3(ii), *and* 4.3(iii), *respectively. If* $G \cong \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$, *then the* $\mathbb{G}_{\mathrm{a}}$*-subgroup of* G *acts nontrivially on* J, *and the* $\mathbb{G}_{\mathrm{m}}$*-subgroup preserves each ruling of* $F \to \mathbb{P}^1$;

(c) *Let* $U = \mathrm{R_u} \cap \mathrm{Aut}^0(W, F) \neq \{1\}$. *Then* U *acts trivially on* J, *fiberwise on* $F \to \mathbb{P}^1$, *and either* $U \cong \mathbb{G}_{\mathrm{a}}$, *or* $U \cong (\mathbb{G}_{\mathrm{a}})^2$. *For any* $t \in U$ *the rational twisted cubic curve* $t(\Psi) \subset F$ *is pointwise fixed under the singular torus* $\mathrm{z(L}_t)$, *where* $\mathrm{L}_t = t \cdot \mathrm{L} \cdot t^{-1}$.

Proof. (a) We have $\mathrm{Aut}(W) = \mathrm{R_u} \rtimes L$, see (5.5.1). Let G be as in Proposition 6.3(c). Since $G \subset L \cap \mathrm{Aut}^0(W, F)$, then

$$(\mathrm{R_u} \cap \mathrm{Aut}^0(W, F)) \rtimes G = (\mathrm{R_u} \cap \mathrm{Aut}^0(W, F)) \cdot G \subset \mathrm{Aut}^0(W, F).$$

To show the opposite inclusion we proceed as follows. Let $g = r \cdot l \in \mathrm{Aut}^0(W, F)$, where $r \in \mathrm{R_u}$ and $l \in \mathrm{L}$.

- Since $r|_\Xi = \mathrm{id}_\Xi$, see Proposition 5.6(c), and $g(J) = J$, then $l(J) = J$.
- Since $l(\Psi) = \Psi$ and $g(\Psi) \subset F$, then $r(\Psi) \subset F$.
- Since $r|_\Xi = \mathrm{id}_\Xi$, then $r(\Pi_\gamma) = \Pi_\gamma$ $\forall \gamma \in \Upsilon$.

- Since $r(\Psi) \subset F$ and $r(\Psi \cap \Pi_\gamma) \subset F \cap \Pi_\gamma = \Lambda_\gamma$, then $r(\Lambda_\gamma) = \Lambda_\gamma\ \forall \gamma \in \Upsilon$.
- Thus, $r(F) = F$, and since $g(F) = F$, then $l(F) = F$.
- It follows that $\mathrm{Aut}^0(W, F) = (\mathrm{R_u} \cap \mathrm{Aut}^0(W, F)) \rtimes (\mathrm{L} \cap \mathrm{Aut}^0(W, F))$.
- Since $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F)$ is connected, then $\mathrm{L} \cap \mathrm{Aut}^0(W, F)$ is.
- Since $l \in (\mathrm{L} \cap \mathrm{Aut}^0(W, F))^0$ and $\varrho(l) \in \mathrm{Aut}^0(\Xi, \Upsilon, J)$, then $l \in G$.
- Therefore, $G = \mathrm{L} \cap \mathrm{Aut}^0(W, F)$, and so, (a) follows.

(b) is straightforward from Lemma 6.2 and the definition of G.

(c) Let $g \in (\mathrm{R_u} \cap \mathrm{Aut}^0(W, F)) \setminus \{1\}$, and let U be the unique one-parameter subgroup of $\mathrm{R_u}$ containing g. Then U is the Zariski closure of the group generated by g. Hence U stabilizes F. We have shown before that the action of $U = \mathrm{R_u} \cap \mathrm{Aut}^0(W, F)$ on F preserves each ruling and fixes J pointwise. The restriction $U|_f$ to a general ruling $f \cong \mathbb{P}^1$ acts via the unipotent part of the Borel subgroup of $\mathrm{Aut}(f) \cong \mathrm{PGL}_2(\mathbb{C})$ fixing the point $f \cap J$. Contracting J to a point $p \in \mathbb{P}^2$ yields an embedding of U onto an abelian unipotent group of linear transformations fixing p and preserving each line through p. Now the remaining assertions follow. □

In the following lemma we indicate the case, where the condition $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F) = \{1\}$ is automatically fulfilled.

6.4. **Lemma.** *Let $F \subset R$ be a rational normal quintic scroll which meets Ξ along a smooth conic J. Then the following conditions are equivalent:*

(i) $\mathrm{Aut}(W, F)$ *contains a subgroup* $\mathcal{H} \cong \mathrm{SL}_2(\mathbb{C})$*;*
(ii) $\mathrm{Aut}(W, F) \cong \mathrm{GL}_2(\mathbb{C})$ *and* $\mathrm{R_u} \cap \mathrm{Aut}(W, F) = \{1\}$*;*
(iii) $J = \Upsilon$ *and F is invariant under the action of a singular torus of* $\mathrm{Aut}(W)$*.*

Proof. The implication (iii)⇒(i) is straightforward from Corollary 6.3.1(b). The implication (ii)⇒(iii) is immediate; indeed, Υ is a unique $\mathrm{PGL}_2(\mathbb{C})$-invariant conic in Ξ.

It remains to prove (i)⇒(ii). The $\mathrm{SL}_2(\mathbb{C})$-subgroup $\mathcal{H} \subset \mathrm{Aut}(W, F)$ is contained in a unique reductive Levi subgroup $\mathrm{L} \cong \mathrm{GL}_2(\mathbb{C})$ of $\mathrm{Aut}(W)$. Let $T := \mathrm{z}(\mathrm{L})$. The center $\mathrm{z}(\mathcal{H})$ has an isolated fixed point on $\Pi_\gamma \setminus \Xi$, $\forall \gamma \in \Upsilon$. Therefore, $\mathrm{z}(\mathcal{H})$ has a curve of fixed points $\Psi \subset R \setminus \Xi$. Since $\mathrm{z}(\mathcal{H}) \subset T$, the points of Ψ are fixed by T, see Proposition 5.6(d). The lines joining the corresponding points of J and Ψ are T-orbits. Therefore, Ψ, $J = F \cap \Xi$, and F are L-invariant.

The subgroup $\mathrm{L} \subset \mathrm{Aut}(W, F)$ normalizes both $\mathrm{Aut}(W, F)$ and $\mathrm{R_u} \cong (\mathbb{G}_\mathrm{a})^4$. Hence it normalizes the intersection $K := \mathrm{Aut}(W, F) \cap \mathrm{R_u}$. Since K is normalized by L and $\mathrm{R_u}$, it is a normal subgroup in $\mathrm{Aut}(W)$. However, K cannot be a proper subgroup of $\mathrm{R_u}$. Indeed, the representation of $\mathrm{L} \cong \mathrm{GL}_2(\mathbb{C})$ on $\mathrm{R_u} \cong \mathbb{C}^4$ by conjugation is irreducible, see Remark 5.5.2.1. Therefore, there is an alternative: either $\mathrm{Aut}(W, F) = \mathrm{L}$, or $\mathrm{Aut}(W, F) = \mathrm{Aut}(W)$. The latter is impossible since $R \setminus \Xi$ is an orbit of $\mathrm{Aut}(W)$, see (5.5.5). This proves (ii). □

6.4.1. **Corollary.** *Let $F_1, F_2 \subset R$ be two quintic scrolls meeting Ξ along smooth conics. If* $\mathrm{Aut}(W, F_i) \cong \mathrm{GL}_2(\mathbb{C})$ *for $i = 1, 2$, then $F_2 = g(F_1)$ for some $g \in \mathrm{R_u}$. Consequently, the pairs (V_1, S_1) and (V_2, S_2) linked to (W, F_1) and (W, F_2), respectively, are isomorphic.*

Proof. Due to Lemma 6.4, for $i = 1, 2$ one has $F_i \cap \Xi = \Upsilon$, $\mathrm{L}_i := \mathrm{Aut}(W, F_i)$ is a Levi subgroup in $\mathrm{Aut}(W)$, and so, $\mathrm{z}(L_i)$ stabilizes F_i. Letting Ψ_i be the component of the fixed point set of $\mathrm{z}(\mathrm{L}_i)|_{F_i}$ different from Υ, we see that all the assumptions of Proposition 6.3(b) are satisfied for F_1 and F_2. By this proposition, both F_1 and F_2 can be transformed into $(\rho(\widehat{\Delta}) \cap R)_{\mathrm{red}}$ by suitable automorphisms from $\mathrm{R_u}$. Now the assertions follow. □

In the next lemma we construct some L-invariant cubic cone in W.

6.5. **Lemma.** *Let* L *be a reductive Levi subgroup of* $\mathrm{Aut}(W)$*, let Q be the unique fixed point of the singular torus $T = \mathrm{z(L)}$ in $W \setminus R$, and let $\Psi \subset R \setminus \Xi$ be the one-dimensional component of the fixed point set W^T, see Proposition* 5.6(d)*. Consider the cubic cone* $\mathrm{Cone}(\Psi, Q) \subset \mathbb{P}^7$ *over Ψ with vertex Q. Then* $\mathrm{Cone}(\Psi, Q)$ *is contained in W and* L*-invariant.*

Proof. It is easy to see that there is a one-dimensional family of lines in W passing through Q (cf. [Isk77, Proposition 5.3], [KPS16, Lemma 2.2.6]). The union of these lines form an L-invariant surface S_Q, which is a cone with vertex Q. Moreover, $S_Q = W \cap T_Q W$ because $W = W_5 \subset \mathbb{P}^7$ is an intersection of quadrics. The singular torus T has a curve of fixed points on S_Q, and this curve must coincide with $S_Q \cap R = \Psi$. Therefore, $S_Q = \mathrm{Cone}(\Psi, Q)$. □

6.6**.** Let us fix the following setup. Consider two linked pairs (W, F) and (V, S) as in Corollary 4.5.1. That is, $F \subset R$ is a smooth rational normal quintic scroll such that $F \cong \mathbb{F}_1$, $J = F \cap \Xi$ is a smooth conic touching Υ with even multiplicities, V is a smooth Fano-Mukai fourfold of genus 10, and $S \subset V$ is a cubic cone with $\mathrm{Aut}^0(V, S) \cong \mathrm{Aut}^0(W, F)$. Suppose that F is invariant under a singular torus $\mathrm{z(L)}$, where $\mathrm{L} \subset \mathrm{Aut}(W)$ is a reductive Levi subgroup, see Remark 5.5.2.2. Via the above isomorphism, the identity component G_{L} of $\mathrm{L} \cap \mathrm{Aut}(W, F)$ acts effectively on V stabilizing S (cf. Corollary 6.3.1). In the following lemma we construct a pair of disjoint G_{L}-invariant cubic cones in V.

6.6.1. **Lemma.** *In the setting of* 6.6 *the following hold.*

- (a) *There exists a unique G_{L}-invariant cubic cone $S' \subset V$ disjoint with S.*
- (b) *If $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F) = \{1\}$, then S' in* (a) *is $\mathrm{Aut}^0(V, S)$-invariant.*
- (c) *If $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F) \neq \{1\}$, then there exists a one-parameter family (S'_t) of cubic cones in V disjoint with S.*

Proof. (a) Let Q be the unique fixed point of $\mathrm{z(L)}$ in $W \setminus R$, and let $\Psi \subset R \setminus \Xi$ be the 1-dimensional component of the fixed point set of $\mathrm{z(L)}$. Consider the L-invariant cubic cone $S'_W := \mathrm{Cone}(\Psi, Q) \subset W$, see Lemma 6.5. Since S'_W meets the hyperplane section R transversely along Ψ, the proper transform $S'_{\widetilde{W}}$ of S'_W in $\widetilde{W}$ is isomorphic to S'_W and disjoint with $\widetilde{R} \subset \widetilde{W}$. Then the image $S' := \xi(S'_{\widetilde{W}}) \subset V$ is disjoint with $S = \xi(\widetilde{R})$ and G-invariant. The linear projection $\theta : \mathbb{P}^{12} \dashrightarrow \mathbb{P}^7$ with center $\langle S \rangle$ sends S' isomorphically onto S'_W. Therefore, S' is a cubic cone disjoint with S. This yields the existence in (a).

Let $S'' \subset V$ be a G_{L}-invariant cubic cone disjoint with S. Then $S''_W = \theta(S'') \subset W$ is a G_{L}-invariant cubic cone in W. The hyperplane section $A = A_S \subset V$ as in 3.1.1 meets S'' along a G_{L}-invariant rational twisted cubic curve. Its image $\Psi'' \subset F$ under θ is also a G_{L}-invariant rational twisted cubic curve and a section of the ruling $F \to J$. Since $\mathrm{z(L)} \subset G_{\mathrm{L}}$, see 6.6, Ψ'' is $\mathrm{z(L)}$-invariant. It follows that Ψ'' is a component of the fixed point set of $\mathrm{z(L)}$ in $R \setminus \Xi$, hence $\Psi'' = \Psi$. The vertex $Q'' \in W \setminus R$ of S''_W coincides with the unique isolated fixed point Q of $\mathrm{z(L)}$. Thus, $S''_W = \mathrm{Cone}(\Psi'', Q'') = \mathrm{Cone}(\Psi, Q) = S'_W$. Therefore, $S'' = S'$. This proves the uniqueness in (a).

(b) If $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F) = \{1\}$, then $\mathrm{Aut}^0(W, F) = G_{\mathrm{L}}$, see Corollary 6.3.1(a). By Lemma 6.5, the cubic cone $S'_W = \mathrm{Cone}(\Psi, Q) \subset W$ is $\mathrm{Aut}^0(W, F)$-invariant. Then S' is $\mathrm{Aut}^0(V, S)$-invariant, because diagram (3.1.1) is $\mathrm{Aut}^0(W, F)$-equivariant.

(c) If $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F) \neq \{1\}$, then the Sarkisov link (3.1.1) provides a one-parameter family of cubic cones in V disjoint with S, see Corollary 6.3.1(c). □

6.7. **Construction.** Let us investigate more closely the rational normal quintic scrolls $F \subset R$ with $F \cong \mathbb{F}_1$ and $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F) \neq \{1\}$. We use the notation from Lemma 2.3 and Proposition 5.6. The proper transform $\widehat{F}$ of F in $\widehat{W}$ (see diagram (2.2.1)) is a smooth scroll which meets $\widehat{\Xi} \cong \mathbb{P}^1 \times \mathbb{P}^1$ along a nondegenerate conic, say, $\widehat{J}$. The morphism $\varphi|_{\widehat{F}} : \widehat{F} \to \Gamma$ induces on $\widehat{F}$ a structure of a $\mathbb{P}^1$-subbundle of the $\mathbb{P}^2$-bundle $\varphi|_{\widehat{R}} : \widehat{R} \to \Gamma$, see Lemma 2.3(a). This is the projectivization of a rank 2 vector subbundle $\mathcal{L} \to \Gamma$ of the normal bundle $\mathscr{N}_{\Gamma/\mathbb{P}^4}$. In each fiber of $\mathscr{N}_{\Gamma/\mathbb{P}^4}$, $\mathcal{L}$ is transversal to the rank 2 vector subbundle $\mathscr{N}_{\Gamma/\mathbb{P}^3}$, where $\mathbb{P}^3 = \mathbb{P}(M_3 \oplus \{0\})$. The pullback $\widetilde{\mathcal{L}}$ of $\mathcal{L}$ in the tangent bundle $T\mathbb{P}^4|_\Gamma$ is a vector subbundle of corank 1 containing the tangent bundle of Γ.

Consider the canonical projection $\pi : (M_3 \oplus \mathbb{C}) \setminus \{0\} \to \mathbb{P}(M_3 \oplus \mathbb{C}) \cong \mathbb{P}^4$ along with the affine cone (with the vertex removed) $\mathrm{Cone}(\Gamma) \subset (M_3 \oplus \mathbb{C}) \setminus \{0\}$, cf. the proof of Proposition 5.6(b). The pullback $\widetilde{\mathcal{L}}^* := \pi^* \widetilde{\mathcal{L}}$ is a vector subbundle of corank 1 of the trivial vector bundle $T(M_3 \oplus \mathbb{C})|_{\mathrm{Cone}(\Gamma)} \cong (M_3 \oplus \mathbb{C} \setminus \{0\}) \times \mathrm{Cone}(\Gamma)$ over $\mathrm{Cone}(\Gamma)$. This subbundle is invariant under the natural $\mathbb{G}_{\mathrm{m}}$-action on $(M_3 \oplus \mathbb{C}) \setminus \{0\}$ and contains the tangent bundle $T(\mathrm{Cone}(\Gamma))$. The fibers of $\widetilde{\mathcal{L}}^*$ define a one-parameter family of hyperplanes $(\widetilde{\mathcal{L}}^*_\gamma)_{\gamma \in \Gamma}$ of the vector 5-space $M_3 \oplus \mathbb{C}$ transversal to $\widetilde{E} := M_3 \oplus \{0\} = \langle \mathrm{Cone}(\Gamma) \rangle$.

In the following lemma we provide a criterion as to when F is invariant under the action of a nontrivial subgroup of the unipotent radical $\mathrm{R_u} = \mathrm{R_u}(\mathrm{Aut}(W))$.

6.7.1. **Lemma.** *In the notation as in* 6.7*, consider the family* $(H_\gamma)_{\gamma \in \Gamma}$ *of projective planes in the projective* 3*-space* $\mathbb{P}(\widetilde{E}) = \mathbb{P}(M_3)$*, where* $H_\gamma := \mathbb{P}(\widetilde{\mathcal{L}}^*_\gamma \cap \widetilde{E})$*. Then the following holds.*

- $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F) \neq \{1\}$ *if and only if the planes* H_γ *pass through a common point;*
- $\dim(\mathrm{R_u} \cap \mathrm{Aut}^0(W, F)) \geq 2$ *if and only if the planes* H_γ *vary in a pencil.*

Proof. The unipotent radical $\mathrm{R_u}$ of $\mathrm{Aut}(W)$ can be identified with the group of translations $(M_3, +) \cong \mathrm{Transl}(\mathbb{C}^4)$ acting on $M_3 \oplus \mathbb{C}$ via

$$(M_3, +) \ni g : (f, z) \longmapsto (f + zg, z) \in M_3 \oplus \mathbb{C},$$

see (5.6.2). This action is trivial on the tangent space $T_{(f,0)} \mathrm{Cone}(\Gamma)$, where $(f, 0) = ((\alpha x + \beta y)^3, 0) \in \mathrm{Cone}(\Gamma)$. The tangent action on $T_{(f,0)}(M_3 \oplus \mathbb{C}) \cong M_3 \oplus \mathbb{C}$ is given by $dg : (u, v) \longmapsto (u + vg, v)$, see (5.6.3).

Let $\gamma = \pi(f, 0) \in \Gamma$. We claim that $dg|(T_{(f,0)}(M_3 \oplus \mathbb{C}))$ preserves the hyperplane $\widetilde{\mathcal{L}}^*_\gamma \subset T_{(f,0)}(M_3 \oplus \mathbb{C}) \cong M_3 \oplus \mathbb{C}$ if and only if $(g, 0) \in \widetilde{\mathcal{L}}^*_\gamma \cap \widetilde{E}$. Indeed, $\widetilde{\mathcal{L}}^*_\gamma$ can be given in $T_{(f,0)}(M_3 \oplus \mathbb{C}) = M_3 \oplus \mathbb{C}$ by equation of the form $\langle u, w \rangle + \delta v = 0$, where $w \in \widetilde{E} = M_3 \oplus \{0\}$ is a nonzero vector and $\delta \in \mathbb{C}$. Let $(u, v) \in \widetilde{\mathcal{L}}^*_\gamma$. Then $dg(u, v) = (u + vg, v) \in \widetilde{\mathcal{L}}^*_\gamma$ if and only if $v \cdot \langle g, w \rangle = 0$. Since $\widetilde{\mathcal{L}}^*_\gamma \neq \widetilde{E}$ then $v \neq 0$ for a general $(u, v) \in \widetilde{\mathcal{L}}^*_\gamma$. So, $dg(u, v) \in \widetilde{\mathcal{L}}^*_\gamma$ for any $(u, v) \in \widetilde{\mathcal{L}}^*_\gamma$ if and only if $(g, 0) \in \widetilde{\mathcal{L}}^*_\gamma \cap \widetilde{E}$. Now the claim follows.

Therefore, dg preserves the subbundle $\widetilde{\mathcal{L}}^* \subset T(M_3 \oplus \mathbb{C})|_{\mathrm{Cone}(\Gamma)}$ if and only if $(g, 0) \in \left(\bigcap_{\gamma\in\Gamma} \widetilde{\mathcal{L}}^*_\gamma\right) \cap \widetilde{E}$. Applying this with $g \neq 0$ yields the first assertion. To show the second, if suffices to apply the same argument to a pair of non-collinear vectors $(g_1, 0)$ and $(g_2, 0)$ from $\widetilde{E} = M_3 \oplus \{0\} \cong \mathrm{R_u}$. □

The following simple lemma should be classically known. For a lack of reference, we provide an elementary argument.

6.7.2. **Lemma.** *Consider the tangent developable quartic surface $\Delta_0 \subset \mathbb{P}^3$ of the twisted cubic curve Γ, cf.* 5.1*. Then any line l on Δ_0 is a tangent line to Γ.*

Proof. Assuming the contrary, consider the projection $\pi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with center at a general point of l. The image $\pi(\Gamma)$ is a rational plane cubic such that the tangent lines to $\pi(\Gamma)$ at smooth points pass all through the same point $\pi(l)$. However, this cannot happen in characteristic zero by the duality argument. □

6.7.3. **Corollary.** *Let $F \subset R$ be a rational normal quintic scrolls such that $J = F \cap \Xi$ is a smooth conic. Then* $\dim(\mathrm{R_u} \cap \mathrm{Aut}^0(W, F)) \leq 1$.

Proof. Assuming the contrary, by Lemma 6.7.1 the planes H_γ, $\gamma \in \Gamma$, contain a common line l. Besides, each plane H_γ contains the tangent line l_γ to Γ at the point $\gamma \in \Gamma$, see 6.7. Hence l meets each l_γ. Since distinct tangent lines l_γ are disjoint, $l \subset \Delta_0$ is different from any l_γ. Now Lemma 6.7.2 gives a contradiction. □

7. G_2-construction

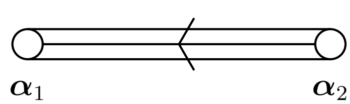
In this section we exploit the description of the Fano-Mukai fourfolds of genus 10 based on a beautiful construction of S. Mukai ([Muk88]), see Theorem 7.1. Using a lemma due to Kapustka and Ranestad [KR13, Lem. 1] we find the stabilizers of elements in the Lie algebra of the exceptional group G_2, and interpret these in terms of the automorphism groups of Fano-Mukai fourfolds V_{18}. Let us start by recalling the following result.

7.1. **Theorem** ([Muk89])**.** *Let G_2 be the simple algebraic group of exceptional type G_2. Consider the adjoint variety $\Omega = \mathrm{G}_2/P \subset \mathbb{P}(\mathfrak{g}_2) = \mathbb{P}^{13}$, where $P \subset \mathrm{G}_2$ is a parabolic subgroup of dimension 9 corresponding to a long root, and $\mathfrak{g}_2$ is the Lie algebra of G_2. Then any Fano-Mukai fourfold of genus 10 is isomorphic to a hyperplane section of the homogeneous fivefold Ω.*

7.2. **Notation.** We let $\mathfrak{g}_2$ be the Lie algebra of G_2, $\mathfrak{h}$ be a fixed Cartan subalgebra of $\mathfrak{g}_2$, and $\Delta \subset \mathfrak{h}^\vee$ be the corresponding root system. Choose a base of simple roots $\boldsymbol{\alpha}_1$, $\boldsymbol{\alpha}_2$ of Δ satisfying ([Bou02])

$$(\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_1) = 2, \quad (\boldsymbol{\alpha}_2, \boldsymbol{\alpha}_2) = 6, \quad (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2) = -3.$$

The Dynkin diagram G_2 looks like



(7.2.1)

7.3**.** Recall that G_2 has dimension 14, rank 2, and its center is trivial. It can be realized as the automorphism group of the octonion algebra $\mathbb{O}$ over $\mathbb{C}$. Let $\mathbb{O}_0 \subset \mathbb{O}$ be the hyperplane of pure octonions. Then the adjoint variety Ω can be realized as the subvariety in the

Grassmannian $\mathrm{Gr}(2, \mathbb{O}_0)$ of the isotropic two-dimensional subspaces $\Lambda \subset \mathbb{O}_0$ with respect to the multiplication in $\mathbb{O}$. The latter means that the multiplication restricts as zero to such a subspace.

In the presentation $\Omega = \mathrm{G}_2/P$ (cf. Theorem 7.1) one can choose for P the parabolic subgroup of G_2 with Lie algebra

$$\mathfrak{p} = \mathfrak{h} \oplus \left(\bigoplus_{\boldsymbol{\alpha} \in \Delta^+} \mathfrak{g}_{\boldsymbol{\alpha}} \right) \oplus \mathfrak{g}_{-\boldsymbol{\alpha}_1} = \mathfrak{b} \oplus \mathfrak{g}_{-\boldsymbol{\alpha}_1},$$

where $\mathfrak{b}$ stands for a Borel subalgebra of $\mathfrak{g}_2$. A reductive Levi subgroup of P is, e.g., the $\mathrm{GL}_2(\mathbb{C})$-subgroup with Lie algebra $\mathfrak{h} \oplus \mathfrak{g}_{\boldsymbol{\alpha}_1} \oplus \mathfrak{g}_{-\boldsymbol{\alpha}_1}$. Respectively, the $\mathrm{SL}_2(\mathbb{C})$-subgroup with Lie algebra $[\mathfrak{g}_{\boldsymbol{\alpha}_1}, \mathfrak{g}_{-\boldsymbol{\alpha}_1}] \oplus \mathfrak{g}_{\boldsymbol{\alpha}_1} \oplus \mathfrak{g}_{-\boldsymbol{\alpha}_1}$ is a semisimple Levi subgroup of P (cf. [Muk88, § 1], [HM02, § 1], and also [LM03, § 2.3]).

7.4. By [Muk88], Ω is a Fano-Mukai fivefold of Fano index 3 with $\mathrm{rk}\,\mathrm{Pic}(\Omega) = 1$. The linear system $\left|-\frac{1}{3}K_\Omega\right|$ defines an embedding $\Omega \hookrightarrow \mathbb{P}^{13}$ onto a variety of degree $\deg \Omega = \left(-\frac{1}{3}K_\Omega\right)^5 = 18$. By the adjunction formula and the Lefschetz hyperplane section theorem, any smooth hyperplane section $V = \Omega \cap \mathbb{P}^{12} \subset \mathbb{P}^{13}$ is a Fano-Mukai fourfold of genus 10. According to Theorem 7.1 any such variety occurs to be a hyperplane section of $\Omega \subset \mathbb{P}^{13}$.

By definition, the adjoint variety Ω is the orbit of the highest weight vector in the projectivized adjoint representation of G_2. This is the projectivized minimal (nonzero) nilpotent orbit of the $\mathrm{Ad}(\mathrm{G}_2)$-action on $\mathfrak{g}_2$ ([CMG93, Thm. 4.3.3]). The adjoint variety Ω contains the points $\mathbb{P}(\mathfrak{g}_{\boldsymbol{\alpha}})$, where $\boldsymbol{\alpha}$ runs over the set $\Delta_\ell \subset \Delta$ of long roots ([Tev05, § 8.5]). The dual variety $D_\ell := \Omega^* \subset \mathbb{P}(\mathfrak{g}_2)^\vee$ of Ω, that is, the locus of hyperplanes in $\mathbb{P}(\mathfrak{g}_2)$ which define singular hyperplane sections of Ω, is $\mathrm{Ad}(\mathrm{G}_2)$-invariant. In fact, D_ℓ is an irreducible hypersurface in $\mathbb{P}(\mathfrak{g}_2^\vee)$ of degree 6. The latter follows, e.g., from the classification of defective varieties of small dimension (see [Tev05, § 7.3.3]), or, alternatively, by comparing [Tev05, Thm. 7.47] with Proposition 8.1(e) below.

We let δ_ℓ be a homogeneous polynomial of degree 6 defining D_ℓ. There exists a second $\mathrm{Ad}(\mathrm{G}_2)$-invariant sextic hypersurface D_{s} in $\mathbb{P}(\mathfrak{g}_2^\vee)$ given by a homogeneous polynomial δ_{s} of degree 6, where δ_ℓ and δ_{s} are elements of the graded subalgebra $\mathbb{C}[\mathfrak{g}_2]^{\mathrm{Ad}(\mathrm{G}_2)}$ of $\mathrm{Ad}(\mathrm{G}_2)$-invariant functions on $\mathfrak{g}_2$, see [Tev05, § 8.5]. Indeed, let $\mathbb{C}[\mathfrak{h}]^{\mathbb{W}}$ be the graded subalgebra of $\mathbb{W}$-invariant functions on $\mathfrak{h}$, where $\mathbb{W}$ is the Weyl group of G_2. It is known (see, e.g., [Bou02, Ch. VI, § 12, Table IX, Summary 32-33]) that $\mathbb{C}[\mathfrak{h}]^{\mathbb{W}} = \mathbb{C}[\mathrm{f}_2, \mathrm{f}_6]$, where $\deg f_2 = 2$ and $\deg f_6 = 6$. In particular, the graded piece $(\mathbb{C}[\mathfrak{h}]^{\mathbb{W}})_6$ of dimension 2 is spanned by f_2^3 and f_6. By the Chevalley restriction theorem ([CG97, Thm. 3.1.38]) the restriction map gives an isomorphism of graded algebras

$$\mathbb{C}[\mathfrak{g}_2]^{\mathrm{Ad}(\mathrm{G}_2)} \cong \mathbb{C}[\mathfrak{h}]^{\mathbb{W}}. \tag{7.4.1}$$

Hence $(\mathbb{C}[\mathfrak{g}_2]^{\mathrm{Ad}(\mathrm{G}_2)})_6 \cong \langle f_2^3, f_6 \rangle$ is spanned by a pair $(\delta_\ell,\ \delta_{\mathrm{s}})$ of irreducible invariants chosen so that ([Tev05, Thm. 8.25])

$$\delta_\ell|_{\mathfrak{h}} = \prod_{\boldsymbol{\alpha} \in \Delta_\ell} \boldsymbol{\alpha} \quad \text{and} \quad \delta_{\mathrm{s}}|_{\mathfrak{h}} = \prod_{\boldsymbol{\beta} \in \Delta_{\mathrm{s}}} \boldsymbol{\beta}, \tag{7.4.2}$$

where Δ_{s} stands for the set of short roots.

7.5**.** Let G be a simple algebraic group of rank r with Lie algebra $\mathrm{Lie}(G)$. For an element $g \in \mathrm{Lie}(G)$ we let $\mathrm{Stab}_G(g)$ stand for the stabilizer of g in G under the adjoint representation. The Lie algebra of $\mathrm{Stab}_G(g)$ is the centralizer of g in $\mathrm{Lie}(G)$. The element g is called *regular* if $\dim \mathrm{Stab}_G(g) = r$ and *singular* otherwise. We have the following facts.

7.5.1. **Proposition.** *For an element $g \in \mathrm{Lie}(G)$ the following hold.*

(a) ([SS70, 1.3]) *The stabilizer $\mathrm{Stab}_G(g)$ contains an abelian subgroup of dimension $r = \mathrm{rk}\, G$.*
(b) ([Kur83, Thms. A and α]; *cf.* [SS70, 1.4, 1.16-1.18]) *g is regular if and only if the stabilizer $\mathrm{Stab}_G(g)$ is abelian.*
(c) ([Ste65, Cor. 3.4]) *If g is singular, then $\dim \mathrm{Stab}_G(g) \geq r + 2$.*
(d) ([SS70, 1.7, 3.9]) *If g is semisimple, then $\mathrm{Stab}_G(g)$ is a reductive group. A semisimple g is regular if and only if $\mathrm{Stab}_G(g)^0$ is a maximal torus of G.*

7.5.2**.** An element $g \in \mathrm{Lie}(G)$ with $\dim \mathrm{Stab}_G(g) = r + 2$ is called *subregular*. The orbit of g in $\mathrm{Lie}(G)$ is also called *subregular*. It should be noted that an orbit $\mathcal{O}$ of the adjoint G-action on $\mathrm{Lie}(G)$ is conic if and only if $\mathcal{O}$ is a nilpotent orbit ([CMG93, Lem. 4.3.1]). The image of a nilpotent orbit $\mathcal{O}$ in $\mathbb{P}(\mathrm{Lie}(G))$ is an orbit of the same codimension as the one of $\mathcal{O}$ in $\mathrm{Lie}(G)$. In contrast, for a non-nilpotent orbit $\mathcal{O} \subset \mathrm{Lie}(G)$, its image in $\mathbb{P}(\mathrm{Lie}(G))$ is an orbit of the same dimension as the one of $\mathcal{O}$.

7.6. **Notation.** We identify $\mathfrak{g}_2$ and $\mathfrak{g}_2^\vee$ via the Killing form. Under this identification, the adjoint representation $\mathrm{Ad}(\mathrm{G}_2)$ and its dual have the same orbits.

For a nonzero element $g \in \mathfrak{g}_2$ we let $[g]$ be the image of g in $\mathbb{P}(\mathfrak{g}_2)$. Let $V_{18}^g = \Omega \cap \mathbb{P}(g_2^\perp)$ be the section of Ω by the projectivized hyperplane $g^\perp$ orthogonal to g with respect to the Killing form.

Using Proposition 7.5.1 and Lemma 1 in [KR13] we deduce the following results.

7.7. **Proposition** ([KR13])**.** *Consider the pencil $\mathcal{D}$ of $\mathrm{Ad}(\mathrm{G}_2)$-invariant sextic hypersurfaces in $\mathbb{P}(\mathfrak{g}_2^\vee) = \mathbb{P}(\mathfrak{g}_2)$ generated by D_ℓ and D_{s}. Then the following hold.*

(a) *For any $D \in \mathcal{D}$ different from $D_\ell = \delta_\ell^*(0)$ and $D_{\mathrm{s}} = \delta_{\mathrm{s}}^*(0)$ the complement $D \setminus D_\ell$ is the $\mathrm{Ad}(\mathrm{G}_2)$-orbit of $[g]$, where $g \in \mathfrak{g}_2$ is regular semisimple with an abelian group $\mathrm{Stab}_{\mathrm{G}_2}(g)$ and with $\mathrm{Stab}_{\mathrm{G}_2}(g)^0 \cong (\mathbb{G}_{\mathrm{m}})^2$.*
(b) *The complement $D_{\mathrm{s}} \setminus D_\ell$ is a union of exactly two $\mathrm{Ad}(\mathrm{G}_2)$-orbits, the orbit of $[g_{\mathrm{s}}]$ and the orbit of $[g_{\mathrm{s}} + g_{\mathrm{n}}]$, where $g_{\mathrm{s}}, g_{\mathrm{n}} \in \mathfrak{g}_2$ are commuting elements such that*
    - *g_{s} is subregular semisimple, $g_{\mathrm{n}} \neq 0$ is nilpotent, and $g_{\mathrm{s}} + g_{\mathrm{n}}$ is regular;*
    - *the orbit $\mathrm{Ad}(\mathrm{G}_2).[g_{\mathrm{s}} + g_{\mathrm{n}}]$ is an open, dense subvariety of $D_{\mathrm{s}} \setminus D_\ell$;*
    - *the orbit $\mathrm{Ad}(\mathrm{G}_2).[g_{\mathrm{s}}]$ is a closed subvariety of $D_{\mathrm{s}} \setminus D_\ell$ of codimension 2;*
    - *the stabilizer $\mathrm{Stab}_{\mathrm{G}_2}(g_{\mathrm{s}} + g_{\mathrm{n}})$ is abelian, and $\mathrm{Stab}_{\mathrm{G}_2}(g_{\mathrm{s}} + g_{\mathrm{n}})^0 \cong \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$;*
    - *the stabilizer $\mathrm{Stab}_{\mathrm{G}_2}(g_{\mathrm{s}})$ is a non-abelian reductive group of dimension 4 and of rank 2.*

Proof. (a) By [KR13, Lem. 1(2)], for $D \neq D_{\mathrm{s}}, D_\ell$, the complement $D \setminus D_\ell$ coincides with the orbit of some $[g] \in \mathbb{P}(\mathfrak{g}_2)$. On the other hand ([Tev05, § 8.5]), $g \in \mathfrak{g}_2$ is regular semisimple if and only if $[g] \notin D_\ell \cup D_{\mathrm{s}}$. Now (a) follows by Proposition 7.5.1(b) and (d).

(b) By [KR13, Lem. 1(1)] and its proof, $D \setminus D_\ell$ is a union of a semisimple subregular orbit $O_1 = \mathrm{Ad}(\mathrm{G}_2).[g_{\mathrm{s}}]$ and a mixed regular orbit $O_2 = \mathrm{Ad}(\mathrm{G}_2).[g_{\mathrm{s}} + g_{\mathrm{n}}]$, where $g_{\mathrm{n}} \neq 0$ is nilpotent and commutes with g_{s}. Thus, $\mathrm{codim}_{\mathbb{P}(\mathfrak{g}_2)} O_1 = 3$ and $\mathrm{codim}_{\mathbb{P}(\mathfrak{g}_2)} O_2 = 1$, see 7.5.2. In particular, O_2 is open and dense in $D_{\mathrm{s}} \setminus D_\ell$, and its complement O_1 in $D_{\mathrm{s}} \setminus D_\ell$

is closed of codimension 2, see [KR13, Rem. 1]. Since g_s is subregular, $\mathrm{Stab}_{\mathrm{G}_2}(g_s)$ is a non-abelian reductive group of dimension 4 and of rank 2 with semisimple part $\mathrm{SL}_2(\mathbb{C})$, see ([CMG93]) and Proposition 7.5.1 (d).

Since $g_s + g_n$ is regular, $\mathrm{Stab}_{\mathrm{G}_2}(g_s + g_n)$ is an abelian group of dimension 2, see Proposition 7.5.1 (b). The unipotent radical of $\mathrm{Stab}_{\mathrm{G}_2}(g_s + g_n)^0$ contains the one-parameter unipotent subgroup $\exp(tg_n)$, $t \in \mathbb{C}$. On the other hand, the semisimple element g_s centralizes $g_s + g_n$, hence $\mathrm{Stab}_{\mathrm{G}_2}(g_s + g_n)^0$ is not unipotent. It follows that $\mathrm{Stab}_{\mathrm{G}_2}(g_s + g_n)^0 \cong \mathbb{G}_a \times \mathbb{G}_m$. □

7.8. **Lemma.** *The stabilizer* $\mathrm{Stab}_{\mathrm{G}_2}(g)$ *acts effectively on the hyperplane section* $V = V_{18}^g$.

The proof starts with the following claim.

7.8.1. **Claim.** *The kernel of the natural homomorphism* $\mathrm{Stab}_{\mathrm{G}_2}(g) \to \mathrm{Aut}(V)$ *is contained in the unipotent radical of* $\mathrm{Stab}_{\mathrm{G}_2}(g)$.

Proof. The simple group G_2 acts effectively on Ω. If $h \in \mathrm{Stab}_{\mathrm{G}_2}(g)$ acts trivially on V, then it acts trivially on the hyperplane $\langle V \rangle \subset \mathbb{P}(\mathfrak{g}_2)$. Hence the adjoint action of h on $\mathfrak{g}_2$ has at most two distinct eigenvalues λ_1 and λ_2 of multiplicity 13 and 1, respectively. Since $h(g) = g$, one has $\lambda_2 = 1$. The Lie algebra $\mathfrak{g}_2$ being semisimple, one has $\det h = 1 = \lambda_1^{13}$. Up to equivalence, the adjoint representation is a unique irreducible representation of G_2 of dimension 14. Hence it is equivalent to the coadjoint representation. It follows that h and h^{-1} are conjugated, and so, $\mathrm{tr}(h) = \mathrm{tr}(h^{-1})$. Thus, $13\lambda_1 + 1 = 13\lambda_1^{-1} + 1$, which implies $\lambda_1 = 1$. Then the semisimple part of h equals 1, and h is unipotent. Finally, the kernel of the representation $\mathrm{Stab}_{\mathrm{G}_2}(g) \to \mathrm{Aut}(V)$ is a normal closed subgroup of $\mathrm{Stab}_{\mathrm{G}_2}(g)$ consisting of unipotent elements. □

The next claim ends the proof of Lemma 7.8.

7.8.2. **Claim.** *The unipotent radical of* $\mathrm{Stab}_{\mathrm{G}_2}(g)$ *acts effectively on* V.

Proof. It suffices to show that any one-parameter unipotent subgroup $U = \{\exp(tN)\}_{t\in\mathbb{C}} \subset \mathrm{Stab}_{\mathrm{G}_2}(g)$, where $N \in \mathfrak{g}_2$ is nilpotent, acts effectively on V. Suppose this is not the case. Then N vanishes on the affine hyperplane $g^\perp \subset \mathfrak{g}_2$. Since it also vanishes on the line $\mathbb{C}g$, one has $N = 0$. □

7.8.3. **Corollary.** *Let* $V = V_{18}^g$, *where* $g \in \mathfrak{g}_2 \setminus \{0\}$ *and* $[g] \notin D_\ell$. *Assume that* V *contains an* $\mathrm{Aut}^0(V)$*-invariant cubic cone* S. *Then the following hold.*

(i) *If* g *is singular semisimple, then* $\mathrm{Aut}^0(V) \cong \mathrm{GL}_2(\mathbb{C})$;
(ii) *if* g *is regular non-semisimple, then* $\mathrm{Aut}^0(V) \supset \mathbb{G}_a \times \mathbb{G}_m$;
(iii) *if* g *is regular semisimple, then* $\mathrm{Aut}^0(V) \supset (\mathbb{G}_m)^2$.

In particular, if $\mathrm{rk}(\mathrm{Aut}^0(V)) = 1$ *then we are in case* (ii). *Any two Fano-Mukai fourfolds satisfying* (i) (*resp.,* (ii)) *are isomorphic via an automorphisms of* Ω *provided by the* $\mathrm{Ad}(\mathrm{G}_2)$*-action on* Ω.

Proof. Statements (ii) and (iii) are straightforward from Propositions 7.5.1(a) -(b) and 7.7 and Lemma 7.8. By Propositions 7.5.1(b) -(c) and 7.7(b) and Lemma 7.8, in case (i) $\mathrm{Aut}^0(V)$ contains a non-abelian reductive subgroup of dimension 4 and of rank 2. Hence V admits a nontrivial $\mathrm{SL}_2(\mathbb{C})$-action. Let (W, F) be the pair linked to (V, S) via an $\mathrm{Aut}^0(V)$-equivariant Sarkisov link (3.1.1). Due to Lemma 6.4 one has $\mathrm{Aut}^0(V) \cong \mathrm{Aut}^0(W, F) \cong \mathrm{GL}_2(\mathbb{C})$.

By (i)–(iii) the equality $\mathrm{rk}(\mathrm{Aut}^0(V)) = 1$ implies that g is regular non-semisimple. According to Proposition 7.7(b) any two such elements $g_1, g_2 \in \mathfrak{g}_2$ belong to the same $\mathrm{Ad}(\mathrm{G}_2)$-orbit. Hence the corresponding smooth hyperplane sections $V_{18}^{g_1}$ and $V_{18}^{g_2}$ of Ω are isomorphic under the $\mathrm{Ad}(\mathrm{G}_2)$-action on Ω. The same argument applies in case (i) (alternatively, see Lemma 6.4.1). □

7.8.4. **Notation.** We let V_{18}^{s} and V_{18}^{a}, respectively denote a unique, up to isomorphism, Fano-Mukai fourfold V_{18}^{g} as in Corollary 7.8.3(i) and (ii), respectively. The existence of these fourfolds is guaranteed by Proposition 7.7(b). Thus, $\mathrm{Aut}^0(V_{18}^{\mathrm{s}}) \cong \mathrm{GL}_2(\mathbb{C})$ and $\mathrm{Aut}^0(V_{18}^{\mathrm{a}}) \supset \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$.

8. Lines in V_{18}

To study the lines in a Fano-Mukai fourfold V_{18}, let us first investigate the Fano variety of lines in the homogeneous space $\Omega = \mathrm{G}_2/P$.

8.1. **Proposition.** *Let $\Omega = \mathrm{G}_2/P$ be as above. Then the following hold.*

(a) *Ω contains a line.*

(b) *For any line $l \subset \Omega$ one has*

$$\mathscr{N}_{l/\Omega} \cong \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(1). \tag{8.1.1}$$

(c) *The Hilbert scheme $\Sigma(\Omega)$ of lines on Ω is a non-singular variety of dimension 5.*

(d) *Consider the universal family of lines on Ω*

$$\begin{array}{ccc} & \mathscr{L}(\Omega) & \\ \swarrow & & \searrow \\ \Sigma(\Omega) & & \Omega \end{array} \tag{8.1.2}$$

where $\mathscr{L}(\Omega) \to \Sigma(\Omega)$ is a $\mathbb{P}^1$-bundle. Then any fiber of $\mathscr{L}(\Omega) \to \Omega$ is one-dimensional.

(e) *For any point $p \in \Omega$ the union $L\Omega_p$ of the lines in Ω passing through p is a cone over a rational twisted cubic curve.*

Proof. (a) It suffices to show that a general section $\Omega \cap \mathbb{P}^{11}$ contains a line. However, the latter follows from Shokurov's theorem [Sho80].

(b) Let l be a line on $\Omega \subset \mathbb{P}^{13}$. Since Ω is a homogeneous space, the tangent bundle T_Ω is generated by global sections (that is, the vector fields from the corresponding Lie algebra). Hence the vector bundles $T_\Omega|_l$ and $\mathscr{N}_{l/\Omega} = T_\Omega|_l/T_l$ are also generated by global sections. This means that the integers a, b, c, d in the decomposition

$$\mathscr{N}_{l/\Omega} = \mathscr{O}_{\mathbb{P}^1}(a) \oplus \mathscr{O}_{\mathbb{P}^1}(b) \oplus \mathscr{O}_{\mathbb{P}^1}(c) \oplus \mathscr{O}_{\mathbb{P}^1}(d)$$

are all non-negative. On the other hand,

$$\deg \mathscr{N}_{l/\Omega} = -K_\Omega \cdot l - 2 + 2\,\mathrm{g}(l) = 1.$$

This implies (8.1.1).

(c) follows from (8.1.1) and the standard facts of the deformation theory. Indeed, we have $H^1(V, \mathscr{N}_{l/\Omega}) = 0$ and $\dim H^0(V, \mathscr{N}_{l/\Omega}) = 5$. Statement (d) follows from the facts that Ω is homogeneous and the diagram (8.1.2) is equivariant. To show (e) we note that $L\Omega_p = T_p\Omega \cap \Omega$ is the tangent cone and $T_p\Omega \cap \Omega$ is a cone over a rational twisted cubic curve, see [HM02, § 1, Prop. 1] or [KR13, Proof of Lemma 3]. □

Next we study the lines in a Fano-Mukai fourfold V_{18}.

8.2. **Proposition.** *Let $V = V_{18} \subset \mathbb{P}^{12}$ be a smooth Fano-Mukai fourfold of genus* 10. *Then the following hold.*

(a) *V contains a line.*

(b) *For any line $l \subset V$ one has*

$$\mathscr{N}_{l/V} \cong \begin{cases} \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} & (*) \quad \text{or} \\ \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-1) \oplus \mathscr{O}_{\mathbb{P}^1}(1) & (**) \end{cases} \tag{8.2.1}$$

(c) *The Hilbert scheme $\Sigma(V)$ of lines on V is a non-singular variety of dimension* 3.

(d) *Consider the universal family of lines on V*

$$\begin{array}{ccc} & \mathscr{L}(V) & \\ {}^{r}\swarrow & & \searrow^{s} \\ \Sigma(V) & & V \end{array} \tag{8.2.2}$$

where $r : \mathscr{L}(V) \to \Sigma(V)$ is a $\mathbb{P}^1$-bundle. Then s is a generically finite morphism of degree 3*. Consequently, V is covered by lines.*

(e) *For $p \in V$, let LV_p be the union of all lines in V passing through p. If $\dim LV_p = 2$, then LV_p is a cubic cone. Otherwise, LV_p is nonempty and consists of at most three lines.*

(f) *A general line l on V has normal bundle of type $(*)$ in* (8.2.1)*. Any line l on V with normal bundle of type $(**)$ is contained in the branching divisor $\mathcal{B} \subset V$ of s.*

(g) *Any line l on V contained in a cubic cone has normal bundle of type $(**)$.*

Proof. (a) The proof is similar to the one of 8.1 (a).

(b) From the standard exact sequence

$$0 \longrightarrow \mathscr{N}_{l/V} \longrightarrow \mathscr{N}_{l/\Omega} \longrightarrow \mathscr{N}_{V/\Omega}|_l \longrightarrow 0, \qquad \mathscr{N}_{V/\Omega}|_l = \mathscr{O}_{\mathbb{P}^1}(1)$$

one deduces that all the summands in a decomposition of $\mathscr{N}_{l/V}$ into a sum of line bundles are of degree ≤ 1, and there is at most one summand of degree 1. Thus (8.1.1) implies (8.2.1).

(c) follows from (8.2.1).

(d)-(e) Recall that V is a section of $\Omega \subset \mathbb{P}^{13}$ by a hyperplane H. For $p \in V$, the subvariety LV_p is a hyperplane section of $L\Omega_p$, that is,

$$LV_p = L\Omega_p \cap V = L\Omega_p \cap H.$$

By Proposition 8.1(e), $L\Omega_p$ is a cubic cone, that is, a cone over a rational twisted cubic curve. Hence, LV_p is non-empty, $\dim LV_p \le 2$, and $\dim LV_p = 2$ if and only if $LV_p = L\Omega_p$. Thus, s is dominant. So, this is a generically finite morphism of degree 3. Indeed, for a general point $p \in V$, H cuts the cubic cone $L\Omega_p$ on V with vertex p along three distinct lines. The last assertion in (e) is now immediate.

(f) To any line $l \in \Sigma(V)$ there corresponds a smooth rational curve $l' \subset \mathscr{L}(V)$. Clearly, the restriction $s|_{l'} : l' \to l \subset V$ is an isomorphism. Consider the differential

$$ds : \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} = \mathscr{N}_{l'/\mathscr{L}(V)} \longrightarrow \mathscr{N}_{l/V}. \tag{8.2.3}$$

In case $(**)$ the map ds has cokernel of rank 1 along l'. Hence s is ramified along l'.

(g) Consider a line $l \subset S$, where $S \subset V$ is a cubic cone. Then l lifts to a ruling l' of a scroll $S' \cong \mathbb{F}_3$ in $\mathscr{L}(V)$. The map $s|_{S'} : S' \to S$ contracts the exceptional section of S' to the vertex p of S. Therefore, the cokernel of ds in (8.2.3) in nontrivial at p, and so, $\mathscr{N}_{l/V} \not\cong \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}$. □

9. Cubic scrolls in V_{18}

In this section we study the cubic scrolls in a Fano-Mukai fourfold $V = V_{18}$ of genus 10. The following facts are proven in [KR13, Prop. 1 and 2 and the proof of Prop. 4].

9.1. **Theorem.** *For any Fano-Mukai fourfold $V = V_{18}$ of genus 10 the following hold.*

(a) *Let $\Sigma(V)$ be the Hilbert scheme of lines in V. Then $\Sigma(V)$ is isomorphic to a smooth divisor of bidegree $(1,1)$ on $\mathbb{P}^2 \times \mathbb{P}^2$.*

(b) *Let $\mathscr{S}(V)$ be the Hilbert scheme of cubic scrolls in V. Then $\mathscr{S}(V)$ is isomorphic to a disjoint union of two projective planes.*

9.1.1. **Remarks.** (i) By the Lefschetz hyperplane section theorem we have $\mathrm{Pic}(\Sigma(V)) = \mathbb{Z} \cdot [\mathcal{F}_1] \oplus \mathbb{Z} \cdot [\mathcal{F}_2]$, where $\mathcal{F}_i$, $i = 1, 2$, are the pull-backs of lines on the corresponding factors of $\mathbb{P}^2 \times \mathbb{P}^2$.

(ii) Note that the embedding $\Sigma(V) \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ onto a smooth divisor D of bidegree $(1,1)$ guaranteed by Theorem 9.1(a) is defined canonically. Indeed, the projections $\mathrm{pr}_i : \Sigma(V) \to \mathbb{P}^2$, $i = 1, 2$, to the factors of $\mathbb{P}^2 \times \mathbb{P}^2$ are extremal contractions. By (i), the projections pr_1 and pr_2 are the only extremal contractions of the threefold $\Sigma(V)$. It follows that the $\mathrm{Aut}^0(\Sigma(V))$-action on $\Sigma(V)$ induces an action of $\mathrm{Aut}^0(\Sigma(V))$ on the factors of $\mathbb{P}^2 \times \mathbb{P}^2$ making the morphisms pr_i, $i = 1, 2$, equivariant.

(iii) One can treat $\Sigma(V) \subset (\mathbb{P}^2)^\vee \times \mathbb{P}^2$ as the variety of full flags in $\mathbb{P}^2$. Using this representation, it can be easily seen that the maps $\mathrm{Aut}^0(\Sigma(V)) \to \mathrm{Aut}((\mathbb{P}^2)^\vee))$ and $\mathrm{Aut}^0(\Sigma(V)) \to \mathrm{Aut}(\mathbb{P}^2)$ are isomorphisms.

We use below the following facts.

9.2. **Lemma.** *Given a cubic scroll $S \in \mathscr{S}(V)$, there is a unique hyperplane section A_S of V with $\mathrm{Sing}(A_S) = S$. This hyperplane section A_S coincides with the union of lines in V which meet S. Given a point $P \in A_S \setminus S$, there is a unique line in A_S through P which meets S.*

Proof. By Proposition 3.1 there is a hyperplane section $A = A_S$ of V with $\mathrm{Sing}(A_S) = S$. The linear projection $\theta : V \dashrightarrow W$ in diagram (3.1.1) sends A to a smooth quintic scroll $F \subset W$ contracting the lines meeting S and not contained in S. These lines correspond to the rulings of $\eta|_{\widetilde{A}} : \widetilde{A} \to F$ in (3.1.1). It follows that $A = \xi(\widetilde{A})$ is covered by such lines. Since through any point in $\widetilde{A} \setminus \widetilde{B}$ passes a unique ruling of $\eta|_{\widetilde{A}} : \widetilde{A} \to F$, the last statement follows. □

9.3. **Lemma.** *A line l in V can be a common exceptional section for at most two smooth cubic scrolls in V, and can be contained in at most finite number of cubic cones. Consequently, the morphism $s^{-1}(l) \to l$ (see diagram (8.2.2)) is generically finite.*

Proof. We claim that l cannot be a common exceptional section of three or more smooth cubic scrolls. Indeed, assuming the contrary, through a general point p of l pass at least 4 distinct lines in V. By Proposition 8.2(e) this implies that p is a vertex of a cubic cone in V. So, the cubic cones with vertices on l form a one-parameter family, say, $\mathcal{C}(l)$. By

Lemma 9.2 for any pair of cubic cones S', S'' through l one has $S'' \subset A_{S'}$ and $S' \subset A_{S''}$. Hence for any $S' \in \mathcal{C}(l)$ there is a Zariski dense open subset of $A_{S'}$ swept up by the cubic cones in $\mathcal{C}(l)$. Therefore, $A_{S'}$ and $S' = \mathrm{Sing}(A_{S'})$ do not depend on the choice of S', a contradiction. □

By virtue of Theorem 9.1 and the next lemma, a general cubic scroll in V is smooth.[3]

9.4. **Lemma.** *Any cubic cone in V is contained in the branching divisor $\mathcal{B}$ of $s : \mathscr{L}(V) \to V$. The family of cubic cones in V is at most one-dimensional.*

Proof. The first statement is immediate from Proposition 8.2(f)–(g).

To show the second, suppose the contrary. Then an entire component of $\mathscr{S}(V)$, say, $\mathscr{S}_i(V)$, $i \in \{1, 2\}$, consists of cubic cones. By Proposition 8.2(e) the vertices of these cones are all distinct and cover a surface, say, $\mathcal{T}_i \subset \mathcal{B}$.

The pull-back $\widetilde{\mathcal{T}}_i = s^{-1}(\mathcal{T}_i) \subset \mathscr{L}(V)$ is $\mathbb{P}^1$-fibered over $\mathcal{T}_i$. Indeed, let $S_t \subset \mathcal{B}$ be the cubic cone with vertex $t \in \mathcal{T}_i$. Then the fiber $s_t = s^{-1}(t) \cong \mathbb{P}^1$ parameterizes the family of lines in V through t, that is, the family of rulings of S_t.[4] Thus, $\widetilde{\mathcal{T}}_i$ is a component of the ramification divisor $\mathcal{R}$ of s.

By Proposition 8.2(e), a general line in V is not contained in $\mathcal{B}$. Hence the image $r(\widetilde{\mathcal{T}}_i)$ is a proper subvariety in $\Sigma(V)$, see diagram (8.2.2). So, $r|_{\widetilde{\mathcal{T}}_i} : \widetilde{\mathcal{T}}_i \to r(\widetilde{\mathcal{T}}_i)$ is a $\mathbb{P}^1$-fibration over a surface $r(\widetilde{\mathcal{T}}_i) \subset \Sigma(V)$. The points of this surface parametrize a family of lines in V contained in $\mathcal{T}_i$. So, there is a one-parameter family of lines passing through a general point $t \in \mathcal{T}_i$. These lines sweep up the cubic cone S_t with vertex t, see again Proposition 8.2(e). Thus, $S_t = \mathcal{T}_i$ for a general $t \in \mathcal{T}_i$. Hence S_t does not depend on t, a contradiction. □

9.5. **Proposition.** *Each component $\mathscr{S}_i(V)$, $i = 1, 2$, of the Hilbert scheme $\mathscr{S}(V)$ of cubic scrolls in V (see Theorem 9.1(b)) contains at least one $\mathrm{Aut}^0(V)$-invariant cubic cone.*

In the proof we use the following auxiliary results.

9.5.1. **Lemma.** *Any morphism $f : \mathbb{P}^2 \to \Sigma(V)$ is constant.*

Proof. We regard $\Sigma(V)$ as a smooth divisor in $\mathbb{P}^2 \times \mathbb{P}^2$ of bidegree $(1, 1)$. Suppose to the contrary that $\mathcal{T} := f(\mathbb{P}^2) \subset \Sigma(V)$ is not a point. Then $\dim \mathcal{T} = 2$ and $f : \mathbb{P}^2 \to \mathcal{T}$ is a finite morphism of degree, say, d. By Remark 9.1.1(i) one has $\mathcal{T} \sim a_1\mathcal{F}_1 + a_2\mathcal{F}_2$ for some integers $a_1, a_2 \geq 0$. There are relations

$$\mathcal{F}_1^3 = \mathcal{F}_2^3 = 0, \quad \mathcal{F}_1^2 \cdot \mathcal{F}_2 = \mathcal{F}_1 \cdot \mathcal{F}_2^2 = 1. \tag{9.5.2}$$

Since $\Sigma(V)$ is smooth, in suitable bihomogeneous coordinates $(x_0 : x_1 : x_2 \,;\, y_0 : y_1 : y_2)$ in $\mathbb{P}^2 \times \mathbb{P}^2$ the equation of $\Sigma(V)$ can be written as

$$x_0y_0 + x_1y_1 + x_2y_2 = 0. \tag{9.5.3}$$

Hence any fiber of the projections to the factors $\mathrm{pr}_i|_{\Sigma(V)} : \Sigma(V) \to \mathbb{P}^2$, $i = 1, 2$, is isomorphic to $\mathbb{P}^1$. It follows that the restrictions $\pi_i = \mathrm{pr}_i|_{\mathcal{T}} : \mathcal{T} \to \mathbb{P}^2$, $i = 1, 2$, are finite morphisms. Indeed, $\pi_i(\mathcal{T})$ is neither a point, nor a curve, since $\mathbb{P}^2$ does not admit any dominant morphism to a curve. In particular, $\mathcal{T} \cdot \mathcal{F}_i^2 > 0$, $i = 1, 2$, hence $a_1, a_2 > 0$.

[3]In [PZ15, Cor. 5.13] we constructed a pair (V, S) such that $S \in \mathscr{S}(V)$ is a smooth cubic scroll.

[4]In fact, s_t is the exceptional section of a surface $\widetilde{S}_t \cong \mathbb{F}_3$ in $\mathscr{L}(V)$. The restriction $s|_{\widetilde{S}_t} : \widetilde{S}_t \to S_t$ is the minimal resolution of singularity of the cubic cone S_t.

The degree of $\pi_1 \circ f : \mathbb{P}^2 \to \mathbb{P}^2$ equals $(\deg f)(\deg \pi_1) = d(\mathcal{T} \cdot \mathcal{F}_1^2) = da_2$, while $\deg(\pi_2 \circ f) = da_1$. Identifying $\Sigma(V)$ with a smooth hyperplane section $D_6 \subset \mathbb{P}^7$ of the image of $\mathbb{P}^2 \times \mathbb{P}^2$ under the Segre embedding, we obtain $\mathscr{O}_{\mathcal{T}}(1) = \mathscr{O}_{\mathcal{T}}(\mathcal{F}_1 + \mathcal{F}_2)$. It follows that

$$\deg \mathcal{T} = (\mathcal{F}_1 + \mathcal{F}_2)^2 \cdot \mathcal{T} = 3(a_1 + a_2).$$

On the other hand, $f^*\mathscr{O}_{\mathcal{T}}(\mathcal{F}_1) = \mathscr{O}_{\mathbb{P}^2}(da_2)$ and $f^*\mathscr{O}_{\mathcal{T}}(\mathcal{F}_2) = \mathscr{O}_{\mathbb{P}^2}(da_1)$. So, $f^*\mathscr{O}_{\mathcal{T}}(1) = \mathscr{O}_{\mathbb{P}^2}(da_1 + da_2)$, and

$$\deg \mathcal{T} = \frac{1}{d}(da_1 + da_2)^2 = d(a_1 + a_2)^2.$$

Thus, $d(a_1 + a_2) = 3$. Up to a transposition in indices, the only possibility is $d = 1$, $a_1 = 2$, $a_2 = 1$. In particular, f and π_1 are birational, hence $\mathcal{T} \cong \mathbb{P}^2$. Since $\mathcal{T}$ is an ample divisor on $\Sigma(V)$, by the Lefschetz theorem, the restriction map $\operatorname{Pic}(\Sigma(V)) \to \operatorname{Pic}(\mathcal{T})$ is injective, a contradiction. □

Proof of Proposition 9.5. By Lemma 9.4, for $i = 1, 2$ there is an $\operatorname{Aut}^0(V)$-invariant Zariski dense, open subset $\mathcal{U}_i$ of $\mathscr{S}_i(V)$, whose points correspond to smooth cubic scrolls in V. So, each point in $\mathscr{S}_i(V) \setminus \mathcal{U}_i$ corresponds to a cubic cone. Any smooth cubic scroll contains a unique distinguished line, which is its exceptional section. This provides an $\operatorname{Aut}^0(V)$-equivariant rational map

$$\varsigma : \mathscr{S}_i(V) \cong \mathbb{P}^2 \dashrightarrow \Sigma(V) \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2 \tag{9.5.4}$$

sending a smooth cubic scroll to its exceptional section. This map is regular on $\mathcal{U}_i$. By Lemma 9.3 the restriction $\varsigma|_{\mathcal{U}_i} : \mathcal{U}_i \to \varsigma(\mathcal{U}_i)$ has finite fibers. Hence $\dim \varsigma(\mathcal{U}_i) = 2$, $i = 1, 2$.

In particular, ς is non-constant. By Lemma 9.5.1, ς cannot be a morphism. So, its indeterminacy set is a nonempty subset of $\mathscr{S}_i(V) \setminus \mathcal{U}_i$ of dimension zero. It consists of isolated fixed points of $\operatorname{Aut}^0(V)$ that correspond to $\operatorname{Aut}^0(V)$-invariant cubic cones in V. □

9.5.5. **Lemma.** *Two general cubic scrolls in V either are disjoint, or meet transversely in a single point.*

Proof. By Lemma 9.4 it is enough to deal with a pair of smooth cubic scrolls in V. Given a smooth cubic scroll S in V, consider the family $\mathcal{S}(S)$ of all $S' \in \mathscr{S}(V)$ such that $S' \subset A_S$. We claim that $\mathcal{S}(S)$ is at most one-dimensional. Indeed, starting with the pair (V, S) one can produce a diagram (3.1.1). If $\widetilde{S'} \subset \widetilde{A_S}$ is the proper transform of S' in $\widetilde{W}$, then $\eta(\widetilde{S'}) \subset \eta(\widetilde{A_S}) = F$. Suppose that $\widetilde{S'}$ dominates F under η. Then $\widetilde{S'}$ meets each ruling $f_p = \widetilde{\eta}^{-1}(p)$, $p \in F$, of $\eta : \widetilde{A_S} \to F$. Hence S' meets each line $\xi(f_p) \subset A_S$. Thus, $\xi(f_p) \subset A_{S'}$ for any $p \in F$. This family of lines covers A_S, and so, $A_S = A_{S'}$. Then also the singular loci S and S' of $A_S = A_{S'}$ coincide, a contradiction.

Thus, $\eta(\widetilde{S'}) = \theta(S')$ is an irreducible curve in F, which can be either a line, or a smooth conic. Anyway, $\widetilde{S'}$ belongs to a one-parameter family of ruled surfaces in $\widetilde{A_S}$. Hence $\dim \mathcal{S}(S) \le 1$.

It follows that $S' \not\subset A_S$ for a general pair of cubic scrolls (S, S') in V. Then $\theta|_{S'} : S' \to \theta(S') \subset W$ is birational. Hence $\dim(\langle S \rangle \cap \langle S' \rangle) \le 1$. If $\dim(\langle S \rangle \cap \langle S' \rangle) = 1$, then $\theta(\langle S' \rangle)$ is a plane in $\mathbb{P}^7$. Then $\theta(S') \subset W$ is as well a plane, which belongs to a one-parameter family of planes in W. Therefore, for a general $S' \in \mathscr{S}(V)$ one has $\dim(\langle S \rangle \cap \langle S' \rangle) \le 0$. So, either S and S' are disjoint, or they meet transversely in a single point. □

Next we describe the middle cohomology group of V.

9.6. **Proposition.** *The group $H^4(V,\mathbb{Z})$ is generated by the classes of two cubic scrolls S_1 and S_2 from different components of $\mathscr{S}(V)$. In the cohomology ring $H^*(V,\mathbb{Z})$ these classes satisfy the relations $[S_1]^2 = [S_2]^2 = 1$, $[S_1]\cdot[S_2] = 0$.*

The proof is preceded by the following lemma.

9.6.1. **Lemma.** *For any cubic scroll S in V one has $S^2 = 1$ in $H^*(V,\mathbb{Z})$.*

Proof. By Lemma 9.4 one can choose $S \in \mathscr{S}(V)$ to be smooth. Consider the corresponding diagram (3.1.1). For the exceptional divisor $\widetilde{B} \sim H^* - \widetilde{A}$ of $\xi : \widetilde{W} \to V$ one has ([PZ16, Lem. 2.3])

$$\widetilde{B}^4 = c_2(V)\cdot S + K_V|_S \cdot K_S - c_2(S) - K_V^2 \cdot S. \tag{9.6.2}$$

Letting s_0 and f be the exceptional section and a ruling of $S \cong \mathbb{F}_1 \to \mathbb{P}^1$, respectively, one computes

$$K_V^2\cdot S = (2H^2)\cdot S = 12, \quad c_2(S) = 4, \quad K_V|_S\cdot K_S = -2(s_0+2f)(-2s_0-3f) = 10. \tag{9.6.3}$$

Plugging (9.6.3) in (9.6.2) gives

$$B^4 = S^2 - 6. \tag{9.6.4}$$

From the exact sequence

$$0 \longrightarrow T_S \longrightarrow T_V|_S \longrightarrow \mathscr{N}_{V/S} \longrightarrow 0$$

one deduces a relation for the Chern classes

$$(1 - K_S t + c_2(S)t^2)(1 + c_1(\mathscr{N}_{V/S})t + c_2(\mathscr{N}_{V/S})t^2) = 1 - K_V|_S t + c_2(V)|_S t^2.$$

This yields a system

$$\begin{aligned} c_1(\mathscr{N}_{V/S}) &= K_S - K_V|_S, \\ c_2(V)\cdot S &= c_2(S) - K_S\cdot c_1(\mathscr{N}_{V/S}) + c_2(\mathscr{N}_{V/S}) \\ &= c_2(S) - K_S^2 + K_S\cdot K_V|_S + S^2. \end{aligned}$$

Using (9.6.3) one obtains the relation $c_2(V)\cdot S = S^2+6$. It follows by (9.6.4) that $\widetilde{B}^4 = S^2$. Finally, from [PZ15, Lem. 5.5] one gets

$$S^2 = \widetilde{B}^4 = (H^* - \widetilde{A})^4 = (H^*)^4 + 6(H^*)^2\cdot\widetilde{A}^2 - 4H^*\cdot\widetilde{A}^3 + \widetilde{A}^4 = 1.$$

□

9.6.5. **Lemma.** *For any two cubic scrolls $S_i \in \mathscr{S}_i(V)$, $i = 1, 2$, one has $S_1\cdot S_2 = 0$.*

Proof. By Lemma 9.6.1 any two disjoint cubic scrolls belong to different components of $\mathscr{S}(V)$. According to Proposition 6.3(b) and Lemma 6.6.1, there is a Fano-Mukai fourfold $V = V_{18}$ of genus 10, which contains two disjoint cubic cones. By Theorem 7.1, any Fano-Mukai fourfold of genus 10 appears as a smooth hyperplane section of the adjoint variety $\Omega \subset \mathbb{P}^{13}$. Since the family of such hyperplane sections of Ω is irreducible, the lemma follows. □

Proof of Proposition 9.6. Using diagram (3.1.1) one derives isomorphisms

$$H^4(V,\mathbb{Z}) \cong H^4(W,\mathbb{Z}) \cong \mathbb{Z}\oplus\mathbb{Z},$$

see [PZ15, Lem. 3.4]. By Lemmas 9.6.1 and 9.6.5 the classes $[S_1]$ and $[S_2]$ are independent and generate $H^4(V,\mathbb{Z})$. □

Let us introduce the following notation.

9.7. **Notation.** Given a cubic scroll $S \in \mathscr{S}(V)$, consider the following objects:

- the variety $\Lambda(S) \subset \Sigma(V)$ of rulings of S (clearly, $\Lambda(S) \cong \mathbb{P}^1$);
- the variety $\Sigma(S) \subset \Sigma(V)$ of lines in V which meet S.

9.7.1. **Lemma.** *$\Sigma(S)$ is a divisor on $\Sigma(V)$.*

Proof. The statement is immediate from Lemma 9.2. □

9.7.2. **Lemma.** *Consider the standard projections* $\mathrm{pr}_i : \Sigma(V) \to \mathbb{P}^2$, $i = 1, 2$ (*see Theorem* 9.1(a)). *Then for any $S \in \mathscr{S}(V)$ the variety $\Lambda(S) \subset \Sigma(V)$ is a fiber of one of these projections, while $\Sigma(S)$ is the pull-back of a line in $\mathbb{P}^2$ under the other one. In particular, $\Sigma(S) \cong \mathbb{F}_1$, and $\Lambda(S) \subset \Sigma(S)$ is the exceptional section.*

Proof. Let $S_1 \in \mathscr{S}_1(V)$. For a general cubic scroll $S_2 \in \mathscr{S}_2(V)$, by Lemma 9.5.5 and Proposition 9.6 one has $S_1 \cap S_2 = \varnothing$. Therefore, $\Sigma(S_1)\cdot\Lambda(S_2) = 0$. Similarly, for a general $S_1' \in \mathscr{S}_1(V)$ one has $\Sigma(S_1)\cdot\Lambda(S_1') = 1$. In the notation of Remark 9.1.1(i), modulo reindexing $\mathscr{S}_i(V)$ one obtains $\Sigma(S_1) \sim \mathcal{F}_1$. Thus, $\Sigma(S_1)$ is the pull-back of a line under a projection $\Sigma(V) \to \mathbb{P}^2$. This shows as well that $\Lambda(S_1)$ is a fiber of the other projection. □

The following corollary of (9.5.2) and 9.7.2 is immediate, cf. [KR13, proof of Prop. 2].

9.7.3. **Corollary.** (i) *Any line $l \in \Sigma(V)$ is a common ruling of exactly two cubic scrolls $S_1(l)$ and $S_2(l)$, where $S_i(l) \in \mathscr{S}_i(V)$, $i = 1, 2$. In particular, V is covered by the cubic scrolls contained in V.*

(ii) *If l is contained in a third cubic scroll $S \neq S_i(l)$, $i = 1, 2$, then S is smooth, and l is the exceptional line of S.*

(iii) *Two distinct cubic scrolls $S' \neq S$ in V can have at most one common ruling.*

The possible intersections of two cubic cones are as follows.

9.7.4. **Corollary.** *Two distinct cubic cones S, S' in V either are disjoint, or meet transversally in one point, or finally contain a common ruling which is the only one-dimensional component of their intersection.*

Proof. By Proposition 9.6, if the intersection $S \cap S'$ is finite, then either it is empty, or S and S' meet transversally at a single point, depending on whether S, S' belong to different components of $\mathscr{S}(V)$ or not.

Assume that $\dim(S \cap S') = 1$. Then any ruling of S' meets S and so it is contained in A_S. Therefore, $S' \subset A_S$. Since through any point of $A_S \setminus S$ passes exactly one line in A_S meeting S (see Lemma 9.2), the vertex $v(S')$ lies on S. By symmetry, $v(S)$ lies on S'. Thus, the line l joining $v(S)$ and $v(S')$ is a common ruling.

It follows that $v(S') \in \langle S\rangle$, and so, $\langle S\rangle \cap S'$ is a union of lines in V passing through $v(S')$. By symmetry, $\langle S'\rangle \cap S$ is a union of lines passing through $v(S)$. One concludes that

$$S \cap S' = (\langle S'\rangle \cap S) \cap (\langle S\rangle \cap S')$$

is a union of lines in V passing through both $v(S)$ and $v(S')$. Since $v(S) \neq v(S')$, see Proposition 8.2(e), l is the only one-dimensional component of $S \cap S'$. $\square$

We can now reinterpret the embedding $\Sigma(V) \hookrightarrow \mathbb{P}^2 \times \mathbb{P}^2$ as follows.

9.7.5. **Corollary.** *In the notation of* 9.7.3(i)*, the morphism*

$$\varrho : \Sigma(V) \longrightarrow \mathscr{S}_1(V) \times \mathscr{S}_2(V) \cong \mathbb{P}^2 \times \mathbb{P}^2, \quad l \longmapsto (S_1(l), S_2(l)), \tag{9.7.6}$$

realizes $\Sigma(V)$ *as a smooth* $(1,1)$*-divisor in* $\mathbb{P}^2 \times \mathbb{P}^2$*.*

We use below the following notation and terminology.

9.8. **Notation.** Given a line l in V, consider the following objects:

- the family $\Sigma(l) \subset \Sigma(V)$ of lines in V which meet l;
- the union $\Theta(l) \subset V$ of lines from $\Sigma(l)$;
- the morphism $\pi : \Sigma(l) \setminus \{l\} \to l$, $l' \longmapsto l' \cap l$.

It is known ([KR13, proof of Prop. 2]) that $\Sigma(l) \setminus \{l\}$ is a curve in $\Sigma(V)$ of bidegree $(1,1)$. Due to Proposition 8.2(e) and Lemma 9.3, π is generically $2:1$ or $1:1$. So, $\Sigma(l)$ consists of at most two components. Then also $\Theta(l)$ consists of at most two components of pure dimension 2.

A line $l \in \Sigma(V)$ will be called a *splitting line*, or an *s-line* for short, if $\Theta(l)$ splits into two components $\Theta_1(l)$ and $\Theta_2(l)$. We let $\Sigma_{\mathrm{s}}(V) \subset \Sigma(V)$ denote the subvariety of s-lines in V. Clearly, $\Sigma_{\mathrm{s}}(V)$ is a closed subset in $\Sigma(V)$. We consider it with its reduced structure.

The proof of Proposition 2 in [KR13] shows that $\Theta_1(l)$ and $\Theta_2(l)$ belong to different components of $\mathscr{S}(V)$. We adopt the convention $\Theta_i(l) \in \mathscr{S}_i(V)$, $i = 1, 2$. Adopting a terminology from [KR13], we call a line l in a cubic scroll S *exceptional* if either S is smooth and l is the exceptional section of S, or S is a cone and l is a ruling of S.

Since $(\Theta(l))_{l\in\Sigma(V)}$ is a flat family over an irreducible base, the function $l \longmapsto \deg \Theta(l)$ on $\Sigma(V)$ is constant. In statement (a) of the following lemma we determine this constant.

9.9. **Lemma.** (a) *For any line* $l \in \Sigma(V)$ *one has* $\deg \Theta(l) = 6$*. If* l *is an* s*-line with* $\Theta(l) = \Theta_1(l) \cup \Theta_2(l)$*, then* $\Theta_i(l)$*,* $i = 1, 2$*, are cubic scrolls.*

(b) *Let* $l \in \Sigma(V)$*. Then* $l \in \Sigma_{\mathrm{s}}(V)$*, that is,* l *is an* s*-line, if and only if* l *is an exceptional line of a cubic scroll.*

(c) *An* s*-line* l *is contained in* $\mathcal{B}$ *if and only if* l *is a ruling of a cubic cone.*

Proof. (a) Consider the rational surfaces $T_i = \overline{\varsigma(\mathcal{U}_i)} \subset \Sigma(V)$, $i = 1, 2$, where $\mathcal{U}_i \subset \mathscr{S}_i(V)$ is the subset of smooth scrolls, and $\varsigma : \mathcal{U}_i \to \Sigma(V)$ sends a scroll $S \in \mathcal{U}_i$ to its exceptional section, cf. the proof of Proposition 9.5. It is easy to see that any two exceptional divisors on $\Sigma(V)$ meet. Hence, $T_1 \cap T_2 \neq \varnothing$.

Let $[l] \in T_1 \cap T_2 \subset \Sigma(V)$. We claim that $\Theta(l) = S_1 \cup S_2$ for some $S_i \in \mathscr{S}_i(V)$, $i = 1, 2$. Indeed, one has $l = \lim_{j\to\infty} l_{i,j}$, where for any $j \geq 1$, $l_{i,j} \in \varsigma(\mathcal{U}_i)$ is the exceptional line of a smooth cubic scroll $S_{i,j} \in \mathscr{S}_i$, $i = 1, 2$. Passing to subsequences one may assume that $\lim_{j\to\infty} S_{i,j} = S_i \in \mathscr{S}_i$, $i = 1, 2$. If S_i is smooth, then $l \subset S_i$ is the exceptional line of S_i. In any case, $S_1 \cup S_2 \subset \Theta(l)$. Since $\Theta(l)$ consists of at most two components, see 9.8, one has $\Theta(l) = S_1 \cup S_2$, as claimed. So, $\deg \Theta(l) = 6$ $\forall l \in \Sigma(V)$. Now the second assertion follows as well. Indeed, Ω (and then also V) contains neither a plane, nor a quadric surface ([KR13, Lem. 3]). Moreover, since V is an intersection of quadrics ([Isk77, Lem. 2.10]), V does not contain any cubic surface F with $\langle F \rangle \cong \mathbb{P}^3$.

(b) To show the "if" part, observe that in both cases the corresponding cubic scroll is a component of $\Theta(l)$. Due to (a), l is an s-line. The "only if" part follows from the definition.

(c) A line l on V is contained in $\mathcal{B}$ if and only if through a general point of l pass at most 2 lines in V including l itself. For an s-line l this is the case if and only if at least one of the components $\Theta_1(l)$ and $\Theta_2(l)$ is a cubic cone. □

9.9.1. **Corollary.** *One has* $\dim \Sigma_{\mathrm{s}}(V) = 2$.

Proof. This follows from Lemma 9.9(a)–(b). Indeed, $\Sigma_{\mathrm{s}}(V)$ consists of the exceptional lines of smooth cubic scrolls and of the rulings of cubic cones. The first family of lines is purely two-dimensional, and the second is at most two-dimensional due to Lemma 9.4. □

9.10. **Proposition.** *For any* $S \in \mathscr{S}(V)$ *one has*

(a) $\Lambda(S) \subset \Sigma_{\mathrm{s}}(V)$ *if and only if* S *is a cubic cone;*
(b) $\Lambda(S) \cdot \Sigma_{\mathrm{s}}(V) = 1$ *for any* $S \in \mathscr{S}(V)$*;*
(c) $\Sigma_{\mathrm{s}}(V) \sim \mathcal{F}_1 + \mathcal{F}_2$.

Proof. (a) is immediate from Lemma 9.9(a)–(b).

(b) Let $S \in \mathscr{S}(V)$ be smooth. Let us show that exactly one ruling of S is an s-line. For such a ruling l one has $\Theta(l) = S_1 \cup S_2$, where $S_i \in \mathscr{S}_i(V)$, $S_i \neq S$, $i = 1, 2$. Due to Corollary 9.7.3(i), at least one of S_1, S_2 is smooth. Since each ruling of S_i meets S, one has $S_i \subset A_S$, $i = 1, 2$.

Starting with the pair (V, S) we construct diagram (3.1.1). The proper transform $\widetilde{S}_i$ of S_i in $\widetilde{W}$ is a ruled surface contained in $\widetilde{A}_S$, and its image $\theta(S_i) \subset W$ under the linear projection $\theta : \mathbb{P}^{12} \dashrightarrow \mathbb{P}^7$ with center $\langle S \rangle$ is a curve contained in F.

Assume first that $\langle S \rangle \cap \langle S_i \rangle = l$ for some $i \in \{1, 2\}$. Then $J := \theta(S_i)$ is a smooth conic. Indeed, if S_i is smooth, then $\theta(S_i)$ is the image of a conic section of $S_i \to \mathbb{P}^1$ disjoint with l, and if S_i is a cubic cone, then $\theta(S_i)$ is the image of a twisted cubic in S_i, which meets l once and transversely. The rational normal quintic scroll F, which contains a nondegenerate conic J, is isomorphic to $\mathbb{F}_1$, and J is its exceptional section. Hence S_i is the only component of $\Theta(l)$ with $\langle S \rangle \cap \langle S_i \rangle = l$. Moreover, l is a unique splitting line in S, because S_i does not contain two distinct rulings of S, see Corollary 9.7.3(iii). Thus, in this case $\Lambda(S) \cap \Sigma_{\mathrm{s}}(V)$ consists of a single point.

Assume further that $\dim(\langle S \rangle \cap \langle S_i' \rangle) = 2$ for $i = 1, 2$. Then $f_i := \theta(S_i)$ is a line in F for $i = 1, 2$. These lines cannot be both rulings of F. Indeed, otherwise $\eta^{-1}(f_i)$, $i = 1, 2$, would be ruled surfaces in $\widetilde{W}$ from the same irreducible family of ruled surfaces. Hence also S_1 and S_2 would belong to the same family of surfaces in V over an irreducible base. However, S_1 and S_2 belong to different components of $\mathscr{S}(V)$ ([KR13, proof of Prop. 2]). Due to Proposition 9.6 we obtain a contradiction.

So, one of the f_i, say, f_1 should be the exceptional line of $F \cong \mathbb{F}_3$, while f_2 should be a ruling of F. Once again, S_1 is uniquely determined and does not contain two distinct rulings of S. Hence $\Lambda(S) \cap \Sigma_{\mathrm{s}}(V)$ is a single point in this case as well.

(c) This follows by (b) and 9.7.2. □

9.10.1. **Corollary.** *If two distinct cubic cones* S *and* S' *in* V *have a common ruling* l*, then they belong to different components of* $\mathscr{S}(V)$.

Proof. Indeed, $\Lambda(S)$ and $\Lambda(S')$ are two lines in $\Sigma_{\rm s}(V)$, which meet in a point $[l]$. If S and S' were in the same component, say, $\mathscr{S}_i(V)$ of $\mathscr{S}(V)$, then these lines would project to the same point $\mathrm{pr}_j([l]) \in \mathbb{P}^2$, where $j \neq i$. The latter means that $S = S'$, a contradiction. $\square$

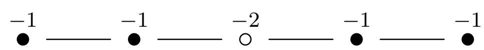
10. Cubic cones in V_{18} and lines on singular del Pezzo sextics

In this section we associate to each Fano-Mukai fourfold $V = V_{18}$ of genus 10 a certain (singular) del Pezzo sextic surface. The cubic cones in V occur to be in a one-to-one correspondence with the lines on this sextic.

10.1. **Lemma** ([Dol12, § 8.1.1, § 8.4.2], [CT88, Prop. 8.3]). *Let $X \subset \mathbb{P}^6$ be a linearly nondegenerate normal surface of degree 6, and let $\widetilde{X}$ be the minimal desingularization of X. Let $\Gamma(\widetilde{X})$ be the dual graph of the configuration of (-1)- and (-2)-curves in $\widetilde{X}$. Suppose that X admits two different birational contractions onto $\mathbb{P}^2$. Then the following hold.*

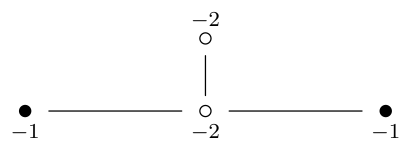
(a) *X has at worst Du Val singularities.*
(b) *X admits a canonical embedding in $\mathbb{P}^2 \times \mathbb{P}^2$.*
(c) *X is one of the following:*
   (i) *a smooth del Pezzo sextic surface, with $\Gamma(\widetilde{X})$ being a cycle of six (-1)-vertices;*
   (ii) *a del Pezzo sextic surface with a unique singular point of type A_1 and with*

(10.1.1) $\Gamma(\widetilde{X})$:



where the black vertices correspond to the lines in X;
   (iii) *a del Pezzo sextic surface with a unique singular point of type A_2 and with*

(10.1.2) $\Gamma(\widetilde{X})$:



(d) *The number of lines in X is 6 for type (c)(i), 4 for type (c)(ii), and 2 for type (c)(iii). There exists a unique divisor $\mathfrak{G}_X \in |-K_X|$ whose support coincides with the union of lines on X.*

Abusing the language we say that a del Pezzo sextic X in (c)(ii) (resp., X in (c)(iii)) is *of type A_1* (resp., *of type A_2*).

One has the following result.

10.2. **Proposition.** *$\Sigma_{\rm s}(V)$ is a sextic surface. If it is reducible, then this is the union $\mathcal{F}_1 \cup \mathcal{F}_2$ of two cubic scrolls, where $\mathcal{F}_i$ is the pullback of a line in $\mathbb{P}^2$ under the projection $\mathrm{pr}_i : \Sigma(V) \to \mathbb{P}^2$, $i = 1, 2$. If it is irreducible, then $\Sigma_{\rm s}(V)$ is a del Pezzo sextic which is either smooth, or of type A_1, or of type A_2.*

Proof. By Proposition 9.10(c) $\Sigma_{\rm s}(V)$ is a sextic hyperplane section of $\Sigma(V)$ with respect to the polarization on $\Sigma(V)$ induced by the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$. Moreover, if the surface $\Sigma_{\rm s}(V)$ is reducible, then $\Sigma_{\rm s}(V) = \mathcal{F}_1 \cup \mathcal{F}_2$. Assume that $\Sigma_{\rm s}(V)$ is irreducible. Again by Proposition 9.10(c) the restrictions $p_i|_{\Sigma_{\rm s}(V)} : \Sigma_{\rm s}(V) \to \mathscr{S}_i(V) = \mathbb{P}^2$, $i = 1, 2$, yield two birational contractions, whose exceptional divisors do not possess any common component. Moreover, any ruling of p_i passing through a singular point of $\Sigma_{\rm s}(V)$ is contained in $\Sigma_{\rm s}(V)$. Therefore $\Sigma_{\rm s}(V)$ is normal. The rest of the proof is straightforward from Lemma 10.1(c). $\square$

10.2.1. **Corollary.** *The set of cubic cones in V is finite if and only if $\Sigma_{\rm s}(V)$ is irreducible. If it is finite, then any cubic cone $S \subset V$ is invariant under the $\mathrm{Aut}^0(V)$-action.*

10.3. **Notation.** Up to an automorphism of $(\mathbb{P}^2)^\vee \times \mathbb{P}^2$, which does not interchange the factors, one may assume that $\Sigma(V)$ coincides with the variety Σ of complete flags in $\mathbb{P}^2$ given in $(\mathbb{P}^2)^\vee \times \mathbb{P}^2$ by equation (9.5.3), cf. 9.1.1. Up to an automorphism of $\Sigma(V)$ one may suppose that $(\mathcal{F}_1, \mathcal{F}_2)$ is one of the following:

(i) $\mathcal{F}_1 = \{x_0 = 0\}$ and $\mathcal{F}_2 = \{y_0 = 0\}$;
(ii) $\mathcal{F}_1 = \{x_0 = 0\}$ and $\mathcal{F}_2 = \{y_1 = 0\}$.

We say that a pair $(\mathcal{F}_1, \mathcal{F}_2)$ is *of the first* (resp., *second*) *kind* if it is equivalent to a pair (i) (resp., (ii)). For a pair $(\mathcal{F}_1, \mathcal{F}_2)$ of the first kind in (i), the intersection $C = \mathcal{F}_1 \cap \mathcal{F}_2$ is the $(1,1)$-conic in $(\mathbb{P}^1)^\vee \times \mathbb{P}^1$ given by equation $x_1y_1 + x_2y_2 = 0$. The projection $p_2|_{\mathcal{F}_1} : \mathcal{F}_1 \to \mathscr{S}_2(V) \cong \mathbb{P}^2$ sends birationally $\mathcal{F}_1$ onto $\mathbb{P}^2$ contracting the exceptional section s_1 of $\mathcal{F}_1 \cong \mathbb{F}_1$ to the point $q_2 = (1 : 0 : 0)$, and sends the conic section $C \subset \mathcal{F}_1$ (disjoint with s_1) to the line $h_2 = \{y_0 = 0\}$, and symmetrically for the projection $p_1|_{\mathcal{F}_2} : \mathcal{F}_2 \to \mathscr{S}_1(V) \cong \mathbb{P}^2$.

For a pair $(\mathcal{F}_1, \mathcal{F}_2)$ of the second kind the intersection $C = \mathcal{F}_1 \cap \mathcal{F}_2$ is reducible and consists of two lines $s_1 \cup s_2$, where s_1 (resp. s_2) is the exceptional section of $\mathcal{F}_1$ (resp. $\mathcal{F}_2$) and a ruling of $\mathcal{F}_2$ (resp. $\mathcal{F}_1$). Thus, $p_2(s_1) = q_2 \in h_2 = \{y_1 = 0\}$, and $p_1(s_2) = q_1 \in h_1 = \{x_0 = 0\}$. The lines s_1 and s_2 meet at a single point.

10.3.1. **Lemma.** *Assume that $\Sigma_{\rm s}(V) = \mathcal{F}_1 \cup \mathcal{F}_2$. Let (W, F) be the pair linked to (V, S_2). Suppose also that the group $\mathrm{Aut}^0(W, F)$ contains a singular torus $\mathrm{z}(\mathrm{L})$ of $\mathrm{Aut}(W)$, where $\mathrm{L} \subset \mathrm{Aut}(W)$ is a Levi subgroup. Then $(\mathcal{F}_1, \mathcal{F}_2)$ is a pair of the first kind. This holds, in particular, if $\mathrm{Aut}(V) \supset (\mathbb{G}_{\rm m})^2$.*

Proof. Suppose to the contrary that $(\mathcal{F}_1, \mathcal{F}_2)$ is a pair of the second kind. By Corollary 10.3.2(b) the cubic cones S_1 and S_2 are $\mathrm{Aut}^0(V)$-invariant. Since $\mathrm{z}(\mathrm{L}) \subset \mathrm{Aut}^0(W, F)$, then $\mathscr{S}_1(V)$ contains a cubic cone S_1' disjoint with S_2, see Lemma 6.6.1. The cones S_1 and S_2 have a common ruling, hence $S_1' \neq S_1$. So, S_1' corresponds to a rulings $\Lambda(S_{1,t})$ of $\mathcal{F}_1$ for some $t \in \mathbb{P}^1$, see Corollary 10.3.2(b). However, the cones S_2 and $S_1' = S_{1,t}$ as well have a common ruling. This gives a desired contradiction. $\square$

From Proposition 10.2 we deduce

10.3.2. **Corollary.** (a) *If $\Sigma_{\rm s}(V)$ is irreducible, then every component $\mathscr{S}_i(V)$, $i = 1, 2$, contains exactly 3, 2 or 1 cubic cone(s) according to types (c)(i), (c)(ii), and (c)(iii) in Lemma 10.1, respectively.*

(b) *If $\Sigma_{\rm s}(V) = \mathcal{F}_1 \cup \mathcal{F}_2$, then the following hold.*
- *For $i = 1, 2$ the subvariety $\mathcal{C}_i(V)$ of $\mathscr{S}_i(V) \cong \mathbb{P}^2$ whose points correspond to cubic cones in V consists of a line $h_i \subset \mathbb{P}^2$ and a reduced $\mathrm{Aut}^0(V)$-invariant point $q_i = [S_i]$. For a pair $(\mathcal{F}_1, \mathcal{F}_2)$ of the first kind, $q_i \notin h_i$, otherwise $q_i \in h_i$.*
- *Each ruling $l_{i,t}$ of S_i, $t \in \mathbb{P}^1$, is a common ruling of S_i and of a unique cubic cone $S_{j,t} \in \mathcal{C}_j(V)$, $j \neq i$.*
- *If $(\mathcal{F}_1, \mathcal{F}_2)$ is a pair of the first kind, then S_j does not contain the vertex of S_i, $i \neq j$.*

Proof. Statement (a) is straightforward from Proposition 9.10(a).

(b) By Proposition 9.10(a) the cubic cones in $\mathscr{S}_i(V)$ correspond to the lines in $\Sigma_s(V) = \mathcal{F}_1 \cup \mathcal{F}_2$ contracted under the projection $p_i : \Sigma_s(V) \to \mathscr{S}_i(V) = \mathbb{P}^2$, $i, j = 1, 2$. Hence they correspond to the rulings of $\mathcal{F}_i$ and the exceptional sections s_i of $\mathcal{F}_i$, $i = 1, 2$. This gives an isomorphism $\mathcal{C}_i(V) \cong h_i \cup \{q_i\}$ and the equalities $[S_i] = \{q_i\}$ and $s_j = \Lambda(S_i) = p_i^{-1}(q_i)$, $i, j = 1, 2$, $j \neq i$. Any ruling of $p_i : \mathcal{F}_i \to h_i$ meets the exceptional section s_i in a point, which corresponds to the unique common ruling of the assigned cubic cones, cf. Corollary 9.7.3. This proves the first two assertions of (b).

To show the last one, we let v_i be the vertex of the cone S_i, $i = 1, 2$. Suppose to the contrary that $v_i \in S_j$ for some choice of $i, j \in \{1, 2\}$, $i \neq j$. Let l be the unique ruling of S_j passing through $v_i \neq v_j$. Since any line in V through v_i is a ruling of S_i, see Proposition 8.2(e), l is a common ruling of S_i and S_j. However, for a pair $(\mathcal{F}_1, \mathcal{F}_2)$ of the first kind both $\Lambda(S_1) \subset \mathcal{F}_2$ and $\Lambda(S_2) \subset \mathcal{F}_1$ are disjoint with the common conic section $C = \mathcal{F}_1 \cap \mathcal{F}_2$. Hence, S_1 and S_2 do not have any ruling in common, a contradiction. □

11. Automorphism groups of singular del Pezzo sextics

In this section we prove the following proposition.

11.1. **Proposition.** *For any Fano-Mukai fourfold $V = V_{18}$ of genus 10 one of the following cases 1^o–4^o occurs.*

	$\Sigma_s(V)$	$\mathrm{Aut}(V)$
1^o	a union of two smooth cubic scrolls meeting along a smooth conic	$\mathrm{GL}_2(\mathbb{C}) \subset \mathrm{Aut}(V) \subset \mathrm{GL}_2(\mathbb{C}) \rtimes (\mathbb{Z}/2\mathbb{Z})$
2^o	a del Pezzo sextic of type A_1	$\mathbb{G}_a \times \mathbb{G}_m \subset \mathrm{Aut}(V) \subset (\mathbb{G}_a \times \mathbb{G}_m) \rtimes (\mathbb{Z}/2\mathbb{Z})$
3^o	a smooth del Pezzo sextic	$(\mathbb{G}_m)^2 \subset \mathrm{Aut}(V) \subset (\mathbb{G}_m)^2 \rtimes (\mathbb{Z}/6\mathbb{Z})$
4^o	a union of two smooth cubic scrolls meeting along a pair of intersecting lines	$\mathbb{G}_a \times \mathbb{G}_m \subset \mathrm{Aut}(V) \subset B(\mathrm{PGL}_3(\mathbb{C})) \rtimes (\mathbb{Z}/2\mathbb{Z})$, where $B(\mathrm{PGL}_3(\mathbb{C}))$ is a Borel subgroup of $\mathrm{PGL}_3(\mathbb{C})$

In Cases 1^o and 4^o the variety V contains two one-parameter families of cubic cones and two $\mathrm{Aut}^0(V)$-invariant cubic cones $S_i \in \mathscr{S}_i(V)$, $i = 1, 2$. The number of cubic cones in V equals 4 in Case 2^o and 6 in Case 3^o; all of these cones are $\mathrm{Aut}^0(V)$-invariant.

In fact, Case 4^o does not occur, see Corollary 12.5.2 in the next section. The proof of Proposition 11.1 is done in 11.5. It is preceded by some preliminary facts and constructions.

11.2. **Lemma.** (a) *The induced $\mathrm{Aut}(V)$-action on $\Sigma(V)$ is effective and leaves invariant the divisor $\Sigma_s(V) \subset \Sigma(V)$.*
(b) *$\mathrm{Aut}^0(V)$ acts effectively on any component of $\Sigma_s(V)$.*
(c) *The action of $\mathrm{Aut}(V)$ on $\Sigma_s(V)$ is effective.*

Proof. (a) By Proposition 8.2(d), there are exactly three lines passing through a general point of V. This allows to reconstruct the $\mathrm{Aut}(V)$-action on V from the induced $\mathrm{Aut}(V)$-action on $\Sigma(V)$. So, the latter action is effective. Now the assertion is straightforward.

(b) One may identify $\Sigma(V) \subset (\mathbb{P}^2)^\vee \times \mathbb{P}^2$ with the variety Σ of complete flags in $\mathbb{P}^2$, see 10.3. The duality permutes the factors $(\mathbb{P}^2)^\vee$ and $\mathbb{P}^2$ inducing an involution ι of Σ. Thus, $\mathrm{Aut}(\Sigma) = \mathrm{Aut}^0(\Sigma) \rtimes (\mathbb{Z}/2\mathbb{Z})$, where $\mathrm{Aut}^0(\Sigma) \cong \mathrm{Aut}(\mathbb{P}^2)$.

For $i = 1, 2$ the action of $\mathrm{Aut}^0(\Sigma(V))$ on the ruling $p_i : \Sigma(V) \to \mathscr{S}_i(V)$ induces an isomorphism $\mathrm{Aut}^0(\Sigma(V)) \cong \mathrm{Aut}(\mathscr{S}_i(V))$ and also an injection $\mathrm{Aut}^0(V) \hookrightarrow \mathrm{Aut}(\mathscr{S}_i(V))$. For any irreducible component $\mathcal{T}$ of the divisor $\Sigma_{\mathrm{s}}(V)$, at least one of the projections $p_i|_{\mathcal{T}} : \mathcal{T} \to \mathscr{S}_i(V)$, $i = 1, 2$, is dominant. Hence the representation $\mathrm{Aut}^0(V) \to \mathrm{Aut}(\mathcal{T})$ is faithful.

(c) Assume that $\alpha \in \mathrm{Aut}(V)$ induces the identity on $\Sigma_{\mathrm{s}}(V)$. In particular, any ruling of $p_i : \Sigma(V) \to \mathscr{S}_i(V)$ contained in $\Sigma_{\mathrm{s}}(V)$ is invariant under α. The rulings on $\Sigma_{\mathrm{s}}(V)$ correspond to the cubic cones in V, and they do exist, see, e.g., Corollary 10.3.2. It follows that $\alpha(\mathscr{S}_i(V)) = \mathscr{S}_i(V)$, $i = 1, 2$, that is, α does not interchange the factors of $(\mathbb{P}^2)^\vee \times \mathbb{P}^2$. Since the projections $p_i|_{\Sigma_{\mathrm{s}}(V)} : \Sigma_{\mathrm{s}}(V) \to \mathscr{S}_i(V)$, $i = 1, 2$, are dominant and α-equivariant, α acts identically on the factors of $(\mathbb{P}^2)^\vee \times \mathbb{P}^2$, hence also on Σ. By (a), $\alpha = \mathrm{id}_V$. □

The following corollary is straightforward.

11.2.1. **Corollary.** *There is an embedding*

$$\mathrm{Aut}(V) \hookrightarrow \mathrm{Aut}((\mathbb{P}^2)^\vee \times \mathbb{P}^2,\, \Sigma(V), \Sigma_{\mathrm{s}}(V)) = \mathrm{Aut}(\Sigma(V), \Sigma_{\mathrm{s}}(V)).$$

Proof. Indeed, the pair of projections of $(\mathbb{P}^2)^\vee \times \mathbb{P}^2$ to the factors being canonical (see 9.1.1), the inclusion $\mathrm{Aut}((\mathbb{P}^2)^\vee \times \mathbb{P}^2,\, \Sigma(V)) \subset \mathrm{Aut}(\Sigma(V))$ is in fact the equality. □

The Hilbert scheme of lines $\Sigma(V)$ admits the following alternative description.

11.2.2. **Construction. 1.** For a vector $\mathbf{y} \neq 0$ in a vector space $U = \mathbb{C}^{n+1}$, we let $[\mathbf{y}]$ denote the image of $\mathbf{y}$ in $\mathbb{P}(U) = \mathbb{P}^n$. Consider the Segre embedding

$$\nu : (\mathbb{P}^n)^\vee \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{n^2-1}, \quad ([\mathbf{x}^*], [\mathbf{y}]) \longmapsto [\mathbf{x}^* \otimes \mathbf{y}], \quad \mathbf{x}^* \in (U)^\vee,\ \mathbf{y} \in U,\ \mathbf{x}^*, \mathbf{y} \neq 0.$$

Using the standard isomorphism $U^\vee \otimes U \cong \mathrm{End}(U) = \mathfrak{gl}(U)$, we regard $U^\vee \otimes U$ as the vector space of square matrices of order $n+1$ over $\mathbb{C}$. We let $(e_0, \ldots, e_n)$ be the standard basis of $U = \mathbb{C}^{n+1}$ and $(e_0^*, \ldots, e_n^*)$ be its dual, so that $e_i^* \otimes e_j = E_{i,j}$ is the elementary matrix with the only nonzero entry $e_{i,j} = 1$. Under this identification, the image of $U^\vee \times U$ in $\mathfrak{gl}(U)$ consists of matrices of rank 1:

$$\mathbf{x}^* \otimes \mathbf{y} = \begin{pmatrix} x_0 y_0 & \ldots & x_0 y_n \\ \vdots & \vdots & \vdots \\ x_n y_0 & \ldots & x_n y_n \end{pmatrix}, \tag{11.2.3}$$

where $\mathbf{x}^* = x_0 e_0^* + \ldots + x_n e_n^*$ and $\mathbf{y} = y_0 e_0 + \ldots + y_n e_n$.

In the sequel we let $n = 2$, so that $\mathfrak{gl}(U) = \mathfrak{gl}_3(\mathbb{C})$. Assuming that $\Sigma(V)$ is realized as the subvariety of complete flags in $\mathbb{P}^2$ embedded in $(\mathbb{P}^2)^\vee \times \mathbb{P}^2$ with equation (9.5.3)

$$x_0 y_0 + x_1 y_1 + x_2 y_2 = 0,$$

its image Σ in $\mathbb{P}^8 = \mathbb{P}(\mathfrak{gl}(U))$ lies in the hyperplane $\mathfrak{sl}(U) = \mathfrak{sl}_3(\mathbb{C})$ of matrices with zero trace. We identify $\Sigma(V)$ with Σ. In this way, $\Sigma(V)$ is realized as a smooth hyperplane section of $\nu((\mathbb{P}^2)^\vee \times \mathbb{P}^2) \subset \mathbb{P}^8$, where ν stands for the Segre embedding, and, alternatively, as the projectivization of the cone of (3×3)-matrices of rank 1 with zero trace.

For a square matrix M of order 3 and of rank 1 one has:

$$\mathrm{tr}(M) = 0 \Longleftrightarrow M^2 = 0 \Longleftrightarrow \mathrm{im}(M) \subset \ker(M).$$

Thus, for $[M] \in \Sigma(V)$, $\mathrm{im}(M)$ is a line in the plane $\ker(M)$, and

$$\mathbb{P}(\mathrm{im}(M)) \subset \mathbb{P}(\ker(M)) \subset \mathbb{P}^2$$

is a complete flag in $\mathbb{P}^2$. The maps $M \longmapsto \ker(M)$ and $M \longmapsto \mathrm{im}(M)$ yield the projections $\Sigma(V) \to (\mathbb{P}^2)^\vee$ and $\Sigma(V) \to \mathbb{P}^2$, respectively.

2. The $\mathrm{GL}_3(\mathbb{C})$-action on $\mathfrak{gl}_3(\mathbb{C})$ by conjugation: $(A, M) \longmapsto AMA^{-1}$ descends to a $\mathrm{PGL}_3(\mathbb{C})$-action on $\mathbb{P}^8 = \mathbb{P}(\mathfrak{gl}_3(\mathbb{C}))$. The Segre embedding is equivariant with respect to the latter $\mathrm{PGL}_3(\mathbb{C})$-action on $\mathbb{P}^8$ and the $\mathrm{PGL}_3(\mathbb{C})$-action on $(\mathbb{P}^2)^\vee \times \mathbb{P}^2$ given by

$$(A, ([\mathbf{x}^*], [\mathbf{y}])) \longmapsto ([(A^{-1})^t \mathbf{x}^*], [A\mathbf{y}]).$$

Any square matrix M of order 3 and of rank 1 with zero trace has Jordan form $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. Therefore, the induced $\mathrm{PGL}_3(\mathbb{C})$-action on Σ is transitive. In fact, $\Sigma \cong \mathrm{PGL}_3(\mathbb{C})/B_0$, where $B_0 \subset \mathrm{PGL}_3(\mathbb{C})$ is the Borel subgroup of upper triangular matrices.

Recall that $\mathrm{Aut}(\Sigma) \cong \mathrm{PGL}_3(\mathbb{C}) \rtimes (\mathbb{Z}/2\mathbb{Z})$, see the proof of Lemma 11.2. One can take the matrix transposition for the generator of $\mathbb{Z}/2\mathbb{Z}$ interchanging the factors of $(\mathbb{P}^2)^\vee \times \mathbb{P}^2$. The (maximal) diagonal torus $\mathbb{T}_0 \cong (\mathbb{G}_\mathrm{m})^2$ of B_0 acts effectively on Σ. Any $(\mathbb{G}_\mathrm{m})^2$-subgroup of $\mathrm{Aut}(\Sigma)$ is conjugate to $\mathbb{T}_0$ in $\mathrm{PGL}_3(\mathbb{C})$.

3. To a square matrix $C \neq 0$ of order 3 one can associate a hyperplane H_C in the vector space $\mathfrak{gl}_3(\mathbb{C})$, where

$$H_C = \{M \in \mathfrak{gl}_3(\mathbb{C}) \mid \mathrm{tr}(M \cdot C) = 0\}.$$

If $C \neq 0$ is a scalar matrix, then H_C coincides with the subspace $\mathfrak{sl}_3(\mathbb{C}) \subset \mathfrak{gl}_3(\mathbb{C})$ of matrices with zero trace. Since the bilinear form $(A, B) \longmapsto \mathrm{tr}(A \cdot B)$ on $\mathfrak{sl}_3(\mathbb{C})$ is nondegenerate, any hyperplane in $\mathfrak{sl}_3(\mathbb{C})$ coincides with $H_C \cap \mathfrak{sl}_3(\mathbb{C})$ for a suitable $C \in \mathfrak{sl}_3(\mathbb{C})$.

4. According to Proposition 9.10(c), $\Sigma_\mathrm{s}(V)$ is a hyperplane section of $\Sigma(V) = \Sigma$ in $\mathbb{P}^8$. Therefore, there exists a (3×3)-matrix $C = C(V)$ with zero trace such that

$$\Sigma_\mathrm{s}(V) = \Sigma \cap \mathbb{P}(H_C) = \nu\left((\mathbb{P}^2)^\vee \times \mathbb{P}^2\right) \cap \mathbb{P}\left(\mathfrak{sl}_3(\mathbb{C}) \cap H_C\right).$$

Such a matrix C is defined uniquely up to a nonzero scalar factor. The $\mathrm{PGL}_3(\mathbb{C})$-action on $\mathbb{P}^8 = \mathbb{P}(\mathfrak{gl}_3(\mathbb{C}))$ by conjugation leaves the pair $(\nu((\mathbb{P}^2)^\vee \times \mathbb{P}^2), \Sigma)$ invariant. Replacing C (defined up to a scalar factor) by its Jordan form, any hyperplane section X of Σ can be sent by an automorphism of Σ to one of the sections

(11.2.4) $$X_{(a:b)} = \Sigma \cap H_{C_{(a,b)}},\ (a, b) \neq (0, 0), \quad \text{and} \quad X_i = \Sigma \cap H_{C_i},\ i = 1, 2, 3,$$

where

$$C_{(a,b)} = \mathrm{diag}(a, b, -a-b),$$

(11.2.5) $$C_1 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad C_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad C_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

5. Consider the group $\widetilde{\mathrm{GL}_n(\mathbb{C})} = \mathrm{GL}_n(\mathbb{C}) \rtimes (\mathbb{Z}/2\mathbb{Z})$, where the generator τ of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathrm{GL}_n(\mathbb{C})$ via the Cartan involution $A \longmapsto (A^t)^{-1}$. The action of $\mathrm{GL}_n(\mathbb{C})$ via conjugation on $\mathfrak{gl}(U)$ with $U = \mathbb{C}^n$ extends to an action of $\widetilde{\mathrm{GL}_n(\mathbb{C})}$, where τ acts on $\mathfrak{gl}(U)$ via $C \longmapsto C^t$. Let also

$$\begin{aligned} \mathrm{PCent}_{\widetilde{\mathrm{GL}_n(\mathbb{C})}}(C) &= \left\{\widetilde{A} \in \widetilde{\mathrm{GL}_n(\mathbb{C})} \,\middle|\, \widetilde{A}.C = \alpha C \quad \text{for some} \quad \alpha \in \mathbb{C}^*\right\}, \\ \widetilde{\mathrm{PGL}_n(\mathbb{C})} &= \widetilde{\mathrm{GL}_n(\mathbb{C})}/\,\mathrm{z}(\widetilde{\mathrm{GL}_n(\mathbb{C})}) \cong \mathrm{PGL}_n(\mathbb{C}) \rtimes (\mathbb{Z}/2\mathbb{Z}), \\ \mathrm{PCent}_{\widetilde{\mathrm{PGL}_n(\mathbb{C})}}(C) &= \mathrm{PCent}_{\widetilde{\mathrm{GL}_n(\mathbb{C})}}(C)/\,\mathrm{z}(\widetilde{\mathrm{GL}_n(\mathbb{C})}). \end{aligned}$$

11.3. **Lemma.** *For a hyperplane section* $X_C = \Sigma \cap H_C$*, where* $C \in \mathfrak{sl}_3(\mathbb{C})$*,* $C \neq 0$*, one has*

$$\mathrm{Aut}(\Sigma, X_C) = \mathrm{PCent}_{\widetilde{\mathrm{PGL}_3(\mathbb{C})}}(C) \subset \widetilde{\mathrm{PGL}_3(\mathbb{C})}.$$

Proof. We have $\mathrm{Aut}(\Sigma) = \widetilde{\mathrm{PGL}_3(\mathbb{C})}$. Clearly, $g \in \widetilde{\mathrm{GL}_3(\mathbb{C})}$ induces an automorphism of (Σ, X_C) if and only if $g(H_C)$ is a member of the pencil generated by H_C and H_0, if and only if $g.C = \alpha C + \beta\,\mathrm{E}$ for some $\alpha, \beta \in \mathbb{C}$. For $A \in \mathrm{GL}_3(\mathbb{C})$ one has $A.C = ACA^{-1}$ and $A\tau.C = AC^tA^{-1}$. Anyway, under the latter condition, $\mathrm{tr}(\alpha C + \beta\,\mathrm{E}) = \mathrm{tr}(C) = 0$, hence $\beta = 0$. Now the claim follows. □

11.4. **Lemma.** *Given a matrix* C *as in* (11.2.5)*, the following hold.*

	C	$\mathrm{Sing}(X_C)$	$\mathrm{Aut}(\Sigma, X_C)$
(a)	$C_{(1,1)}$, $C_{(1,-2)}$, $C_{(-2,1)}$	a smooth conic	$\widetilde{\mathrm{GL}_2(\mathbb{C})} = \mathrm{GL}_2(\mathbb{C}) \rtimes (\mathbb{Z}/2\mathbb{Z})$
(b)	$C_{(1,\zeta)}$, $\zeta \neq 1$, $\zeta^3 = 1$	$\varnothing$	$(\mathbb{G}_\mathrm{m})^2 \rtimes (\mathbb{Z}/6\mathbb{Z})$
(c)	$C_{(1,b)}$, $b \notin \{-2, -1/2\}$, $b^3 \neq 1$	$\varnothing$	$(\mathbb{G}_\mathrm{m})^2 \rtimes (\mathbb{Z}/2\mathbb{Z})$
(d)	C_1	A_1	$(\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}) \rtimes (\mathbb{Z}/2\mathbb{Z})$
(e)	C_2	A_2	$((\mathbb{G}_\mathrm{a})^2 \rtimes \mathbb{G}_\mathrm{m}) \rtimes (\mathbb{Z}/2\mathbb{Z})$
(f)	C_3	two intersecting lines	$B(\mathrm{PGL}_3(\mathbb{C})) \rtimes (\mathbb{Z}/2\mathbb{Z})$

where $B(\mathrm{PGL}_3(\mathbb{C}))$ *stands for a Borel subgroup of* $\mathrm{PGL}_3(\mathbb{C})$*, in Cases* (c) *and* (d) *the generator of* $\mathbb{Z}/2\mathbb{Z}$ *acts by the inversion* $g \mapsto g^{-1}$ *on the first factor, and in Case* (e) *the group* $\mathrm{Aut}(\Sigma, X_C)$ *does not contain the product* $\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$*.*

Proof. (a) Let $C \in \{\mathrm{diag}(1,1,-2), \mathrm{diag}(1,-2,1), \mathrm{diag}(-2,1,1)\}$. The hyperplane section $X_{(1:1)} = \{x_2y_2 = 0\} \cap \Sigma$ has two irreducible components given by $x_2 = 0$ and $y_2 = 0$, respectively. Hence $X_{(1:1)}$ is a union $\mathcal{F}_1 \cup \mathcal{F}_2$ of the first kind (see 10.3), and similarly $X_{(1:-2)}$ and $X_{(-2:1)}$ are. By the spectral mapping theorem ([EL04]) one obtains

$$\mathrm{PCent}_{\widetilde{\mathrm{GL}_3(\mathbb{C})}}(C) = \mathrm{Cent}_{\widetilde{\mathrm{GL}_3(\mathbb{C})}}(C) \cong \widetilde{\mathrm{GL}_2(\mathbb{C})} \times \mathrm{z}(\widetilde{\mathrm{GL}_3(\mathbb{C})}).$$

Now the assertion follows from Lemma 11.3.

(b) For $C = \mathrm{diag}(1, \zeta, \zeta^2)$, where $\zeta \neq 1$ and $\zeta^3 = 1$, one has

$$\mathrm{PCent}_{\widetilde{\mathrm{GL}_3(\mathbb{C})}}(C) = \mathbb{T}(3) \rtimes ((\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})),$$

where $\mathbb{T}(3) = \mathrm{Cent}_{\mathrm{GL}_3(\mathbb{C})}(C)$ is the diagonal 3-torus of $\mathrm{GL}_3(\mathbb{C})$, the factor $\mathbb{Z}/3\mathbb{Z}$ is generated by the cyclic permutation matrix $\left(\begin{smallmatrix} 0&0&1\\ 1&0&0\\ 0&1&0 \end{smallmatrix}\right)$, and $\mathbb{Z}/2\mathbb{Z} = \langle \tau \rangle$ acts on $\mathbb{T}(3)$ by the inversion. Passing to the quotient by the center $\mathrm{z}(\widetilde{\mathrm{GL}_3(\mathbb{C})}) \subset \mathbb{T}(3)$ gives the result, see Lemma 11.3. One can easily check that X_C is smooth.

The proof of (c) is similar.

(d) For C_1 as in (11.2.5) one has

$$\mathrm{PCent}_{\widetilde{\mathrm{GL}_3(\mathbb{C})}}(C_1) = \mathrm{Cent}_{\mathrm{GL}_3(\mathbb{C})}(C_1) = \left\{ \begin{pmatrix} \lambda & a & 0\\ 0 & \lambda & 0\\ 0 & 0 & \mu \end{pmatrix}, \begin{pmatrix} a & \lambda & 0\\ \lambda & 0 & 0\\ 0 & 0 & \mu \end{pmatrix} \cdot \tau \;\middle|\; \lambda, \mu \in \mathbb{C}^*,\ a \in \mathbb{C} \right\}.$$

This group contains the center $z(\widetilde{\mathrm{GL}_3(\mathbb{C})}) \cong \mathbb{G}_{\mathrm{m}}$ and an element $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix}\right) \cdot \tau$ of order 2. The quotient by the center is isomorphic to $(\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}) \rtimes (\mathbb{Z}/2\mathbb{Z})$. The generator of the factor $\mathbb{Z}/2\mathbb{Z}$ acts by the inversion $g \mapsto g^{-1}$ on $\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$.

We use the notation $X_i = X_{C_i}$, $i = 1, 2, 3$, see (11.2.4). The hyperplane section $\mathfrak{G}_{X_1} \in |-K_{X_1}|$ as in Lemma 10.1(d) is given by $X_1 \cap H_{C_{(1,1)}}$. It is easy to see that $X_1 \cap H_{C_{(1,1)}}$ consists of four lines. Then by Lemma 10.1(d) the surface X_1 is a singular del Pezzo sextic of type A_1.

(e) Similarly, for C_2 as in (11.2.5) one has

$$\mathrm{PCent}_{\widetilde{\mathrm{GL}_3(\mathbb{C})}}(C_2) = \left\langle \begin{pmatrix} \mu^2\lambda & \mu a & b \\ 0 & \mu\lambda & a \\ 0 & 0 & \lambda \end{pmatrix}, \begin{pmatrix} b & \mu a & \mu^2\lambda \\ a & \mu\lambda & 0 \\ \lambda & 0 & 0 \end{pmatrix} \cdot \tau \,\middle|\, \lambda, \mu \in \mathbb{C}^*, \ a, b \in \mathbb{C} \right\rangle.$$

This group contains the center $z(\widetilde{\mathrm{GL}_3(\mathbb{C})}) = \{\lambda\,\mathrm{E}\}_{\lambda \in \mathbb{C}^*}$ and an element $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix}\right) \cdot \tau$ of order 2. The quotient $\mathrm{PCent}_{\widetilde{\mathrm{GL}_3(\mathbb{C})}}(C_2)/\,z(\widetilde{\mathrm{GL}_3(\mathbb{C})})$ is isomorphic to $((\mathbb{G}_{\mathrm{a}})^2 \rtimes \mathbb{G}_{\mathrm{m}}) \rtimes (\mathbb{Z}/2\mathbb{Z})$. None of the one-parameter unipotent subgroups is centralized by a $\mathbb{G}_{\mathrm{m}}$-subgroup, as stated. The corresponding surface X_2 contains exactly two lines given by $X_2 \cap H_{C_2^2}$. We conclude as in (d) that X_2 is an A_2-surface.

(f) The matrix C_3 is conjugate to C_2^2 with centralizer

$$\mathrm{PCent}_{\widetilde{\mathrm{GL}_3(\mathbb{C})}}(C_2^2) = \left\{ \begin{pmatrix} \lambda_0 & a_0 & a_1 \\ 0 & \lambda_1 & a_2 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \begin{pmatrix} a_1 & a_0 & \lambda_0 \\ a_2 & \lambda_1 & 0 \\ \lambda_2 & 0 & 0 \end{pmatrix} \cdot \tau \,\middle|\, \lambda_i \in \mathbb{C}^*, \, a_i \in \mathbb{C}, \ i = 0, 1, 2 \right\}.$$

Thus, $\mathrm{PCent}_{\widetilde{\mathrm{PGL}_3(\mathbb{C})}}(C_3) \cong B(\mathrm{PGL}_3(\mathbb{C})) \rtimes (\mathbb{Z}/2\mathbb{Z})$. By Lemma 11.3 the latter group is isomorphic to $\mathrm{Aut}(\Sigma, X_3)$. The hyperplane section $X_3 = \{x_1 y_0 = 0\} \cap \Sigma$ has two irreducible components given by $x_1 = 0$ and $y_0 = 0$, respectively. Hence X_3 is a union $\mathcal{F}_1 \cup \mathcal{F}_2$ of the second kind, see 10.3. □

11.5. *Proof of Proposition* 11.1. According to 11.2.2.4, the pair $(\Sigma(V), \Sigma_{\mathrm{s}}(V))$ is equivalent to one of the pairs $(\Sigma, X_{(a:b)})$ and (Σ, X_i), $i = 1, 2, 3$. The automorphism groups of the latter pairs are described in Lemma 11.4. By Corollary 11.2.1, $\mathrm{Aut}(V)$ embeds in one of this groups. On the other hand, due to Corollary 7.8.3 and Proposition 9.5, $\mathrm{Aut}(V)$ contains one of the groups $\mathrm{GL}_2(\mathbb{C})$, $\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$, and $(\mathbb{G}_{\mathrm{m}})^2$, which is not the case for $\mathrm{Aut}(\Sigma, X_2)$. Hence either $\mathrm{Aut}(V) \hookrightarrow \mathrm{Aut}(\Sigma, X_{(a:b)})$, or $\mathrm{Aut}(V) \hookrightarrow \mathrm{Aut}(\Sigma, X_i)$, $i \in \{1, 3\}$.

Inspecting Lemma 11.4 we see that, if $\mathrm{GL}_2(\mathbb{C}) \subset \mathrm{Aut}(V)$, then there is an embedding $\mathrm{Aut}(V) \hookrightarrow \mathrm{Aut}(\Sigma, X_{(1:1)}) \cong \widetilde{\mathrm{GL}_2(\mathbb{C})}$. Furthermore, in this case $\mathrm{Aut}^0(V) \cong \mathrm{GL}_2(\mathbb{C})$, see Lemma 6.4, and $\Sigma_{\mathrm{s}}(V)$ is a union $\mathcal{F}_1 \cup \mathcal{F}_2$ of the first kind, see Lemma 11.4 (a). The $\mathrm{GL}_2(\mathbb{C})$-action on $\Sigma_{\mathrm{s}}(V) = \mathcal{F}_1 \cup \mathcal{F}_2$ preserves two disjoint lines, which are the exceptional sections of the cubic scrolls $\mathcal{F}_i \cong \mathbb{F}_1$, $i = 1, 2$. These lines correspond to the families of rulings $\Lambda(S_i)$, $i = 1, 2$, where the cubic cones $S_1 \in \mathscr{S}_1(V)$ and $S_2 \in \mathscr{S}_2(V)$ are $\mathrm{Aut}^0(V)$-invariant and have no common ruling, see Proposition 9.10. Since $S_1 \cdot S_2 = 0$ in $H^*(V, \mathbb{Z})$, see Proposition 9.6, then $S_1 \cap S_2 = \varnothing$ due to Corollary 9.7.4. This corresponds to Case 1^o of Proposition 11.1.

Suppose now that $\mathrm{rk}\,\mathrm{Aut}(V) \geq 2$, but $\mathrm{GL}_2(\mathbb{C}) \not\subset \mathrm{Aut}(V)$. By Lemma 11.4 one of the following holds: either, for a suitable $(a : b) \in \mathbb{P}^1$,

$$(\mathbb{G}_{\mathrm{m}})^2 \subset \mathrm{Aut}(V) \subset \mathrm{Aut}(\Sigma, X_{(a:b)}) \subset (\mathbb{G}_{\mathrm{m}})^2 \times (\mathbb{Z}/6\mathbb{Z}),$$

or

$$(11.5.1) \qquad (\mathbb{G}_m)^2 \subset \mathrm{Aut}(V) \subset \mathrm{Aut}(\Sigma, X_3) = B(\mathrm{PGL}_3(\mathbb{C})) \rtimes (\mathbb{Z}/2\mathbb{Z}).$$

In the latter case, by Lemma 11.4 (d) one has $\Sigma_s(V) \cong X_3 = Y_{(0:1)} = \mathcal{F}_1 \cup \mathcal{F}_2$ is a union of the second kind. However, the latter contradicts Lemma 10.3.1. In the former case, by Lemma 11.4 (b) (c), $\Sigma_s(V)$ is a smooth del Pezzo sextic. This corresponds to Case 3^o of Proposition 11.1.

Assume further that $\mathrm{rk}\,\mathrm{Aut}(V) = 1$. By Lemma 11.4 either (11.5.1) holds and we are in Case 4^o, or there are embeddings

$$\mathbb{G}_a \times \mathbb{G}_m \hookrightarrow \mathrm{Aut}(V) \hookrightarrow \mathrm{Aut}(\Sigma, X_1) = (\mathbb{G}_a \times \mathbb{G}_m) \rtimes (\mathbb{Z}/2\mathbb{Z}),$$

and $\Sigma_s(V) \cong X_1$ is a normal del Pezzo sextic of type A_1, see Lemma 11.4(d). The latter corresponds to Case 2^o of Proposition 11.1. □

11.5.2. **Remark.** According to Lemma 11.4(b), for the matrix $C = \mathrm{diag}(1, \zeta, \zeta^2)$, where ζ is a primitive cubic root of unity, one has $\mathrm{Aut}(\Sigma, X_C) \cong (\mathbb{G}_m)^2 \times (\mathbb{Z}/6\mathbb{Z})$. However, we do not know whether this group can be realized as the automorphism group $\mathrm{Aut}(V)$ for some Fano-Mukai fourfold $V = V_{18}$. In other words, we ignore whether X_C is equivalent to some $\Sigma_s(V)$ under the $\mathrm{Aut}(\Sigma)$-action on Σ.

12. Automorphism groups of V_{18}

Let $V = V_{18}$ be a Fano-Mukai fourfold of genus 10. By Proposition 11.1, $\mathrm{Aut}^0(V)$ is one of the groups $\mathrm{GL}_2(\mathbb{C})$, $\mathbb{G}_a \times \mathbb{G}_m$, and $(\mathbb{G}_m)^2$. The central result of this section is the following theorem.

12.1. **Theorem.** *Given an* $\mathrm{Aut}^0(V)$*-invariant cubic cone* $S \subset V$*, consider the pair* (W, F_S) *linked to* (V, S)*, and let* $J_S = F_S \cap \Xi$*. With this notation, Case* 4^o *of Proposition* 11.1 *does not occur, that is,* $\Sigma_s(V)$ *cannot be a union* $\mathcal{F}_1 \cup \mathcal{F}_2$ *of the second type. Moreover, one of the following possibilities occurs:*

	$\mathrm{Aut}^0(V)$	V	$\Sigma_s(V)$	(Υ, J_S)
(i)	$\mathrm{GL}_2(\mathbb{C})$	V_{18}^{s}	$\mathcal{F}_1 \cup \mathcal{F}_2$ of the first kind	$J_S = \Upsilon$
(ii)	$\mathbb{G}_a \times \mathbb{G}_m$	V_{18}^{a}	an A_1-del Pezzo sextic	4.3(ii) for a suitable choice[5] of S
(iii)	$(\mathbb{G}_m)^2$	$\not\cong V_{18}^{\mathrm{s}}, V_{18}^{\mathrm{a}}$	a smooth del Pezzo sextic	4.3(iii) for any choice of S

For the proof of the first assertion see Corollary 12.5.2. Then the equivalence between the first and the third properties in (i)–(iii) follows from Proposition 11.1. The proof of the remaining statements is done in 12.9 after a certain preparation and several auxiliary results.

12.2. **Lemma.** *Suppose that* $\mathrm{Aut}(V) \supset \mathrm{SL}_2(\mathbb{C})$*. Then any component* $\mathscr{S}_i(V)$*,* $i = 1, 2$*, of* $\mathscr{S}(V)$ *contains a unique* $\mathrm{Aut}^0(V)$*-fixed point, and this point corresponds to an* $\mathrm{Aut}^0(V)$*-invariant cubic cone* $S_i \in \mathscr{S}_i(V)$*. Furthermore, one has* $S_1 \cap S_2 = \varnothing$*.*

[5]The cubic cones $S \subset V$ such that (Υ, J_S) is of type 4.3(ii) correspond actually to the two outer (-1)-vertices in diagram (10.1.1). The two inner (-1)-vertices in (10.1.1) correspond to the cubic cones $S \subset V$ with (Υ, J_S) of type 4.3(iii); see Remark 13.3.2.

Proof. The existence of an $\mathrm{Aut}^0(V)$-fixed point in $\mathscr{S}_i(V)$ that corresponds to a cubic cone is established in Proposition 9.5. We claim that an $\mathrm{SL}_2(\mathbb{C})$-fixed point in $\mathscr{S}_i(V)$ is unique; then, of course, this point is a unique $\mathrm{Aut}^0(V)$-fixed point as well.

Suppose to the contrary that the $\mathrm{SL}_2(\mathbb{C})$-action on $\mathscr{S}_i(V) \cong \mathbb{P}^2$ admits at least two fixed points. Since the group $\mathrm{SL}_2(\mathbb{C})$ is simply connected, the induced $\mathrm{SL}_2(\mathbb{C})$-action on $\mathbb{P}^2$ can be lifted to a linear $\mathrm{SL}_2(\mathbb{C})$-action on $\mathbb{C}^3$ trivial on the one-dimensional subspaces which correspond to the fixed points. Due to the complete reducibility, such an $\mathrm{SL}_2(\mathbb{C})$-action on $\mathbb{C}^3$ is trivial. This is a contradiction.

By Lemma 6.4, under our assumptions one has $\mathrm{Aut}^0(V) = \mathrm{GL}_2(\mathbb{C})$.

Due to the uniqueness, the unordered pair (S_1, S_2) coincides with the pair (S, S') of $\mathrm{Aut}^0(V)$-invariant cubic cones as in Lemma 6.6.1(a). However, the latter cones are disjoint. [6] □

12.3. **Lemma.** *Assume that* $\Sigma_\mathrm{s}(V) = \mathcal{F}_1 \cup \mathcal{F}_2$. *For* $i = 1, 2$ *let* $S_i \in \mathscr{S}_i(V)$ *be the* $\mathrm{Aut}^0(V)$-*invariant cubic cone provided by the exceptional section of the ruling* $\mathcal{F}_j \to \mathbb{P}^1$, $j \neq i$, *see Corollary* 10.3.2(b). *Let* (W, F) *be the pair linked to* (V, S_2). *Suppose that* $\mathrm{Aut}^0(W, F)$ *contains a singular torus of* $\mathrm{Aut}^0(W)$. *Then the following hold.*

(a) $(\mathcal{F}_1, \mathcal{F}_2)$ *is a pair of the first kind;*
(b) $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F) = \{1\}$, *and* $\mathrm{Aut}^0(V)$ *is isomorphic to one of the groups* $\mathrm{GL}_2(\mathbb{C})$, $\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$, *and* $(\mathbb{G}_\mathrm{m})^2$.

Proof. (a) By Lemma 6.6.1(a) there exists a cubic cone $S' \subset V$ disjoint with S_2. Since $S' \cdot S_2 = 0$, these cubic cones belong to different components of $\mathscr{S}(V)$, see Proposition 9.6. Thus, $S' \in \mathscr{S}_1(V)$, and so, $\Lambda(S') \subset \Sigma_\mathrm{s}(V)$ is either a ruling of $\mathcal{F}_1$, or the exceptional section of $\mathcal{F}_2$. However, if $(\mathcal{F}_1, \mathcal{F}_2)$ were a pair of the second kind, then in both cases S' and S_2 would possess a common ruling, a contradiction. This proves (a).

(b) Therefore, $(\mathcal{F}_1, \mathcal{F}_2)$ is a pair of the first kind. Since the cubic cone S_1 is $\mathrm{Aut}^0(V)$-invariant, its vertex v_1 is fixed under the action of $\mathrm{Aut}^0(V)$ on V. By Corollary 10.3.2(b), $v_1 \notin S_2$. It follows that $\theta(v_1) \notin R$, see diagram (3.1.1), where $B = R$. The projection $\theta : V \dashrightarrow W$ with center $\langle S_2 \rangle$ as in diagram (3.1.1) sends v_1 to a fixed point $\theta(v_1)$ of $\mathrm{Aut}^0(W, F)$. However, by Proposition 5.6(c) the unipotent radical $\mathrm{R_u}$ of $\mathrm{Aut}^0(W)$ acts freely in $W \setminus R$. Thus, $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F) = \{1\}$. The last assertion follows from Corollary 6.3.1(a)-(b). □

12.3.1. **Corollary.** *Suppose that* $\Sigma_\mathrm{s}(V) = \mathcal{F}_1 \cup \mathcal{F}_2$ *is a pair of the second kind. Let* (W, F) *be linked to* (V, S), *where* $S \subset V$ *is an* $\mathrm{Aut}^0(V)$-*invariant cubic cone. Then the following hold.*

- $\mathrm{rk}\,\mathrm{Aut}(V) = 1$;
- $\mathrm{Aut}(V) \supset \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$;
- *Any* $\mathbb{G}_\mathrm{m}$-*subgroup of* $\mathrm{Aut}(W, F)$ *acts nontrivially on* $J = F \cap \Xi$.

Proof. Suppose to the contrary that $\mathrm{rk}\,\mathrm{Aut}(V) = 2$. Then $\mathrm{Aut}^0(W, F)$ contains a singular torus of $\mathrm{Aut}^0(W)$, and so, by Lemma 12.3(a), $(\mathcal{F}_1, \mathcal{F}_2)$ is a pair of the first kind, a contradiction.

[6] Alternatively, one can notice that the cones S_1 and S_2 correspond to the disjoint exceptional sections of the components $\mathcal{F}_1$ and $\mathcal{F}_2$ of $\Sigma_\mathrm{s}(V)$. It follows that $S_1 \cap S_2$ is zero-dimensional, hence empty since $S_1 \cdot S_2 = 0$ in $H^*(V, \mathbb{Z})$, see Proposition 9.6 and Corollary 9.7.4.

Since $\mathrm{Aut}(V)$ of rank 1 contains one of the groups $\mathrm{GL}_2(\mathbb{C})$, $\mathbb{G}_\mathrm{a}\times\mathbb{G}_\mathrm{m}$, $(\mathbb{G}_\mathrm{m})^2$, it contains $\mathbb{G}_\mathrm{a}\times\mathbb{G}_\mathrm{m}$.

Suppose that there is a $\mathbb{G}_\mathrm{m}$-subgroup, say, $Z\subset\mathrm{Aut}(W,F)$ acting trivially on J. Then Z is contained in the kernel $\mathrm{R_u}\rtimes\mathrm{z(L)}$ of the homomorphism $\varrho:\mathrm{Aut}(W,F)\to\mathrm{PGL}_2(\mathbb{C})$ in (5.4.1). Hence Z is a singular torus of $\mathrm{Aut}(W)$. Since F is Z-invariant, this leads again to a contradiction with Lemma 12.3(a). □

12.4. **Lemma.** *Let $S\subset V$ be a cubic cone, let (W,F) be linked to (V,S), and let $J=F\cap\Xi$ be the exceptional section of F. Then the following hold.*

(a)

$$\dim\mathrm{Aut}^0(W,F)=\dim\mathrm{Aut}^0(V,S)\geq\begin{cases}2 & \textit{if}\quad J\neq\Upsilon,\\ 4 & \textit{if}\quad J=\Upsilon.\end{cases}\tag{12.4.1}$$

(b) *Let $\mathfrak{F}_J$ be the family of all rational normal quintic scrolls $F'\subset R$ with exceptional section J. Then the equality in* (12.4.1) *holds if and only the orbit of $F\in\mathfrak{F}_J$ under the natural* $\mathrm{Stab}_{\mathrm{Aut}(W)}(J)$*-action on $\mathfrak{F}_J$ is open.*

Proof. We claim that $\mathfrak{F}_J$ is of pure dimension 4. Indeed, by a deformation argument the family of twisted cubics in $R\setminus\Xi=R\setminus\mathrm{Sing}(R)$ has dimension 6. Given a scroll $F\in\mathfrak{F}_J$, the twisted cubic curves Ψ contained in F are the sections of $F\to\mathbb{P}^1$ disjoint with J, that is, the sections with $\Psi^2=1$. Therefore, the family of twisted cubic curves $\Psi\subset F$ is two-dimensional. Now the claim follows.

The group $\mathrm{Aut}(\Upsilon)\cong\mathrm{PGL}_2(\mathbb{C})$ acts naturally on the plane $\Xi\cong\mathbb{P}^2$ viewed as the symmetric square of $\Upsilon\cong\mathbb{P}^1$. By Lemma 6.2(a) one has

$$\dim\mathrm{Stab}_{\mathrm{Aut}(\Xi,\Upsilon)}(J)=\begin{cases}1 & \text{if}\quad J\neq\Upsilon,\\ 3 & \text{if}\quad J=\Upsilon.\end{cases}$$

The stabilizer $\mathcal{G}:=\mathrm{Stab}_{\mathrm{Aut}(W)}(J)$ acts on $\mathfrak{F}_J$. From (5.4.1) and Proposition 5.6(c) one can deduce the equalities

$$\dim\mathcal{G}=5+\dim\mathrm{Stab}_{\mathrm{Aut}(\Xi,\Upsilon)}(J)=\begin{cases}6 & \text{if}\quad J\neq\Upsilon,\\ 8 & \text{if}\quad J=\Upsilon.\end{cases}\tag{12.4.2}$$

Notice that $\mathrm{Aut}(W,F)$ is the stabilizer of $F\in\mathfrak{F}_J$ under the $\mathcal{G}$-action on $\mathfrak{F}_J$. For the dimension of the orbit $\mathcal{G}.F$ of F under this action one has

$$\dim\mathcal{G}.F=\dim\mathcal{G}-\dim\mathrm{Stab}_{\mathcal{G}}(F)=\dim\mathcal{G}-\dim\mathrm{Aut}^0(W,F)\leq\dim\mathfrak{F}_J=4.\tag{12.4.3}$$

Then (12.4.2) and (12.4.3) imply (12.4.1). This gives (a). Now (b) is straightforward. □

12.5. **Lemma.** *Suppose that $\Sigma_\mathrm{s}(V)$ is a union $\mathcal{F}_1\cup\mathcal{F}_2$ of the second kind. Then* $\mathrm{Aut}^0(V)\cong\mathbb{G}_\mathrm{a}\times\mathbb{G}_\mathrm{m}$.

Proof. According to Proposition 9.5 there exists an $\mathrm{Aut}^0(V)$-invariant cubic cone $S\subset V$. Let (W,F_S) be the pair linked to (V,S), and let $J_S=F_S\cap\Xi$. Assume to the contrary that $\mathrm{Aut}^0(V)\cong\mathrm{Aut}^0(W,F_S)\not\cong\mathbb{G}_\mathrm{a}\times\mathbb{G}_\mathrm{m}$. By Corollary 12.3.1, $\mathrm{Aut}^0(W,F_S)\cong\mathrm{Aut}^0(V)$ has rank 1, and any $\mathbb{G}_\mathrm{m}$-subgroup of $\mathrm{Aut}^0(W,F_S)$ acts nontrivially on J_S. Hence either $J_S=\Upsilon$ is of type 4.3(i), or (Υ,J_S) is of type 4.3(iii). Furthermore, by Corollary 12.3.1, $\mathrm{Aut}^0(W,F_S)$ contains $\mathbb{G}_\mathrm{a}\times\mathbb{G}_\mathrm{m}$.

Suppose first that $J_S = \Upsilon$. Consider the unipotent radical $\mathrm{R_u}(\mathrm{Aut}^0(W, F_S))$ of the solvable group $\mathrm{Aut}^0(W, F_S) \cong \mathrm{Aut}^0(V) \subset B(\mathrm{GL}_3(\mathbb{C}))/\,\mathrm{z}(\mathrm{GL}_3(\mathbb{C}))$ of rank 1, see Case 4^o of Proposition 11.1. By Lemma 12.4, in our case $\dim \mathrm{Aut}^0(W, F_S) \geq 4$. Hence $\dim \mathrm{R_u}(\mathrm{Aut}^0(W, F_S)) \geq 3$. The image of $\mathrm{R_u}(\mathrm{Aut}^0(W, F_S))$ in $\mathrm{Aut}(\Upsilon) = \mathrm{PGL}_2(\mathbb{C})$ is contained in the unipotent radical $\mathrm{R_u}(B(\mathrm{PGL}_2(\mathbb{C})) \cong \mathbb{G}_\mathrm{a}$ of a Borel subgroup. The exact sequence (5.4.1) reads:

$$1 \longrightarrow \mathrm{R_u} \rtimes \mathrm{z}(\mathrm{L}) \longrightarrow \mathrm{Aut}(W) \xrightarrow{\varrho} \mathrm{Aut}(\Upsilon) = \mathrm{PGL}_2(\mathbb{C}) \longrightarrow 1. \qquad (12.5.1)$$

It follows that the kernel of $\varrho|_{\mathrm{R_u}(\mathrm{Aut}^0(W,F_S))}$ has dimension ≥ 2 and is contained in $\mathrm{R_u}$. Therefore, $\dim(\mathrm{R_u} \cap \mathrm{Aut}^0(W, F_S)) \geq 2$. The latter contradicts Corollary 6.7.3.

Thus, (Υ, J_S) is a pair of type 4.3(iii), $\varrho(\mathrm{Aut}^0(W, F_S)) = \mathrm{Aut}^0(\Upsilon, J_S) \cong \mathbb{G}_\mathrm{m}$, and $\ker(\varrho|_{\mathrm{Aut}^0(W,F_S)}) \cong \mathbb{G}_\mathrm{a}$, see Corollary 6.7.3. Finally, one has $\mathrm{Aut}^0(V) \cong \mathrm{Aut}^0(W, F_S) \cong \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$. □

12.5.2. **Corollary.** *Case* 4^o *of Proposition* 11.1 *does not occur.*

Proof. Assume to the contrary that $\Sigma_\mathrm{s}(V)$ is a union $\mathcal{F}_1 \cup \mathcal{F}_2$ of the second kind. By Lemma 12.5 we have $\mathrm{Aut}^0(V) \cong \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$. This implies that any cubic cone S in V is $\mathrm{Aut}^0(V)$-invariant. Indeed, notice that $\mathrm{Aut}(V, S) = \mathrm{Aut}(V, v(S))$ is the stabilizer in $\mathrm{Aut}(V)$ of the vertex $v(S)$ of S, see Proposition 8.2(e). If S were not $\mathrm{Aut}^0(V)$-invariant, then the $\mathrm{Aut}^0(V)$-orbit of $v(S)$ would be a curve in V, see Lemma 9.4. Hence the stabilizer of S in $\mathrm{Aut}^0(V)$ would have codimension 1, so $\dim \mathrm{Aut}^0(V, S) = 1$. The latter contradicts (12.4.1). Thus, the cubic cones in V are $\mathrm{Aut}^0(V)$-invariant.

It follows that the $\mathrm{Aut}^0(V)$-action on $\Sigma_\mathrm{s}(V)$ preserves any ruling $f_{i,t}$ of $\mathcal{F}_i \to \mathbb{P}^1$, $i = 1, 2$, see Proposition 9.10(a). If the factor $\mathbb{G}_\mathrm{m}$ of $\mathrm{Aut}^0(V) = \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$ acts non-trivially on a general ruling $f_{i,t}$ of $\mathcal{F}_i$, then its two fixed points must be fixed by the $\mathbb{G}_\mathrm{a}$-subgroup. Anyway, at least one of the factors $\mathbb{G}_\mathrm{a}$ and $\mathbb{G}_\mathrm{m}$ of $\mathrm{Aut}^0(V) \cong \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$ acts trivially on $\mathcal{F}_i$. The latter contradicts Lemma 11.2. □

We can remove now an extra assumption in Lemma 6.4(iii).

12.5.3. **Corollary.** *Let S be an* $\mathrm{Aut}^0(V)$*-invariant cubic cone in V, and (W, F_S) be the pair linked to (V, S). Let $J_S = F_S \cap \Xi$. Then* $\mathrm{Aut}^0(V) \cong \mathrm{GL}_2(\mathbb{C})$ *if and only if $J_S = \Upsilon$.*

Proof. The "only if" part follows from Lemma 6.4. The "if" part follows from Proposition 11.1 and Corollary 12.5.2. □

In the next lemma we examine the cubic cones in V which are not $\mathrm{Aut}^0(V)$-invariant.

12.6. **Lemma.** *Let $S \subset V$ be a cubic cone, and let (W, F_S) be the pair linked to (V, S). Let $J_S = F_S \cap \Xi$. Assume that S is not* $\mathrm{Aut}^0(V)$*-invariant. Then* $\mathrm{Aut}^0(V) \cong \mathrm{GL}_2(\mathbb{C})$ *and (Υ, J_S) is a pair of type* 4.3(iii)*, see Figure* 4.3.1*. Furthermore,*

$$\mathrm{Aut}^0(V, S) \cong \mathrm{Aut}^0(W, F_S) \cong \mathbb{G}_\mathrm{a} \rtimes (\mathbb{G}_\mathrm{m})^2 \cong (\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}) \rtimes \mathbb{G}_\mathrm{m}, \qquad (12.6.1)$$

where $\mathbb{G}_\mathrm{a} = \mathrm{R_u} \cap \mathrm{Aut}^0(W, F_S)$*, a non-abelian subgroup* $\mathbb{G}_\mathrm{a} \rtimes \mathbb{G}_\mathrm{m} \subset \mathrm{Aut}^0(V, S)$ *acts on F_S preserving the rulings, and the* $\mathbb{G}_\mathrm{m}$*-subgroup centralizing the* $\mathbb{G}_\mathrm{a}$*-subgroup acts effectively on J_S.*

Proof. By Proposition 11.1 and Corollary 12.5.2 one has $\Sigma_\mathrm{s}(V) = \mathcal{F}_1 \cup \mathcal{F}_2$, where $(\mathcal{F}_1, \mathcal{F}_2)$ is a pair of the first kind, and $\mathrm{Aut}^0(V) \cong \mathrm{GL}_2(\mathbb{C})$. By Lemma 12.2 there are exactly two $\mathrm{Aut}^0(V)$-invariant cubic cones S_1, S_2 in V; these correspond to the exceptional sections of $\mathcal{F}_1$ and $\mathcal{F}_2$. Under the $\mathrm{Aut}^0(V)$-action, the cone $S \neq S_1, S_2$ varies in a one-parameter family of cubic cones, which correspond to the rulings of a scroll $\mathcal{F}_i$, $i \in \{1,2\}$. It follows that $\dim \mathrm{Aut}^0(V,S) = \dim \mathrm{Aut}^0(V) - 1 = 3$. By Corollary 12.5.3, $J_S \neq \Upsilon$. Since $\mathrm{Aut}^0(V)/\mathrm{Aut}^0(V,S) \cong \mathbb{P}^1$, then $\mathrm{Aut}^0(V,S) =: B$ is a Borel subgroup of $\mathrm{Aut}^0(V) \cong \mathrm{GL}_2(\mathbb{C})$. Hence (12.6.1) holds, and $\mathrm{Aut}^0(W, F_S) \cong B$ contains a singular torus of $\mathrm{Aut}(W)$.

The maximal torus of $\mathrm{Aut}^0(W, F_S)$ acts on J_S non-identically with exactly two fixed points. It follows by Lemma 6.2(a) that $J_S \neq \Upsilon$ is of type 4.3(iii), and the $\mathbb{G}_\mathrm{a}$-subgroup of $\mathrm{Aut}^0(W, F_S)$ acts identically on J_S. Hence it preserves each ruling of F_S. In the notation of Corollary 6.3.1 one has $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F_S) \cong \mathbb{G}_\mathrm{a}$ and $G \cong (\mathbb{G}_\mathrm{m})^2$. Now the remaining assertions are immediate. $\square$

12.7. **Lemma.** *Let $S \subset V$ be a cubic cone, let (W, F_S) be the pair linked to (V,S), and let $J_S = F_S \cap \Xi$. Assume that (Υ, J_S) is of type* 4.3(ii) *and F_S is Z-invariant, where $Z \subset \mathrm{Aut}(W)$ is a singular torus. Then the cone S is $\mathrm{Aut}^0(V)$-invariant, $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F_S) = \{1\}$, and $\mathrm{Aut}^0(V) \cong \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$.*

Proof. Due to Corollary 6.3.1(a)-(b) and its proof one has

$$\mathrm{Aut}^0(V,S) \cong \mathrm{Aut}^0(W,F_S) \cong (\mathrm{R_u} \cap \mathrm{Aut}^0(W,F_S)) \rtimes (\mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}), \tag{12.7.1}$$

where the $\mathbb{G}_\mathrm{m}$-subgroup Z preserves the rulings of F_S and the $\mathbb{G}_\mathrm{a}$-subgroup centralized by the $\mathbb{G}_\mathrm{m}$-subgroup acts effectively on J_S. In particular, $\mathrm{rk}\,\mathrm{Aut}^0(V,S) = 1$.

If the cone S is $\mathrm{Aut}^0(V)$-invariant, then $\mathrm{rk}\,\mathrm{Aut}^0(V) = 1$, and so, $\mathrm{Aut}^0(V) \cong \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$ by Proposition 11.1. Hence $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F_S) = \{1\}$ due to (12.7.1).

Otherwise, $\dim \mathrm{Aut}^0(V) = \dim \mathrm{Aut}^0(V,S) + 1 \geq 3$. Hence by Proposition 11.1 $\mathrm{Aut}^0(V) = \mathrm{GL}_2(\mathbb{C})$. So, Lemma 12.6 applies to the pair (V,S). According to this lemma, $\mathrm{rk}\,\mathrm{Aut}^0(V,S) = 2$ contrary to (12.7.1). $\square$

12.8**.** We use the notation V_{18}^{s} and V_{18}^{a} introduced in 7.8.4. Let as before $\mathfrak{g}_2 = \mathrm{Lie}(\mathrm{G}_2)$. Recall that $V_{18}^{\mathrm{s}} = V^{g_\mathrm{s}}$ ($V_{18}^{\mathrm{a}} = V^{g_\mathrm{a}}$, respectively) for a singular semisimple element $g_\mathrm{s} \in \mathfrak{g}_2$ (a regular non-semisimple element $g_\mathrm{a} \in \mathfrak{g}_2$, respectively). Due to Proposition 7.7(b), such nonzero elements g_s and g_a do exist, and their images $[g_\mathrm{s}]$ and $[g_\mathrm{a}]$ in $\mathbb{P}(\mathfrak{g}_2)$ form two distinct orbits of the induced $\mathrm{Ad}(\mathrm{G}_2)$-action on $\mathbb{P}(\mathfrak{g}_2)$. Therefore, the Fano-Mukai fourfolds V_{18}^{s} and V_{18}^{a} of genus 10 do exist and are unique up to isomorphism. By Corollary 7.8.3 one has $\mathrm{Aut}^0(V_{18}^{\mathrm{s}}) \cong \mathrm{GL}_2(\mathbb{C})$ and $\mathrm{Aut}^0(V_{18}^{\mathrm{a}}) \supset \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$. Moreover, the following hold.

12.8.1. **Lemma.** *Assume that $V \cong V_{18}^{\mathrm{a}}$. Then $\mathrm{Aut}^0(V) = \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$, and any cubic cone $S \subset V$ is $\mathrm{Aut}^0(V)$-invariant. Let (W, F_S) be the pair linked to (V,S). Then, for a suitable choice of S, $J_S = F_S \cap \Xi$ is of type* 4.3(ii)*, and F_S is invariant under a singular torus $\mathrm{z(L)} \subset \mathrm{Aut}(W)$.*

Proof. Fix a smooth conic $J' \subset \Xi$ of type 4.3(ii) and a Levi subgroup L of $\mathrm{Aut}(W)$. By Proposition 6.3(a) there exists a $\mathrm{z(L)}$-invariant quintic scroll $F' \subset R$ such that $J' = F' \cap \Xi$. Let (V', S') be the pair linked to (W, F'). Then by Lemma 12.7 the cone S' is $\mathrm{Aut}^0(V')$-invariant, and $\mathrm{Aut}^0(V') \cong \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$.

By Theorem 7.1, $V' = V_{18}^{g}$ for some $g \in \mathfrak{g}_2$. By Corollary 7.8.3, g is regular non-semisimple. Hence $V' \cong V_{18}^{\mathrm{a}} \cong V$, and so, $\mathrm{Aut}(V) \cong \mathrm{Aut}(V') \cong \mathbb{G}_\mathrm{a} \times \mathbb{G}_\mathrm{m}$. We identify V and V' via this isomorphism.

By Proposition 11.1, $\Sigma_\mathrm{s}(V)$ is irreducible and contains exactly 4 lines. Therefore, V contains exactly 4 cubic cones, which are all $\mathrm{Aut}^0(V)$-invariant. By construction, the cone $S \subset V$ which corresponds to $S' \subset V'$ satisfies the conditions of the lemma. $\square$

12.9. *Proof of Theorem* 12.1. Corollary 12.5.3 yields the equivalence

$$J_S = \Upsilon \Longleftrightarrow \mathrm{Aut}^0(V) \cong \mathrm{GL}_2(\mathbb{C}).$$

If $V \cong V_{18}^{\mathrm{s}}$, then $\mathrm{Aut}^0(V) \cong \mathrm{GL}_2(\mathbb{C})$ by Corollary 7.8.3(i). Conversely, suppose that $\mathrm{Aut}^0(V) \cong \mathrm{GL}_2(\mathbb{C})$. By Theorem 7.1, $V = V_{18}^{g}$ for some $g \in \mathfrak{g}_2$, where g is singular semisimple, see Proposition 7.5.1(a)–(c). Hence $V \cong V_{18}^{\mathrm{s}}$, see Proposition 7.7. This yields the equivalence

$$\mathrm{Aut}^0(V) \cong \mathrm{GL}_2(\mathbb{C}) \quad \Longleftrightarrow \quad V \cong V_{18}^{\mathrm{s}}. \tag{12.9.1}$$

Combining with Proposition 11.1 this ends the proof of 12.1(i).

Suppose further that $\mathrm{Aut}^0(V) \cong \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$, and let $V = V_{18}^{g}$ for some $g \in \mathfrak{g}_2$. By Proposition 7.5.1 and Corollary 7.8.3, g is regular non-semisimple, that is, $V \cong V_{18}^{\mathrm{a}}$. By Lemma 12.8.1 this gives the equivalence

$$\mathrm{Aut}^0(V) \cong \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}} \quad \Longleftrightarrow \quad V \cong V_{18}^{\mathrm{a}}. \tag{12.9.2}$$

From (12.9.1) and (12.9.2) we deduce

$$\mathrm{Aut}^0(V) \cong (\mathbb{G}_{\mathrm{m}})^2 \quad \Longleftrightarrow \quad V \not\cong V_{18}^{\mathrm{s}}, V_{18}^{\mathrm{a}}.$$

If $\mathrm{Aut}^0(V) \cong (\mathbb{G}_{\mathrm{m}})^2$, then the pair (Υ, J_S) is of type 4.3(iii) for any cubic cone $S \subset V$. Thus, if for some $\mathrm{Aut}^0(V)$-invariant cubic cone $S \subset V$ the corresponding pair (Υ, J_S) is of type 4.3(ii), then $\mathrm{rk}\,\mathrm{Aut}^0(V) = 1$ and $\mathrm{Aut}^0(V) \cong \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$. By Lemma 12.8.1 we have the converse implication, and so, the equivalence

$$\mathrm{Aut}^0(V) \cong \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}} \quad \Longleftrightarrow \quad (\Upsilon, J_S) \text{ is of type 4.3(ii) for a cubic cone } S \subset V.$$

Together with Proposition 11.1 and Corollary 12.5.2 this proves 12.1(ii).

From 12.1(i)–12.1(ii) we deduce the equivalence

$$\mathrm{Aut}^0(V) \cong (\mathbb{G}_{\mathrm{m}})^2 \quad \Longleftrightarrow \quad (\Upsilon, J_S) \text{ is of type 4.3(iii) for any cubic cone } S \subset V.$$

Due to Proposition 11.1 and Corollary 12.5.2 this completes the proof of 12.1(iii). □

12.9.3. **Remark.** By virtue of Proposition 6.3(b), for a conic $J \subset \Xi$ touching Υ with even multiplicities and a Levi subgroup L of $\mathrm{Aut}(W)$ there is a unique z(L)-invariant scroll $F \subset R$ with $J = F \cap \Xi$ if $J = \Upsilon$, and exactly two distinct such scrolls otherwise. Since the singular tori in $\mathrm{Aut}(W)$ are conjugated under the $\mathrm{R_u}$-action, the quintic scrolls $F \subset R$ with given $J = F \cap R$ are equivalent under the $\mathrm{R_u}$-action on W up to passing to the "conjugate" scroll in the case $J \neq \Upsilon$. The variety $\mathcal{F}_J$ of such rational quintic scrolls is isomorphic to $\mathbb{C}^4$ if $J = \Upsilon$. For J of type 4.3(ii), $\mathcal{F}_J$ consists of two disjoint components isomorphic to $\mathbb{C}^4$. For J of type 4.3(iii), $\mathcal{F}_J$ has two disjoint components isomorphic to $\mathbb{C}^4$, and then eventually also some number of lower-dimensional components. Anyway, the automorphism $\widetilde{\varkappa}$ as in Proposition 6.3 (d) interchanges these two $\mathbb{C}^4$-components. Cf. also Lemma 12.4(b).

13. Proofs of the main theorems and beyond

In this section we prove Theorems 1.1 – 1.3 from the Introduction. Besides, Remark 13.4 and Theorem 13.5 complement the main results.

13.1**.** *Proof of Theorem* 1.1. By Proposition 9.5, V contains two distinct $\mathrm{Aut}^0(V)$-invariant cubic cones $S_i \in \mathscr{S}_i(V)$, $i = 1, 2$. To the pair (V, S_i) there corresponds an $\mathrm{Aut}^0(V)$-equivariant Sarkisov link (3.1.1). Now Theorem 1.1 follows immediately from Corollary 4.5.2. □

13.2**.** *Proof of Theorem* 1.2. By Proposition 7.7 and Theorem 12.1, for any $g \in \mathfrak{g}_2$ such that $[g] \notin D_\ell$ one has $\mathrm{Aut}^0(V^g) = \mathrm{Stab}_{\mathrm{G}_2}(g)^0$. Now the result follows. □

From the proof we deduce the following corollary (cf. 15.1 below).

13.2.1. **Corollary.** *For each* $g \in \mathfrak{g}_2$ *such that* $[g] \notin D_\ell$ *the group* $\mathrm{Aut}^0(V^g)$ *is the identity component of the stabilizer of* V^g *in* $\mathrm{Aut}^0(\Omega) \cong \mathrm{G}_2$.

13.3**.** *Proof of Theorem* 1.3. The existence and the uniqueness in Theorem 1.3(i) and (ii) follow from 12.8 by virtue of Theorem 12.1. Since $\mathrm{Aut}^0(V_{18}^{\mathrm{s}}) \cong \mathrm{GL}_2(\mathbb{C})$ comes from a Levi subgroup $\mathrm{L} \subset \mathrm{Aut}(W)$, and L acts on W with a principal open orbit, see Proposition 5.5(c), then the latter holds as well for the induced $\mathrm{GL}_2(\mathbb{C})$-action on V_{18}^{s}. As for the description of the fixed points in (i) see Theorem 13.5(d) below. Inclusions (1.3.3) follow from Proposition 11.1 and Theorem 12.1. Let us show (1.3.1) and (1.3.2).

Suppose that $\mathrm{Aut}^0(V) \cong \mathrm{GL}_2(\mathbb{C})$, that is, $V \cong V_{18}^{\mathrm{s}}$. We claim that there is an exact sequence

$$1 \longrightarrow \mathrm{Aut}^0(V) \longrightarrow \mathrm{Aut}(V) \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow 0. \tag{13.3.1}$$

Indeed, let (S_1, S_2) be the unique pair of $\mathrm{Aut}^0(V)$-invariant cubic cones, see Lemma 12.2. By Corollary 6.4.1 there exists an isomorphism $\tau_V : (V, S_1) \xrightarrow{\cong} (V, S_2)$. Due to the uniqueness of the pair (S_1, S_2) one has $\tau_V(S_2) = S_1$. Therefore, $\tau_V^2 \in \mathrm{Aut}(V, S_1)$. Since the group $\mathrm{Aut}(V, S_1) \cong \mathrm{GL}_2(\mathbb{C})$ is connected, one obtains $\tau_V^2 \in \mathrm{Aut}^0(V)$. Moreover, $\mathrm{Aut}^0(V)$ and τ_V generate $\mathrm{Aut}(V)$. This proves our claim.

Thus, $\mathrm{Aut}(V)/\mathrm{Aut}^0(V) \cong \mathbb{Z}/2\mathbb{Z}$. It follows that the embedding $\mathrm{Aut}(V) \hookrightarrow \mathrm{GL}_2(\mathbb{C}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ in Case 1^o of Proposition 11.1 is an isomorphism. The last assertion of Theorem 1.3(i) is straightforward.

Similarly, we claim that (13.3.1) still holds in the case where $V \cong V_{18}^{\mathrm{a}}$, and so, (1.3.2) follows by Case 2^o of Proposition 11.1.

Indeed, among the four cubic cones contained in V, there is a unique pair of disjoint cones. Namely, these are the cubic cones S_1, S_2 which do not contain the ruling corresponding to the singular point of $\Sigma_{\mathrm{s}}(V)$, see diagram (10.1.1).

Consider the pair (W, F_2) linked to (V, S_2). The image of S_1 in W is a cubic cone $S_{1,W}$, which meets F_2 along a twisted cubic section Ψ. The construction being $\mathrm{Aut}^0(V)$-equivariant, the image of $\mathrm{Aut}^0(V) \cong \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$ in $\mathrm{Aut}(W)$ is contained in the stabilizer of the vertex $v(S_{1,W}) \in W \setminus R$.

The latter stabilizer is a Levi subgroup, say, $\mathrm{L}_2 \subset \mathrm{Aut}(W)$, see Proposition 5.5(c). The $\mathbb{G}_{\mathrm{a}}$-subgroup of $\mathrm{Aut}^0(V)$ cannot preserve the rulings of $S_{1,W}$ and F_2, respectively. Indeed, otherwise on such a ruling l it would have two fixed points $l \cap \Psi$ and $v(S_{1,W})$ ($l \cap \Psi$ and $l \cap \Xi$, respectively). In particular, $\mathrm{Aut}^0(V)$ would act identically on S_1, which is impossible. It follows that the $\mathbb{G}_{\mathrm{m}}$-subgroup of $\mathrm{Aut}^0(V)$ corresponds to the singular torus $\mathrm{z}(\mathrm{L}_2)$ fixing Ψ and $J_2 := F_2 \cap \Xi$ pointwise and preserving the rulings of both $S_{1,W}$ and F_2. By contrast, the $\mathbb{G}_{\mathrm{a}}$-subgroup of $\mathrm{Aut}^0(W, F_2) \cong \mathrm{Aut}^0(V)$ acts effectively on J_2. Hence (Υ, J_2) is a pair of type 4.3(ii).

Symmetrically, for the pair (W, F_1) linked to (V, S_1) the $\mathbb{G}_{\mathrm{m}}$-subgroup of $\mathrm{Aut}^0(W, F_1) \cong \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$ acts trivially on $J_1 = F_1 \cap \Xi$, while the $\mathbb{G}_{\mathrm{a}}$-subgroup acts effectively. So, (Υ, J_1) is as well a pair of type 4.3(ii).

We claim that the pairs (W, F_1) and (W, F_2) are isomorphic. Indeed, the corresponding Levi subgroups L_1 and L_2 are conjugate in $\mathrm{Aut}(W)$. Hence, up to an automorphism of W, one may suppose that $\mathrm{L}_1 = \mathrm{L}_2 =: \mathrm{L}$. By Lemma 6.2(b) the pairs (Υ, J_1) and (Υ, J_2) of type 4.3(ii) are isomorphic under the $\mathrm{Aut}(\Xi, \Upsilon)$-action. This isomorphism can be realized by an element of L. Hence one may suppose also that $J_1 = J_2 =: J$. By virtue of Proposition 6.3(b) and (d), under these assumptions there is an isomorphism of pairs $(W, F_1) \cong (W, F_2)$, as claimed.

The latter isomorphism induces an isomorphism of linked pairs $\tau_V : (V, S_1) \xrightarrow{\cong} (V, S_2)$. Since $\{S_1, S_2\}$ is the only pair of disjoint cubic cones in V, one has $\tau_V^2 \in \mathrm{Aut}(V, S_i)$, $i = 1, 2$. Since $\mathrm{Aut}^0(V) \subset \mathrm{Aut}(V, S_i)$ and $[\mathrm{Aut}(V) : \mathrm{Aut}^0(V)] \le 2$, see Proposition 11.1.2^o, we deduce (13.3.1), as desired. □

13.3.2. **Remark.** Let again $V \cong V_{18}^{\mathrm{a}}$, and let $\{S_1', S_2'\}$ be the pair of cubic cones in V with a common ruling, which corresponds to the unique singular point of $\Sigma_{\mathrm{s}}(V)$, cf. diagram (10.1.1). For the corresponding linked pairs (W, F_j'), $j = 1, 2$, one has $\mathrm{Aut}^0(W, F_j') \cong \mathrm{Aut}^0(V, S_j') \cong \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$. The argument in the proof above shows that the $\mathbb{G}_{\mathrm{m}}$-subgroup of $\mathrm{Aut}^0(W, F_j')$ acts nontrivially on $J_j' = F_j' \cap \Xi$, while the $\mathbb{G}_{\mathrm{a}}$-subgroup acts identically on J_j'. It follows that $\mathrm{R_u} \cap \mathrm{Aut}^0(W, F_j') \cong \mathbb{G}_{\mathrm{a}}$, and (Υ, J_j') is a pair of type 4.3(iii) for $j = 1, 2$.

13.4. **Remark.** It is a folklore that the moduli space $\mathcal{M}_{18}$ of the Fano-Mukai fourfolds of genus 10 is one-dimensional. Indeed, this can be seen as follows.

Identify G_2 with its image in $\mathrm{Aut}(\mathbb{P}(\mathfrak{g}_2^\vee)) \cong \mathrm{PGL}_{14}(\mathbb{C})$ under the dual of the adjoint representation. The open set $\mathcal{U} = \mathbb{P}(\mathfrak{g}_2^\vee) \setminus (D_\ell \cup D_{\mathrm{s}})$ is G_2-invariant. Each point $[g] \in \mathcal{U}$ corresponds to the hyperplane section $V^g = \Omega \cap g^\perp$; the latter is a Fano-Mukai fourfolds of genus 10 with $\mathrm{Aut}^0(V^g) \cong (\mathbb{G}_{\mathrm{m}})^2$, see Theorem 1.3(iii). Then $\mathcal{M}_{18}$ is dominated by the one-dimensional quotient $\mathcal{U}/\,\mathrm{G}_2$, see Proposition 7.7. The fiber of $\mathcal{U} \to \mathcal{U}/\,\mathrm{G}_2$ through a point $[g] \in \mathcal{U}$ is isomorphic to $\mathrm{G}_2 / \mathrm{Stab}_{\mathrm{G}_2}(V^g)$. By Corollary 13.2.1 one has

$$\mathrm{Aut}^0(V^g) = \mathrm{Stab}_{\mathrm{G}_2}(V^g)^0 \subset \mathrm{Stab}_{\mathrm{G}_2}(V^g) \subset \mathrm{Aut}(V^g).$$

The fiber of the morphism $\mathcal{U}/\,\mathrm{G}_2 \to \mathcal{M}_{18}$ though the image of $[g]$ in $\mathcal{U}/\,\mathrm{G}_2$ is isomorphic to $\mathrm{Aut}(V^g)/\,\mathrm{Stab}_{\mathrm{G}_2}(V^g)$. The latter is a cyclic group whose order is a factor of 6, see (1.3.3). Hence $\mathcal{U}/\,\mathrm{G}_2 \to \mathcal{M}_{18}$ is a finite morphism, and so, $\dim \mathcal{M}_{18} = 1$.

Notice, however, that the moduli space $\mathcal{M}_{18}$ does not exist as a separated scheme. Indeed, the points $[V_{18}^{\mathrm{s}}]$ and $[V_{18}^{\mathrm{a}}]$ do not admit disjoint neighborhoods in $\mathcal{M}_{18}$, as follows from Proposition 7.7.

In addition to Theorem 1.3(i) we have the following results. Item (f) will be used in the next section.

13.5. **Theorem.** *Let $V = V_{18}^{\mathrm{s}}$ be the Fano-Mukai fourfold as in Theorem 1.3(i) and let S_1, $S_2 \subset V$ be the unique $\mathrm{Aut}^0(V)$-invariant cubic cones (see Lemma 12.2). Let $A_i = A_{S_i}$, $i \in \{1, 2\}$ be the unique $\mathrm{Aut}^0(V)$-invariant hyperplane section with $\mathrm{Sing}(A_i) = S_i$. Then the following hold.*

(a) *For $i \neq j$, A_j cuts S_i along a rational twisted cubic curve $\Gamma_j \subset S_i$. The curves Γ_1 and Γ_2 are disjoint, the union $\Gamma_1 \cup \Gamma_2$ is an* $\mathrm{Aut}(V)$*-orbit contained in $A_1 \cap A_2$ and pointwise fixed under the* $\mathrm{z}(\mathrm{Aut}^0(V))$*-action.*

(b) *For $j = 1, 2$ consider the family $(S_{j,t})_{t\in\mathbb{P}^1}$ of cubic cones on V such that $\Lambda(S_{j,t})$ is a ruling of $\mathcal{F}_j \to \mathbb{P}^1$ (see Corollary* 10.3.2(b)*). Then each point of Γ_j is the vertex $v_{j,t} = v(S_{j,t})$, and for $j \neq i$ one has $A_i = \bigcup_{t\in\mathbb{P}^1} S_{j,t}$.*

(c) *The twisted cubics Γ_1 and Γ_2 are sections of a rational normal scroll D of degree 6 which is the image of $\mathbb{P}^1 \times \mathbb{P}^1$ embedded to $\mathbb{P}^7 \subset \mathbb{P}^{12}$ by the linear system of bidegree $(1, 3)$. One has $(A_1 \cap A_2)_{\mathrm{red}} = D$.*

(d) *The vertices $v(S_i)$, $i = 1, 2$, are the unique* $\mathrm{GL}_2(\mathbb{C})$*-fixed points in V. Furthermore, $D \setminus (\Gamma_1 \cup \Gamma_2)$ is an orbit of* $\mathrm{GL}_2(\mathbb{C})$*.*

(e) *A_1 and A_2 are two components of the branching divisor $\mathcal{B}$ of the morphism $s : \mathscr{L}(V) \to V$ in* (8.2.2)*.*

(f) *The Fano fourfold V is covered by the affine charts*

$$U_i = V \setminus A_i \cong \mathbb{C}^4 \quad \textit{and} \quad U_{i,t} = V \setminus A_{i,t} \cong \mathbb{C}^4, \quad t \in \mathbb{P}^1, \ i = 1, 2\,, \tag{13.5.1}$$

where $A_{i,t} = A_{S_{i,t}}$ is the unique hyperplane section of V with $\mathrm{Sing}(A_{i,t}) = S_{i,t}$*.*

Proof. (a) Since $S_1 \cap S_2 = \varnothing$ one has $v(S_j) \notin A_i$ for $j \neq i$. This yields the first assertion. The cones S_1 and S_2 being disjoint (see Lemma 12.2) also the curves Γ_1 and Γ_2 are. Since $\Gamma_i \subset S_j \subset A_j$ one has $\Gamma_i \subset A_i \cap A_j$, $i = 1, 2$.

The factor $\mathbb{Z}/2\mathbb{Z}$ of $\mathrm{Aut}(V)$ in Theorem 1.3 (i) is generated by an involution $\tau \in \mathrm{Aut}(V) \setminus \mathrm{Aut}^0(V)$ interchanging S_1 and S_2. Then τ switches also Γ_1 and Γ_2. Since both A_i and S_j are $\mathrm{Aut}^0(V)$-invariant then $\Gamma_i = A_i \cap S_j$ is. The curve Γ_i is an orbit of $\mathrm{Aut}^0(V)$. It is pointwise fixed under the $\mathrm{z}(\mathrm{Aut}^0(V))$-action, and $\Gamma_1 \cup \Gamma_2$ is an orbit of $\mathrm{Aut}(V)$.

(b) Since $S_{j,t}$ and S_i contain a common ruling one has $v(S_{j,t}) \in S_i$, and so, $S_{j,t} \subset A_i$ (recall that A_i is the union of lines in V meeting S_i, see Lemma 9.2). The union $\bigcup_{t\in\mathbb{P}^1} S_{j,t} \subset A_i$ is a closed subvariety of dimension 3. Since A_i is irreducible of dimension 3 one has $A_i = \bigcup_{t\in\mathbb{P}^1} S_{j,t}$. The set of vertices $\{v(S_{j,t})\}_{t\in\mathbb{P}^1} \subset S_i$ is $\mathrm{Aut}^0(V)$-invariant, hence, $\{v(S_{j,t})\}$ is a closed one-dimensional $\mathrm{Aut}^0(V)$-orbit in S_i. The center $\mathrm{z}(\mathrm{Aut}^0(V))$ acts nontrivially on any ruling l of S_i with just two fixed points, $v(S_i)$ and $\Gamma_i \cap l$. Therefore, $\Gamma_j = S_i \cap A_j$ is the only one-dimensional $\mathrm{Aut}^0(V)$-orbit in S_i. It follows that $\{v(S_{j,t})\} = \Gamma_j$.

(c) The conic $C = \mathcal{F}_1 \cap \mathcal{F}_2 \subset \Sigma_{\mathrm{s}}(V)$ parameterizes a family of lines (l_t) in V, where $l_t \subset S_{1,t} \cap S_{2,t}$. Since $v(S_{j,t}) \in \Gamma_j \cap l_t$, $j = 1, 2$, the line l_t meets both S_1 and S_2, hence is contained in $A_1 \cap A_2$. Thus, $D = \bigcup_{t\in C} l_t \subset A_1 \cap A_2$ is a rational normal scroll.

According to Corollary 10.3.2(b) the exceptional section $\Lambda(S_2)$ of $\mathcal{F}_1$ and a ruling $\Lambda(S_{2,t})$ of $\mathcal{F}_2$ project to two distinct points of $\mathscr{S}_2(V) \cong \mathbb{P}^2$. Since S_2 and $S_{2,t}$ are members of the same family $\mathscr{S}_2(V)$ one has $S_2 \cdot S_{2,t} = 1$, see Proposition 9.6. Since they have no common ruling, they meet transversally in one point, say, $P_{2,t}$, which is smooth on both S_2 and $S_{2,t}$, see Corollary 9.7.4. Hence there is a unique ruling $l_{2,t} \ni P_{2,t}$ of $S_{2,t}$ which meets S_2. By symmetry, there is a unique ruling $l_{1,t}$ of $S_{1,t}$ which meets S_1. Notice that the unique common ruling l_t of $S_{1,t}$ and $S_{2,t}$ meets both S_1 and S_2, hence $l_{1,t} = l_{2,t} = l_t \subset D \subset A_1 \cap A_2$, see the proof of (b).

The hyperplane section A_1 passes through the vertex $v_{1,t} \in \Gamma_1$ of $S_{1,t} \subset A_2$ cutting $S_{1,t}$ along a union of at most 3 rulings. Since $v_{1,t} \in l_t \subset \Gamma_1 = A_1 \cap S_2$ each of these rulings meets S_2. It follows by the preceding that the ruling of $S_{1,t}$ which meets S_2 is unique and coincides with l_t. We conclude that l_t is a triple intersection of $S_{1,t} \subset A_2$ and A_1.

Therefore, $A_1 \cap A_2 = 3D$, and so, $(A_1 \cap A_2)_{\mathrm{red}} = D$, as stated. We leave to the reader to check the remaining statements of (c).

(d) Since S_1, S_2 are $\mathrm{Aut}^0(V)$-invariant their vertices $v(S_1)$ and $v(S_2)$ are two distinct fixed points of $\mathrm{Aut}^0(V)$. The hyperplane sections A_1 and A_2 are $\mathrm{Aut}^0(V)$-invariant. For $i = 1, 2$ the center $\mathrm{z}(\mathrm{Aut}^0(V))$ acts on $V \setminus A_i \cong \mathbb{C}^4$ via homotheties with a unique fixed point $v(S_i)$. According to (a) and (b), $\mathrm{z}(\mathrm{Aut}^0(V))$ fixes also the vertex $v_{j,t}$ of each cone $S_{j,t}$ leaving the cone invariant.

We claim that $\mathrm{z}(\mathrm{Aut}^0(V))$ acts nontrivially on each ruling l_t of D. Indeed, suppose to the contrary that $\mathrm{z}(\mathrm{Aut}^0(V))$ acts identically on l_t. The latter is true for any $t \in \mathbb{P}^1$ due to the rigidity of the reductive group actions. Since l_t is a ruling of the cone $S_{1,t}$, then also $\mathrm{z}(\mathrm{Aut}^0(V))$ acts identically on any ruling of $S_{1,t}$. Hence it acts identically on the cone $S_{1,t}$ for any $t \in \mathbb{P}^1$. By (b) one has $A_2 = \bigcup_{t \in \mathbb{P}^1} S_{1,t}$. It follows that $\mathrm{z}(\mathrm{Aut}^0(V))$ acts identically on $S_2 \subset A_2$. However, $\mathrm{z}(\mathrm{Aut}^0(V))$ acts via homotheties in $V \setminus A_1 \cong \mathbb{C}^4$ with the unique fixed point $v_2 = v(S_2)$. This gives a contradiction.

Hence each ruling l_t of D is $\mathrm{z}(\mathrm{Aut}^0(V))$-invariant and contains just two $\mathrm{z}(\mathrm{Aut}^0(V))$-fixed points $l_t \cap (\Gamma_1 \cup \Gamma_2)$. Now the absence of fixed points of $\mathrm{Aut}^0(V) \cong \mathrm{GL}_2(\mathbb{C})$ both in D and in $V \setminus (D \cup \{v(S_1), v(S_2)\})$ follows. The second assertion is now straightforward.

(e) is also straightforward by virtue of (b) and Lemma 9.4.

(f) Notice first of all that any affine chart in (13.5.1) is isomorphic indeed to $\mathbb{C}^4$ by Corollary 4.5.2. Suppose to the contrary that there is a point $P \in V$ which is not covered by any of the affine charts in (13.5.1). Thus, all the hyperplane sections A_i, $A_{i,t}$ pass through P. In particular, $P \in D = (A_1 \cap A_2)_{\mathrm{red}}$, see (c), and so,

$$P \in B := D \cap \bigcap_{i=1,2,\, t \in \mathbb{P}^1} A_{i,t}\,.$$

Let us show that B should contain a point of $\Gamma_1 \cup \Gamma_2$. Indeed, $D \setminus (\Gamma_1 \cup \Gamma_2)$ is an orbit of $\mathrm{Aut}^0(V)$, see the proof of (d). If this orbit contains a point $P \in B$ then B, being $\mathrm{Aut}^0(V)$-invariant, contains the whole orbit $D \setminus (\Gamma_1 \cup \Gamma_2)$. Thus, also $B \supset D$ since B is closed.

Therefore, one may assume that, say, $P \in \Gamma_1 \subset B$. By (b) the curve Γ_1 is filled in by the vertices of the cones $S_{1,t}$. We claim that for any $t_1 \neq t_2$ the cones $S_{1,t_1} \in \mathscr{S}_1(V)$ and $S_{2,t_2} \in \mathscr{S}_2(V)$ are disjoint. Indeed, by Proposition 9.6 one has $S_{1,t_1} \cdot S_{2,t_2} = 0$. Since these cones have no common ruling, our claim follows by Corollary 9.7.4.

Let us show that the vertex v_{1,t_1} of S_{1,t_1} cannot lie on A_{2,t_2}, hence also in B, which gives a desired contradiction. Indeed, otherwise there is a line through v_{1,t_1} meeting S_{2,t_2}. The only lines through v_{1,t_1} are the rulings of S_{1,t_1}. However, as we have seen, the latter rulings are dijoint with S_{2,t_2}. □

14. Flexibility of affine cones over Fano-Mukai fourfolds V_{18}

14.1. An affine variety X of dimension at least 2 is called *flexible* (in the sense of ([AFK+13]) if the subgroup $\mathrm{SAut}(X) \subset \mathrm{Aut}(X)$ generated by all the unipotent algebraic subgroups of $\mathrm{Aut}(X)$ acts transitively on the smooth locus X_{reg}. We say that X is *flexible in codimension one* if $\mathrm{SAut}(X)$ admits an open orbit O_X whose complement has codimension at least 2 in X. In the latter case $\mathrm{SAut}(X)$ acts m-transitively on O_X for any natural m ([AFK+13, Theorem 2.2]).

Actions of unipotent groups on affine cones over Fano varieties was recently a subject of intensive studies. Several families of smooth Fano fourfolds were examined from this viewpoint; see, e.g., [PZ15], [PZ16] and the references therein. Using a criterion in [KPZ13], one can deduce from [PZ15] and Theorem 1.1 such a corollary.

14.1.1. **Corollary.** *Let $V = V_{18} \subset \mathbb{P}^{12}$ be a Fano-Mukai fourfold of genus 10, and let A be a hyperplane section of V containing either a smooth cubic scroll, or a cubic cone. Then $V \setminus A$ contains a principal open cylinder $U \cong Z \times \mathbb{C}$, where $\dim Z = 3$. Consequently, any affine cone $X = \mathrm{Cone}_H(V)$, where H is an ample polarization of V, admits an effective $\mathbb{G}_{\mathrm{a}}$-action.*

Flexibility of affine cones over Fano varieties was studied recently in [MPS16]. Theorem 1.4 and Remark 1.2 in [*ibid*] lead to the following criterion.

14.2. **Proposition.** *Let V be a projective variety with a very ample polarization H. Suppose that V contains a Zariski open subset U whose complement $V \setminus U$ has codimension at least 2 in V and such that $U = \bigcup_\alpha U_\alpha$ is covered by a collection (U_α) of smooth, flexible principal open subsets $U_\alpha = V \setminus H_\alpha$ where $H_\alpha \in |H|$. Then the corresponding affine cone $X = \mathrm{Cone}_H(V)$ is flexible in codimension one. More precisely, the pullback of U in X is contained in the open orbit of $\mathrm{SAut}(X)$. In particular, if $U = V$ then X is flexible.*

Applying this criterion in the setting of Theorem 1.1 one obtains the following result.

14.3. **Theorem.** *The affine cone over any polarized Fano-Mukai fourfold $V = V_{18}$ of genus 10 is flexible in codimension one. For $V = V_{18}^{\mathrm{s}}$ this cone is flexible.*

Proof. Let A_i be the hyperplane sections of V with $\mathrm{Sing}(A_i) = S_i$, $i = 1, 2$, where S_1, S_2 are two distinct cubic cones in V, see Theorem 1.1. Then the principal open subsets $U_i := V \setminus A_i \cong \mathbb{C}^4$, $i = 1, 2$, are flexible, and $\mathrm{codim}_V(V \setminus (U_1 \cup U_2)) = \mathrm{codim}_V(A_1 \cap A_2) = 2$. Since $\mathrm{Pic}(V) \cong \mathbb{Z}$ the criterion of Proposition 14.2 applies and gives the result.

If $V = V_{18}^{\mathrm{s}}$ then by Theorem 13.5(f), V is covered by the affine charts isomorphic to $\mathbb{C}^4$. Once again, the flexibility of the cone $\mathrm{Cone}_H(V)$ follows from Proposition 14.2. □

15. Final remarks and open questions

15.1. Given a simple affine algebraic group G over $\mathbb{C}$ of adjoint type and a parabolic subgroup $P \subset G$, consider the flag variety G/P. It is known (see [Dem77], [Akh95, Theorem 2 in § 3.3 and the subsequent remark]) that $\mathrm{Aut}^0(G/P) \cong G$ except in certain three cases. In the exceptional cases $G' := \mathrm{Aut}^0(G/P)$ is again a simple affine algebraic group of adjoint type, and $G/P = G'/P'$ for a parabolic subgroup $P' \subset G'$. In particular, for $G = \mathrm{G}_2$ and $G/P = \Omega$ as in Section 7 one has $\mathrm{Aut}^0(G/P) \cong \mathrm{G}_2$. A similar phenomenon might occur as well for smooth hyperplane sections of adjoint varieties.

15.2. **Problem.** *Consider a flag variety G/P, where G is a semisimple affine algebraic group with trivial center and $P \subset G$ is a parabolic subgroup. Choose G and P suitable so that $\mathrm{Aut}^0(G/P) \cong G$. Let $\iota\colon G/P \hookrightarrow \mathbb{P}^n$ be a G-equivariant embedding with linearly nondegenerate image. Since $\mathrm{Pic}(G/P)$ is discrete, one may identify $G \cong \mathrm{Aut}^0(G/P)$ with $\mathrm{Aut}^0(\mathbb{P}^n, \iota(G/P))$. We wonder as to when for any smooth hyperplane section H of $\iota(G/P)$ one has $\mathrm{Aut}^0(H) = \mathrm{Stab}^0_G(H)$.*

Theorem 1.2 says that this is indeed the case for $G = \mathrm{G}_2$ and for the adjoint orbit $\Omega = \mathrm{G}_2/P$. By [FH16, Lem. 8.2(i)] this is the case for any irreducible hermitian symmetric space G/P of compact type, provided the embedding $\iota\colon G/P \hookrightarrow \mathbb{P}^n$ is given by the ample generator of the Picard group $\mathrm{Pic}(G/P) \cong \mathbb{Z}$. See [FH16, Prop. 8.4 and Thm. 8.5] for concrete examples.

One can ask the same question more generally for the smooth linear sections L of $\iota(G/P)$ provided the Picard group of L is isomorphic to $\mathbb{Z}$. The answer is known to be affirmative for *general* linear sections of codimension $l \le N-2$ and $l=3$ in $N=4$ of the Grassmannians of lines in $\mathbb{P}^N$, $N \ge 4$, see [PdV99, Thm. 1.2, Cor. 1.3 and its proof].

15.3. **Question.** It is known that for any compactification (V, A) of $\mathbb{C}^3$ with $\mathrm{b}_2(V) = 1$ the middle Betti number $\mathrm{b}_3(V)$ vanishes. In all known examples in dimension 4, that is, for $\mathbb{P}^4$, $Q^4 \subset \mathbb{P}^5$, the examples in [Pro94], and the ones in Theorem 1.1, the middle Betti number satisfies the inequality $\mathrm{b}_4(V) \le 2$. *We wonder whether this inequality still holds for any compactification* (V, A) *of* $\mathbb{C}^4$ *with* $\mathrm{b}_2(V) = 1$.

The next problem arises naturally regarding Theorem 1.3, cf. Remark 11.5.2.

15.4. **Problem.** *Describe explicitly the involution acting on* $V = V_{18}^{\mathrm{s}}$ ($V = V_{18}^{\mathrm{a}}$, *respectively*) *and interchanging the* $\mathrm{Aut}^0(V)$*-invariant cubic cones* S_1, $S_2 \subset V$. *Determine the discrete groups* $\mathrm{Aut}(V)/\mathrm{Aut}^0(V)$ *for the Fano-Mukai fourfolds* $V \not\cong V_{18}^{\mathrm{s}}, V_{18}^{\mathrm{a}}$ *of genus* 10.

15.5. **Remark.** The group $\mathrm{Aut}(V_{18}^{\mathrm{a}}) \cong (\mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}) \rtimes (\mathbb{Z}/2\mathbb{Z})$ being non-abelian, the involution of V_{18}^{a} interchanging the $\mathrm{Aut}^0(V)$-invariant cubic cones S_1, S_2 does not admit an extension to an element of G_2 acting on Ω, see Propositions 7.5.1(b) and 7.7(b). However, we ignore if it can be extended to an automorphism of Ω.

15.6. **Problem.** *Which Fano-Mukai fourfolds* V_{18} *of genus* 10 *admit a Kähler-Einstein metric* ([Tia15])*?* Notice that the group $\mathrm{Aut}^0 V_{18}^{\mathrm{a}} \cong \mathbb{G}_{\mathrm{a}} \times \mathbb{G}_{\mathrm{m}}$ is not reductive, see Theorem 1.3(ii). Hence, according to Matsushima's theorem ([Mat57]), the variety V_{18}^{a} does not admit such a metric.

15.7. **Problem.** *Study* singular *Fano-Mukai fourfolds of type* V_{18} (*cf.* [Pro15]*,* [Pro16]).

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