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An extension of Lobachevsky formula

Hassan Jolany

Abstract. In this paper we extend the Dirichlet integral formula of Lobachevsky. Let $f(x)$ be a continuous function and satisfy in the π -periodic assumption $f(x + \pi) = f(x)$, and $f(\pi - x) = f(x)$, $0 \leq x < \infty$. If the integral $\int_0^\infty \frac{\sin^4 x}{x^4} f(x) dx$ defined in the sense of the improper Riemann integral, then we show the following equality

$$\int_0^\infty \frac{\sin^4 x}{x^4} f(x) dx = \int_0^{\frac{\pi}{2}} f(t) dt - \frac{2}{3} \int_0^{\frac{\pi}{2}} \sin^2 t f(t) dt$$

hence if we take $f(x) = 1$, then we have

$$\int_0^\infty \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}$$

Moreover, we give a method for computing $\int_0^\infty \frac{\sin^{2n} x}{x^{2n}} f(x) dx$ for $n \in \mathbb{N}$

1. INTRODUCTION Dirichlet integral play an important role in distribution theory. We can see the Dirichlet integral in terms of distribution. The following classical Dirichlet integral has drawn lots of attention.

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

We can use the theory of residues to evaluate this Dirichlet's integral formula. G.H. Hardy and A. C. Dixon gave a lot of different proofs for it. See [5-7]. In this paper we give an elegant method to generalize this Lobachevsky formula.

We start with the following elementary lemma. See [1-4]

Lemma. For $\alpha \notin \mathbb{Z}\pi$, we have

$$\frac{1}{\sin \alpha} = \frac{1}{\alpha} + \sum_{m=1}^{\infty} (-1)^m \left(\frac{1}{\alpha - m\pi} + \frac{1}{\alpha + m\pi} \right)$$

Proof. For every positive integer N , denote by C_N the positively-oriented square in the complex plane with vertices $(N + \frac{1}{2})(\pm 1 \pm i)$. On the one hand, since the function $1/\sin(z)$ is bounded on C_N by a constant which is independent of N , one has

$$\oint_{C_N} \frac{2\pi\alpha dz}{(z^2\pi^2 - \alpha^2) \sin(\pi z)} \rightarrow 0$$

as $N \rightarrow \infty$. On the other hand, by the Residue Theorem, one also gets

$$\oint_{C_N} \frac{2\pi\alpha dz}{(z^2\pi^2 - \alpha^2) \sin(\pi z)} = \sum_{n=-N}^N (-1)^n \frac{2\alpha}{n^2\pi^2 - \alpha^2} + \frac{1}{\sin(\alpha)}$$

which proves the claim as $N \rightarrow \infty$. ■

Lemma. For $\alpha \notin \mathbb{Z}\pi$, we have the following identity

$$\frac{1}{\sin^2 \alpha} = \frac{1}{\alpha^2} + \sum_{m=1}^{\infty} \left(\frac{1}{(\alpha - m\pi)^2} + \frac{1}{(\alpha + m\pi)^2} \right)$$

Proof. The identity follows by differentiating termwise the classical formula ,

$$\cot(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{2z}{z^2 - m^2\pi^2}$$

■

2. LOBACHEVSKY FORMULA Now, we present the Lobachevsky formula.

Theorem 1. *Let f satisfy the conditions of the beginning of the article. Then we have the following Lobachevsky identity*

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} f(x) dx = \int_0^{\infty} \frac{\sin x}{x} f(x) dx = \int_0^{\frac{\pi}{2}} f(x) dx$$

Proof. Take

$$I = \int_0^{\infty} \frac{\sin x}{x} f(x) dx$$

we can write I as follows

$$I = \sum_{v=0}^{\infty} \int_{v\frac{\pi}{2}}^{(v+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x) dx$$

where $v = 2\mu - 1$ or $v = 2\mu$, by changing $x = \mu\pi + t$ or $x = \mu\pi - t$ we get

$$\int_{2\mu\frac{\pi}{2}}^{(2\mu+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x) = (-1)^{\mu} \int_0^{\frac{\pi}{2}} \frac{\sin t}{\mu\pi + t} f(t) dt$$

and

$$\int_{(2\mu-1)\frac{\pi}{2}}^{(2\mu)\frac{\pi}{2}} \frac{\sin x}{x} f(x) = (-1)^{\mu-1} \int_0^{\frac{\pi}{2}} \frac{\sin t}{\mu\pi - t} f(t) dt$$

so we get

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} f(t) dt + \sum_{\mu=1}^{\infty} \int_0^{\frac{\pi}{2}} (-1)^{\mu} f(t) \left(\frac{1}{t + \mu\pi} + \frac{1}{t - \mu\pi} \right) \sin t dt$$

Consequently we can write I in the following form

$$I = \int_0^{\frac{\pi}{2}} \sin t \left(\frac{1}{t} + \sum_{\mu=1}^{\infty} (-1)^\mu \left(\frac{1}{t + \mu\pi} + \frac{1}{t - \mu\pi} \right) \right) f(t) dt$$

Hence

$$I = \int_0^{\frac{\pi}{2}} f(t) dt$$

and proof of the identity $\int_0^{\infty} \frac{\sin x}{x} f(x) dx = \int_0^{\frac{\pi}{2}} f(x) dx$ is complete. Now we prove the second part of identity. Take

$$J = \int_0^{\infty} \frac{\sin^2 x}{x^2} f(x) dx$$

we can write J as follows

$$J = \sum_{v=0}^{\infty} \int_{v\frac{\pi}{2}}^{(v+1)\frac{\pi}{2}} \frac{\sin^2 x}{x^2} f(x) dx$$

where $v = 2\mu - 1$ or $v = 2\mu$, by changing $x = \mu\pi + t$ or $x = \mu\pi - t$ we get

$$\int_{2\mu\frac{\pi}{2}}^{(2\mu+1)\frac{\pi}{2}} \frac{\sin^2 x}{x^2} f(x) = \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{(\mu\pi + t)^2} f(t) dt$$

and

$$\int_{(2\mu-1)\frac{\pi}{2}}^{(2\mu)\frac{\pi}{2}} \frac{\sin^2 x}{x^2} f(x) = \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{(\mu\pi - t)^2} f(t) dt$$

so we get

$$J = \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{t^2} f(t) dt + \sum_{\mu=1}^{\infty} \int_0^{\frac{\pi}{2}} f(t) \left(\frac{1}{(t + \mu\pi)^2} + \frac{1}{(t - \mu\pi)^2} \right) \sin^2 t dt$$

consequently we can write J in the following form

$$J = \int_0^{\frac{\pi}{2}} \sin^2 t \left(\frac{1}{t^2} + \sum_{\mu=1}^{\infty} \left(\frac{1}{(t + \mu\pi)^2} + \frac{1}{(t - \mu\pi)^2} \right) \right) f(t) dt$$

Hence from Lemma 1.2, we get

$$J = \int_0^{\frac{\pi}{2}} f(t) dt$$

and proof is complete ■

3. EXTENSION OF THE LOBACHEVSKY FORMULA Now we give a general method for calculating the following Dirichlet integral.

$$\int_0^{\infty} \frac{\sin^{2n} x}{x^{2n}} f(x) dx$$

where $f(\pi + x) = f(x)$, and $f(\pi - x) = f(x)$, $0 \leq x < \infty$. Here we have assumed f is continuous and $\int_0^{\infty} \frac{\sin^{2n} x}{x^{2n}} f(x) dx$ defined in the sense of the improper Riemann integral. We start with $n = 2$. As we did in the previous section, take

$$I = \int_0^{\infty} \frac{\sin^4 x}{x^4} f(x) dx$$

By a direct computation

$$\frac{d^2}{dx^2} \left(\frac{1}{\sin^2 x} \right) = \frac{6}{\sin^4(x)} - \frac{4}{\sin^2(x)}$$

Next, differentiating twice termwise the right-hand side of identity of Lemma 1.2, we get the following identity

$$\frac{1}{\sin^4 \alpha} - \frac{2}{3 \sin^2 \alpha} = \frac{1}{\alpha^4} + \sum_{m=1}^{\infty} \left(\frac{1}{(\alpha - m\pi)^4} + \frac{1}{(\alpha + m\pi)^4} \right)$$

From the previous method which we explained in section 2, we can write I as follows

$$I = \int_0^{\frac{\pi}{2}} \sin^4 t \left(\frac{1}{\sin^4 t} - \frac{2}{3 \sin^2 t} \right) f(t) dt$$

Hence

$$I = \int_0^{\frac{\pi}{2}} f(t) dt - \frac{2}{3} \int_0^{\frac{\pi}{2}} \sin^2 t f(t) dt$$

So we proved the following theorem

Theorem 2. Let $f(x)$ satisfies in $f(x + \pi) = f(x)$, and $f(\pi - x) = f(x)$, $0 \leq x < \infty$. If the following integral

$$\int_0^{\infty} \frac{\sin^4 x}{x^4} f(x) dx$$

defined in the sense of the improper Riemann integral, then we have the following equality

$$\int_0^{\infty} \frac{\sin^4 x}{x^4} f(x) dx = \int_0^{\frac{\pi}{2}} f(t) dt - \frac{2}{3} \int_0^{\frac{\pi}{2}} \sin^2 t f(t) dt$$

As remark if we take $f(x) = 1$, then we have

Remark. We have

$$\int_0^{\infty} \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}$$

We have also the following remark from Lobachevsky formula

Remark. If $f(x)$ satisfy in the condition $f(x + \pi) = f(x)$, and $f(\pi - x) = f(x)$, $0 \leq x < \infty$, take

$$I = \int_0^{\infty} \frac{\sin^{2n+1} x}{x} f(x) dx = \int_0^{\infty} \sin^{2n} x \frac{\sin x}{x} f(x) dx$$

If we set $\sin^{2n} x f(x) = g(x)$, we get $g(x + \pi) = g(x)$, $g(\pi - x) = g(x)$, now if we take $f(x) = 1$, then

$$\int_0^{\infty} \frac{\sin^{2n+1} x}{x} dx = \int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \frac{(2n-1)!! \pi}{(2n)!! 2}$$

Now, by the following important remark, we can calculate the Lobachevsky formula for any $n \geq 3$. Let $f(z)$ satisfy the conditions of the beginning of the article.

Remark. In fact, the Dirichlet integral

$$\int_0^{\infty} \frac{\sin^{2n} z}{z^{2n}} f(z) dz$$

has the form (for $n \geq 3$)

$$\alpha_1 \int_0^{\frac{\pi}{2}} f(z) dz + \alpha_2 \int_0^{\frac{\pi}{2}} \cot^{2n-2}(z) \sin^{2n-2}(z) f(z) dz + \dots + \alpha_k \int_0^{\frac{\pi}{2}} \cot^2(z) \sin^2(z) f(z) dz$$

where the constants α_i can be computed by the use of the following formulas (and the help of the engine Wolfram Alpha for instance) : For every positive integer n , one can compute

$$\frac{d^n}{dz^n} \left(\frac{1}{\sin^2(z)} \right) = \frac{d^n}{dz^n} (1 + \cot^2 z) = \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k}}{dz^{n-k}} (\cot z) \frac{d^k}{dz^k} (\cot z)$$

by Leibnitz rule and then apply the closed formula

$$\frac{d^m}{dz^m} (\cot z) = (2i)^m (\cot(z) - i) \sum_{j=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} (i \cot(z) - 1)^k$$

of Lemma 2.1. of [8], where $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$ are the Stirling numbers of the second kind. Now by applying the identity

$$\frac{d^n}{dz^n} \left(\frac{1}{\sin^2 z} \right) = \sum_{k=-\infty}^{\infty} \frac{(-1)^n (n+1)!}{(z + k\pi)^{n+2}}$$

we can find a closed formula for such Dirichlet integral formula for any n . For example, when $n = 3$ we have $\{\alpha_1 = \frac{2}{15}, \alpha_2 = \frac{2}{15}, \alpha_3 = \frac{11}{15}\}$ and for $n = 4$, $\{\alpha_1 = \frac{272}{7!}, \alpha_2 = \frac{64}{7!}, \alpha_3 = \frac{1824}{7!}, \alpha_4 = \frac{2880}{7!}\}$.

Remark. We have the following formulas

$$\begin{aligned} 1) \int_0^\infty \frac{\sin^6 z}{z^6} dz &= \frac{11\pi}{40} \\ 2) \int_0^\infty \frac{\sin^8 z}{z^8} dz &= \frac{151\pi}{630} \\ 3) \int_0^\infty \frac{\sin^{10} z}{z^{10}} dz &= \frac{15619\pi}{72576} \end{aligned}$$

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