An extension of Lobachevsky formula

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In this paper we extend the Dirichlet integral formula of Lobachevsky. Let $f(x)$ be a continuous function and satisfy in the assumption $f(x+\pi)=f(x)$, and $f(\pi-x)=f(x)$, $0\leqslant x<\infty$. If the integral $\int_0^\infty \frac{\sin^4 x}{x^4}f(x)dx$ converges and defined in the sense of the improper Riemann integral, then we show the following equality

$$\int_0^\infty \frac{\sin^4 x}{x^4}f(x)dx=\int_0^{\frac{\pi}{2}} f(t)dt-\frac{2}{3}\int_0^{\frac{\pi}{2}} \sin^2 t f(t)dt$$

hence if we take $f(x)=1$, then we have

$$\int_0^\infty \frac{\sin^4 x}{x^4}dx=\frac{\pi}{3}$$

Moreover,we give an explicit formula for $\int_0^\infty \frac{\sin^n x}{x^n}f(x)dx$ where $n=6,8$ and for general natural number $n\in\mathbb{N}$, we give a method to find such explicit formula.

1. Introduction

The following classical Dirichlet integral has drawn lots of attention.

$$\int_0^\infty \frac{\sin x}{x}dx=\frac{\pi}{2}$$

We can use the theory of residues to evaluate this Dirichlet's integral formula. G.H. Hardy and A. C. Dixon gave a lot of different proofs for it. See [5-7]. In this paper we generalize this Lobachevsky formula.

We start with the following elementary lemma. See [1-4]

LEMMA 1.1 We have

$$\frac{1}{\sin\alpha}=\frac{1}{\alpha}+\sum_{m=1}^{\infty}(-1)^m\left(\frac{1}{\alpha-m\pi}+\frac{1}{\alpha+m\pi}\right)$$

Proof. Let

$$f(x)=1-\frac{\sin x}{\sin\alpha}$$

where $\alpha\neq k\pi$, then we have

$$f(x)=\left(1-\frac{x}{\alpha}\right)\prod_{n=1}^{\infty}\left(1-\frac{x}{(2n-1)\pi-\alpha}\right)\left(1+\frac{x}{(2n-1)\pi+\alpha}\right)\left(1-\frac{x}{2n\pi+\alpha}\right)\left(1+\frac{x}{2n\pi-\alpha}\right)$$

By applying Taylor expansion on $\sin x$,

$$f(x)=1-\frac{x}{\sin\alpha}+\frac{x^3}{3!\sin\alpha}-\frac{x^5}{5!\sin\alpha}+\cdots$$

Now let the right hand side of the above equality as an infinite series. If $a_1,\cdots,a_n,\cdots$ are the roots of $f(x)$, then we can write

$$f(x)=\left(1-\frac{x}{a_1}\right)\left(1-\frac{x}{a_2}\right)\cdots\left(1-\frac{x}{a_n}\right)\cdots=\prod_{k=1}^{\infty}\left(1-\frac{x}{a_k}\right)$$

Now since the roots of $f(x)$ are

$$\begin{cases} 2n\pi+\alpha \\ 2n\pi-\alpha+\pi \quad n=0,\pm1,\pm2,\cdots \end{cases}$$

□

so

$$f(x)=\prod_{n=-\infty}^{\infty}\left(1-\frac{x}{2n\pi+\alpha}\right)\left(1-\frac{x}{2n\pi-\alpha+\pi}\right)$$

hence we have

$$f(x)=\left(1-\frac{x}{\alpha}\right)\prod_{n=1}^{\infty}\left(1-\frac{x}{(2n-1)\pi-\alpha}\right)\left(1+\frac{x}{(2n-1)\pi+\alpha}\right)\left(1-\frac{x}{2n\pi+\alpha}\right)\left(1+\frac{x}{2n\pi-\alpha}\right)$$

By writing the right hand side of the equality $f(x)=1-\frac{\sin x}{\sin\alpha}$ as a power series and equate the coefficients of both sides, the right hand side is infinite elementary symmetric functions, since in general

$$\prod_{j=1}^{n}(\lambda-X_j)=\lambda^n-e_1(X_1,\ldots,X_n)\lambda^{n-1}+e_2(X_1,\ldots,X_n)\lambda^{n-2}+\cdots+(-1)^n e_n(X_1,\ldots,X_n).$$

where for $k\geqslant 0$ we define

$$e_k(X_1,\ldots,X_n)=\sum_{1\leqslant j_1<j_2<\cdots<j_k\leqslant n}X_{j_1}\cdots X_{j_k},$$

so that $e_k(X_1,\cdots,X_n)=0$ if $k>n$. Then we get the desired result.

LEMMA 1.2 We have the following identity

$$\frac{1}{\sin^2\alpha}=\frac{1}{\alpha^2}+\sum_{m=1}^{\infty}\left(\frac{1}{(\alpha-m\pi)^2}+\frac{1}{(\alpha+m\pi)^2}\right)$$

Proof. By the same idea from the proof of Lemma 1.1,by taking $f(x)=1-\frac{\sin^2 x}{\sin^2\alpha}$ we get

$$\frac{1}{\sin^2\alpha}=\frac{1}{\alpha^2}+\sum_{m=1}^{\infty}\left(\frac{1}{((2m-1)\pi-\alpha)^2}+\frac{1}{((2m-1)\pi+\alpha)^2}+\frac{1}{(2m\pi+\alpha)^2}+\frac{1}{(2m\pi-\alpha)^2}\right)$$

so we have

$$\frac{1}{\sin^2\alpha}=\frac{1}{\alpha^2}+\sum_{m=1}^{\infty}\left(\frac{1}{(\alpha-m\pi)^2}+\frac{1}{(\alpha+m\pi)^2}\right)$$

It is worth to mention that we can get the identity of Lemma 1.2, by differentiating termwise the following classical formula,

$$\cot(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{2z}{z^2 - m^2\pi^2}$$

□

2. Lobachevsky formula

Now we present Lobachevsky formula with new technique.

THEOREM 2.1 Let $f(x)$ satisfies in the condition $f(x+\pi) = f(x)$, $f(\pi - x) = f(x)$, $0 \leqslant x < \infty$ and the following integral exists

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} f(x)dx$$

and defined in the sense of the improper Riemann integral, then we have the following Lobachevsky identity

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} f(x)dx = \int_0^{\infty} \frac{\sin x}{x} f(x)dx = \int_0^{\frac{\pi}{2}} f(x)dx$$

Proof. Take

$$I = \int_0^{\infty} \frac{\sin x}{x} f(x)dx$$

we can writ I as follows

$$I = \sum_{v=0}^{\infty} \int_{v\frac{\pi}{2}}^{(v+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x)dx$$

where $v = 2\mu - 1$ or $v = 2\mu$, by changing $x = \mu\pi + t$ or $x = \mu\pi - t$ we get

$$\int_{2\mu\frac{\pi}{2}}^{(2\mu+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x) = (-1)^{\mu} \int_0^{\frac{\pi}{2}} \frac{\sin t}{\mu\pi + t} f(t)dt$$

and

$$\int_{(2\mu-1)\frac{\pi}{2}}^{(2\mu)\frac{\pi}{2}} \frac{\sin x}{x} f(x) = (-1)^{\mu-1} \int_0^{\frac{\pi}{2}} \frac{\sin t}{\mu\pi - t} f(t)dt$$

so we get

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} f(t)dt + \sum_{\mu=1}^{\infty} \int_0^{\frac{\pi}{2}} (-1)^{\mu} f(t) \left(\frac{1}{t + \mu\pi} + \frac{1}{t - \mu\pi} \right) \sin t dt$$

□

Consequently we can write I in the following form

$$I = \int_0^{\frac{\pi}{2}} \sin t \left(\frac{1}{t} + \sum_{\mu=1}^{\infty} (-1)^{\mu} \left(\frac{1}{t + \mu\pi} + \frac{1}{t - \mu\pi} \right) \right) f(t)dt$$

Hence

$$I=\int_0^{\frac{\pi}{2}} f(t)dt$$

and proof of the identity $\int_0^{\infty}\frac{\sin x}{x}f(x)dx=\int_0^{\frac{\pi}{2}}f(x)dx$ is complete. Now we prove the second part of identity. Take

$$J=\int_0^{\infty}\frac{\sin^2 x}{x^2}f(x)dx$$

we can writ J as follows

$$J=\sum_{v=0}^{\infty}\int_{v\frac{\pi}{2}}^{(v+1)\frac{\pi}{2}}\frac{\sin^2 x}{x^2}f(x)dx$$

where $v=2\mu-1$ or $v=2\mu$, by changing $x=\mu\pi+t$ or $x=\mu\pi-t$ we get

$$\int_{2\mu\frac{\pi}{2}}^{(2\mu+1)\frac{\pi}{2}}\frac{\sin^2 x}{x^2}f(x)=\int_0^{\frac{\pi}{2}}\frac{\sin^2 t}{(\mu\pi+t)^2}f(t)dt$$

and

$$\int_{(2\mu-1)\frac{\pi}{2}}^{(2\mu)\frac{\pi}{2}}\frac{\sin^2 x}{x^2}f(x)=\int_0^{\frac{\pi}{2}}\frac{\sin^2 t}{(\mu\pi-t)^2}f(t)dt$$

so we get

$$J=\int_0^{\frac{\pi}{2}}\frac{\sin^2 t}{t^2}f(t)dt+\sum_{\mu=1}^{\infty}\int_0^{\frac{\pi}{2}}f(t)\left(\frac{1}{(t+\mu\pi)^2}+\frac{1}{(t-\mu\pi)^2}\right)\sin^2 t dt$$

Consequently we can write J in the following form

$$J=\int_0^{\frac{\pi}{2}}\sin^2 t\left(\frac{1}{t^2}+\sum_{\mu=1}^{\infty}\left(\frac{1}{(t+\mu\pi)^2}+\frac{1}{(t-\mu\pi)^2}\right)\right)f(t)dt$$

Hence from Lemma 1.2, we get

$$J=\int_0^{\frac{\pi}{2}}f(t)dt$$

and proof is complete

3. Extension of Lobachevsky formula

Now we give a general method for calculating the generalized Dirichlet integral, as follows

$$\int_0^{\infty}\frac{\sin^{2n}x}{x^{2n}}f(x)dx$$

where $f(\pi+x)=f(x)$, and $f(\pi-x)=f(x)$, $0\leqslant x<\infty$. Here we have assumed f is continuous and $\int_0^{\infty}\frac{\sin^{2n}x}{x^{2n}}f(x)dx$ converges and defined in the sense of the improper Riemann integral. We start with $n=2$. As we did in previous section, take

$$I=\int_0^{\infty}\frac{\sin^4 x}{x^4}f(x)dx$$

By a direct computation

$$\frac{d^2}{dx^2}\left(\frac{1}{\sin^2 x}\right)=\frac{6}{\sin^4(x)}-\frac{4}{\sin^2(x)}$$

Next, differentiating twice termwise the right-hand side of identity of Lemma 1.2, we get the following identity

$$\frac{1}{\sin^4\alpha}-\frac{2}{3\sin^2\alpha}=\frac{1}{\alpha^4}+\sum_{m=1}^{\infty}\left(\frac{1}{(\alpha-m\pi)^4}+\frac{1}{(\alpha+m\pi)^4}\right)$$

From previous method which we explained in section 2, we can write I as follows

$$I=\int_0^{\frac{\pi}{2}}\sin^4 t\left(\frac{1}{\sin^4 t}-\frac{2}{3\sin^2 t}\right)f(t)dt$$

Hence

$$I=\int_0^{\frac{\pi}{2}}f(t)dt-\frac{2}{3}\int_0^{\frac{\pi}{2}}\sin^2 t f(t)dt$$

So we proved the following theorem

THEOREM 3.1 Let $f(x)$ satisfies in $f(x+\pi)=f(x)$, and $f(\pi-x)=f(x)$, $0\leqslant x<\infty$. If the following integral

$$\int_0^{\infty}\frac{\sin^4 x}{x^4}f(x)dx$$

exists and defined in the sense of the improper Riemann integral, then we have the following equality

$$\int_0^{\infty}\frac{\sin^4 x}{x^4}f(x)dx=\int_0^{\frac{\pi}{2}}f(t)dt-\frac{2}{3}\int_0^{\frac{\pi}{2}}\sin^2 t f(t)dt$$

As remark if we take $f(x)=1$, then we have

REMARK 3.1 We have

$$\int_0^{\infty}\frac{\sin^4 x}{x^4}dx=\frac{\pi}{3}$$

We have also the following remark from Lobachevsky formula

REMARK 3.2 If $f(x)$ satisfies in the condition $f(x+\pi)=f(x)$, and $f(\pi-x)=f(x)$, $0\leqslant x<\infty$, then

$$I=\int_0^{\infty}\frac{\sin^{2n+1}x}{x}f(x)dx=\int_0^{\infty}\sin^{2n}x f(x)\frac{\sin x}{x}dx$$

Now if we set $\sin^{2n}x f(x)=g(x)$, we get $g(x+\pi)=g(x)$, $g(\pi-x)=g(x)$, now if we take $f(x)=1$, then

$$\int_0^{\infty}\frac{\sin^{2n+1}x}{x}dx=\int_0^{\frac{\pi}{2}}\sin^{2n}x dx=\frac{(2n-1)!!}{(2n)!!}\frac{\pi}{2}$$

We can have the following theorem, which is the generalization of the previous theorem.

THEOREM 3.2 Let the continuous function $f(x)$ satisfies in $f(x+\pi)=f(x)$, and $f(\pi-x)=f(x)$, $0\leqslant x<\infty$. If the following integral

$$\int_0^\infty \frac{\sin^6 x}{x^6} f(x)dx$$

converges and defined in the sense of the improper Riemann integral, then we have the following equality

$$\int_0^\infty \frac{\sin^6 x}{x^6} f(x)dx = \frac{2}{15}\int_0^{\frac{\pi}{2}} f(x)dx + \frac{2}{15}\int_0^{\frac{\pi}{2}} \cot^4(x)\sin^4(x)f(x)dx + \frac{11}{15}\int_0^{\frac{\pi}{2}} \sin^2(x)\cot^2(x)f(x)dx$$

Moreover, if the following integral

$$\int_0^\infty \frac{\sin^8 x}{x^8} f(x)dx$$

converges, then

$$\int_0^\infty \frac{\sin^8 x}{x^8} f(x)dx = \frac{272}{7!}\int_0^{\frac{\pi}{2}} f(x)dx + \frac{64}{7!}\int_0^{\frac{\pi}{2}} \cot^6(x)\sin^6(x)f(x)dx + \frac{1824}{7!}\int_0^{\frac{\pi}{2}} \sin^4(x)\cot^4(x)f(x)dx + \frac{2880}{7!}\int_0^{\frac{\pi}{2}} \sin^2(x)\cot^2(x)f(x)$$

Proof. The proof is as same as previous method, and we just need to compute

$$\frac{d^4}{dx^4}\left(\frac{1}{\sin^2 x}\right) = 16\csc^6(x) + 16\cot^4(x)\csc^2(x) + 88\cot^2(x)\csc^4(x)$$

and

$$\frac{d^6}{dx^6}\left(\frac{1}{\sin^2 x}\right) = 272\csc^8(x) + 64\cot^6(x)\csc^2(x) + 1824\cot^4(x)\csc^4(x) + 2880\cot^2(x)\csc^6(x)$$

and using the identity

$$\frac{d^n}{dx^n}\left(\frac{1}{\sin^2 x}\right) = \sum_{k=-\infty}^{\infty} \frac{(-1)^n(n+1)!}{(x+k\pi)^{n+2}}$$

we get the desired result. □

Now from the previous method, the following Dirichlet integral

$$\int_0^\infty \frac{\sin^{2n} x}{x^{2n}} f(x)dx$$

can have the following expression

$$\alpha_1\int_0^{\frac{\pi}{2}} f(x)dx + \alpha_2\int_0^{\frac{\pi}{2}} \cot^{2n-2}(x)\sin^{2n-2}(x)f(x)dx + \cdots + \alpha_k\int_0^{\frac{\pi}{2}} \cot^2(x)\sin^2(x)f(x)dx$$

for finding the constants α_i, since $1+\cot^2 x = \frac{1}{\sin^2 x}$, by using Leibnitz rule we have

$$\frac{d^n}{dx^n}(1+\cot^2 x) = \sum_{k=0}^{n}\binom{n}{k}\frac{d^{n-k}}{dx^{n-k}}(\cot x)\frac{d^k}{dx^k}(\cot x)$$

and then apply the closed formula

$$\frac{d^m}{dx^m}(\cot x) = (2i)^m(\cot(x)-i)\sum_{j=1}^{m}\left\{ {m \atop k} \right\}(i\cot(x)-1)^k$$

of Lemma 2.1. of [8], where $\left\{ {n \atop k} \right\}$ are the Stirling numbers of the second kind. Now by applying the identity

$$\frac{d^n}{dx^n}\left(\frac{1}{\sin^2 x}\right) = \sum_{k=-\infty}^{\infty}\frac{(-1)^n(n+1)!}{(x+k\pi)^{n+2}}$$

we can find a closed formula for such Dirichlet integral formula for any n.

REMARK 3.3 We have the following formulas

1) $\int_0^\infty \frac{\sin^6 x}{x^6}dx = \frac{11\pi}{40}$

2) $\int_0^\infty \frac{\sin^8 x}{x^8}dx = \frac{151\pi}{630}$

3) $\int_0^\infty \frac{\sin^{10} x}{x^{10}}dx = \frac{15619\pi}{72576}$

4. Acknowledgment

I would like to thank of the anonymous reviewers for their careful reading of my manuscript and their many insightful comments and suggestions and also correcting my explicit formula.

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