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## An extension of Lobachevsky formula

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In this paper we extend the Dirichlet integral formula of Lobachevsky. Let  $f(x)$  be a continuous function and satisfy in the assumption  $f(x + \pi) = f(x)$ , and  $f(\pi - x) = f(x)$ ,  $0 \leq x < \infty$ . If the integral  $\int_0^\infty \frac{\sin^4 x}{x^4} f(x) dx$  converges and defined in the sense of the improper Riemann integral, then we show the following equality

$$\int_0^\infty \frac{\sin^4 x}{x^4} f(x) dx = \int_0^{\frac{\pi}{2}} f(t) dt - \frac{2}{3} \int_0^{\frac{\pi}{2}} \sin^2 t f(t) dt$$

hence if we take  $f(x) = 1$ , then we have

$$\int_0^\infty \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}$$

Moreover, we give an explicit formula for  $\int_0^\infty \frac{\sin^n x}{x^n} f(x) dx$  where  $n = 6, 8$  and for general natural number  $n \in \mathbb{N}$ , we give a method to find such explicit formula.

### 1. Introduction

The following classical Dirichlet integral has drawn lots of attention.

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}$$

We can use the theory of residues to evaluate this Dirichlet's integral formula. G.H. Hardy and A. C. Dixon gave a lot of different proofs for it. See [5-7]. In this paper we generalize this Lobachevsky formula.

We start with the following elementary lemma. See [1-4]

LEMMA 1.1 We have

$$\frac{1}{\sin \alpha} = \frac{1}{\alpha} + \sum_{m=1}^{\infty} (-1)^m \left( \frac{1}{\alpha - m\pi} + \frac{1}{\alpha + m\pi} \right)$$

*Proof.* Let

$$f(x) = 1 - \frac{\sin x}{\sin \alpha}$$

where  $\alpha \neq k\pi$ , then we have

$$f(x) = \left(1 - \frac{x}{\alpha}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{(2n-1)\pi - \alpha}\right) \left(1 + \frac{x}{(2n-1)\pi + \alpha}\right) \left(1 - \frac{x}{2n\pi + \alpha}\right) \left(1 + \frac{x}{2n\pi - \alpha}\right)$$

By applying Taylor expansion on  $\sin x$ ,

$$f(x) = 1 - \frac{x}{\sin \alpha} + \frac{x^3}{3! \sin \alpha} - \frac{x^5}{5! \sin \alpha} + \dots$$

Now let the right hand side of the above equality as an infinite series. If  $a_1, \dots, a_n, \dots$  are the roots of  $f(x)$ , then we can write

$$f(x) = \left(1 - \frac{x}{a_1}\right) \left(1 - \frac{x}{a_2}\right) \dots \left(1 - \frac{x}{a_n}\right) \dots = \prod_{k=1}^{\infty} \left(1 - \frac{x}{a_k}\right)$$

Now since the roots of  $f(x)$  are

$$\begin{cases} 2n\pi + \alpha \\ 2n\pi - \alpha + \pi \end{cases} \quad n = 0, \pm 1, \pm 2, \dots$$

□

so

$$f(x) = \prod_{n=-\infty}^{\infty} \left(1 - \frac{x}{2n\pi + \alpha}\right) \left(1 - \frac{x}{2n\pi - \alpha + \pi}\right)$$

hence we have

$$f(x) = \left(1 - \frac{x}{\alpha}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{(2n-1)\pi - \alpha}\right) \left(1 + \frac{x}{(2n-1)\pi + \alpha}\right) \left(1 - \frac{x}{2n\pi + \alpha}\right) \left(1 + \frac{x}{2n\pi - \alpha}\right)$$

By writing the right hand side of the equality  $f(x) = 1 - \frac{\sin x}{\sin \alpha}$  as a power series and equate the coefficients of both sides, the right hand side is infinite elementary symmetric functions, since in general

$$\prod_{j=1}^n (\lambda - X_j) = \lambda^n - e_1(X_1, \dots, X_n) \lambda^{n-1} + e_2(X_1, \dots, X_n) \lambda^{n-2} + \dots + (-1)^n e_n(X_1, \dots, X_n).$$

where for  $k \geq 0$  we define

$$e_k(X_1, \dots, X_n) = \sum_{1 \leq j_1 < j_2 < \dots < j_k \leq n} X_{j_1} \cdots X_{j_k},$$

so that  $e_k(X_1, \dots, X_n) = 0$  if  $k > n$ . Then we get the desired result.

LEMMA 1.2 We have the following identity

$$\frac{1}{\sin^2 \alpha} = \frac{1}{\alpha^2} + \sum_{m=1}^{\infty} \left( \frac{1}{(\alpha - m\pi)^2} + \frac{1}{(\alpha + m\pi)^2} \right)$$

*Proof.* By the same idea from the proof of Lemma 1.1, by taking  $f(x) = 1 - \frac{\sin^2 x}{\sin^2 \alpha}$  we get

$$\frac{1}{\sin^2 \alpha} = \frac{1}{\alpha^2} + \sum_{m=1}^{\infty} \left( \frac{1}{((2m-1)\pi - \alpha)^2} + \frac{1}{((2m-1)\pi + \alpha)^2} + \frac{1}{(2m\pi + \alpha)^2} + \frac{1}{(2m\pi - \alpha)^2} \right)$$

so we have

$$\frac{1}{\sin^2 \alpha} = \frac{1}{\alpha^2} + \sum_{m=1}^{\infty} \left( \frac{1}{(\alpha - m\pi)^2} + \frac{1}{(\alpha + m\pi)^2} \right)$$

It is worth to mention that we can get the identity of Lemma 1.2, by differentiating termwise the following classical formula,

$$\cot(z) = \frac{1}{z} + \sum_{m=1}^{\infty} \frac{2z}{z^2 - m^2\pi^2}$$

□

## 2. Lobachevsky formula

Now we present Lobachevsky formula with new technique.

**THEOREM 2.1** Let  $f(x)$  satisfies in the condition  $f(x + \pi) = f(x)$ ,  $f(\pi - x) = f(x)$ ,  $0 \leq x < \infty$  and the following integral exists

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} f(x) dx$$

and defined in the sense of the improper Riemann integral, then we have the following Lobachevsky identity

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} f(x) dx = \int_0^{\infty} \frac{\sin x}{x} f(x) dx = \int_0^{\frac{\pi}{2}} f(x) dx$$

*Proof.* Take

$$I = \int_0^{\infty} \frac{\sin x}{x} f(x) dx$$

we can write  $I$  as follows

$$I = \sum_{\nu=0}^{\infty} \int_{\nu\frac{\pi}{2}}^{(\nu+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x) dx$$

where  $\nu = 2\mu - 1$  or  $\nu = 2\mu$ , by changing  $x = \mu\pi + t$  or  $x = \mu\pi - t$  we get

$$\int_{2\mu\frac{\pi}{2}}^{(2\mu+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x) = (-1)^{\mu} \int_0^{\frac{\pi}{2}} \frac{\sin t}{\mu\pi + t} f(t) dt$$

and

$$\int_{(2\mu-1)\frac{\pi}{2}}^{(2\mu)\frac{\pi}{2}} \frac{\sin x}{x} f(x) = (-1)^{\mu-1} \int_0^{\frac{\pi}{2}} \frac{\sin t}{\mu\pi - t} f(t) dt$$

so we get

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} f(t) dt + \sum_{\mu=1}^{\infty} \int_0^{\frac{\pi}{2}} (-1)^{\mu} f(t) \left( \frac{1}{t + \mu\pi} + \frac{1}{t - \mu\pi} \right) \sin t dt$$

□

Consequently we can write  $I$  in the following form

$$I = \int_0^{\frac{\pi}{2}} \sin t \left( \frac{1}{t} + \sum_{\mu=1}^{\infty} (-1)^{\mu} \left( \frac{1}{t + \mu\pi} + \frac{1}{t - \mu\pi} \right) \right) f(t) dt$$

Hence

$$I = \int_0^{\frac{\pi}{2}} f(t) dt$$

and proof of the identity  $\int_0^{\infty} \frac{\sin x}{x} f(x) dx = \int_0^{\frac{\pi}{2}} f(x) dx$  is complete. Now we prove the second part of identity. Take

$$J = \int_0^{\infty} \frac{\sin^2 x}{x^2} f(x) dx$$

we can write  $J$  as follows

$$J = \sum_{\nu=0}^{\infty} \int_{\nu \frac{\pi}{2}}^{(\nu+1) \frac{\pi}{2}} \frac{\sin^2 x}{x^2} f(x) dx$$

where  $\nu = 2\mu - 1$  or  $\nu = 2\mu$ , by changing  $x = \mu\pi + t$  or  $x = \mu\pi - t$  we get

$$\int_{2\mu \frac{\pi}{2}}^{(2\mu+1) \frac{\pi}{2}} \frac{\sin^2 x}{x^2} f(x) dx = \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{(\mu\pi + t)^2} f(t) dt$$

and

$$\int_{(2\mu-1) \frac{\pi}{2}}^{(2\mu) \frac{\pi}{2}} \frac{\sin^2 x}{x^2} f(x) dx = \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{(\mu\pi - t)^2} f(t) dt$$

so we get

$$J = \int_0^{\frac{\pi}{2}} \frac{\sin^2 t}{t^2} f(t) dt + \sum_{\mu=1}^{\infty} \int_0^{\frac{\pi}{2}} f(t) \left( \frac{1}{(t + \mu\pi)^2} + \frac{1}{(t - \mu\pi)^2} \right) \sin^2 t dt$$

Consequently we can write  $J$  in the following form

$$J = \int_0^{\frac{\pi}{2}} \sin^2 t \left( \frac{1}{t^2} + \sum_{\mu=1}^{\infty} \left( \frac{1}{(t + \mu\pi)^2} + \frac{1}{(t - \mu\pi)^2} \right) \right) f(t) dt$$

Hence from Lemma 1.2, we get

$$J = \int_0^{\frac{\pi}{2}} f(t) dt$$

and proof is complete

### 3. Extension of Lobachevsky formula

Now we give a general method for calculating the generalized Dirichlet integral, as follows

$$\int_0^{\infty} \frac{\sin^{2n} x}{x^{2n}} f(x) dx$$

where  $f(\pi + x) = f(x)$ , and  $f(\pi - x) = f(x)$ ,  $0 \leq x < \infty$ . Here we have assumed  $f$  is continuous and  $\int_0^{\infty} \frac{\sin^{2n} x}{x^{2n}} f(x) dx$  converges and defined in the sense of the improper Riemann integral. We start with  $n = 2$ . As we did in previous section, take

$$I = \int_0^{\infty} \frac{\sin^4 x}{x^4} f(x) dx$$

By a direct computation

$$\frac{d^2}{dx^2} \left( \frac{1}{\sin^2 x} \right) = \frac{6}{\sin^4(x)} - \frac{4}{\sin^2(x)}$$

Next, differentiating twice termwise the right-hand side of identity of Lemma 1.2, we get the following identity

$$\frac{1}{\sin^4 \alpha} - \frac{2}{3 \sin^2 \alpha} = \frac{1}{\alpha^4} + \sum_{m=1}^{\infty} \left( \frac{1}{(\alpha - m\pi)^4} + \frac{1}{(\alpha + m\pi)^4} \right)$$

From previous method which we explained in section 2, we can write  $I$  as follows

$$I = \int_0^{\frac{\pi}{2}} \sin^4 t \left( \frac{1}{\sin^4 t} - \frac{2}{3 \sin^2 t} \right) f(t) dt$$

Hence

$$I = \int_0^{\frac{\pi}{2}} f(t) dt - \frac{2}{3} \int_0^{\frac{\pi}{2}} \sin^2 t f(t) dt$$

So we proved the following theorem

**THEOREM 3.1** Let  $f(x)$  satisfies in  $f(x + \pi) = f(x)$ , and  $f(\pi - x) = f(x)$ ,  $0 \leq x < \infty$ . If the following integral

$$\int_0^{\infty} \frac{\sin^4 x}{x^4} f(x) dx$$

exists and defined in the sense of the improper Riemann integral, then we have the following equality

$$\int_0^{\infty} \frac{\sin^4 x}{x^4} f(x) dx = \int_0^{\frac{\pi}{2}} f(t) dt - \frac{2}{3} \int_0^{\frac{\pi}{2}} \sin^2 t f(t) dt$$

As remark if we take  $f(x) = 1$ , then we have

**REMARK 3.1** We have

$$\int_0^{\infty} \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}$$

We have also the following remark from Lobachevsky formula

**REMARK 3.2** If  $f(x)$  satisfies in the condition  $f(x + \pi) = f(x)$ , and  $f(\pi - x) = f(x)$ ,  $0 \leq x < \infty$ , then

$$I = \int_0^{\infty} \frac{\sin^{2n+1} x}{x} f(x) dx = \int_0^{\infty} \sin^{2n} x f(x) \frac{\sin x}{x} dx$$

Now if we set  $\sin^{2n} x f(x) = g(x)$ , we get  $g(x + \pi) = g(x)$ ,  $g(\pi - x) = g(x)$ , now if we take  $f(x) = 1$ , then

$$\int_0^{\infty} \frac{\sin^{2n+1} x}{x} dx = \int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2}$$

We can have the following theorem, which is the generalization of the previous theorem.

**THEOREM 3.2** Let the continuous function  $f(x)$  satisfies in  $f(x + \pi) = f(x)$ , and  $f(\pi - x) = f(x)$ ,  $0 \leq x < \infty$ . If the following integral

$$\int_0^{\infty} \frac{\sin^6 x}{x^6} f(x) dx$$

converges and defined in the sense of the improper Riemann integral, then we have the following equality

$$\int_0^{\infty} \frac{\sin^6 x}{x^6} f(x) dx = \frac{2}{15} \int_0^{\frac{\pi}{2}} f(x) dx + \frac{2}{15} \int_0^{\frac{\pi}{2}} \cot^4(x) \sin^4(x) f(x) dx + \frac{11}{15} \int_0^{\frac{\pi}{2}} \sin^2(x) \cot^2(x) f(x) dx$$

Moreover, if the following integral

$$\int_0^{\infty} \frac{\sin^8 x}{x^8} f(x) dx$$

converges, then

$$\int_0^{\infty} \frac{\sin^8 x}{x^8} f(x) dx = \frac{272}{7!} \int_0^{\frac{\pi}{2}} f(x) dx + \frac{64}{7!} \int_0^{\frac{\pi}{2}} \cot^6(x) \sin^6(x) f(x) dx + \frac{1824}{7!} \int_0^{\frac{\pi}{2}} \sin^4(x) \cot^4(x) f(x) dx + \frac{2880}{7!} \int_0^{\frac{\pi}{2}} \sin^2(x) \cot^2(x) f(x) dx$$

*Proof.* The proof is as same as previous method, and we just need to compute

$$\frac{d^4}{dx^4} \left( \frac{1}{\sin^2 x} \right) = 16 \csc^6(x) + 16 \cot^4(x) \csc^2(x) + 88 \cot^2(x) \csc^4(x)$$

and

$$\frac{d^6}{dx^6} \left( \frac{1}{\sin^2 x} \right) = 272 \csc^8(x) + 64 \cot^6(x) \csc^2(x) + 1824 \cot^4(x) \csc^4(x) + 2880 \cot^2(x) \csc^6(x)$$

and using the identity

$$\frac{d^n}{dx^n} \left( \frac{1}{\sin^2 x} \right) = \sum_{k=-\infty}^{\infty} \frac{(-1)^k (n+1)!}{(x+k\pi)^{n+2}}$$

we get the desired result. □

Now from the previous method, the following Dirichlet integral

$$\int_0^{\infty} \frac{\sin^{2n} x}{x^{2n}} f(x) dx$$

can have the following expression

$$\alpha_1 \int_0^{\frac{\pi}{2}} f(x) dx + \alpha_2 \int_0^{\frac{\pi}{2}} \cot^{2n-2}(x) \sin^{2n-2}(x) f(x) dx + \dots + \alpha_k \int_0^{\frac{\pi}{2}} \cot^2(x) \sin^2(x) f(x) dx$$

for finding the constants  $\alpha_i$ , since  $1 + \cot^2 x = \frac{1}{\sin^2 x}$ , by using Leibnitz rule we have

$$\frac{d^n}{dx^n} (1 + \cot^2 x) = \sum_{k=0}^n \binom{n}{k} \frac{d^{n-k}}{dx^{n-k}} (\cot x) \frac{d^k}{dx^k} (\cot x)$$

and then apply the closed formula

$$\frac{d^m}{dx^m}(\cot x) = (2i)^m(\cot(x) - i) \sum_{j=1}^m \left\{ \begin{matrix} m \\ k \end{matrix} \right\} (i \cot(x) - 1)^k$$

of Lemma 2.1. of [8], where  $\left\{ \begin{matrix} n \\ k \end{matrix} \right\}$  are the Stirling numbers of the second kind. Now by applying the identity

$$\frac{d^n}{dx^n} \left( \frac{1}{\sin^2 x} \right) = \sum_{k=-\infty}^{\infty} \frac{(-1)^n (n+1)!}{(x+k\pi)^{n+2}}$$

we can find a closed formula for such Dirichlet integral formula for any  $n$ .

**REMARK 3.3** We have the following formulas

$$\begin{aligned} 1) \int_0^{\infty} \frac{\sin^6 x}{x^6} dx &= \frac{11\pi}{40} \\ 2) \int_0^{\infty} \frac{\sin^8 x}{x^8} dx &= \frac{151\pi}{630} \\ 3) \int_0^{\infty} \frac{\sin^{10} x}{x^{10}} dx &= \frac{15619\pi}{72576} \end{aligned}$$

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