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► **To cite this version:**

| Hassan Jolany. An extension of Lobachevsky formula. 2017. hal-01539895v1

**HAL Id: hal-01539895**

**<https://hal.science/hal-01539895v1>**

Preprint submitted on 15 Jun 2017 (v1), last revised 13 Sep 2017 (v3)

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## An extension of Lobachevsky formula

HASSAN JOLANY

In this paper we extend the Dirichlet integral formula of Lobachevsky. Let  $f(x)$  satisfies in  $f(x + \pi) = f(x)$ , and  $f(\pi - x) = f(x)$ ,  $0 \leq x < \infty$ . If the integral  $\int_0^\infty \frac{\sin^4 x}{x^4} f(x) dx$  exists then we show the following equality

$$\int_0^\infty \frac{\sin^4 x}{x^4} f(x) dx = \int_0^{\frac{\pi}{2}} f(t) dt + \frac{2}{3} \int_0^{\frac{\pi}{2}} \sin^2 t f(t) dt$$

hence if we take  $f(x) = 1$ , then we have

$$\int_0^\infty \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}$$

Extension of Lobachevsky formula was a part of my bachelor project in 2007.

### 1. Introduction

We start with the following elementary lemma. See [1-4]

LEMMA 1.1 Let

$$f(x) = 1 - \frac{\sin x}{\sin \alpha}$$

where  $\alpha \neq k\pi$ , then we have

$$f(x) = \left(1 - \frac{x}{\alpha}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{(2n-1)\pi - \alpha}\right) \left(1 + \frac{x}{(2n-1)\pi + \alpha}\right) \left(1 - \frac{x}{2n\pi + \alpha}\right) \left(1 + \frac{x}{2n\pi - \alpha}\right)$$

*Proof.* By applying Taylor expansion on  $\sin x$ ,

$$f(x) = 1 - \frac{x}{\sin \alpha} + \frac{x^3}{3! \sin \alpha} - \frac{x^5}{5! \sin \alpha} + \dots$$

Now let the right hand side of the above equality as a polynomial of infinite degree. If  $a_1, \dots, a_n, \dots$  are the roots of  $f(x)$ , then we can write

$$f(x) = \left(1 - \frac{x}{a_1}\right) \left(1 - \frac{x}{a_2}\right) \dots \left(1 - \frac{x}{a_n}\right) \dots = \prod_{k=1}^{\infty} \left(1 - \frac{x}{a_k}\right)$$

Now since the roots of  $f(x)$  are

$$\begin{cases} 2n\pi + \alpha \\ 2n\pi - \alpha + \pi \end{cases} \quad n = 0, \pm 1, \pm 2, \dots$$

□

so

$$f(x) = \prod_{n=-\infty}^{\infty} \left(1 - \frac{x}{2n\pi + \alpha}\right) \left(1 - \frac{x}{2n\pi - \alpha + \pi}\right)$$

hence we have

$$f(x) = \left(1 - \frac{x}{\alpha}\right) \prod_{n=1}^{\infty} \left(1 - \frac{x}{(2n-1)\pi - \alpha}\right) \left(1 + \frac{x}{(2n-1)\pi + \alpha}\right) \left(1 - \frac{x}{2n\pi + \alpha}\right) \left(1 + \frac{x}{2n\pi - \alpha}\right)$$

LEMMA 1.2 We have the following two identities

$$\frac{1}{\sin \alpha} = \frac{1}{\alpha} + \sum_{m=1}^{\infty} (-1)^m \left( \frac{1}{\alpha - m\pi} + \frac{1}{\alpha + m\pi} \right)$$

and

$$\frac{1}{\sin^2 \alpha} = \frac{1}{\alpha^2} + \sum_{m=1}^{\infty} \left( \frac{1}{(\alpha - m\pi)^2} + \frac{1}{(\alpha + m\pi)^2} \right)$$

*Proof.* By writing the right hand side of the equality  $f(x) = 1 - \frac{\sin x}{\sin \alpha}$  as a power series and equate the coefficients of both sides, the right hand side is infinite elementary symmetric functions. Now we use the Newton relations in infinite case. Put

$$\sigma_m = \sum_{i_1, \dots, i_m} a_{i_1} a_{i_2} \cdots a_{i_m} \text{ and } S_m = \sum_{i=1}^{\infty} a_i^m$$

Then we get

$$S_1 = \sigma_1,$$

$$S_2 = \sigma_1^2 - 2\sigma_2,$$

$$S_3 = \sigma_1^3 - 3\sigma_2\sigma_1 + 3\sigma_3,$$

$$S_4 = \sigma_1^4 - 4\sigma_2\sigma_1^2 + 4\sigma_3\sigma_1 + 2\sigma_2^2 - 4\sigma_4,$$

$$S_5 = \sigma_1^5 - 5\sigma_2\sigma_1^3 + 5\sigma_3\sigma_1^2 + 5\sigma_2^2\sigma_1 - 5\sigma_4\sigma_1 - 5\sigma_3\sigma_2 + 5\sigma_5,$$

$$S_6 = \sigma_1^6 - 6\sigma_2\sigma_1^4 + 6\sigma_3\sigma_1^3 + 9\sigma_2^2\sigma_1^2 - 6\sigma_4\sigma_1^2 - 12\sigma_3\sigma_2\sigma_1 + 6\sigma_5\sigma_1 - 2\sigma_2^3 + 3\sigma_3^2 + 6\sigma_4\sigma_2 - 6\sigma_6.$$

The general formula (for all non-negative integers "m") is:

$$S_m = \sum_{\substack{r_1+2r_2+\dots+mr_m=m \\ r_1 \geq 0, \dots, r_m \geq 0}} (-1)^m \frac{m(r_1+r_2+\dots+r_m-1)!}{r_1!r_2!\dots r_m!} \prod_{i=1}^m (-\sigma_i)^{r_i}$$

This can be conveniently stated in terms of logarithmic polynomials as

$$S_m = \frac{(-1)^{n-1}}{(n-1)!} L_m(\sigma_1, 2!\sigma_2, 3!\sigma_3, \dots, m!\sigma_m).$$

The logarithmic polynomials are related to the partial exponential Bell polynomials by

$$L_n(x_1, x_2, \dots, x_n) = \sum_{k=1}^n (-1)^{k-1} (k-1)! B_{n,k}(x_1, \dots, x_{n-k+1}).$$

$$\frac{1}{\sin \alpha} = \frac{1}{\alpha} + \sum_{m=1}^{\infty} (-1)^m \left( \frac{1}{\alpha - m\pi} + \frac{1}{\alpha + m\pi} \right)$$

but we can write it as

$$\frac{1}{\sin \alpha} = \frac{1}{\alpha} + \sum_{m=1}^{\infty} \left( \frac{1}{(2m-1)\pi - \alpha} - \frac{1}{(2m-1)\pi + \alpha} + \frac{1}{2m\pi + \alpha} - \frac{1}{2m\pi - \alpha} \right)$$

and hence

$$\frac{1}{\sin^2 \alpha} = \frac{1}{\alpha^2} + \sum_{m=1}^{\infty} \left( \frac{1}{((2m-1)\pi - \alpha)^2} + \frac{1}{((2m-1)\pi + \alpha)^2} + \frac{1}{(2m\pi + \alpha)^2} + \frac{1}{(2m\pi - \alpha)^2} \right)$$

so we get

$$\frac{1}{\sin^2 \alpha} = \frac{1}{\alpha^2} + \sum_{m=1}^{\infty} \left( \frac{1}{(\alpha - m\pi)^2} + \frac{1}{(\alpha + m\pi)^2} \right)$$

□

## 2. Lobachevsky formula

Now we present Lobachevsky formula with new technique.

**THEOREM 2.1** Let  $f(x)$  satisfies in the condition  $f(x + \pi) = f(x)$ ,  $f(\pi - x) = f(x)$ ,  $0 \leq x < \infty$  and the following integral exists

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} f(x) dx$$

then we have the following Lobachevsky identity

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} f(x) dx = \int_0^{\infty} \frac{\sin x}{x} f(x) dx = \int_0^{\frac{\pi}{2}} f(x) dx$$

*Proof.* Take

$$I = \int_0^{\infty} \frac{\sin x}{x} f(x) dx$$

we can write  $I$  as follows

$$I = \sum_{v=0}^{\infty} \int_{v\frac{\pi}{2}}^{(v+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x) dx$$

where  $v = 2\mu - 1$  or  $v = 2\mu$ , by changing  $x = \mu\pi + t$  or  $x = \mu\pi - t$  we get

$$\int_{2\mu\frac{\pi}{2}}^{(2\mu+1)\frac{\pi}{2}} \frac{\sin x}{x} f(x) = (-1)^{\mu} \int_0^{\frac{\pi}{2}} \frac{\sin t}{\mu\pi + t} f(t) dt$$

and

$$\int_{(2\mu-1)\frac{\pi}{2}}^{(2\mu)\frac{\pi}{2}} \frac{\sin x}{x} f(x) = (-1)^{\mu-1} \int_0^{\frac{\pi}{2}} \frac{\sin t}{\mu\pi - t} f(t) dt$$

so we get

$$I = \int_0^{\frac{\pi}{2}} \frac{\sin t}{t} f(t) dt + \sum_{\mu=1}^{\infty} \int_0^{\frac{\pi}{2}} (-1)^{\mu} f(t) \left( \frac{1}{t + \mu\pi} + \frac{1}{t - \mu\pi} \right) \sin t dt$$

□

Consequently we can write  $I$  in the following form

$$I = \int_0^{\frac{\pi}{2}} \sin t \left( \frac{1}{t} + \sum_{\mu=1}^{\infty} (-1)^{\mu} \left( \frac{1}{t + \mu\pi} + \frac{1}{t - \mu\pi} \right) \right) f(t) dt$$

Hence

$$I = \int_0^{\frac{\pi}{2}} f(t) dt$$

and proof is complete

### 3. Extension of Lobachevsky formula

Now we give a general method for calculating the generalized Dirichlet integral (which could be written as Bell polynomials), as follows

$$\int_0^{\infty} \frac{\sin^{2n} x}{x^{2n}} f(x) dx$$

where  $f(\pi + x) = f(x)$ , and  $f(\pi - x) = f(x)$ ,  $0 \leq x < \infty$

We start with  $n = 2$ . As we did in previous section, take

$$I = \int_0^{\infty} \frac{\sin^4 x}{x^4} f(x) dx$$

we can have the following identity from Newton formula

$$\frac{1}{\sin^4 \alpha} + \frac{2}{3 \sin^2 \alpha} = \frac{1}{\alpha^4} + \sum_{m=1}^{\infty} \left( \frac{1}{(\alpha - m\pi)^4} + \frac{1}{(\alpha + m\pi)^4} \right)$$

From previous method which we explained in section 2, we can write  $I$  as follows

$$I = \int_0^{\frac{\pi}{2}} \sin^4 t \left( \frac{1}{\sin^4 t} + \frac{2}{3 \sin^2 t} \right) f(t) dt$$

Hence

$$I = \int_0^{\frac{\pi}{2}} f(t) dt + \frac{2}{3} \int_0^{\frac{\pi}{2}} \sin^2 t f(t) dt$$

So we proved the following theorem

**THEOREM 3.1** Let  $f(x)$  satisfies in  $f(x + \pi) = f(x)$ , and  $f(\pi - x) = f(x)$ ,  $0 \leq x < \infty$ . If the following integral

$$\int_0^{\infty} \frac{\sin^4 x}{x^4} f(x) dx$$

exists then we have the following equality

$$\int_0^{\infty} \frac{\sin^4 x}{x^4} f(x) dx = \int_0^{\frac{\pi}{2}} f(t) dt + \frac{2}{3} \int_0^{\frac{\pi}{2}} \sin^2 t f(t) dt$$

As remark if we take  $f(x) = 1$ , then we have

REMARK 3.1 We have

$$\int_0^{\infty} \frac{\sin^4 x}{x^4} dx = \frac{\pi}{3}$$

We have also the following remark from Lobachevsky formula

REMARK 3.2 If  $f(x)$  satisfies in the condition  $f(x + \pi) = f(x)$ , and  $f(\pi - x) = f(x)$ ,  $0 \leq x < \infty$ , then

$$I = \int_0^{\infty} \frac{\sin^{2n+1} x}{x} f(x) dx = \int_0^{\infty} \sin^{2n} x f(x) \frac{\sin x}{x} dx$$

Now if we set  $\sin^{2n} x f(x) = g(x)$ , we get  $g(x + \pi) = g(x)$ ,  $g(\pi - x) = g(x)$ , now if we take  $f(x) = 1$ , then

$$\int_0^{\infty} \frac{\sin^{2n+1} x}{x} dx = \int_0^{\frac{\pi}{2}} \sin^{2n} x dx = \frac{(2n-1)!!}{(2n)!!} \frac{\pi}{2}$$

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- [4] P. Stanley, Richard, Enumerative Combinatorics, vol. 2. Cambridge University Press

#### REFERENCES

I would like to thanks of my Bachelor project teacher for introducing me this question in 2007. He informed me that professor **Mohsen Hachtroudi** proposed this question to him long times ago when he was his student in Tabriz university. It is worth to mention that Mohsen Hachtroudi was Iranian PhD student of **Elie Cartan** and wrote his thesis entitling Les espaces d'elements a connexion projective normale in Universite Paris IV-Sorbonne in 1936