BRANCHED HOLOMORPHIC CARTAN GEOMETRIES AND CALABI-YAU MANIFOLDS

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Abstract. We introduce the concept of a *branched holomorphic Cartan geometry*. It generalizes to higher dimension the definition of branched (flat) complex projective structure on a Riemann surface introduced by Mandelbaum. This new framework is much more flexible than that of the usual holomorphic Cartan geometries. We show that all compact complex projective manifolds admit branched flat holomorphic projective structure. We also give an example of a non-flat branched holomorphic normal projective structure on a compact complex surface. It is known that no compact complex surface admits such a structure with empty branching locus. We prove that non-projective compact simply connected Kähler Calabi-Yau manifolds do not admit branched holomorphic projective structures. The key ingredient of its proof is the following result of independent interest: *If E is a holomorphic vector bundle over a compact simply connected Kähler Calabi-Yau manifold, and E admits a holomorphic connection, then E is a trivial holomorphic vector bundle equipped with the trivial connection.*

Contents

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1. Introduction

The uniformization theorem for Riemann surfaces asserts that any Riemann surface is isomorphic either to the projective line $\mathbb{C}P^1$, or to a quotient of $\mathbb{C}$ or of the unit disk in $\mathbb{C}$ by a discrete group of projective transformations (lying in the Möbius group $\mathrm{PSL}(2, \mathbb{C})$). In particular, any Riemann surface X admits a holomorphic atlas with coordinates in $\mathbb{C}P^1$ and transition maps in $\mathrm{PSL}(2, \mathbb{C})$. This defines a *(flat) complex projective structure* on X. Complex projective structures on Riemann surfaces were introduced on the study of the second order ordinary differential equations in the complex domain and had a very major role in understanding the framework of uniformization theorem [Gu, StG].

The complex projective line acted on by the Möbius group is a geometry in the sense of Klein's Erlangen program in which he proposed to study Euclidean, affine and projective geometries in the unifying frame of the homogeneous model spaces G/H, where G is a (finite dimensional) Lie group and H a closed subgroup in G.

Following Ehresmann [Eh], a manifold X is locally modelled on a homogeneous space G/H, if X admits an atlas with charts in G/H and transition maps given by elements in G using its left-translation action on G/H. Any G-invariant geometric feature of G/H will have an intrinsic meaning on X.

Elie Cartan generalized Klein's homogeneous model spaces to *Cartan geometries* (or *Cartan connections*) (see definition in Section 2.1). We recall that these are geometrical structures infinitesimally modelled on homogeneous spaces G/H. The Cartan geometry associated to the affine (respectively, projective) geometry is classically called an affine (respectively, projective) connection. A Cartan geometry on a manifold X is equipped with a curvature tensor (see definition in Section 2.1) which vanishes exactly when X is locally modelled on G/H in the sense of Ehresmann [Eh]. In such a situation the Cartan geometry is called *flat*.

In this article we study holomorphic Cartan geometries on compact complex manifolds of complex dimension at least two. Contrary to the situation of Riemann surfaces, holomorphic Cartan geometries in higher dimension are not always flat. Moreover, for a compact complex manifold, to admit a holomorphic Cartan geometry is a very stringent condition: most of the compact complex manifolds do not admit any holomorphic Cartan geometry.

In [KO], Kobayashi and Ochiai proved that compact complex surfaces admitting a holomorphic projective connection are biholomorphic either to the complex projective plane $\mathbb{C}P^2$, or to a quotient of an open set in $\mathbb{C}P^2$ by a discrete group of projective transformations acting properly and discontinuously on it. In particular, they also admit a flat complex projective structure (modelled on $\mathbb{C}P^2$). In this list of compact complex surfaces admitting (flat) complex projective structures, the only *projective* ones are $\mathbb{C}P^2$, abelian varieties (and their unramified finite quotients) and quotients of the ball (complex hyperbolic plane).

Another inspiring source of this paper is the work of Mandelbaum [Ma1, Ma2] who introduced and studied *branched affine and projective structures* on Riemann surfaces. According to his definition, branched projective structures on Riemann surfaces are given by holomorphic atlas where local charts are finite branched coverings on open sets in $\mathbb{C}P^1$ and transition maps lie in $\mathrm{PSL}(2,\mathbb{C})$). Such structures arise naturally in the study of conical hyperbolic structures, and also when one consider ramified coverings.

Here we define a more general notion of *branched holomorphic Cartan geometry* on a complex manifold X (see Definition 2.1), which is valid also in higher dimension and for non-flat geometries. We show that the notion of curvature continues to hold, and in fact the curvature vanishes exactly when there is a holomorphic atlas where local charts are branched holomorphic maps to the model G/H. Two local charts agree up to the action on G/H of an element in G. The geometric description of the flat case follows the usual one: there exists a branched holomorphic developing map from the universal cover of X to the model G/H which is a local biholomorphism away from a divisor. This developing map is equivariant with respect to the monodromy homomorphism (which is a group homomorphism from the fundamental group of X into G, unique up to post-composition by inner automorphisms of G).

This new notion of branched Cartan geometry is much more flexible than the usual one. For example, all compact complex projective manifolds admit branched flat holomorphic projective structures (see Proposition 3.1).

We also prove that there exists branched *normal* holomorphic projective connections (see definition in Section 2.2) on compact surfaces which are not flat (see Proposition 3.4). This is not the case for holomorphic projective connections with empty branching set, meaning any normal projective structure on a compact complex surface is automatically flat [Du3].

The following is proved in Theorem 6.2: *if E is a holomorphic vector bundle over a compact simply connected Kähler Calabi-Yau manifold, and E admits a holomorphic connection, then E is a trivial holomorphic vector bundle equipped with the trivial connection.*

This result, which is of independent interest, is related to the classification of branched holomorphic Cartan geometries on Calabi-Yau manifolds. It yields Corollary 6.3 asserting that *non-projective compact simply connected Kähler Calabi-Yau manifolds do not admit branched holomorphic projective structures.*

The structure of this paper is as follows. Section 2 introduces the main notations and definitions. Section 3 gives interesting examples of branched holomorphic Cartan geometries and contains the proofs of Proposition 3.1 and Proposition 3.4. In Section 4 we give a criterion (Theorem 4.1) for the existence of branched holomorphic Cartan geometries. In Section 5 we study holomorphic projective structures on compact parallelizable manifolds. Section 6 deals with branched holomorphic Cartan geometries on Calabi-Yau manifolds and it contains the proofs of Theorem 6.2 and Corollary 6.3.

2. Holomorphic Cartan geometry and branched holomorphic Cartan geometry

2.1. **Holomorphic Cartan geometry.** We first recall the definition of a holomorphic Cartan geometry.

Let G be a connected complex Lie group and $H \subset G$ a connected complex Lie subgroup. The Lie algebras of H and G will be denoted by $\mathfrak{h}$ and $\mathfrak{g}$ respectively.

Let X be a connected complex manifold and

$$f : E_H \longrightarrow X \tag{2.1}$$

a holomorphic principal H–bundle on X. Let

$$E_G := E_H \times^H G \xrightarrow{f_G} X \tag{2.2}$$

be the holomorphic principal G–bundle on X obtained by extending the structure group of E_H using the inclusion of H in G. So E_G is the quotient of $E_H \times G$ where two points $(c_1, g_1), (c_2, g_2) \in E_H \times G$ are identified if there is an element $h \in H$ such that $c_2 = c_1 h$ and $g_2 = h^{-1}g_1$. The projection f_G in (2.2) is induced by the map $E_H \times G \longrightarrow X$, $(c, g) \longmapsto f(c)$, where f is the projection in (2.1). The action of G on E_G is induced by the action of G on $E_H \times G$ given by the right–translation action of G on itself. Let $\text{ad}(E_H) = E_H \times^H \mathfrak{h}$ and $\text{ad}(E_G) = E_G \times^G \mathfrak{g}$ be the adjoint vector bundles for E_H and E_G respectively. We have a short exact sequence of holomorphic vector bundles on X

$$0 \longrightarrow \text{ad}(E_H) \xrightarrow{\iota_1} \text{ad}(E_G) \longrightarrow \text{ad}(E_G)/\text{ad}(E_H) \longrightarrow 0\,. \tag{2.3}$$

The holomorphic tangent bundle of a complex manifold Y will be denoted by TY. Let

$$\text{At}(E_H) = (TE_H)/H \longrightarrow X \quad \text{and} \quad \text{At}(E_G) = (TE_G)/G \longrightarrow X$$

be the Atiyah bundles for E_H and E_G respectively; see [At]. Let

$$0 \longrightarrow \text{ad}(E_H) \xrightarrow{\iota_2} \text{At}(E_H) \xrightarrow{q_H} TX \longrightarrow 0 \tag{2.4}$$

and

$$0 \longrightarrow \text{ad}(E_G) \longrightarrow \text{At}(E_G) \xrightarrow{q_G} TX \longrightarrow 0 \tag{2.5}$$

be the Atiyah exact sequences for E_H and E_G respectively; see [At]. The projection q_H (respectively, q_G) is induced by the differential of the map f (respectively, f_G). A holomorphic connection on a holomorphic principal bundle is a holomorphic splitting of the Atiyah exact sequence associated to the principal bundle [At].

A holomorphic Cartan geometry on X of type G/H is a pair (E_H, θ), where E_H is a holomorphic principal H–bundle on X and

$$\theta : \text{At}(E_H) \longrightarrow \text{ad}(E_G)$$

is a holomorphic isomorphism of vector bundles such that $\theta \circ \iota_2 = \iota_1$ (see (2.4) and (2.3)). So we have the following commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \text{ad}(E_H) & \xrightarrow{\iota_2} & \text{At}(E_H) & \xrightarrow{q_H} & TX & \longrightarrow & 0\\
 & & \| & & \big\downarrow\theta & & \big\downarrow\phi & & \\
0 & \longrightarrow & \text{ad}(E_H) & \xrightarrow{\iota_1} & \text{ad}(E_G) & \longrightarrow & \text{ad}(E_G)/\text{ad}(E_H) & \longrightarrow & 0
\end{array}$$

[Sh, Ch. 5]; the above homomorphism ϕ induced by θ is evidently an isomorphism.

We can embed $\text{ad}(E_H)$ in $\text{At}(E_H) \oplus \text{ad}(E_G)$ by sending any v to $(\iota_2(v), -\iota_1(v))$ (see (2.4), (2.3)). The Atiyah bundle $\text{At}(E_G)$ is the quotient $(\text{At}(E_H) \oplus \text{ad}(E_G))/\text{ad}(E_H)$ for this embedding. The inclusion of $\mathrm{ad}(E_G)$ in $\mathrm{At}(E_G)$ in (2.5) is given by the inclusion of $\mathrm{ad}(E_G)$ in $\mathrm{At}(E_H)\oplus \mathrm{ad}(E_G)$.

Given a holomorphic Cartan geometry $(E_H,\, \theta)$ of type G/H on X, the homomorphism

$$\mathrm{At}(E_H)\oplus \mathrm{ad}(E_G) \longrightarrow \mathrm{ad}(E_G), \quad (v,\, w) \longmapsto \theta(v)+w$$

produces a homomorphism

$$\theta' \,:\, \mathrm{At}(E_G) \longrightarrow \mathrm{ad}(E_G)$$

which is a holomorphic splitting of (2.5). Therefore, θ' is a holomorphic connection on the principal G–bundle E_G.

The curvature $\mathrm{Curv}(\theta')$ of θ' is a holomorphic section

$$\mathrm{Curv}(\theta') \,\in\, H^0(X,\, \mathrm{ad}(E_G)\otimes \Omega^2_X)\,,$$

where $\Omega^i_X \,:=\, \bigwedge^i (TX)^*$.

The Cartan geometry $(E_H,\, \theta)$ is called *normal* if

$$\mathrm{Curv}(\theta') \,\in\, H^0(X,\, \mathrm{ad}(E_H)\otimes \Omega^2_X)$$

[Sh, Ch. 8, § 2, p. 338].

The Cartan geometry $(E_H,\, \theta)$ is called *flat* if

$$\mathrm{Curv}(\theta') \,=\, 0$$

[Sh, Ch. 5, § 1, p. 177]. So flat Cartan geometries are normal.

If $(E_H,\, \theta)$ is a holomorphic Cartan geometry, then the isomorphism θ can be interpreted as a $\mathfrak{g}$–valued holomorphic 1–form β on E_H satisfying the following three conditions:

(1) the homomorphism $\beta \,:\, TE_H \longrightarrow E_H\times \mathfrak{g}$ is an isomorphism,
(2) β is H–equivariant with H acting on $\mathfrak{g}$ via conjugation, and
(3) the restriction of β to each fiber of f coincides with the Maurer–Cartan form associated to the action of H on E_H.

2.2. **Branched holomorphic Cartan geometry.**

Definition 2.1. A *branched* holomorphic Cartan geometry on X of type G/H is a pair $(E_H,\, \theta)$, where E_H is a holomorphic principal H–bundle on X and

$$\theta \,:\, \mathrm{At}(E_H) \longrightarrow \mathrm{ad}(E_G)$$

is a holomorphic homomorphism of vector bundles, such that following three conditions hold:

(1) θ is an isomorphism over a nonempty open subset of X, and
(2) $\theta\circ\iota_2 \,=\, \iota_1$ (see (2.4) and (2.3)).

In other words, we have a commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{ad}(E_H) & \stackrel{\iota_2}{\longrightarrow} & \mathrm{At}(E_H) & \stackrel{q_H}{\longrightarrow} & TX & \longrightarrow & 0\\
 & & \| & & \Big\downarrow\theta & & \Big\downarrow\phi & & \\
0 & \longrightarrow & \mathrm{ad}(E_H) & \stackrel{\iota_1}{\longrightarrow} & \mathrm{ad}(E_G) & \longrightarrow & \mathrm{ad}(E_G)/\mathrm{ad}(E_H) & \longrightarrow & 0
\end{array} \tag{2.6}$$

of holomorphic vector bundles on X, where ϕ is induced by θ.

Let $U \subset X$ be the nonempty open subset over which θ is an isomorphism. From the commutativity of (2.6) it follows that ϕ is an isomorphism exactly over U.

Lemma 2.2. *The complement $X \setminus U$ is a divisor.*

Proof. Let d be the complex dimension of X. The homomorphism ϕ in (2.6) produces a homomorphism

$$\bigwedge^d \phi \,:\, \bigwedge^d TX \,\longrightarrow\, \bigwedge^d (\mathrm{ad}(E_G)/\mathrm{ad}(E_H))\,, \tag{2.7}$$

so $\bigwedge^d \phi$ is a holomorphic section of the line bundle $(\bigwedge^d(\mathrm{ad}(E_G)/\mathrm{ad}(E_H))) \otimes \Omega^d_X$. The complement $X \setminus U$ coincides with the support of the divisor associated to this section $\bigwedge^d \phi$ of $(\bigwedge^d(\mathrm{ad}(E_G)/\mathrm{ad}(E_H))) \otimes \Omega^d_X$. $\square$

Definition 2.3. The divisor associated to the above section $\bigwedge^d \phi$ of $(\bigwedge^d(\mathrm{ad}(E_G)/\mathrm{ad}(E_H)))\otimes \Omega^d_X$ will be called the *branching divisor* for the branched holomorphic Cartan geometry $(E_H,\, \theta)$ on X.

Just as in the case of usual Cartan geometries, the homomorphism

$$\mathrm{At}(E_H) \oplus \mathrm{ad}(E_G) \,\longrightarrow\, \mathrm{ad}(E_G), \quad (v,\, w) \,\longmapsto\, \theta(v) + w$$

produces a holomorphic connection

$$\theta' \,:\, \mathrm{At}(E_G) \,\longrightarrow\, \mathrm{ad}(E_G) \tag{2.8}$$

on E_G.

We will call a branched holomorphic Cartan geometry $(E_H,\, \theta)$ to be *normal* if

$$\mathrm{Curv}(\theta') \,\in\, H^0(X,\, \mathrm{ad}(E_H) \otimes \Omega^2_X)\,.$$

We will call a branched holomorphic Cartan geometry $(E_H,\, \theta)$ to be *flat* if

$$\mathrm{Curv}(\theta') = 0.$$

If $(E_H,\, \theta)$ is a branched holomorphic Cartan geometry, then the homomorphism θ can be interpreted as a $\mathfrak{g}$–valued holomorphic 1–form β on E_H satisfying the following three conditions:

(1) the homomorphism $\beta \,:\, TE_H \,\longrightarrow\, E_H \times \mathfrak{g}$ is an isomorphism over a nonempty open subset of E_H,
(2) β is H–equivariant with H acting on $\mathfrak{g}$ via conjugation, and
(3) the restriction of β to each fiber of f coincides with the Maurer–Cartan form associated to the action of H on E_H.

3. Examples of branched holomorphic Cartan geometries

3.1. **The standard model.** We recall the standard (flat) Cartan geometry of type G/H.

Set $X = G/H$. Let F_H be the holomorphic principal H–bundle on X defined by the quotient map $G \longrightarrow G/H$; we use the notation F_H instead of E_H because it is a special case which will be used later. Identify the Lie algebra $\mathfrak{g}$ with the right–invariant vector fields on G. This produces an isomorphism of $\mathrm{At}(F_H)$ with $\mathrm{ad}(F_G)$ and hence a Cartan geometry of type G/H on X (we use the notation F_G instead of E_G for the same reason as above). Equivalently, the tautological holomorphic $\mathfrak{g}$–valued 1–form on $G = F_H$ satisfies the three conditions needed to define a Cartan geometry of type G/H (see the last paragraph of Section 2.1).

The above holomorphic $\mathfrak{g}$–valued 1–form on $G = F_H$ will be denoted by $\theta_{G,H}$.

3.2. **Flat Cartan geometries.** A (E_H, θ) holomorphic Cartan geometry of type G/H is flat if and only if it is locally isomorphic to $(F_H, \theta_{G,H})$ [Sh, Ch. 5, § 5, Theorem 5.1].

If a complex manifold X admits a flat holomorphic Cartan geometry of type G/H, then X admits an open cover by sets U_i and local biholomorphisms $\phi_i : U_i \longrightarrow G/H$ such that all transition maps

$$\phi_i \circ \phi_j^{-1} : \phi_j(U_i \cap U_j) \longrightarrow \phi_i(U_i \cap U_j)$$

are, on each connected component, the restriction of automorphism of G/H given by the action of a unique element $g_{ij} \in G$ [Sh, Ch. 5, § 5, Theorem 5.2].

Following Ehresmann [Eh] one classically defines then a *monodromy homomorphism* ρ from the fundamental group $\pi_1(X)$ of X into G and a developing map $\delta : \widetilde{X} \longrightarrow G/H$ which is a $\pi_1(X)$-equivariant local biholomorphism from the universal cover $\widetilde{X}$ into the model G/H.

3.3. **Construction of branched holomorphic Cartan geometries.** Let X be a connected complex manifold and

$$\gamma : X \longrightarrow G/H$$

a holomorphic map such that the differential

$$d\gamma : TX \longrightarrow T(G/H)$$

is an isomorphism over a nonempty subset of X.

The above condition on $d\gamma$ is equivalent to the condition that $\dim X = \dim(G/H)$ with $\gamma(X)$ containing a nonempty open subset of G/H. The homomorphism $d\gamma$ is always an isomorphism over an open subset of X, which may be empty.

Set E_H to be the pullback $\gamma^* F_H$. So we have a holomorphic map $\eta : E_H \longrightarrow F_H$ which is H–equivariant and fits in the commutative diagram

$$\begin{array}{ccc} E_H & \stackrel{\eta}{\longrightarrow} & F_H \\ \Big\downarrow & & \Big\downarrow \\ X & \stackrel{\gamma}{\longrightarrow} & G/H \end{array}$$

Then $(E_H, \eta^*\theta_{G,H})$ defines a branched Cartan geometry of type G/H on X.

To describe the above branched Cartan geometry in terms of the Atiyah bundle, first note that $\mathrm{At}(\gamma^* F_H)$ coincides with the subbundle of the vector bundle $\gamma^*\mathrm{At}(F_H) \oplus TX$ given by the kernel of the homomorphism

$$\gamma^*\mathrm{At}(F_H) \oplus TX \longrightarrow \gamma^* T(G/H), \quad (v, w) \longmapsto \gamma^* q_{G,H}(v) - d\gamma(w),$$

where $q_{G,H} : \text{At}(F_H) \longrightarrow T(G/H)$ is the natural projection (see (2.4)). Consider the standard Cartan geometry $\theta_{G,H} : \text{At}(F_H) \xrightarrow{\sim} \text{ad}(F_G)$ of type G/H on the quotient G/H. The restriction of the homomorphism

$$\gamma^*\text{At}(F_H) \oplus TX \longrightarrow \gamma^*\text{ad}(F_G)\,, \quad (a,\, b) \longmapsto \gamma^*\theta_{G,H}(a)$$

to $\text{At}(\gamma^* F_H)$ is a homomorphism

$$\text{At}(\gamma^* F_H) \longrightarrow \text{ad}(\gamma^* F_G) = \gamma^*\text{ad}(F_G) = \text{ad}(E_G)\,,$$

which defines a branched holomorphic Cartan geometry of type G/H on X.

The divisor of X over which the above branched Cartan geometry of type G/H on X fails to be a Cartan geometry evidently coincides with the divisor over which the differential $d\gamma$ fails to be an isomorphism.

The curvature of the holomorphic connection on E_G associated to the above branched Cartan geometry of type G/H on X vanishes identically. Indeed, this follows immediately from the fact that the standard Cartan geometry $\theta_{G,H}$ is flat. In particular, this branched Cartan geometry on X is normal.

Conversely, let X be a complex manifold endowed with a branched flat holomorphic Cartan geometry with branching divisor D. Then the proof of Theorem 5.2 in [Sh, Ch. 5, § 5] shows that X admits an open cover by sets U_i such that there exist holomorphic maps $\phi_i : U_i \longrightarrow G/H$ which are local biholomorphisms on $U_i \setminus (U_i \cap D)$; moreover, for each connected component of the overlaps $U_i \cap U_j$, there exists a unique $g_{ij} \in G$ such that $g_{ij} \circ f_j = f_i$ on the entire connected component.

Then the Ehresmann method [Eh] (based by analytic continuation of charts along paths) defines a monodromy morphism $\rho : \pi_1(X) \longrightarrow G$ and a developing map $\delta : \widetilde{X} \longrightarrow G/H$ which is a local biholomorphism away from the pull-back of D to the universal cover.

3.4. **Branched flat affine and projective structures.** Let us recall the standard model G/H of the affine geometry.

Consider the semi-direct product $\mathbb{C}^d \rtimes \text{GL}(d, \mathbb{C})$ for the standard action of $\text{GL}(d, \mathbb{C})$ on $\mathbb{C}^d$. This group $\mathbb{C}^d \rtimes \text{GL}(d, \mathbb{C})$ is identified with the group of all affine transformations of $\mathbb{C}^d$. Set $H = \text{GL}(d, \mathbb{C})$ and $G = \mathbb{C}^d \rtimes \text{GL}(d, \mathbb{C})$.

A *holomorphic affine structure* (or equivalently *holomorphic affine connection*) on a complex manifold X of dimension d is a holomorphic Cartan geometry of type G/H. This terminology comes from the fact that the bundle F_H will be automatically isomorphic to the holomorphic frame bundle of X and the form $\theta_{G,H}$ defines a holomorphic connection in the holomorphic tangent bundle of X. Conversely, any holomorphic connection in the holomorphic tangent bundle of X uniquely defines a holomorphic Cartan geometry of type G/H. This connection is torsionfree exactly when the Cartan geometry is normal [MM]. For details about the equivalence between the several definitions of a holomorphic affine connection (especially with the one seeing the connection as an operator ∇ acting on local holomorphic vector fields and satisfying the Leibniz rule), the reader could refer to [MM, Sh].

A branched holomorphic Cartan geometry of type $\mathbb{C}^d \rtimes \mathrm{GL}(d,\mathbb{C})/\mathrm{GL}(d,\mathbb{C})$ will be called a *branched holomorphic affine structure* or a *branched holomorphic affine connection.*

We also recall that a *holomorphic projective structure* (or a *holomorphic projective connection*) on a complex manifold X of dimension d is a holomorphic Cartan geometry of type $\mathrm{PGL}(d+1,\mathbb{C})/Q$, where $Q \subset \mathrm{PGL}(d+1,\mathbb{C})$ is the maximal parabolic subgroup that fixes a given point for the standard action of $\mathrm{PGL}(d+1,\mathbb{C})$ on $\mathbb{C}P^d$ (the space of lines in $\mathbb{C}^{d+1}$). In particular, there is a standard holomorphic projective structure on $\mathrm{PGL}(d+1,\mathbb{C})/Q = \mathbb{C}P^d$. Locally a holomorphic projective connection is an equivalence class of holomorphic affine connections, where two affine connections are considered to be equivalent if they admit the same unparametrized geodesics. The projective connection is normal exactly when it admits a local representative which is a torsionfree affine connection [MM, OT].

We will call a branched holomorphic Cartan geometry of type $\mathrm{PGL}(d+1,\mathbb{C})/Q$ a *branched holomorphic projective structure* or a *branched holomorphic projective connection.*

Proposition 3.1. *Every compact complex projective manifold admits a branched flat holomorphic projective structure.*

Proof. Let X be a compact complex projective manifold of complex dimension d. Then there exists a finite surjective morphism

$$\gamma : X \longrightarrow \mathbb{C}P^d .$$

Indeed, one proves that the smallest integer N for which there exists a finite morphism f from X to $\mathbb{C}P^N$ is d. If $N > d$, then there exists $P \in \mathbb{C}P^N \setminus f(X)$ and consider the projection $\pi : \mathbb{C}P^N \setminus \{P\} \longrightarrow \mathbb{C}P^{N-1}$. The fibers of $\pi \circ f$ must be finite (otherwise $f(X)$ would contain a line through P, hence P). Since $\pi \circ f$ is a proper morphism with finite fibers, it must be finite.

Now we can pull back the standard holomorphic projective structure on $\mathbb{C}P^d$ using the map γ to get a branched holomorphic projective structure on X. □

Proposition 3.2.

(i) *Simply connected compact complex manifolds do not admit branched flat holomorphic affine structures.*

(ii) *Simply connected compact complex manifolds admitting branched flat holomorphic projective structures are Moishezon.*

Proof. (i) If, by contradiction, a simply connected compact complex manifold X admits a branched flat holomorphic affine structure, then the developing map $\delta : X \longrightarrow \mathbb{C}^d$ is holomorphic and nonconstant: a contradiction.

(ii) If X is a simply connected manifold of complex dimension d admitting a branched flat holomorphic projective structure, then its developing map is a holomorphic map $\delta : X \longrightarrow \mathbb{C}P^d$ which is a local biholomorphism away from a divisor D in X. Thus, the algebraic dimension of X must be d and, consequently, X is Moishezon. □

Since compact Kähler manifolds are Moishezon if and only if they are projective, one gets then the following

Corollary 3.3. *Non-projective simply connected Kähler manifolds do not admit branched flat holomorphic projective structures.*

In particular, non projective $K3$ surfaces do not admit branched flat holomorphic projective structures.

3.5. **Branched normal holomorphic projective structure on complex surfaces.** In [KO], Kobayashi and Ochiai classified all compact complex surfaces admitting holomorphic projective structures (connections). All of them happen to be isomorphic to quotients of open sets in $\mathbb{C}P^2$ by discrete subgroups of $\mathrm{PGL}(3,\mathbb{C})$ acting properly and discontinuously. Consequently, all of them also admit flat holomorphic projective structures. Among those surfaces, the only projective ones are the following : $\mathbb{C}P^2$, surfaces covered by the ball and abelian varieties (and their finite unramified quotients).

Moreover, it is known that every normal projective structure (connection) on a compact complex surface is automatically flat [Du3].

Proposition 3.1 shows that the class of compact complex surfaces admitting branched holomorphic projective structures is much broader. Moreover, we have the following

Proposition 3.4. *There exists branched holomorphic projective structures on compact complex surfaces which are normal, but not flat.*

Proof. Let Y be a compact connected Riemann surface of genus at least two. Fix two holomorphic 1–forms $\alpha_1,\, \alpha_2 \,\in\, H^0(X,\, \Omega^1_X)$ that are linearly independent. Set $X \,=\, Y\times Y$. Let $Q \,\subset\, \mathrm{PGL}(3,\mathbb{C})$ be the maximal parabolic subgroup that fixes the point $(1,\, 0,\, 0) \,\in\, \mathbb{C}P^2$ for the standard action of $\mathrm{PGL}(3,\mathbb{C})$ on $\mathbb{C}P^2$. Set $H \,=\, Q$ and $G \,=\, \mathrm{PGL}(3,\mathbb{C})$.

Let $E_H \,=\, X\times H \,\stackrel{f}{\longrightarrow}\, X$ be the trivial holomorphic principal H–bundle on X. So, the corresponding holomorphic principal G–bundle E_G is the trivial holomorphic principal G–bundle $X\times G$. The adjoint vector bundles $\mathrm{ad}(E_H)$ and $\mathrm{ad}(E_G)$ are the trivial vector bundles $X\times\mathfrak{h}$ and $X\times\mathfrak{g}$ respectively. The trivialization of E_H produces a trivial holomorphic connection on E_H. This connection defines a holomorphic splitting of the Atiyah exact sequence in (2.4). So we have

$$\mathrm{At}(E_H) \,=\, \mathrm{ad}(E_G)\oplus TX \,=\, (X\times\mathfrak{h})\oplus TX\, .$$

Now let

$$\theta\,:\, \mathrm{At}(E_H)\,\longrightarrow\, \mathrm{ad}(E_G) \,=\, X\times\mathfrak{g}$$

be the holomorphic homomorphism which over any point $(y_1,\, y_2)\,\in\, Y\times Y \,=\, X$ is defined by

$$(w,\, (v_1,\, v_2))\,\longmapsto\, w+\begin{pmatrix} 0 & 0 & \alpha_1(y_1)(v_1)\\ \alpha_1(y_1)(v_1) & 0 & 0\\ \alpha_2(y_2)(v_2) & 0 & 0\end{pmatrix}\, ,\ w\,\in\,\mathfrak{h}\, ,\ v_i\,\in\, T_{y_i}Y\, .$$

Note that the Lie algebra $\mathfrak{g}$ is the space of 3×3 complex matrices of trace zero, while $\mathfrak{h}$ is the subalgebra of $\mathfrak{g}$ consisting of matrices $(a_{i,j})_{i,j=1}^3$ such that $a_{2,1} \,=\, 0 \,=\, a_{3,1}$. Therefore, θ

is an isomorphism over the nonempty open subset of X consisting of all $(y_1, y_2) \in Y \times Y$ such that both $\alpha_1(y_1)$ and $\alpha_2(y_2)$ are nonzero.

Let θ' be the holomorphic connection on $E_G = X \times G$ associated to θ (see (2.8)). To describe θ', let D_0 denote the trivial holomorphic connection on $E_G = X \times G$ given by its trivialization. Let

$$p_i : X = Y \times Y \longrightarrow Y , \quad i = 1, 2$$

be the projection to the i–th factor. Then we have

$$\theta' = D_0 + \begin{pmatrix} 0 & 0 & p_1^*\alpha_1 \\ p_1^*\alpha_1 & 0 & 0 \\ p_2^*\alpha_2 & 0 & 0 \end{pmatrix} ;$$

note that $\mathrm{ad}(E_G) = X \times \mathfrak{g}$, and

$$\begin{pmatrix} 0 & 0 & p_1^*\alpha_1 \\ p_1^*\alpha_1 & 0 & 0 \\ p_2^*\alpha_2 & 0 & 0 \end{pmatrix} \in H^0(X, \mathrm{ad}(E_G) \otimes \Omega^1_X)$$

because the diagonal terms are zero. Therefore, the curvature $\mathrm{Curv}(\theta')$ of the connection θ' has the following expression:

$$\mathrm{Curv}(\theta') = \begin{pmatrix} 0 & 0 & p_1^*\alpha_1 \\ p_1^*\alpha_1 & 0 & 0 \\ p_2^*\alpha_2 & 0 & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & p_1^*\alpha_1 \\ p_1^*\alpha_1 & 0 & 0 \\ p_2^*\alpha_2 & 0 & 0 \end{pmatrix} = \begin{pmatrix} (p_1^*\alpha_1) \wedge (p_2^*\alpha_2) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (p_2^*\alpha_2) \wedge (p_1^*\alpha_1) \end{pmatrix}$$

Hence we have

$$\mathrm{Curv}(\theta') \in H^0(X, \mathrm{ad}(E_H) \otimes \Omega^2_X) .$$

So the branched projective structure (E_H, θ) constructed above in normal. But we have $\mathrm{Curv}(\theta') \neq 0$. $\square$

We don't know whether (non-projective) compact complex surfaces admitting branched holomorphic projective structures are exactly those admitting branched flat holomorphic projective structures.

4. A criterion

Let X be a compact connected Kähler manifold of complex dimension d equipped with a Kähler form ω. Chern classes will always mean ones with real coefficients. For a torsionfree coherent analytic sheaf V on X, define

$$\mathrm{degree}(V) := (c_1(V) \cup \omega^{d-1}) \cap [X] \in \mathbb{R} . \tag{4.1}$$

The degree of a divisor D on X is defined to be $\mathrm{degree}(\mathcal{O}_X(D))$.

Fix an effective divisor D on X. Fix a holomorphic principal H–bundle E_H on X.

Proposition 4.1. *If* $\mathrm{degree}(\Omega^1_X) - \mathrm{degree}(D) \neq \mathrm{degree}(\mathrm{ad}(E_H))$, *then there is no branched holomorphic Cartan geometry of type G/H on X with branching divisor D (see Definition 2.3). In particular, if $D \neq 0$ and* $\mathrm{degree}(\Omega^1_X) \leq \mathrm{degree}(\mathrm{ad}(E_H))$, *then there is no branched holomorphic Cartan geometry of type G/H on X with branching divisor D.*

Proof. Let (E_H, θ) be a branched holomorphic Cartan geometry of type G/H on X with branching divisor D. Consider the homomorphism $\bigwedge^d \phi$ in (2.7). Since D is the divisor for the corresponding holomorphic section of the line bundle $(\bigwedge^d(\text{ad}(E_G)/\text{ad}(E_H))) \otimes \Omega_X^d$, we have

$$\begin{aligned}\text{degree}(D) \,&=\, \text{degree}((\bigwedge^d(\text{ad}(E_G)/\text{ad}(E_H))) \otimes \Omega_X^d) \\ &=\, \text{degree}(\text{ad}(E_G)) - \text{degree}(\text{ad}(E_H)) + \text{degree}(\Omega_X^1)\,. \end{aligned} \tag{4.2}$$

Recall that E_G has a holomorphic connection θ' corresponding to θ. It induces a holomorphic connection on $\text{ad}(E_G)$. Hence we have $c_1(\text{ad}(E_G)) \,=\, 0$ [At, Theorem 4], which implies that $\text{degree}(\text{ad}(E_G)) \,=\, 0$. Therefore, from (4.2) it follows that

$$\text{degree}(\Omega_X^1) - \text{degree}(D) \,=\, \text{degree}(\text{ad}(E_H))\,. \tag{4.3}$$

If $D \,\neq\, 0$, then $\text{degree}(D) \,>\, 0$. Hence in that case (4.3) fails if we have $\text{degree}(\Omega_X^1) \,\leq\, \text{degree}(\text{ad}(E_H))$. ☐

Corollary 4.2.

(i) *If* $\text{degree}(\Omega_X^1) \,<\, 0$*, then there is no branched holomorphic affine structure on* X*.*
(ii) *If* $\text{degree}(\Omega_X^1) \,=\, 0$*, then all branched holomorphic affine structures on* X *are actually holomorphic affine structures.*

Proof. Set $H \,=\, \text{GL}(d, \mathbb{C})$ and $G \,=\, \mathbb{C}^d \rtimes \text{GL}(d, \mathbb{C})$. Recall that a branched holomorphic affine structure on X is a branched holomorphic Cartan geometry on X of type G/H, where H and G are as above. Let (E_H, θ) be a branched holomorphic affine structure on the compact Kähler manifold (X, ω) of dimension d. The homomorphism

$$\text{M}(d, \mathbb{C}) \otimes \text{M}(d, \mathbb{C}) \longrightarrow \mathbb{C}\,, \quad A \otimes B \longmapsto \text{Trace}(AB)$$

is nondegenerate and $\text{GL}(d, \mathbb{C})$–invariant. In other words, the Lie algebra $\mathfrak{h}$ of $H \,=\, \text{GL}(d, \mathbb{C})$ is self-dual as an H–module. Hence $\text{ad}(E_H) \,=\, \text{ad}(E_H)^*$, in particular, we have

$$\text{degree}(\text{ad}(E_H)) \,=\, 0\,.$$

Therefore, the corollary follows from Proposition 4.1. ☐

Remark 4.3. Let X be a rationally connected compact complex manifold. The proof of Theorem 4.1 in [BM] extends to branched Cartan geometries on X. In other words, any branched Cartan geometry of type G/H on X is flat and it is given by a holomorphic map $X \longrightarrow G/H$. This implies that G/H is compact.

5. Holomorphic projective structure on parallelizable manifolds

Recall that, by a theorem of Wang [Wa], compact complex parallelizable manifolds (i.e., manifolds with a trivial holomorphic tangent bundle) are isomorphic to quotients G/Γ of complex Lie groups G by *cocompact* lattices $\Gamma \,\subset\, G$ (recall that cocompact (or normal) lattices are those for which the quotient is compact). Such a quotient is known to be Kähler if and only if G is abelian.

All the compact complex parallelizable manifolds admit a holomorphic affine structure (connections) given by the trivialization of the holomorphic tangent bundle (by right-invariant vector fields). As soon as G is non-abelian the holomorphic affine connection for which right-invariant vector fields are parallel have non vanishing torsion and, consequently, it is not flat.

Moreover we have the following:

Proposition 5.1. *Let G be a complex semi-simple Lie group and Γ a cocompact lattice in G. Then the quotient G/Γ does not admit any branched flat affine structure.*

Let us first prove the following:

Lemma 5.2. *Let G be a complex semi-simple Lie group and Γ a cocompact lattice in G. Then any branched holomorphic Cartan geometry on $X = G/\Gamma$ has an empty branching set.*

Proof. Assume, by contradiction, that the branching set is not empty. Then, by Lemma 2.2 the branching set must be a divisor in X. On the other hand, it is known, [HM], that G/Γ contains no divisor: a contradiction. □

Now we go back to the proof of Proposition 5.1.

Proof. Assume, by contradiction, that $X = G/\Gamma$ admits a branched flat affine structure. Using Lemma 5.2 the branching set must be empty. Consider then the holomorphic affine connection ∇ in the holomorphic tangent bundle TX associated the holomorphic flat affine structure. If d is the complex dimension of X, denote by $(V_1, V_2, \ldots, V_d)$ a family of globally defined holomorphic vector fields on X trivializing TX (the V_i's descend from right-invariant vector fields on G). For any i, j, the holomorphic vector field $\nabla_{V_i} V_j$ is also globally defined on X and must be a linear combination of V_i's with constant coefficients. It follows that the pull-back of $\nabla_{V_i} V_j$ to G is a right-invariant vector field. This implies that the pull-back to G of ∇ is right-invariant. But it is known, [Du5], that a semi-simple complex Lie algebra does not admit translation invariant holomorphic flat affine structures: a contradiction. □

The simplest example is that of compact quotients of $\mathrm{SL}(2, \mathbb{C})$ by lattices Γ: they do not admit (branched) flat holomorphic affine structures. But we will see that they admit flat holomorphic projective structures.

Indeed, the Killing quadratic form on the Lie algebra of $\mathrm{SL}(2, \mathbb{C})$ is nondegenerate. It endows the complex manifold $\mathrm{SL}(2, \mathbb{C})$ with a right-invariant *holomorphic Riemannian metric* in the sense of the following definition.

Definition 5.3. A holomorphic Riemannian metric on X is a holomorphic section

$$g \in H^0(X, \mathrm{S}^2((TX)^*))$$

such that for every point $x \in X$ the quadratic form $g(x)$ on the fiber T_xX is nondegenerate.

A holomorphic Riemannian metric on a complex manifold of dimension d is a holomorphic Cartan geometry of the type G/H, with $G = \mathbb{C}^d \rtimes \mathrm{O}(d, \mathbb{C})$ and $H = \mathrm{O}(d, \mathbb{C})$, with $\mathrm{O}(d, \mathbb{C})$ the complex orthogonal group [Sh, Ch. 6].

As in the Riemannian or pseudo-Riemannian setting, one associates to a holomorphic Riemannian metric g a unique holomorphic affine connection ∇. This connection ∇, called the Levi-Civita connection of g, is uniquely determined by the following properties: ∇ is torsionfree and the holomorphic tensor g is parallel with respect to ∇. Using ∇ one can compute the curvature tensor of g which vanishes identically if and only if g is locally isomorphic to the standard flat model $dz_1^2 + \ldots + dz_n^2$, seen as a homogeneous space of $G = \mathbb{C}^d \rtimes \mathrm{O}(d, \mathbb{C})$.

The holomorphic Riemannian metric on $\mathrm{SL}(2, \mathbb{C})$ coming from the Killing quadratic form is bi-invariant (since the Killing quadratic form is invariant under the adjoint action of $\mathrm{SL}(2, \mathbb{C})$). It has nonzero constant sectional curvature [Gh]. Since the Levi-Civita connection of a metric of constant sectional curvature is known to be projectively flat, this endows $\mathrm{SL}(2, \mathbb{C})$ with a bi-invariant flat holomorphic projective structure. For more details about the geometry of holomorphic Riemannian metrics one can see [Gh, Du1, DZ].

Interesting exotic deformations of parallelizable manifolds $\mathrm{SL}(2, \mathbb{C})/\Gamma$ was constructed by Ghys in [Gh].

Those deformations are constructed choosing a group homomorphism $u : \Gamma \longrightarrow \mathrm{SL}(2, \mathbb{C})$ and considering the embedding $\gamma \longmapsto (u(\gamma), \gamma)$ of Γ into $\mathrm{SL}(2, \mathbb{C}) \times \mathrm{SL}(2, \mathbb{C})$ (acting on $\mathrm{SL}(2, \mathbb{C})$ by left and right translations). Algebraically, the action is given by:

$$(\gamma, x) \in \Gamma \times \mathrm{SL}(2, \mathbb{C}) \longrightarrow u(\gamma^{-1}) x \gamma \in \mathrm{SL}(2, \mathbb{C}) .$$

It is proved in [Gh] that, for u close enough to the trivial morphism, Γ acts properly and freely on $\mathrm{SL}(2, \mathbb{C})$ such that the quotient $M(u, \Gamma)$ is a complex compact manifold (covered by $\mathrm{SL}(2, \mathbb{C})$). In general, these examples do not admit parallelizable manifolds as finite covers. Moreover, for generic u the space of holomorphic global vector fields is trivial. All manifolds $M(u, \Gamma)$ inherit a flat holomorphic projective structure (coming from the bi-invariant projective structure constructed above). Moreover, any small deformation of the manifold $\mathrm{SL}(2, \mathbb{C})$ is isomorphic to $M(u, \Gamma)$ for some u [Gh]. Therefore we get the following:

Theorem 5.4 (Ghys)**.** *Complex compact parallelizable manifolds* $\mathrm{SL}(2, \mathbb{C})/\Gamma$ *and their small deformations admit flat holomorphic projective structures.*

It is not not known whether for generic homomorphisms u, complex manifolds $M(u, \Gamma)$ admit other flat holomorphic projective structures than the standard one (that descends from the bi-invariant flat holomorphic projective structure on $\mathrm{SL}(2, \mathbb{C})$ constructed above).

For some non-generic homomorphisms u, complex manifolds $M(u, \Gamma)$ also admit holomorphic Riemannian metrics with nonconstant sectional curvature [Gh]. The associated holomorphic projective structures on those manifolds are not flat.

Recall here the main result in [DZ]:

Theorem 5.5. *Let M be a compact complex threefold endowed with a holomorphic Riemannian metric. Then M admits a finite unramified covering bearing a holomorphic Riemannian metric of constant sectional curvature (and hence the associated flat holomorphic projective structure).*

Now we will describe the global geometry of holomorphic projective structures on complex parallelizable manifolds. Let us first prove the following.

Lemma 5.6. *Consider a holomorphic projective connection on a compact complex manifold X with trivial canonical bundle. Then X admits a holomorphic affine connection ∇ which is projectively isomorphic to the given holomorphic projective connection.*

Proof. Let $X = \bigcup U_i$ be an open cover of X such that on each U_i there exist a holomorphic affine connection ∇_i projectively equivalent to the given projective connection. Let ω be a global nontrivial holomorphic section of the canonical bundle. On each U_i, there exists a unique holomorphic affine connection $\widetilde{\nabla}_i$ projectively equivalent to ∇_i and such that ω is parallel with respect to $\widetilde{\nabla}_i$ [OT, Appendix A.3]. By uniqueness, these $\widetilde{\nabla}_i$'s agree on the overlaps of the U_i's and define a global holomorphic affine connection on X projectively equivalent to the original holomorphic projective connection (for a different proof one can also combine two results: [Gu, p. 96] and [KO, p. 78–79]). □

A corollary of this result is the following:

Proposition 5.7. *Let G be a complex Lie group of dimension d and Γ a lattice in G. Then $X = G/\Gamma$ admits a flat holomorphic projective structure if and only if there exists a Lie group homomorphism $i : \widetilde{G} \longrightarrow \mathrm{PGL}(d+1, \mathbb{C})$ such that $i(\widetilde{G})$ acts with an open orbit on the standard model $\mathbb{C}P^d$, where $\widetilde{G}$ is the universal cover of G.*

Note that the condition in the statement of Proposition 5.7 is equivalent to the existence of a Lie algebra homomorphism $\widetilde{i}$ from the Lie algebra of G into the Lie algebra of $\mathrm{PGL}(d{+}1, \mathbb{C})$, such that the image of $\widetilde{i}$ intersects trivially the Lie subalgebra of the stabilizer Q of a point in $\mathbb{C}P^d$. A classification of those complex Lie algebras admitting such homomorphisms is done in [Ka] (see also [Ag] for the real case).

Proof. First assume that there exists a group homomorphism $i : \widetilde{G} \longrightarrow \mathrm{PGL}(d{+}1, \mathbb{C})$ such that $i(\widetilde{G})$ acts on $\mathbb{C}P^d$ with an open orbit $O \subset X$. Fix a point $o \in O$ and consider the map $\pi : \widetilde{G} \longrightarrow O$ defined by $\pi(g) = i(g) \cdot o$, for all $g \in \widetilde{G}$. The map π is a covering and the pull-back of the flat holomorphic projective structure of O through π is a right-invariant flat holomorphic projective structure on $\widetilde{G}$. This flat holomorphic projective structure descends to the quotient $X = \widetilde{G}/\widetilde{\Gamma}$, where $\widetilde{\Gamma}$ is the lift of Γ to the universal covering $\widetilde{G}$ of G.

To prove the converse, assume that G/Γ is equipped with a flat holomorphic projective structure. By Lemma 5.6, there exists a holomorphic affine connection ∇ on G/Γ which is projectively equivalent to the given flat holomorphic projective structure. The proof of Proposition 5.1 shows that the pull-back of ∇ to G is a right-invariant holomorphic affine connection. In particular, the pull-back of the initial flat holomorphic projective structure to G is right-invariant. It follows that the Lie algebra of G acts locally projectively on the standard projective model $\mathbb{C}P^d$. Since the model is simply connected, this local action extends to a projective locally free global action of $\widetilde{G}$ on $\mathrm{PGL}(d + 1, \mathbb{C})$. This gives the required Lie group homomorphism i. □

It may be remarked that the Lie group morphism i in the statement of Proposition 5.7 extends the monodromy morphism $\rho : \widetilde{\Gamma} \longrightarrow \mathrm{PGL}(d+1, \mathbb{C})$ to a Lie group homomorphism i. Those projective structures are called *homogeneous*.

In order to see that $\mathrm{SL}(2, \mathbb{C})$ admits actions as in the statement of Proposition 5.7, consider the $\mathrm{SL}(2, \mathbb{C})$ irreducible linear action on the vector space of homogeneous polynomials of degree 3 in two variables (by linear changing of variables). The projectivization of this linear action gives a projective action of $\mathrm{SL}(2, \mathbb{C})$ on $\mathbb{C}P^3$ admitting an open orbit: the $\mathrm{SL}(2, \mathbb{C})$-orbit of those polynomials which are a product of three distinct linear forms (recall that the projective action of $\mathrm{SL}(2, \mathbb{C})$ on the projective line $\mathbb{C}P^1$ is transitive on the set of triples of distinct points).

6. Calabi–Yau manifolds and branched Cartan geometries

We are interested here in understanding branched holomorphic Cartan geometries on Calabi-Yau manifolds.

Recall that Kähler Calabi-Yau manifolds are compact complex Kähler manifolds such that the first Chern class (with real coefficients) of the holomorphic tangent bundle vanishes. By Yau's proof of Calabi's conjecture those manifolds admit Kähler metrics with vanishing Ricci curvature [Ya]. Compact Kähler manifolds admitting holomorphic affine connections have vanishing real Chern classes [At]; it was proved in [IKO] that Yau's result implies that they must admit finite unramified coverings which are complex tori.

It was proved in [BM] (see also [Du2, Du4]) that Calabi-Yau manifolds bearing holomorphic Cartan geometries admit finite unramified covers by complex tori. We extend here this result to branched holomorphic Cartan geometries.

Theorem 6.1. *A compact (Kähler) Calabi-Yau manifold X bearing a branched holomorphic affine structure admits a finite unramified covering by a complex torus.*

Proof. Since $c_1(TX) = 0$, part (ii) in Corollary 4.2 implies that the branched holomorphic affine structure on X is actually a holomorphic affine structure (connection). Hence X admits a finite unramified covering by a complex torus [IKO]. ☐

Theorem 6.2. *Let X be a compact simply connected Kähler manifold such that $c_1(TX) = 0$. Let E be a holomorphic vector bundle on X equipped with a holomorphic connection. Then E is a trivial holomorphic vector bundle and D is the trivial connection on it.*

Proof. The theorem of Yau says that X admits a Ricci–flat Kähler metric [Ya]. Fix a Ricci–flat Kähler form ω on X. The degree of a torsionfree coherent analytic sheaf on X will be defined using ω (as in (4.1)). Since ω is Ricci–flat, the tangent bundle TX is polystable. Since TX is polystable with $c_1(TX) = 0$, and E admits a holomorphic connection, it follows that E is semistable [Bi, p. 2830].

Note that $c_i(E), = 0$, $i \geq 1$, because E admits a holomorphic connection [At, Theorem 4]. In particular, we have $\mathrm{degree}(E) = 0$.

Notice that, for *projective* Calabi-Yau manifolds X, Corollary 1.3 in [Si] implies then that E admits a flat holomorphic connection. Since X is simply connected, this implies that E is trivial if X is projective.

We will address now the general Kähler case.

Let $V \subset E$ be a polystable subsheaf such that

- degree$(V) = 0$, and
- the quotient E/V is torsionfree.

The second condition implies that V is reflexive. Since E is semistable, and V is polystable with degree$(V) = 0 =$ degree(E), it follows that E/V is semistable with degree$(E/V) = 0$.

Let d be the complex dimension of X. Let the ranks of V and E/V be r and s respectively. Since V and E/V are semistable, we have the Bogomolov inequality

$$((2r \cdot c_2(V) - (r-1)c_1(V)^2) \cup \omega^{d-2}) \cap [X] \geq 0\,, \tag{6.1}$$

$$((2s \cdot c_2(E/V) - (s-1)c_1(E/V)^2) \cup \omega^{d-2}) \cap [X] \geq 0 \tag{6.2}$$

[BM, Lemma 2.1].

We will show that the inequalities in (6.1) and (6.2) are equalities. Denote the sheaf E/V by W. We have

$$\begin{aligned} & 2(r+s)c_2(V\oplus W) - (r+s-1)c_1(V\oplus W)^2 \\ = \ & 2(r+s)(c_2(V)+c_2(W)+c_1(V)c_1(W)) - (r+s-1)(c_1(V)^2+c_1(W)^2+2c_1(V)c_1(W)) \\ = \ & \frac{r+s}{r}(2rc_2(V)-(r-1)c_1(V)^2) + \frac{r+s}{s}(2rc_2(W)-(s-1)c_1(W)^2) \\ & -\frac{s}{r}c_1(V)^2 - \frac{r}{s}c_1(W)^2 + 2c_1(V)c_1(W) \\ = \ & \frac{r+s}{r}(2rc_2(V)-(r-1)c_1(V)^2) + \frac{r+s}{s}(2rc_2(W)-(s-1)c_1(W)^2) - \frac{1}{sr}(s\cdot c_1(V) - r\cdot c_1(W))^2 . \end{aligned}$$

On the other hand, $c_i(V \oplus W) = c_i(E) = 0$, so

$$\begin{aligned} & \frac{r+s}{r}((2r\cdot c_2(V)-(r-1)c_1(V)^2)\cup\omega^{d-2})\cap[X] + \frac{r+s}{s}((2rc_2(W)-(s-1)c_1(W)^2)\cup\omega^{d-2})\cap[X] \\ & -\frac{1}{sr}((s\cdot c_1(V) - r\cdot c_1(W))^2 \cup \omega^{d-2}) \cap [X] = 0\,. \end{aligned} \tag{6.3}$$

From Hodge index theorem, [Vo] (Section 6.3), it follows that

$$-\frac{1}{sr}((s\cdot c_1(V) - r\cdot c_1(W))^2 \cup \omega^{d-2}) \cap [X] \geq 0\,.$$

Therefore, from (6.3) we conclude that the inequalities in (6.1) and (6.2) are equalities.

Since $((2r \cdot c_2(V) - (r-1)c_1(V)^2) \cup \omega^{d-2}) \cap [X] = 0$, from [BS, p. 40, Corollary 3] we conclude that V is a polystable *vector bundle* admitting a projectively flat unitary connection. Therefore, the projective bundle $P(V)$ is given by a representation of $\pi_1(X)$ in $\mathrm{PU}(r)$. As X is simply connected, we conclude that the projective bundle $P(V)$ is trivial. Hence

$$V = L^{\oplus r}\,, \tag{6.4}$$

where L is a holomorphic line bundle on X. We have

$$\text{degree}(L) = 0\,,$$

because $\text{degree}(V) = 0$.

Now assume that V is preserved by the connection D on E. Then V is a subbundle of E, and the quotient E/V has a holomorphic connection D_1 induced by D. So we may repeat the above arguments for $(E/V,\, D_1)$ and get a subsheaf $V_1 \subset E/V$ which is a direct sum of line bundles of degree zero (as in (6.4)). Again assume that V_1 is preserved by D_1 and repeat the argument. In this way we get a filtration of E by subbundles such that each successive quotient is a polystable vector bundle of degree zero. Now Theorem 2 in [Si] implies that E admits a flat holomorphic connection. Since X is simply connected, this implies that E is holomorphically trivial. A trivial holomorphic vector bundle on X has exactly one holomorphic connection because $H^0(X,\, \Omega^1_X) = 0$ (recall that X is simply connected). Therefore, a trivial holomorphic vector bundle on X has only the trivial connection.

Now assume, by contradiction, that V is not preserved by the connection D on E. Consider the holomorphic section of $\text{Hom}(V,\, E/V) \otimes \Omega^1_X$ given by D; it is nonzero because V is not preserved by D. Let

$$\delta \,:\, TX \longrightarrow \text{Hom}(V,\, E/V)$$

be the homomorphism given by this section.

The rank of $\text{Hom}(V,\, E/V)$ is rs. We have

$$\text{degree}(\text{Hom}(V,\, E/V)) \,=\, r\cdot \text{degree}(E/V) - s\cdot \text{degree}(V) \,=\, 0\,,$$

and $\text{Hom}(V,\, E/V)$ is semistable because both V and E/V are semistable [AB, Lemma 2.7]. On the other hand TX is a polystable vector bundle of degree zero. Hence the image $\delta(TX)$ is also a polystable vector bundle of degree zero.

Let t be the rank of $U := \delta(TX)$. We have

$$\begin{aligned}(2rs\cdot c_2(\text{Hom}(V,\, E/V)) - (rs-1)c_1(\text{Hom}(V,\, E/V))^2) \cup \omega^{d-2}) \cap [X] \\ =\, ((2r\cdot c_2(V) - (r-1)c_1(V)^2)\cup\omega^{d-2})\cap[X] \\ +((2s\cdot c_2(E/V) - (s-1)c_1(E/V)^2)\cup\omega^{d-2})\cap[X] \,=\, 0\,.\end{aligned}$$

This implies that

$$((2t\cdot c_2(U) - (t-1)c_1(U)^2)\cup\omega^{d-2})\cap[X] \,=\, 0\,,$$

because the Bogomolov inequality holds for both U and $\text{Hom}(V,\, E/V)/U$. Indeed, the Bogomolov inequality holds for all three terms in the short exact sequence

$$0 \longrightarrow U \longrightarrow \text{Hom}(V,\, E/V) \longrightarrow \text{Hom}(V,\, E/V)/U \longrightarrow 0$$

and furthermore it is an equality for $\text{Hom}(V,\, E/V)$; hence the Bogomolov inequality is an equality for both U and $\text{Hom}(V,\, E/V)/U$.

Again from [BS, p. 40, Corollary 3] we conclude that $P(U)$ admits a flat connection. Hence U is of the form

$$U \,=\, N^{\oplus t}\,,$$

where N is a holomorphic line bundle of degree zero.

Since TX is polystable, the quotient bundle U is a direct summand of TX. This implies that U is a subbundle of TX. Hence we have a holomorphic decomposition

$$TX \,=\, N\oplus N'\,, \tag{6.5}$$

where N is a holomorphic line bundle on X of degree zero, and the rank of N' is $d-1$.

A result of Beauville [Be, Theorem A] associates to any holomorphic splitting

$$TX = U_1 \oplus U_2 \oplus \ldots \oplus U_j$$

a corresponding decomposition $X = X_1 \times X_2 \times \ldots \times X_j$, with X_i simply connected Calabi-Yau manifolds, such that $U_i = \pi_i^*(TX_i)$, where $p_i : X \longrightarrow X_i$, $1 \leq i \leq j$, are the canonical projection. Now from (6.5) we conclude that X is a product of Calabi-Yau manifolds with one factor of dimension one. But there is no simply connected Calabi-Yau manifold of complex dimension one. So we get a contradiction. This completes the proof. □

Corollary 6.3.

(i) *Any branched holomorphic Cartan geometry on a compact simply connected (Kähler) Calabi-Yau manifold is flat. Consequently, the model G/H of the Cartan geometry must be compact.*

(ii) *Non-projective compact simply connected (Kähler) Calabi-Yau manifolds do not admit branched holomorphic projective structures.*

Proof. Let X be a simply connected Calabi-Yau manifold endowed with a branched holomorphic Cartan geometry of type G/H.

(i) Theorem 6.2 implies that the associated holomorphic connection θ' of E_H must be flat. Consequently, the Cartan geometry is flat. The developing map $\delta : X \longrightarrow G/H$ is a branched holomorphic map. This implies that $\delta(X) = G/H$ is compact.

(ii) This follows from part (i) and Corollary 3.3. □

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