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WEAK NORMALIZATION AND SEMINORMALIZATION IN REAL ALGEBRAIC GEOMETRY

GOULWEN FICHOU, JEAN-PHILIPPE MONNIER AND RONAN QUAREZ

ABSTRACT. We define the weak normalization and the seminormalization of a central real algebraic variety. The study is related to the properties of the rings of continuous rational functions and regulous functions on real algebraic varieties. We provide in particular several characterizations (algebraic or geometric) of these varieties, and study in full details the case of curves.

The concept of weak normalization of a complex analytic variety has been introduced by Andreotti & Bombieri [3] in order to study the space of analytic cycles associated with a complex algebraic variety. The operation of weak normalization consists in enriching the sheaf of holomorphic functions with those continuous functions which are also meromorphic. Later Andreotti & Norguet [4] defined the notion of weak normalization in the context of schemes. For algebraic varieties, it consists roughly speaking of an intermediate algebraic variety between an algebraic variety X and its normalization, in such a way that the weak normalization of X is in bijection with X . One way to construct it is to identify in the normalization all the points belonging to the pre-image of points in X . It gives rise to a variety satisfying a universal property among those varieties in birational bijection via a universal homeomorphism onto X . The theory of seminormalization, closely related to that of weak normalization, have been developed later by Traverso [33] for commutative rings, with subsequent work notably by Swan [32] or Leahy & Vitulli [23] (see also [34]), with a more particular focus on the algebraic approach or the singularities. Note however that in the geometric context of complex algebraic variety, weak normalization and seminormalization lead to the same notion. We refer to Vitulli [35] for a survey on weak normality and seminormality for commutative rings and algebraic varieties. More recently, the concept of seminormalization is used in the minimal model program of Kollár and Kovács [16] and it appears also in [17].

In the context of real geometry, the first occurrence of weak normality or seminormality is the work by Acquistapace, Broglia and Tognoli [1] in the case of real analytic spaces. In [24] the Traverso seminormalization of real algebraic varieties is studied by considering the ring of regular functions, showing that such notion does not provide natural universal property. Seminormalization in the Nash context is introduced in [28]. Our aim in this paper is to provide appropriate definitions for weak normalization and seminormalization in real algebraic geometry, leading to natural universal properties. Contrarily to the complex setting, it will appear that the notions of weak normalization and seminormalization are distinct, the difference being witnessed by the behaviour of continuous rational functions on real algebraic varieties.

The first focus on continuous rational function in real geometry is due to Kreisel [20] who proved that a positive answer to Hilbert seventeenth problem of representing a positive polynomial as a sum of squares of rational functions, can always be chosen among continuous functions. Besides, Kucharz [21] used this class of functions to approximate as algebraically as possible continuous maps between

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spheres, whereas Kollár & Nowak [19] initiated the proper study of these functions, proving notably that the restriction of a continuous function defined on a (central) real algebraic variety, which is also rational, does not remain rational in general. It is however the case as soon as the ambient variety is nonsingular. As a consequence, on a singular real algebraic variety one may consider the ring $\mathcal{R}_0(X)$ of continuous rational functions, or its subring $\mathcal{R}^0(X)$ consisting of those continuous rational functions which remain rational under restriction. This class, called hereditarily rational in [19], has been systematically studied in [11] under the name of regulous functions. The continuous rational functions are now extensively studied in real geometry, we refer for example to [22, 18, 12, 26] for further readings related to the subject of the paper.

Let X be an irreducible real algebraic variety in the sense of [7]. Assume that X is central, namely that the set of regular points of X is dense in X for the Euclidean topology; this condition guaranties that the continuous extension of a given rational function, if it exists, is unique. The rational functions on X that satisfy a monic polynomial equation with coefficients in the ring $\mathcal{P}(X)$ of polynomial functions on X , form a ring (the integral closure of $\mathcal{P}(X)$ in $\mathcal{K}(X)$) which is a finite module over $\mathcal{P}(X)$. This ring is the polynomial ring of a well-known real algebraic variety, the normalization X' of X , coming with a finite birational morphism onto X . Now, if we require moreover that the rational functions admit a continuous extension to X , the corresponding subring $\mathcal{R}_0(X) \cap \mathcal{P}(X')$ (the integral closure of $\mathcal{P}(X)$ in $\mathcal{R}_0(X)$) of $\mathcal{P}(X')$ is still a finite module over $\mathcal{P}(X)$, and therefore it coincides with the polynomial ring of a real algebraic variety. We call this variety the (real) weak normalization X_* of X . It comes again with a finite birational morphism onto X , which is an homeomorphism both for the Euclidean and regulous topologies. We prove in Theorem 6.8 that it is actually the biggest real algebraic variety lying between X and X' satisfying either of these properties. We provide in the same statement several characterizations of X_* , notably from a geometric point of view that X^* is the biggest intermediate variety between X and X' being in bijection with X , or in an algebraic point of view that X_* is the biggest intermediate variety between X and X' satisfying the strong real lying over property (cf. Definition 2.2). The justification for calling X_* the weak normalization of X comes from Theorem 6.15, which illustrates that the ring $\mathcal{P}(X_*)$ satisfies analogue properties in the real setting as the weak normalization for complex algebraic varieties.

Now, replacing the ring of continuous rational functions with the ring of regulous functions leads similarly to the definition of the (real) seminormalisation X^* of X , whose ring of polynomial functions is given by $\mathcal{R}^0(X) \cap \mathcal{P}(X')$ (the integral closure of $\mathcal{P}(X)$ in $\mathcal{R}^0(X)$). The seminormalization of X is an intermediate variety between X and X_* , so that X^* admits a finite birational morphism onto X which is an homeomorphism both for the Euclidean and regulous topologies. It is moreover the biggest intermediate variety between X and X' whose polynomial functions are regulous on X . It is also the biggest intermediate variety between X and X' satisfying the very strong real lying over property (cf. Definition 2.5).

The normalization, weak normalization and seminormalization of a real algebraic variety are different in general. The latter two coincide on varieties where all continuous rational functions are regulous, for instance in the case of curves. We provide in this particular case a full description of the singularities of a weakly normal curves, in the spirit of [10] in the complex context.

The paper is organized as follows. In the first section, we provide a detailed study of the integral closure of the rings of polynomial and regular functions on a real algebraic varieties. The results presented there are probably well-known to the specialists, however we give them with complete proofs for reason of a lack of reference. The second section is devoted to the study of topological properties of integral morphisms, where by topology we refer either to Zariski topology, the topology of the real spectrum or the real Zariski topology. After reviewing some basics about continuous rational functions in the third section, we study in the fourth section finite birational morphisms between real algebraic varieties, with a particular focus on the case of central varieties (the non central case is a

work under progress to appear in a forthcoming paper). Section five deals with the relation between rational continuous functions with the normalization map, giving in particular in Proposition 5.15 a description of the integral closure of the ring of continuous rational functions on a central curve. The last section is the heart of the paper, where we define and study the weak normalization and the seminormalization of a central real algebraic variety.

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1. NORMALIZATION OF REAL ALGEBRAIC VARIETIES

In this section we review the basic properties of the normalization of a real algebraic variety, defined via the integral closure of the ring of polynomial functions. We put a particular stress on the integral closure of the ring of regular functions.

1.1. Ideals of polynomial and regular functions. We consider in this text only affine real algebraic varieties and more precisely real algebraic sets which are the real closed points of affine real algebraic varieties. This is not an overly restrictive choice since almost all real algebraic varieties are affine [7, Rem. 3.2.12]. Unless specified, all algebraic sets are real. Let $X \subset \mathbb{R}^n$ be an algebraic set. Let $\mathcal{K}(X)$ denotes the ring of rational functions on X , it is a field when X is irreducible. Let $\mathcal{P}(X)$ be the ring of polynomial functions on X defined by $\mathcal{P}(X) = \mathbb{R}[x_1, \dots, x_n]/I$, where $I \subset \mathbb{R}[x_1, \dots, x_n]$ is the ideal of polynomials that vanish identically on X . If f is a real function on X then we denote by $\mathcal{Z}(f)$ the zero set of f in X . If E is a set of real functions on X , we denote by $\mathcal{Z}(E)$ the set $\bigcap_{f \in E} \mathcal{Z}(f)$. Let A be a ring of real functions on X and let Z be a subset of X , we denote by $\mathcal{I}_A(Z)$ the ideal of functions on A that vanish identically on Z . Let \mathcal{O}_X be the sheaf of regular functions on X (see [7, Sect. 3.2]). Let $\mathcal{O}(X)$ be the ring of global sections of \mathcal{O}_X on X , it is called the ring of regular functions on X . Let $\mathcal{S}(X)$ be the multiplicative set of polynomial functions on X that does not vanish on X i.e. $\mathcal{S}(X) = \{p \in \mathcal{P}(X) \mid \mathcal{Z}(p) = \emptyset\}$. By [7, Def. 3.2.1], we have

$$\mathcal{O}(X) = \mathcal{S}(X)^{-1} \mathcal{P}(X).$$

Let A be a commutative ring with unit. An ideal I of A is called real if, for every sequence a_1, \dots, a_k of elements of A , then $a_1^2 + \dots + a_k^2 \in I$ implies $a_i \in I$ for $i = 1, \dots, k$. We denote by $\text{R-Spec } A$ the real part of $\text{Spec } A$ i.e. the set of real prime ideals of A . We denote by $\text{Max } A$ the set of maximal ideals of A .

Proposition 1.1. *We have*

$$\text{Max } \mathcal{O}(X) \subset \text{R-Spec } \mathcal{O}(X)$$

i.e. any maximal ideal of $\mathcal{O}(X)$ is real.

Proof. A maximal ideal of $\mathcal{O}(X)$ can be identified, by intersecting it with $\mathcal{P}(X)$, with a maximal ideal \mathfrak{m} of $\mathcal{P}(X)$ such that $\mathfrak{m} \cap \mathcal{S}(X) = \emptyset$ and it is clearly sufficient to prove that \mathfrak{m} is real. Assume $p_1^2 + \dots + p_k^2 \in \mathfrak{m}$ for some $p_i \in \mathcal{P}(X)$ and suppose moreover that $p_1 \notin \mathfrak{m}$. If $k = 1$ then we get a contradiction since a maximal ideal is radical. So assume $k > 1$. Since \mathfrak{m} is maximal, there exist $q \in \mathcal{P}(X)$ and $r \in \mathfrak{m}$ such that $qp_1 = 1 + r$. We get $q^2 p_1^2 + \dots + q^2 p_k^2 \in \mathfrak{m}$ and $q^2 p_1^2 = (1 + r)^2 = 1 + r'$ with $r' \in \mathfrak{m}$. Hence $1 + \sum_{i=2}^k q^2 p_i^2 \in \mathfrak{m}$ but it is impossible since $1 + \sum_{i=2}^k q^2 p_i^2 \in \mathcal{S}(X)$. \square

Proposition 1.2. *The set $\text{Max } \mathcal{O}(X)$ is in bijection with X . More precisely, for any maximal ideal \mathfrak{m} of $\mathcal{O}(X)$ there exists a unique $x \in X$ such that $\mathfrak{m} = \mathfrak{m}_x$ with $\mathfrak{m}_x = \mathcal{I}_{\mathcal{O}(X)}(\{x\})$.*

Proof. It is sufficient to prove that any maximal ideal \mathbf{m} of $\mathcal{P}(X)$ that does not intersect $\mathcal{S}(X)$ is of the form \mathbf{m}_x for a $x \in X$ with $\mathbf{m}_x = \mathcal{I}_{\mathcal{P}(X)}(\{x\})$. As we have already seen in the proof of Proposition 1.1, \mathbf{m} is a real ideal. By the real Nullstellensatz [7, Thm. 4.1.4], we have $\mathcal{I}_{\mathcal{P}(X)}(\mathcal{Z}(\mathbf{m})) = \mathbf{m}$ and thus $\mathcal{Z}(\mathbf{m}) \neq \emptyset$. Let $x \in \mathcal{Z}(\mathbf{m})$, by [7, Thm. 4.1.4] it follows that $\mathcal{I}_{\mathcal{P}(X)}(\mathcal{Z}(\mathbf{m})) = \mathbf{m} \subset \mathcal{I}_{\mathcal{P}(X)}(\{x\}) = \mathbf{m}_x$ and the proof is done since \mathbf{m} is maximal. \square

Proposition 1.3. *The set $\text{Max } \mathcal{P}(X) \cap \text{R-Spec } \mathcal{P}(X)$ is in bijection with X . More precisely, for any maximal and real ideal \mathbf{m} of $\mathcal{P}(X)$ there exists a unique $x \in X$ such that $\mathbf{m} = \mathbf{m}_x$ with $\mathbf{m}_x = \mathcal{I}_{\mathcal{P}(X)}(\{x\})$.*

Proof. Copy the end of the proof of Proposition 1.2. \square

Proposition 1.4. [25, Thm. 4.7, Sect. 4]

Let A be an integral domain. We have

$$A = \bigcap_{\mathfrak{p} \in \text{Spec } A} A_{\mathfrak{p}} = \bigcap_{\mathfrak{m} \in \text{Max } A} A_{\mathfrak{m}}.$$

Proof. Since A has no zero-divisors then the canonical map $A \rightarrow A_{\mathfrak{p}}$ is injective. It follows that

$$A \subset \bigcap_{\mathfrak{p} \in \text{Spec } A} A_{\mathfrak{p}} \subset \bigcap_{\mathfrak{m} \in \text{Max } A} A_{\mathfrak{m}}.$$

We can conclude using [25, Thm. 4.7]. \square

Let X be an irreducible algebraic set. From Propositions 1.2 and 1.4, we get

$$\mathcal{O}(X) = \bigcap_{\mathfrak{p} \in \text{Spec } \mathcal{O}(X)} \mathcal{O}(X)_{\mathfrak{p}} = \bigcap_{x \in X} \mathcal{O}(X)_{\mathbf{m}_x} = \bigcap_{x \in X} \mathcal{P}(X)_{\mathbf{m}_x} = \bigcap_{x \in X} \mathcal{O}_{X,x}.$$

In the previous formula \mathbf{m}_x is the maximal ideal of polynomial functions or regular functions that vanish at $x \in X$.

1.2. Polynomial and regular maps. In complex affine algebraic geometry, polynomial and regular functions coincide and thus we have a unique and natural definition of morphism between complex algebraic sets. In the real setting no such natural definition exists. Usually, real algebraic geometers prefer working with the ring of regular functions rather than the ring of polynomial functions on a real algebraic set. In this paper, we make the opposite choice and we justify this choice in Remark 6.4. In other words, we see real algebraic sets as affine \mathbb{R} -schemes and we consider the natural class of functions on it.

Let $X \subset \mathbb{R}^n$, $Y \subset \mathbb{R}^m$ be real algebraic sets. A polynomial (resp. regular) map from X to Y is a map whose coordinate functions are polynomial (resp. regular). A polynomial (resp. regular) map $\varphi : X \rightarrow Y$ induces an \mathbb{R} -algebra homomorphism $\varphi^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ (resp. $\varphi^* : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$) defined by $\varphi^*(f) = f \circ \varphi$. The map $\varphi \mapsto \varphi^*$ gives a bijection between the set of polynomial maps from X to Y and the \mathbb{R} -algebra homomorphisms from $\mathcal{P}(Y)$ to $\mathcal{P}(X)$ (resp. the set of regular maps from X to Y and the \mathbb{R} -algebra homomorphisms from $\mathcal{O}(Y)$ to $\mathcal{O}(X)$). We say that a polynomial (resp. regular) map $\varphi : X \rightarrow Y$ is an isomorphism (resp. a regular isomorphism) if φ is bijective with a polynomial (resp. regular) inverse, or in another words if $\varphi^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ (resp. $\varphi^* : \mathcal{O}(Y) \rightarrow \mathcal{O}(X)$) is an isomorphism. See [7, Sec. 3.2] and [9, Ch. 5] for detailed proofs of the previous well known results. In the remainder of the paper, a map designates a polynomial map.

1.3. Complexification of real algebraic sets. Let $X \subset \mathbb{R}^n$ be a real algebraic set. The complexification of X , denoted by $X_{\mathbb{C}}$, is the complex algebraic set $X_{\mathbb{C}} \subset \mathbb{C}^n$, whose ring of polynomial functions is $\mathcal{P}(X_{\mathbb{C}}) = \mathcal{P}(X) \otimes_{\mathbb{R}} \mathbb{C}$. We say that X is geometrically smooth if $X_{\mathbb{C}}$ is smooth. Remark that $X_{\mathbb{C}}$ is automatically irreducible because X is an algebraic set. The situation is different when we consider an algebraic variety X over \mathbb{R} , i.e. a reduced separated scheme of finite type over $\text{Spec } \mathbb{R}$; actually X

can be irreducible and $X \times_{\text{Spec } \mathbb{R}} \text{Spec } \mathbb{C}$ reducible when the set of real points of X is not Zariski dense in the set of complex points ($\text{Spec } \mathbb{R}[x]/(x^2 + 1)$ for example).

Let $\varphi : X \rightarrow Y$ be a polynomial map between real algebraic sets. The tensor product by \mathbb{C} of the morphism of \mathbb{R} -algebras $\varphi^* : \mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ gives a morphism of \mathbb{C} -algebras $\mathcal{P}(Y_{\mathbb{C}}) \rightarrow \mathcal{P}(X_{\mathbb{C}})$ and by duality we get a polynomial map $\varphi_{\mathbb{C}} : X_{\mathbb{C}} \rightarrow Y_{\mathbb{C}}$ called the complexification of φ . We clearly get:

Proposition 1.5. *Let $\varphi : X \rightarrow Y$ be a polynomial map between real algebraic sets. If φ is an isomorphism then $\varphi_{\mathbb{C}}$ is an isomorphism.*

Remark 1.6. Two non-isomorphic real algebraic sets can be isomorphic over the complex numbers: the empty conic $\mathcal{Z}(x^2 + y^2 + 1)$ and the circle $\mathcal{Z}(x^2 + y^2 - 1)$.

It is not possible in general to complexify a regular map between real algebraic sets.

1.4. Normalization. Let A be an integral domain with fraction field denoted by K . An element $b \in K$ is integral over A if b is the root of a monic polynomial with coefficients in A . By [5, Prop. 5.1], b is integral over A if and only if $A[b]$ is a finite A -module. This equivalence allows to prove that $\overline{A} = \{b \in K \mid b \text{ is integral over } A\}$ is a ring called the integral closure of A (in K). The ring A is called integrally closed if $A = \overline{A}$. If A is Noetherian then \overline{A} is a finite A -module [25, Lem. 1 Sec. 33]. In particular, if A is the ring of polynomial functions on an irreducible algebraic set Y over a field k then \overline{A} is a finitely generated k -algebra and so \overline{A} is the ring of polynomial functions of an irreducible algebraic set, denoted by Y' , called the normalization of Y . We recall that a map $X \rightarrow Y$ between two algebraic sets over a field k is said finite if the ring morphism $\mathcal{P}(Y) \rightarrow \mathcal{P}(X)$ makes $\mathcal{P}(X)$ a finitely generated $\mathcal{P}(Y)$ -module. The inclusion $A \subset \overline{A}$ induces a finite map $Y' \rightarrow Y$, called the normalization map, which is a birational equivalence. We say that (an irreducible algebraic set over a field k) Y is a normal if its ring of polynomial functions is integrally closed i.e. Y is its own normalization.

Let X be an irreducible algebraic set. We say that X is geometrically normal if the associated complex algebraic set $X_{\mathbb{C}}$ is normal.

Proposition 1.7. *Let X be a geometrically smooth irreducible algebraic set. Then X is normal.*

Proof. It is well-known that if $X_{\mathbb{C}}$ is smooth then $\mathcal{P}(X_{\mathbb{C}})$ is integrally closed in $\mathcal{K}(X_{\mathbb{C}})$ (see [31, Thm. 1 Ch. 2 Sect. 5] for example). Hence $\mathcal{P}(X)$ is also integrally closed in $\mathcal{K}(X)$. \square

Example 1.8. The following algebraic curve $X = \mathcal{Z}(y^2 - (x^2 + 1)^2(x - 1))$ is smooth but not geometrically smooth. The ring $\mathcal{P}(X)$ (instead of $\mathcal{O}(X)$) is not integrally closed in $\mathcal{K}(X)$ since the non-polynomial but regular function $f = \frac{y}{x^2 + 1}$ is integral over $\mathcal{P}(X)$ since $f^2 - (x - 1) = 0$.

Proposition 1.9. *Let X be a smooth irreducible algebraic set. Then $\mathcal{O}(X)$ is integrally closed in $\mathcal{K}(X)$.*

Proof. By [7, 3.3.6, 3.3.7], for any $x \in X$ the local ring $\mathcal{O}_{X,x}$ of germs of regular functions at x is integrally closed in $\mathcal{K}(X)$. Therefore

$$\overline{\mathcal{O}(X)} = \overline{\bigcap_{x \in X} \mathcal{O}_{X,x}} \subset \bigcap_{x \in X} \overline{\mathcal{O}_{X,x}} = \bigcap_{x \in X} \mathcal{O}_{X,x} = \mathcal{O}(X).$$

\square

Proposition 1.10. *Let X be an irreducible algebraic set. If X is normal then $\mathcal{O}(X)$ is also integrally closed in $\mathcal{K}(X)$.*

Proof. We have already seen that $\mathcal{O}(X) = \mathcal{S}(X)^{-1} \mathcal{P}(X)$ where $\mathcal{S}(X)$ is the multiplicative part of $\mathcal{P}(X)$ of polynomial functions that do not vanish on X . We get the proof using [5, prop. 5.12]. \square

Proposition 1.11. *Let X be an irreducible algebraic set. Then X is normal if and only if X is geometrically normal.*

Proof. One implication is trivial and we prove the other one. Assume X is normal i.e. $\mathcal{P}(X)$ is integrally closed. Let $f \in \mathcal{K}(X_{\mathbb{C}})$ be such that $f \in \overline{\mathcal{P}(X_{\mathbb{C}})}$. We have $\mathcal{K}(X_{\mathbb{C}}) = \mathcal{K}(X) \otimes \mathbb{C}$ and thus we have a complex involution $\sigma : \mathcal{K}(X_{\mathbb{C}}) \rightarrow \mathcal{K}(X_{\mathbb{C}})$ with fixed part equal to $\mathcal{K}(X)$. We can write $f = \frac{f + \sigma(f)}{2} + i \frac{-i(f - \sigma(f))}{2}$. Since $\sigma(\mathcal{P}(X_{\mathbb{C}})) \subset \mathcal{P}(X_{\mathbb{C}})$ and since $\overline{\mathcal{P}(X_{\mathbb{C}})}$ is a ring then we see that $\sigma(f)$ and $f + \sigma(f)$ are integral over $\mathcal{P}(X_{\mathbb{C}})$. Since $f + \sigma(f) \in \mathcal{K}(X)$ then $f + \sigma(f)$ is integral over $\mathcal{P}(X)$, taking the real part of the coefficient of the integral equation, and thus $f + \sigma(f) \in \mathcal{P}(X)$. Since $\overline{\mathcal{P}(X_{\mathbb{C}})}$ is a ring then $\frac{-i(f - \sigma(f))}{2}$ is integral over $\mathcal{P}(X_{\mathbb{C}})$ and thus is also integral over $\mathcal{P}(X)$ (it is fixed by σ). It follows that $f \in \mathcal{P}(X) \otimes \mathbb{C} = \mathcal{P}(X_{\mathbb{C}})$. \square

In the remaining of this section we will compare the rings $\overline{\mathcal{O}(X)}$ and $\mathcal{O}(X')$. We have $\overline{\mathcal{O}(X)} = \mathcal{S}(X)^{-1} \mathcal{P}(X')$ [5, prop. 5.12] and $\mathcal{O}(X') = \mathcal{S}(X')^{-1} \mathcal{P}(X')$. Let $\pi : X' \rightarrow X$ be the normalization map. It is easy to check that if $f \in \mathcal{S}(X)$ then $f \circ \pi \in \mathcal{S}(X')$ and thus

$$\overline{\mathcal{O}(X)} \hookrightarrow \mathcal{O}(X').$$

Lemma 1.12. *Let A be integral domain. Then*

$$\overline{A} = \overline{\bigcap_{\mathfrak{m} \in \text{Max } A} A_{\mathfrak{m}}} = \bigcap_{\mathfrak{m} \in \text{Max } A} \overline{A_{\mathfrak{m}}} = \bigcap_{\mathfrak{m}' \in \text{Max } \overline{A}} \overline{A_{\mathfrak{m}'}}.$$

Proof. By Proposition 1.4, we have $A = \bigcap_{\mathfrak{m} \in \text{Max } A} A_{\mathfrak{m}}$ and thus

$$\overline{A} = \overline{\bigcap_{\mathfrak{m} \in \text{Max } A} A_{\mathfrak{m}}} \subset \bigcap_{\mathfrak{m} \in \text{Max } A} \overline{A_{\mathfrak{m}}}$$

and

$$\overline{A} = \bigcap_{\mathfrak{m}' \in \text{Max } \overline{A}} \overline{A_{\mathfrak{m}'}}.$$

It remains to prove the inclusion

$$\bigcap_{\mathfrak{m} \in \text{Max } A} \overline{A_{\mathfrak{m}}} \subset \bigcap_{\mathfrak{m}' \in \text{Max } \overline{A}} \overline{A_{\mathfrak{m}'}}.$$

Let $\mathfrak{m} \in \text{Max } A$. We have $A_{\mathfrak{m}} \subset \bigcap_{\mathfrak{p}' \in \text{Spec } \overline{A}, \mathfrak{p}' \cap A = \mathfrak{m}} \overline{A_{\mathfrak{p}'}}$ (By [5, Prop. 3.9] the map $A_{\mathfrak{m}} \rightarrow \overline{A_{\mathfrak{m}}} = \overline{A} \otimes_A A_{\mathfrak{m}}$ is injective, the map $\overline{A_{\mathfrak{m}}} \rightarrow \overline{A_{\mathfrak{p}'}}$ is clearly injective). By [5, Prop. 5.13], $\overline{A_{\mathfrak{p}'}}$ is integrally closed for any $\mathfrak{p}' \in \text{Spec } \overline{A}$. Thus we get

$$\overline{A_{\mathfrak{m}}} \subset \bigcap_{\mathfrak{p}' \in \text{Spec } \overline{A}, \mathfrak{p}' \cap A = \mathfrak{m}} \overline{A_{\mathfrak{p}'}}.$$

By [25, Lem. 2, Sect. 9], the map $\varphi : \text{Spec } \overline{A} \rightarrow \text{Spec } A$, $\mathfrak{p}' \mapsto \mathfrak{p}' \cap A$ has only non-empty fibers and $\varphi^{-1}(\text{Max } A) = \text{Max } \overline{A}$. According to the above results the proof is done. \square

We denote by φ the map $\varphi : \text{Spec } \mathcal{P}(X') \rightarrow \text{Spec } \mathcal{P}(X)$, $\mathfrak{p}' \mapsto \mathfrak{p}' \cap \mathcal{P}(X)$. Since $\mathcal{P}(X) \hookrightarrow \mathcal{P}(X')$ is integral, we have already seen at the end of the proof of Lemma 1.12 that the fibers of φ are non-empty and $\varphi^{-1}(\text{Max } \mathcal{P}(X)) = \text{Max } \mathcal{P}(X')$.

We have already seen that the maximal ideals of the ring of regular functions $\mathcal{O}(X')$ of the normalization of X are all real ideals (Proposition 1.1). We prove in the next proposition that the set of maximal ideals of the integral closure $\overline{\mathcal{O}(X)}$ of the ring of regular functions on X corresponds to $\varphi^{-1}(X)$. Therefore, a maximal ideal of $\overline{\mathcal{O}(X)}$ is not necessarily real and consequently the rings $\overline{\mathcal{O}(X)}$ and $\mathcal{O}(X')$ can be distinct. It shows that “taking the integral closure” and “taking the regular functions” are operations that do not commute in the real setting. Proposition 1.13 gives a description of the rings $\overline{\mathcal{O}(X)}$ and $\mathcal{O}(X')$ as the intersection of the localizations at their maximal ideals. In Proposition 1.16, we explain when $\overline{\mathcal{O}(X)} = \mathcal{O}(X')$.

Proposition 1.13. *We have:*

- 1) The set $\text{Max } \overline{\mathcal{O}(X)}$ is in bijection with $\varphi^{-1}(X) = \varphi^{-1}(\{\mathbf{m}_x = \mathcal{I}_{\mathcal{P}(X)}(\{x\}), x \in X\})$.
 2)

$$\overline{\mathcal{O}(X)} = \bigcap_{x \in X} \overline{\mathcal{O}_{X,x}} = \bigcap_{\mathbf{m}' \in \varphi^{-1}(X)} \mathcal{P}(X')_{\mathbf{m}'}$$

3)

$$\mathcal{O}(X') = \bigcap_{\mathbf{m}' \in \text{Max } \mathcal{P}(X') \cap \text{R-Spec } \mathcal{P}(X')} \mathcal{P}(X')_{\mathbf{m}'}$$

Proof. By Proposition 1.2 and [25, Lem. 2, Sect. 9], we get 1). The third statement is a consequence of Propositions 1.2 and 1.4. By Lemma 1.12 and Proposition 1.2 we get

$$\overline{\mathcal{O}(X)} = \overline{\bigcap_{x \in X} \mathcal{O}_{X,x}} = \bigcap_{x \in X} \overline{\mathcal{O}_{X,x}} = \bigcap_{\mathbf{m}' \in \varphi^{-1}(X)} \mathcal{P}(X')_{\mathbf{m}'},$$

that proves the second statement. \square

Example 1.14. Consider the cubic $X = \mathcal{Z}(y^2 - x^2(x-1))$ with an isolated point at the origin. Then $\mathcal{P}(X') = \mathcal{P}(X)[y/x] \simeq \mathbb{R}[x, z]/(z^2 - (x-1))$, setting $z = y/x$. The function $f = 1/(1+z^2) = x^2/(x^2+y^2)$ is regular on X' . However $f \notin \mathcal{P}(X')_{\mathbf{m}'}$ for the non-real maximal ideal $\mathbf{m}' = (1+z^2) = \varphi^{-1}(\mathbf{m})$ of $\mathcal{P}(X')$, where $\mathbf{m} = (x, y)$ is the (real) maximal ideal of the origin in $\mathcal{P}(X)$. Indeed we have $1/f \in \mathbf{m}'$. In particular $f \in \mathcal{O}(X') \setminus \overline{\mathcal{O}(X)}$.

Example 1.15. Consider the surface $X = \mathcal{Z}(x^3 - y^3(1+z^2))$. Then $\mathcal{P}(X') = \mathcal{P}(X)[x/y] \simeq \mathbb{R}[t, y, z]/(t^3 - (1+z^2))$, setting $t = x/y$. The function $f = 1/(t^2+t+1+z^2) = y^2/(x^2+xy+y^2+y^2z^2)$ is regular on X' . Let $\mathbf{m} = (x, y, z)$ be the maximal and real ideal of polynomial functions on X that vanish at the origin. Over \mathbf{m} we have two maximal ideals of $\mathcal{P}(X')$, only one of these two ideals is real, namely $\varphi^{-1}(\mathbf{m}) = \{\mathbf{m}' = (t-1, y, z), \mathbf{m}'' = (t^2+t+1, y, z)\}$. We have $f \notin \mathcal{P}(X')_{\mathbf{m}''}$ since $1/f \in \mathbf{m}''$. It follows that $f \in \mathcal{O}(X') \setminus \overline{\mathcal{O}(X)}$.

Let $\pi : X' \rightarrow X$ denote the normalization map.

Proposition 1.16. *The rings $\overline{\mathcal{O}(X)}$ and $\mathcal{O}(X')$ are isomorphic if and only if the fibers of $\pi_{\mathbb{C}} : X'_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ over the points of X are totally real i.e. $\#(\pi_{\mathbb{C}}^{-1}(x)) = \#(\pi^{-1}(x)) \forall x \in X$.*

Proof. Assume there exists $x \in X$ such that $\pi_{\mathbb{C}}^{-1}(x)$ is not totally real. It forces the existence of a non-real maximal ideal \mathbf{m}' of $\mathcal{P}(X')$ such that $\mathbf{m}' \cap \mathcal{P}(X) = \mathcal{I}_{\mathcal{P}(X)}(\{x\})$. By Propositions 1.13 (first statement) and 1.2 then $\overline{\mathcal{O}(X)}$ and $\mathcal{O}(X')$ cannot be isomorphic.

Assume the fibers of $\pi_{\mathbb{C}} : X'_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ over the points of X are totally real. Since the image by φ of a real prime ideal of $\mathcal{P}(X')$ is a real prime ideal of $\mathcal{P}(X)$. It follows that $\varphi^{-1}(X) = \text{Max } \mathcal{P}(X') \cap \text{R-Spec } \mathcal{P}(X')$ and we conclude the proof using the last two statements of Proposition 1.13. \square

We may wonder if $\overline{\mathcal{O}(X)}$ is the ring of regular functions of an algebraic set. The following proposition gives an answer to this question.

Proposition 1.17. *Assume $\overline{\mathcal{O}(X)}$ is the ring of regular functions of an intermediate algebraic set Y between X and X' i.e. $\mathcal{P}(X) \subset \mathcal{P}(Y) \subset \mathcal{P}(X')$. Then*

$$\overline{\mathcal{O}(X)} \simeq \mathcal{O}(X').$$

Proof. Assume $\overline{\mathcal{O}(X)} = \mathcal{O}(Y)$. By Proposition 1.1, all the maximal ideals of $\overline{\mathcal{O}(X)}$ are real. It follows from Proposition 1.13.(1) that the fibers of $\pi_{\mathbb{C}} : X'_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ over the points of X are totally real. By Proposition 1.16, we get the proof. \square

Remark 1.18. It may happen that two algebraic sets have isomorphic rings of regular functions but non isomorphic rings of polynomial functions. Consider an irreducible algebraic set X of dimension one such that $X_{\mathbb{C}}$ is singular and has only non-real singularities (e.g $X = \overline{\mathcal{Z}(y^2 - (x^2 + 1)^2x)}$). Let X' be the normalization of X . From Proposition 1.9, we get $\mathcal{O}(X) = \mathcal{O}(X')$. By Proposition 1.16, we see that $\mathcal{O}(X)$ and $\mathcal{O}(X')$ are thus isomorphic. By Proposition 1.11, it follows that X is not normal and thus $\mathcal{P}(X)$ and $\mathcal{P}(X')$ are not isomorphic rings. To conclude, for an irreducible algebraic set X of dimension one, the non-real singularities of $X_{\mathbb{C}}$ contribute to the non-normality of $\mathcal{P}(X)$ but do not affect the normality of $\mathcal{O}(X)$.

2. SOME TOPOLOGICAL PROPERTIES OF INTEGRAL MORPHISMS

In real algebraic geometry, it is common to use various topologies, like the Zariski topology or the Euclidean topology. When dealing with algebra, the same situation appears, and in this section with study topological properties of integral morphism with respect to Zariski topology, the topology of the real spectrum, and the real Zariski topology.

2.1. Several topologies on a ring.

2.1.1. *Zariski topology.* Let A be a commutative ring containing \mathbb{Q} and denote by $\text{Spec } A$ (the Zariski spectrum of A) the set of all prime ideals of A . Then, $\text{Spec } A$ can be endowed with the Zariski topology whose basis of open subsets is given by the sets $D(a) = \{\mathfrak{p} \in \text{Spec } A \mid a \notin \mathfrak{p}\}$ for $a \in A$ (and whose closed subsets are given by the sets $V(I) = \{\mathfrak{p} \in \text{Spec } A \mid I \subset \mathfrak{p}\}$ where I is an ideal of A).

2.1.2. *Real spectrum topology.* To a commutative ring A containing \mathbb{Q} one may also associate a topological subspace $\text{Spec}_r A$ which takes into account only prime ideals \mathfrak{p} whose residual field admits an ordering.

Let us detail this construction below. An order α in A is given by a real prime ideal \mathfrak{p} of A (called the support of α and denoted by $\text{supp}(\alpha)$) and an ordering on the residue field $k(\mathfrak{p})$ at \mathfrak{p} or equivalently it is given by a morphism ϕ from A to a real closed field K .

The value $a(\alpha)$ of $a \in A$ at the ordering α is just $\phi(a)$. The set of orders of A is called the real spectrum of A and denoted by $\text{Spec}_r A$. It is empty if and only if -1 is a sum of squares in A . One endows $\text{Spec}_r A$ with a natural topology whose open subsets are generated by the sets $\{\alpha \in \text{Spec}_r A \mid a(\alpha) > 0\}$ where $a \in A$. The set $\text{Spec}_r A$ can be also identified with the set of prime cones of A : a prime cone α of A is a subset of A that satisfies (i) $\alpha + \alpha \subset \alpha$, (ii) $\alpha \cdot \alpha \subset \alpha$, (iii) $a^2 \in \alpha \forall a \in A$, (iv) $-1 \notin \alpha$, (v) $ab \in \alpha \Rightarrow (a \in \alpha \text{ or } -b \in \alpha) \forall (a, b) \in A \times A$. If α is a prime cone of A then the support of α is $\alpha \cap -\alpha$. A subset of A satisfying the conditions (i), (ii) and (iii) is called a cone of A . A cone of A satisfying (iv) is called a proper cone of A . Let α, β be two points of $\text{Spec}_r A$, then we say that β is a specialization of α and that α is generalization of β if $\alpha \subset \beta$ (as prime cones). By [7, Prop. 7.1.18], β is a specialization of α if and only if β is in the closure of the singleton $\{\alpha\}$ for the topology introduced previously.

For more details on the real spectrum, the reader is referred to [7].

One has a natural support mapping $\text{Spec}_r A \rightarrow \text{Spec } A$ which sends α to $\text{supp}(\alpha)$.

2.1.3. *Real Zariski topology.* In this work, we will also consider the set $\text{R-Spec } A$ which is just the image of the support mapping, namely it consists in all the real prime ideals of A . We endow it with the induced Zariski topology.

Let us denote $\text{Max } A \subset \text{Spec } A$ the subsets of all maximal ideals of A . Let us also denote $D_R(a) = D(a) \cap (\text{R-Spec } A)$ and $V_R(I) = V(I) \cap (\text{R-Spec } A)$.

Then, the closed subsets of $\text{R-Spec } A$ have the form $V_R(I)$ where I is an ideal of A and a basis of open subsets is given by the subsets $D_R(a)$ for $a \in A$.

2.1.4. *Functoriality.* Let $\phi : A \rightarrow B$ be a ring morphism. It canonically induces a map $\psi : \text{Spec } B \rightarrow \text{Spec } A$ which is continuous for the Zariski topology.

It also induces a map $\psi_r : \text{Spec}_r B \rightarrow \text{Spec}_r A$ which is continuous for the real spectrum topology. And also,

Proposition 2.1. *The morphism $\phi : A \rightarrow B$ induces a map $\psi_R : \text{R-Spec } B \rightarrow \text{R-Spec } A$ which is continuous for the Zariski topology.*

Proof. Let us see first that this is a well defined map. Indeed, let $\mathbf{q} \in \text{R-Spec } B$ and $\mathbf{p} = \psi(\mathbf{q})$. Then, there exists an ordering on $k(\mathbf{q})$ that one may define by giving a morphism $B/\mathbf{q} \rightarrow R$ into a real closed field R . Hence, one gets the following commutative diagram :

$$\begin{array}{ccc} A & \rightarrow & B \\ \downarrow & & \downarrow \\ A/\mathbf{p} & \rightarrow & B/\mathbf{q} \rightarrow R \end{array}$$

which defines an ordering on $k(\mathbf{p})$ and hence \mathbf{p} is a real prime ideal.

The continuity comes from the following sequence of equalities :

$$\begin{aligned} \psi_R^{-1}(D_R(a)) &= \psi_R^{-1}(D(a) \cap \text{R-Spec } A) = \psi^{-1}(D(a) \cap \text{R-Spec } A) \cap \text{R-Spec } B = \\ &= \psi^{-1}(D(a)) \cap \psi^{-1}(\text{R-Spec } A) \cap \text{R-Spec } B = D(\phi(a)) \cap \text{R-Spec } B = D_R(\phi(a)). \end{aligned}$$

□

From now on, we will deal with ring extensions, namely ϕ will be injective.

2.2. Lying over and going-up.

2.2.1. Lying over.

Definition 2.2. We say that a ring extension $\phi : A \rightarrow B$ satisfies the lying over property if ψ is surjective. Likewise, we say that ϕ satisfies the real lying over (resp. strong real lying over) property if ψ_R is surjective (resp. bijective).

We recall, for instance from [25, Theorem 9.3]) :

Proposition 2.3. *Assume that the ring extension $\phi : A \rightarrow B$ is integral. Then*

- (1) ϕ satisfies the lying over property,
- (2) ψ induces a map from $\text{Max } B$ to $\text{Max } A$ which is surjective.

One has also, induced from ψ_R , a map from $\text{R-Max } B$ to $\text{R-Max } A$ but the real counterpart of the last property is false in general, namely ψ_R is not necessarily surjective.

Indeed, let us consider Example 1.14. Namely, denote by A the ring of polynomial functions of the cubic with an isolated singularity $Z(y^2 - x^2(x - 1))$ and by \bar{A} the ring of polynomial functions of its normalization. Then, $A \rightarrow \bar{A}$ is integral although there is only a non-real maximal ideal of \bar{A} lying over the maximal and real ideal of A corresponding to the real isolated point of the cubic.

Remark 2.4. The lying over property does not imply the real lying-over property.

We define now a very strong real lying over property.

Definition 2.5. We say that a ring extension $\phi : A \rightarrow B$ satisfies the very strong real lying over property if, given $\mathbf{p} \in \text{R-Spec } A$, there exists a unique $\mathbf{q} \in \text{R-Spec } B$ such that $\mathbf{q} \cap A = \mathbf{p}$ (i.e ϕ satisfies the strong real lying over property) and the induced injective map on the residue fields $k(\mathbf{p}) \hookrightarrow k(\mathbf{q})$ is an isomorphism.

2.2.2. Going-up.

Definition 2.6. We say that a ring extension $\phi : A \rightarrow B$ satisfies the going-up property if, for any couple of prime ideals $\mathfrak{p} \subset \mathfrak{p}'$ in $\text{Spec } A$ and a prime ideal $\mathfrak{q} \in \text{Spec } B$ lying over \mathfrak{p} , there exists a prime ideal $\mathfrak{q}' \in \text{Spec } B$ over \mathfrak{p}' and such that $\mathfrak{q} \subset \mathfrak{q}'$.

The going-up property is stronger than the lying over property : it is obvious in the case where A and B are domains and it follows from a theorem by Kaplansky in all generality.

When our ring extension is integral, we recall, for instance from [25, Theorem 9.3] :

Proposition 2.7. *Assume that the ring extension $\phi : A \rightarrow B$ is integral. Then, ϕ satisfies the going-up property.*

One readily deduces from Propositions 2.3 and 2.7:

Proposition 2.8. *Assume that $\phi : A \rightarrow B$ is integral. Then, ψ is a closed mapping.*

To get a real counterpart, one has to increase the assumptions.

One can define also a real going-up property for the real spectrum.

Definition 2.9. We say that a ring extension $\phi : A \rightarrow B$ satisfies the going-up property for the real spectrum if, for any couple of points $\alpha, \alpha' \in \text{Spec}_r A$ such that α' belongs to the closure of α (which we denote by $\alpha \rightarrow \alpha'$) and a point $\beta \in \text{Spec}_r B$ lying over α , there exists a point $\beta' \in \text{Spec}_r B$ over α' and such that $\beta \rightarrow \beta'$.

We recall from [2, Proposition 4.2]:

Proposition 2.10. *Assume that the ring extension $\phi : A \rightarrow B$ is integral. Then, ϕ satisfies the real going-up property for the real spectrum and ψ_r is a closed mapping.*

Likewise, one may define a going-up property for real prime ideals. Example 1.14 shows that integral extensions do not necessarily satisfy the real going-up property since they do not necessarily satisfy the real lying-over property. The going-up property for real prime ideals will be studied more precisely in a forthcoming paper [13].

We define a strong and a very strong real lying over property for ideals.

Definition 2.11. Let $A \rightarrow B$ be a ring extension.

- 1) We say that $A \rightarrow B$ satisfies the strong real lying over property if, given a real prime ideal \mathfrak{p} of A , there exists a unique $\mathfrak{p}' \in \text{R-Spec } B$ lying over \mathfrak{p} .
- 2) We say that $A \rightarrow B$ satisfies the very strong real lying over property if, given a real prime ideal \mathfrak{p} of A , there exists a unique $\mathfrak{p}' \in \text{R-Spec } B$ lying over \mathfrak{p} and the induced injective map on the residue fields $k(\mathfrak{p}) \rightarrow k(\mathfrak{p}')$ is an isomorphism.

2.3. Central rings.

Definition 2.12. We say that a domain A with fractions field K is a central ring if the image of $\text{Spec}_r K$ is dense in $\text{Spec}_r A$.

In the geometric setting, one also has the notion of a central algebraic set:

Definition 2.13. Let X be an irreducible algebraic set. We say that X is central if the set of all its regular points is dense with respect to the euclidean topology.

According to [7, Proposition 7.6.2] and [7, Proposition 7.6.4], one has :

Proposition 2.14. *Let X be an irreducible algebraic set. Then, X is a central if and only if its ring of polynomial functions is a central ring.*

Example 2.15. (1) The cubic with isolated point of Example 1.14 is not central.

(2) The nodal curve $\mathcal{Z}(y^2 - x^2(x - 1))$ is central.

Of central interest is the following result :

Proposition 2.16. *Let $\phi : A \rightarrow B$ be an integral ring extension where A and B are domains with same fraction field. If A is central, then :*

- (1) B is central,
- (2) ψ_R is surjective (ϕ satisfies the real lying over property).

Proof. (1) Let $\beta' \in \text{Spec}_r B$. It lies over $\alpha' \in \text{Spec}_r A$. Since A is central, there exists $\alpha \in \text{Spec}_r A$ coming from $\text{Spec}_r K$ such that $\alpha \rightarrow \alpha'$. Since A and B have same field of fractions, there is $\beta \in \text{Spec}_r B$ coming from $\text{Spec}_r K$ over α . Looking now at α and α' as prime cones, then $\alpha \subset \alpha' = \phi^{-1}(\beta')$. And hence, $\phi(\alpha) \subset \phi(\phi^{-1}(\beta')) \subset \beta'$. One gets $\beta \subset \beta'$, namely $\beta \rightarrow \beta'$. This shows that B is central.

- (2) Let $\mathbf{p}' \in \text{R-Spec } A$. It is the support of a point $\alpha' \in \text{Spec}_r A$. Since A is central, there exists $\alpha \in \text{Spec}_r K \cap \text{Spec}_r A$ such that $\alpha \rightarrow \alpha'$. Since A and B have same field of fractions K , there is $\beta \in \text{Spec}_r B \cap \text{Spec}_r K$ over α . By the real going-up for the real spectrum (Proposition 2.10), one deduces the existence of $\beta' \in \text{Spec}_r B$ over α and such that $\beta \rightarrow \beta'$. It implies that the support of β' is a real prime ideal \mathbf{q}' lying over \mathbf{p}' . Namely, ϕ satisfies the real lying over property or, in other words, ψ_R is surjective. □

Let us end this section with a property which we will need in the following.

Lemma 2.17. *Let $A \rightarrow B$ be an integral extension with A and B domains. Then, $\dim A = \dim B$.*

Proof. Since the morphism is integral, there do not exist two different prime ideals $\mathbf{q} \subset \mathbf{q}'$ in B lying over the same prime ideal \mathbf{p} in A [25, Thm. 9.3, Sect. 9]. Hence, one has $\dim A \geq \dim B$.

Since the morphism is injective and A and B are domains, by the going-up property, one gets that $\dim A \leq \dim B$. □

3. RATIONAL AND CONTINUOUS FUNCTIONS

We introduce three classes of functions on real algebraic varieties, the regulous functions, rational continuous functions and blow-regular functions. We briefly recall the relations between these functions. We refer the reader to [26] for more details.

3.1. Regulous functions. Regulous functions on an algebraic set were introduced in [11] in order to correct the defects of the classical real algebraic geometry i.e. the real algebraic geometry where the usual class of functions is the class of regular functions.

Definition 3.1. We say that a function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is regulous on \mathbb{R}^n if f is continuous on \mathbb{R}^n and f is a rational function on \mathbb{R}^n , i.e. there exists a non-empty Zariski open subset $U \subseteq \mathbb{R}^n$ such that $f|_U$ is regular. We denote by $\mathcal{R}^0(\mathbb{R}^n)$ the ring of regulous functions on \mathbb{R}^n .

The regulous topology of \mathbb{R}^n is defined to be the topology whose closed subsets are generated by the zero sets of regulous functions on \mathbb{R}^n . By [11, Thm. 6.4], the regulous topology on \mathbb{R}^n is the algebraically constructible topology on \mathbb{R}^n (denoted by \mathcal{C} -topology).

Definition 3.2. Let X be an algebraic subset of \mathbb{R}^n . A regulous function on X is the restriction to X of a regulous function on \mathbb{R}^n . The ring of regulous functions on X , denoted by $\mathcal{R}^0(X)$, corresponds to

$$\mathcal{R}^0(X) = \mathcal{R}^0(\mathbb{R}^n) / \mathcal{I}(X)$$

where $\mathcal{I}(X)$ is the ideal of $\mathcal{R}^0(\mathbb{R}^n)$ of regulous functions on \mathbb{R}^n that vanish identically on X .

Let X be an algebraic subset of \mathbb{R}^n . By [19, Prop. 8] or [11, Thm. 4.1], a regulous function on X is always rational on X (coincides with a regular function on a dense Zariski open subset of X). We have a natural ring morphism $\phi^0 : \mathcal{R}^0(X) \rightarrow \mathcal{K}(X)$ which send $f \in \mathcal{R}^0(X)$ to the class $(U, f|_U)$ in $\mathcal{K}(X)$, where $(U, f|_U)$ is a regular presentation of f i.e. U is a dense Zariski open subset of X and $f|_U$ is regular.

In the following, we will denote by \overline{E}^τ the closure of the subset E of \mathbb{R}^n for the topology τ on \mathbb{R}^n . We also denote by X_{reg} (resp. $\text{Sing}(X)$) the smooth (resp. singular) locus of X .

Proposition 3.3. [26, Prop. 2.7]

Let X be an algebraic subset of \mathbb{R}^n . The map $\phi^0 : \mathcal{R}^0(X) \rightarrow \mathcal{K}(X)$ is injective if and only if $\overline{X_{reg}}^c = X$.

3.2. Rational continuous functions. Let X be an algebraic set. Let $f \in \mathcal{K}(X)$ be a rational function on X . The domain of f , denoted by $\text{dom}(f)$, is the biggest dense Zariski open subset of X on which f is regular, namely $f = \frac{p}{q}$ on $\text{dom}(f)$ where p and q are polynomial functions on \mathbb{R}^n such that $\mathcal{Z}(q) \cap X = X \setminus \text{dom}(f)$. The indeterminacy locus or polar locus of f is defined to be the Zariski closed set $\text{indet}(f) = X \setminus \text{dom}(f)$.

Definition 3.4. Let f be a real continuous function on X . We say that f is a rational continuous function on X if f is rational on X i.e. there exists a dense Zariski open subset $U \subseteq X$ such that $f|_U$ is regular.

Let $\mathcal{R}_0(X)$ denote the ring of rational continuous functions on X . We have a natural ring morphism $\phi_0 : \mathcal{R}_0(X) \rightarrow \mathcal{K}(X)$ which send $f \in \mathcal{R}_0(X)$ to the class $(U, f|_U)$ in $\mathcal{K}(X)$, where $(U, f|_U)$ is a regular presentation of f .

Remark 3.5. We have $\mathcal{R}_0(\mathbb{R}^n) = \mathcal{R}^0(\mathbb{R}^n)$.

Recall that X is “central” if $\overline{X_{reg}}^{eucl} = X$ (Definition 2.13).

Proposition 3.6. [26, Prop. 2.14]

The map $\phi_0 : \mathcal{R}_0(X) \rightarrow \mathcal{K}(X)$ is injective if and only if X is central.

Remark 3.7. The use of rational continuous functions in real algebraic geometry is interesting only for real algebraic varieties that are central, indeed without the central hypothesis a rational continuous function is not necessarily semi-algebraic. On the contrary, a regulous function is always semi-algebraic.

By [19, Prop. 8] or [11, Thm. 4.1], any $f \in \mathcal{R}^0(X)$ can be identified with a unique function in $\mathcal{R}_0(X)$. Hence we get the following ring inclusion $\phi_0^0 : \mathcal{R}^0(X) \hookrightarrow \mathcal{R}_0(X)$ and moreover $\phi^0 = \phi_0 \circ \phi_0^0$. In [19] Kollár and Nowak study the differences between rational continuous functions and regulous functions on central real algebraic varieties (“regulous functions” are named “hereditarily rational continuous functions” in [19]), in particular they give the first example of rational continuous function that is not regulous (in the central case) [19, Ex. 2]. In [26] and [18] the comparison between rational continuous and regulous continues.

Theorem 3.8. [19, Prop. 8, Thm. 10] (*the smooth case*) and [26, Thm. 2.22]

Let X be an algebraic set such that $\dim \text{Sing}(X) \leq 0$. Then the map $\phi_0^0 : \mathcal{R}^0(X) \hookrightarrow \mathcal{R}_0(X)$ is an isomorphism.

3.3. Blow-regular functions.

Definition 3.9. Let X be an algebraic set. Let $\mathcal{B}(X)$ denote the ring of real functions f defined on X such that, there exists a resolution of singularities $\pi : \tilde{X} \rightarrow X$ such that the composite $f \circ \pi$ is regular on \tilde{X} . A $f \in \mathcal{B}(X)$ is called a “blow-regular function” on X .

By [11, Thm. 3.11], regulous functions and blow-regular functions coincide in the smooth case.

Theorem 3.10. [11, Thm. 3.11]

Let X be a smooth algebraic set. We have $\mathcal{R}^0(X) = \mathcal{B}(X)$.

Remark 3.11. According to Theorems 3.10 and 3.8, $f \in \mathcal{B}(X)$ if and only if f is a real function defined on X such that for any resolution of singularities $\pi : \tilde{X} \rightarrow X$ such that $f \circ \pi$ is regulous or rational continuous on \tilde{X} .

In [26], Theorem 3.10 is extended to the central case.

Proposition 3.12. [26, Prop. 2.29]

Let X be a central algebraic set. We have

$$\mathcal{B}(X) = \mathcal{R}_0(X).$$

3.4. Zariski spectrum of the ring of rational continuous functions.

Proposition 3.13. Let X be an algebraic set. The maximal ideals of $\mathcal{R}_0(X)$ are real.

Proof. Since $1 + \sum \mathcal{R}_0(X)^2 \subset \mathcal{R}_0(X)^\times$, we can conclude using the arguments given in the proof of Proposition 1.1. \square

Let X be an algebraic set. We say that X is “almost central” if $\dim(X \setminus \overline{X_{reg}}^{eucl}) \leq 0$. Any algebraic curve is almost central.

Remark 3.14. Let X be an almost central algebraic set.

- If $f \in \mathcal{R}_0(X)$ then f is a semi-algebraic function on X : On $X \setminus \overline{X_{reg}}^{eucl}$ it is clear since $X \setminus \overline{X_{reg}}^{eucl}$ is empty or a finite set of points. On the semi-algebraic set $\overline{X_{reg}}^{eucl}$ the graph of f is the euclidean closure of the graph of a regular presentation of f .
- If $f \in \mathcal{R}_0(X)$ then $\mathcal{Z}(f)$ is regulous closed: $\mathcal{Z}(f) = (\mathcal{Z}(f) \cap (X \setminus \overline{X_{reg}}^{eucl})) \amalg (\mathcal{Z}(f) \cap \overline{X_{reg}}^{eucl})$, $\mathcal{Z}(f) \cap (X \setminus \overline{X_{reg}}^{eucl})$ is clearly regulous closed and $\mathcal{Z}(f) \cap \overline{X_{reg}}^{eucl}$ is also regulous closed by [26, Prop. 3.5].

Proposition 3.15. Let X be an almost central algebraic set. Let I be an ideal of $\mathcal{R}_0(X)$. There exists $f \in I$ such that $\mathcal{Z}(f) = \mathcal{Z}(I)$.

Proof. We have $\mathcal{Z}(I) = \bigcap_{f \in I} \mathcal{Z}(f)$. By Remark 3.14, $\mathcal{Z}(I)$ is thus an intersection of regulous closed sets. Since the regulous topology on X is the topology on X induced by the regulous topology on \mathbb{R}^n (by definition of $\mathcal{R}^0(X)$) then the regulous topology on X is Noetherian [11, Thm. 4.3]. Therefore there exists $f_1, \dots, f_t \in I$ such that

$$\mathcal{Z}(I) = \mathcal{Z}(f_1) \cap \dots \cap \mathcal{Z}(f_t) = \mathcal{Z}(f_1^2 + \dots + f_t^2).$$

\square

Proposition 3.16. (Weak Nullstellensatz)

Let X be an almost central algebraic set. Let I be an ideal of $\mathcal{R}_0(X)$. Then $\mathcal{Z}(I) = \emptyset$ if and only if $I = \mathcal{R}_0(X)$.

Proof. By Proposition 3.15 there exists $f \in I$ such that $\mathcal{Z}(f) = \mathcal{Z}(I)$. Therefore $\mathcal{Z}(I) = \emptyset$ if and only if f is a unit in $\mathcal{R}_0(X)$. \square

Corollary 3.17. Let X be an almost central algebraic set. The set $\text{Max } \mathcal{R}_0(X)$ is in bijection with X . More precisely, for any maximal ideal \mathfrak{m} of $\mathcal{R}_0(X)$ there exists a unique $x \in X$ such that $\mathfrak{m} = \mathfrak{m}_x$ with $\mathfrak{m}_x = \mathcal{I}_{\mathcal{R}_0(X)}(\{x\})$.

Proof. Let \mathfrak{m} be a maximal ideal of $\mathcal{R}_0(X)$. By Proposition 3.16 there exists $x \in \mathcal{Z}(\mathfrak{m})$. We have $\mathfrak{m} \subset \mathcal{I}_{\mathcal{R}_0(X)}(\mathcal{Z}(\mathfrak{m})) \subset \mathcal{I}_{\mathcal{R}_0(X)}(\{x\})$. Since $\mathcal{I}_{\mathcal{R}_0(X)}(\{x\}) \neq \mathcal{R}_0(X)$ and \mathfrak{m} is maximal then the previous inclusions are equalities. \square

Lemma 3.18. [7, Prop. 2.6.4] (*Łojasiewicz inequality*)

Let X be an algebraic set. Let f be a continuous semi-algebraic function on X and let g be a continuous semi-algebraic function on $X \setminus \mathcal{Z}(f)$. For $N \in \mathbb{N}$ sufficiently big then $f^N g$ is a continuous semi-algebraic function on X vanishing over the whole set $\mathcal{Z}(f)$.

Theorem 3.19. (*Nullstellensatz*)

Let X be an almost central algebraic set. Let I be an ideal of $\mathcal{R}_0(X)$. Then

$$\mathcal{I}_{\mathcal{R}_0(X)}(\mathcal{Z}(I)) = \sqrt{I}.$$

Proof. We have trivially $\sqrt{I} \subset \mathcal{I}_{\mathcal{R}_0(X)}(\mathcal{Z}(I))$.

Let $g \in \mathcal{I}_{\mathcal{R}_0(X)}(\mathcal{Z}(I))$. By Proposition 3.15 there exists $f \in I$ such that $\mathcal{Z}(f) = \mathcal{Z}(I)$. Since $\mathcal{Z}(f) \subset \mathcal{Z}(g)$ then the function $\frac{1}{f}$ is a continuous semi-algebraic function on $X \setminus \mathcal{Z}(g)$. By Lemma 3.18, for a sufficiently big integer N the function $h_N = \frac{g^N}{f}$ is continuous on X and thus clearly belongs to $\mathcal{R}_0(X)$. Therefore $g^N = h_N f \in I$ and the proof is done. \square

Proposition 3.20. Let X be an almost central algebraic set. Any radical ideal of $\mathcal{R}_0(X)$ is real.

Proof. Let I be a radical ideal of $\mathcal{R}_0(X)$. Assume f_1, \dots, f_t are rational continuous functions on X ($t > 1$) such that $f_1^2 + \dots + f_t^2 \in I$. Let $j \in \{1, \dots, t\}$. We have $\mathcal{Z}(f_1^2 + \dots + f_t^2) \subset \mathcal{Z}(f_j)$ and thus, by Lemma 3.18, $\frac{f_j^N}{f_1^2 + \dots + f_t^2} \in \mathcal{R}_0(X)$ for N sufficiently big. We conclude that for N big then $f_j^N \in I$ and it gives the proof since I is radical. \square

4. PROPERTIES OF FINITE BIRATIONAL MAPS

In this section, we investigate the properties of finite birational maps between algebraic sets in relationship with the rings of rational continuous and regulous functions. In this text, a birational map $\pi : Y \rightarrow X$ between two algebraic sets is a polynomial map that induces an isomorphism from a dense Zariski open subset of Y to a dense Zariski open subset of X . It means that π is defined everywhere, but its inverse may not be. If X and Y are irreducible, π is birational if and only if π induces an isomorphism $\mathcal{K}(X) \simeq \mathcal{K}(Y)$.

Recall that a map $\pi : Y \rightarrow X$ between two topological spaces is called proper if the preimage of every compact subset of X is a compact subset of Y .

Lemma 4.1. Let $\pi : Y \rightarrow X$ be a finite birational map between irreducible algebraic sets. Then

- 1) The ring morphism $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$, $p \mapsto p \circ \pi$ is injective and integral.
- 2) The map π is proper, and thus closed, for the euclidean topology.

Proof. The ring morphism $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is injective and integral since π is respectively birational (and thus dominant) and finite.

By [2, Ch. 2, Prop 4.2-4.3], the map $\psi_r : \text{Spec}_r \mathcal{P}(Y) \rightarrow \text{Spec}_r \mathcal{P}(X)$ is closed for the real spectrum topology. According to [7, Theorem 7.2.3], there is a bijective correspondence between open (resp. closed) semi-algebraic subsets of X (resp. Y) and open (resp. closed) constructible subsets of the real spectrum $\text{Spec}_r \mathcal{P}(X)$ (resp. the real spectrum $\text{Spec}_r \mathcal{P}(Y)$). It follows that the image by π of every closed semi-algebraic subset of Y is a closed semi-algebraic subset of X . Let $K \subset X$ be a compact subset. Let $\bigcup_{i \in I} U_i$ be an open cover of $\pi^{-1}(K)$; since the open balls are semi-algebraic sets, we can assume the U_i 's to be semi-algebraic sets. For any $x \in X$, the fiber $\pi^{-1}(x)$ is finite so there exists a finite subset $\Lambda_x \subset I$ such that $\pi^{-1}(x) \subset \bigcup_{i \in \Lambda_x} U_i$. The set $Y \setminus \bigcup_{i \in \Lambda_x} U_i$ is a closed semi-algebraic subset of Y and thus its image by π is a closed semi-algebraic subset of X . It follows that $V_x = X \setminus \pi(Y \setminus \bigcup_{i \in \Lambda_x} U_i)$ is an open subset of X that clearly contains x . Since K is compact

and since $K \subset \bigcup_{x \in K} V_x$ then there exists $l \in \mathbb{N} \setminus \{0\}$ and $x_1, \dots, x_l \in K$ such that $K \subset \bigcup_{j=1}^l V_{x_j}$. Let Λ be the finite subset of I given by $\Lambda = \bigcup_{j=1}^l \Lambda_{x_j}$. We have proved that $\pi^{-1}(K)$ is compact since

$$\pi^{-1}(K) \subset \pi^{-1}\left(\bigcup_{j=1}^l V_{x_j}\right) \subset \bigcup_{i \in \Lambda} U_i.$$

Since π is proper for the Euclidean topology then π is closed for the Euclidean topology. \square

Lemma 4.2. *Let $\pi : Y \rightarrow X$ be a birational map between irreducible central algebraic sets. The composition by π gives a injective ring morphism*

$$\pi_0 : \mathcal{R}_0(X) \hookrightarrow \mathcal{R}_0(Y), \quad f \mapsto f \circ \pi.$$

Proof. Let $f \in \mathcal{R}_0(X)$. The function $f \circ \pi$ is clearly rational on Y since π is birational. The function $f \circ \pi$ is clearly continuous on Y and thus $f \circ \pi \in \mathcal{R}_0(Y)$. By Proposition 3.6 and since π is birational then $\mathcal{R}_0(X)$ and $\mathcal{R}_0(Y)$ inject into $\mathcal{K}(X)$. It follows that f and $f \circ \pi$ correspond to the same rational function of $\mathcal{K}(X)$. Since Y is central then $f \circ \pi$ is the unique continuous extension to Y of $f \circ \pi = f$ seen as a rational function on Y . \square

Proposition 4.3. *Let $\pi : Y \rightarrow X$ be a finite birational map between irreducible algebraic sets. We denote by ψ the map $\text{Spec } \mathcal{P}(Y) \rightarrow \text{Spec } \mathcal{P}(X)$, $\mathbf{p} \mapsto \mathbf{p} \cap \mathcal{P}(X)$ (see 2.1.4).*

If X is central then:

- 1) π is surjective.
- 2) $\psi(\text{R-Spec } \mathcal{P}(Y)) = \text{R-Spec } \mathcal{P}(X)$ i.e. $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ satisfies the real lying over property i.e. $\psi_R : \text{R-Spec } \mathcal{P}(Y) \rightarrow \text{R-Spec } \mathcal{P}(X)$ is surjective.
- 3) $\psi(\text{R-Spec } \mathcal{P}(Y) \cap \text{Max } \mathcal{P}(Y)) = \text{R-Spec } \mathcal{P}(X) \cap \text{Max } \mathcal{P}(X)$.
- 4) Y is central.
- 5) π is a quotient map for the Euclidean topology.
- 6) *The image of the injective ring morphism $\pi_0 : \mathcal{R}_0(X) \rightarrow \mathcal{R}_0(Y)$, $f \mapsto f \circ \pi$ is the ring of rational continuous functions on Y that are constant on the fibers of π . We denote by $\frac{\mathcal{R}_0(Y)}{\pi}$ this ring and thus*

$$\mathcal{R}_0(X) \simeq \frac{\mathcal{R}_0(Y)}{\pi}.$$

Proof. The first four properties are direct consequences of Proposition 2.16 and 2.3.

By 1) and Lemma 4.1 the map π is continuous, surjective and a closed map for the Euclidean topology; this gives 5).

Let $g \in \mathcal{R}_0(Y)$. By 4) we know that X and Y are central. By Proposition 3.6 and since π is birational, the function g is rational on X . Since π is a quotient map for the Euclidean topology then the continuous function g on Y induces a continuous function on X if and only if g is constant on the fibers of π . \square

The following theorem will lead to the introduction of the real weak normalization of an algebraic set in a forthcoming section.

Theorem 4.4. *Let $\pi : Y \rightarrow X$ be a finite birational map between irreducible algebraic sets. We assume moreover that X is central. The following properties are equivalent:*

- 1) π is a bijection.
- 2) *The ring morphism $\mathcal{R}_0(X) \rightarrow \mathcal{R}_0(Y)$ is an isomorphism.*
- 3) $\mathcal{P}(Y) \subset \mathcal{R}_0(X)$.
- 4) π is an homeomorphism for the Euclidean topology.
- 5) π is an homeomorphism for the regular topology.

- 6) $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ satisfies the strong real lying over property i.e. the induced mapping $\text{R-Spec } \mathcal{P}(Y) \rightarrow \text{R-Spec } \mathcal{P}(X)$ is a bijection.

Proof. By Proposition 4.3, we know that π is surjective and Y is central.

By 6) of Proposition 4.3 then we get that 1) implies 2).

Assume $\mathcal{R}_0(X) \rightarrow \mathcal{R}_0(Y)$ is an isomorphism and π is not bijective. There exists $x \in X$ such that we have $\{y_1, y_2\} \subset \pi^{-1}(x)$ and $y_1 \neq y_2$. There exists $p \in \mathcal{P}(Y)$ such that $p(y_1) \neq p(y_2)$. By Proposition 4.3 we get that $p \in \mathcal{R}_0(Y) \setminus \mathcal{R}_0(X)$ since p is not constant on the fibers of π , a contradiction. We have proved that 2) implies 1).

Clearly, 2) implies 3).

Assume $\mathcal{P}(Y) \subset \mathcal{R}_0(X)$. If π is not a bijection, we can find as above a $p \in \mathcal{P}(Y)$ such that $p \in \mathcal{R}_0(Y) \setminus \mathcal{R}_0(X)$ and it gives a contradiction. We have proved that 3) implies 1). At this level of the proof, 1), 2), and 3) are equivalent.

It is clear that 4) and 5) imply 1). We are going to prove that 1) implies 4) and 5). Assume π is a bijection. By Lemma 4.1, π is closed for the Euclidean topology and thus it is clearly an homeomorphism for the Euclidean topology. Since π is a bijection, we get that $\mathcal{R}^0(X) \simeq \mathcal{R}^0(Y)$ (we already know that 1) implies 2)). Since the regulous closed sets are the zero sets of rational continuous functions (Remark 3.14) then it follows that π is also an homeomorphism for the regulous topology. Indeed, let $F = \mathcal{Z}(f)$ be a regulous closed subset of Y ($f \in \mathcal{R}_0(Y)$) then $\pi(V) = \mathcal{Z}(f) \cap X$ when we see f as a rational continuous function on X .

If we assume $\psi_R : \text{R-Spec } \mathcal{P}(Y) \rightarrow \text{R-Spec } \mathcal{P}(X)$ is a bijection then π is clearly injective by restricting ψ_R to the real and maximal ideals. Since we have already noticed that π is surjective, we have proved that 6) implies 1). We prove now the converse. Assume π is bijective. It means that $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ satisfies the strong real lying over property for maximal and real ideals. So let $\mathfrak{p} \in \text{R-Spec } \mathcal{P}(X)$. By Proposition 4.3 then $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ satisfies the real lying over property and thus there exists $\mathfrak{p}' \in \text{R-Spec } \mathcal{P}(Y)$ such that $\mathfrak{p}' \cap \mathcal{P}(X) = \mathfrak{p}$. Assume there exists $\mathfrak{p}'' \in \text{R-Spec } \mathcal{P}(Y)$ such that $\mathfrak{p}'' \cap \mathcal{P}(X) = \mathfrak{p}$. We denote respectively by V, V' and V'' the irreducible real algebraic sets $\mathcal{Z}(\mathfrak{p}), \mathcal{Z}(\mathfrak{p}')$ and $\mathcal{Z}(\mathfrak{p}'')$. Since the maps $\frac{\mathcal{P}(X)}{\mathfrak{p}} \hookrightarrow \frac{\mathcal{P}(Y)}{\mathfrak{p}'}$ and $\frac{\mathcal{P}(X)}{\mathfrak{p}} \hookrightarrow \frac{\mathcal{P}(Y)}{\mathfrak{p}''}$ are integral then the rings $\frac{\mathcal{P}(X)}{\mathfrak{p}}, \frac{\mathcal{P}(Y)}{\mathfrak{p}'}$ and $\frac{\mathcal{P}(Y)}{\mathfrak{p}''}$ have the same Krull dimension (Lemma 2.17) that we denote by d . Since $\mathfrak{p}, \mathfrak{p}'$ and \mathfrak{p}'' are real ideals then $\dim V = \dim V' = \dim V'' = d$. The integral maps $\frac{\mathcal{P}(X)}{\mathfrak{p}} \hookrightarrow \frac{\mathcal{P}(Y)}{\mathfrak{p}'}$ and $\frac{\mathcal{P}(X)}{\mathfrak{p}} \hookrightarrow \frac{\mathcal{P}(Y)}{\mathfrak{p}''}$ induces two polynomial maps $\varphi' : V' \rightarrow V$ and $\varphi'' : V'' \rightarrow V$ between real algebraic sets. Since π is a bijection then it follows that φ' and φ'' are both injective. By 5) the sets $\varphi'(V')$ and $\varphi''(V'')$ are semi-algebraic subsets of dimension d of V and thus $\overline{\varphi'(V')^{Zar}} = \overline{\varphi''(V'')^{Zar}} = V$. By Newman's Theorem as stated in [6] then $\varphi'(V')$ contains a Zariski dense open subset of V . Of course we get the same property for $\varphi''(V'')$ and thus also for $\varphi'(V') \cap \varphi''(V'')$. Consequently, $\dim(V' \cap V'') = d$. By the real Nullstellensatz [7, Thm. 4.1.4], $\mathcal{I}_{\mathcal{P}(X)}(\mathcal{Z}(\mathfrak{p})) = \mathfrak{p}$, $\mathcal{I}_{\mathcal{P}(Y)}(\mathcal{Z}(\mathfrak{p}')) = \mathfrak{p}'$ and $\mathcal{I}_{\mathcal{P}(Y)}(\mathcal{Z}(\mathfrak{p}'')) = \mathfrak{p}''$ and it follows from above results that $\mathfrak{p}' = \mathfrak{p}''$. We have proved that $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ satisfies the strong real lying-over property. \square

Remark 4.5. Note that a bijective birational polynomial map is not necessarily an isomorphism. For instance, let X be the cuspidal curve given by $y^2 = x^3$ in \mathbb{R}^2 , and X' be its normalization. The normalization map $\pi : X' \rightarrow X$ is birational, finite and bijective. It is even a homeomorphism with respect to the Zariski topology (the curves are irreducible, so the Zariski subsets are just points). However X is singular whereas X' is smooth.

Remark 4.6. Let $\pi : Y \rightarrow X$ be a finite birational map between irreducible algebraic sets satisfying one of six properties of Theorem 4.4. It seems not easy to decide whether π is an homeomorphism for the Zariski topology. It will be the subject of a forthcoming paper [13].

The following theorem will lead to the introduction of the real seminormalization of an algebraic set in a forthcoming section.

Theorem 4.7. *Let $\pi : Y \rightarrow X$ be a finite birational map between irreducible algebraic sets. We assume moreover that X is central. The following properties are equivalent:*

- 1) $\mathcal{P}(Y) \subset \mathcal{R}^0(X)$.
- 2) $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ satisfies the very strong real lying over property.

Proof. We begin with proving that 1) implies 2). Assume $\mathcal{P}(Y) \subset \mathcal{R}^0(X)$. Since $\mathcal{R}^0(X) \subset \mathcal{R}_0(X)$ then the six properties of Theorem 4.4 are satisfied. Let $\mathfrak{p} \in \mathbf{R}\text{-Spec } \mathcal{P}(X)$. Since $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ satisfies the strong real lying over property, there exists a unique real prime ideal \mathfrak{q} of $\mathcal{P}(Y)$ lying over \mathfrak{p} . Since the ring extension $\frac{\mathcal{P}(X)}{\mathfrak{p}} \hookrightarrow \frac{\mathcal{P}(Y)}{\mathfrak{q}}$ is integral then passing to the fraction fields $k(\mathfrak{p}) \hookrightarrow k(\mathfrak{q})$ is a finite algebraic extension of fields. Assume $k(\mathfrak{p}) \not\simeq k(\mathfrak{q})$. It means there exists $f \in \mathcal{P}(Y)$ such that the class of f in $k(\mathfrak{q})$ is not contained in $k(\mathfrak{p})$. Since f is regulous on X thus the restriction of f to $\mathcal{Z}(\mathfrak{p})$ is rational. It means that the image $f(\mathfrak{q})$ of f by the map $\mathcal{P}(Y) \rightarrow \frac{\mathcal{P}(Y)}{\mathfrak{q}} \hookrightarrow k(\mathfrak{q})$ belongs to the fraction field of $\frac{\mathcal{P}(X)}{\mathfrak{p}} = k(\mathfrak{p})$, a contradiction.

We prove that 2) implies 1). Assume $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ satisfies the very strong real lying over property. Since $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ satisfies the strong real lying-over property, according to Theorem 4.4 we have $\mathcal{P}(Y) \subset \mathcal{R}_0(X)$. Let $f \in \mathcal{P}(Y)$. By the above remark we know that f is rational continuous on X . Let V be an irreducible algebraic subset of X . We denote by \mathfrak{p} the real and prime ideal of $\mathcal{P}(X)$ corresponding to $\mathcal{I}_{\mathcal{P}(X)}(V)$. By the very strong lying over property, there exists a unique real prime ideal \mathfrak{q} of $\mathcal{P}(Y)$ lying over \mathfrak{p} and moreover $k(\mathfrak{p}) \simeq k(\mathfrak{q})$. It follows that $f(\mathfrak{q}) \in k(\mathfrak{p})$. Since $k(\mathfrak{p})$ is the fraction field of $\frac{\mathcal{P}(X)}{\mathfrak{p}}$ then the restriction of f to V is rational on V . According to the above f is a rational continuous function on X which remains rational by restriction to any algebraic subset of X . It follows from the main result of [19, Thm. 10] that $f \in \mathcal{R}^0(X)$. Therefore

$$\mathcal{P}(Y) \subset \mathcal{R}^0(X).$$

□

Remark 4.8. Let $\pi : Y \rightarrow X$ be a finite birational map between irreducible algebraic sets. By Lemma 4.1 the corresponding morphism $\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ is injective and integral. Let V' be an irreducible algebraic subset of Y . There exists $\mathfrak{q} \in \mathbf{R}\text{-Spec } \mathcal{P}(Y)$ such that $V' = \mathcal{Z}(\mathfrak{q})$. We denote by \mathfrak{p} the real prime ideal $\mathfrak{q} \cap \mathcal{P}(X)$ and by V the real irreducible algebraic subset of X given by $\mathcal{Z}(\mathfrak{p})$. The restriction of π to V' gives clearly a map $\pi|_{V'} : V' \rightarrow V$ which is finite since the corresponding morphisms of polynomial functions $\frac{\mathcal{P}(X)}{\mathfrak{p}} \hookrightarrow \frac{\mathcal{P}(Y)}{\mathfrak{q}}$ is integral. In general $\pi|_{V'}$ is no longer birational (cf. Example 6.24). In fact, $k(\mathfrak{q})$ is an algebraic extension of $k(\mathfrak{p})$. So, $\pi|_{V'}$ is still birational if and only if $k(\mathfrak{p}) \simeq k(\mathfrak{q})$.

The following definition is inspired by [19].

Definition 4.9. Let $\pi : Y \rightarrow X$ be a finite birational map between irreducible algebraic sets. We say that π is hereditarily birational if the restriction of π to any irreducible algebraic subset of Y (as explained in Remark 4.8) is still birational.

From Theorems 4.4, 4.7 and Remark 4.8, we get:

Proposition 4.10. *Let $\pi : Y \rightarrow X$ be a bijective finite birational map between irreducible algebraic sets. We assume moreover that X is central. The following properties are equivalent:*

- 1) $\mathcal{P}(Y) \subset \mathcal{R}^0(X)$.
- 2) π is hereditarily birational.

5. RATIONAL CONTINUOUS FUNCTIONS AND NORMALIZATION

We are interested in the relationship between the ring of rational continuous functions and the normalization map. We give in particular a criterion, in terms of rational continuous functions, for the normalisation map of an algebraic set to be bijective.

Before this, we begin with investigating the integral closure of the ring of rational continuous functions. Later, we discuss the particular case of curves.

5.1. Integral closure of the ring of rational continuous functions. As a first result, we prove that the ring of rational continuous functions on a smooth algebraic set is integrally closed.

Theorem 5.1. *Let X be a smooth irreducible algebraic set. Then $\mathcal{R}_0(X) = \mathcal{R}^0(X)$ is integrally closed in $\mathcal{K}(X)$.*

Proof. Assume $f \in \mathcal{K}(X)^*$ and there exist $d \in \mathbb{N}^*$ and $a_i \in \mathcal{R}^0(X)$, $i = 0, \dots, d-1$ such that

$$f^d + a_{d-1}f^{d-1} + \dots + a_0 = 0$$

in $\mathcal{K}(X)$. It means that there exists a non-empty Zariski open subset U of X such that $\forall x \in U$ we have $f^d(x) + a_{d-1}(x)f^{d-1}(x) + \dots + a_0(x) = 0$. By Theorem 3.10, there exists a composition of blowings-up with smooth centers $\pi : Y \rightarrow X$ such that $\tilde{a}_i = a_i \circ \pi$ is regular on Y for $i = 0, \dots, d-1$. Then $f \circ \pi$ is a rational function on Y which is integral over $\mathcal{O}(Y)$. Since Y is smooth then $\mathcal{O}(Y)$ is integrally closed in $\mathcal{K}(Y)$ (Proposition 1.9) and thus $f \circ \pi$ can be extended to a regular function \tilde{f} on Y . Obviously, we have $\forall y \in Y$ $\tilde{f}^d(y) + \tilde{a}_{d-1}(y)\tilde{f}^{d-1}(y) + \dots + \tilde{a}_0(y) = 0$. Let $x \in X$. Since each \tilde{a}_i is constant on $\pi^{-1}(x)$ then $\forall y \in \pi^{-1}(x)$ the real number $\tilde{f}(y)$ is a root of $p(t) = t^d + \tilde{a}_{d-1}(x)t^{d-1} + \dots + \tilde{a}_0(x) \in \mathbb{R}[t]$. Since $\pi^{-1}(x)$ is connected and since \tilde{f} is continuous on Y then \tilde{f} must be constant on $\pi^{-1}(x)$. Hence \tilde{f} induces a real continuous function g on X such that $\tilde{f} = g \circ \pi$ and g is a continuous extension to X of f . \square

A rational function which is not continuous on an algebraic set may admit several different behaviours at a indeterminacy point. It can be unbounded like $\frac{1}{x}$ at the origin in \mathbb{R} , bounded with infinitely many limit points like $\frac{x^2}{x^2 + y^2}$ at the origin in \mathbb{R}^2 , or bounded with finitely many limit points like in the case of rational function satisfying an integral equation with rational continuous coefficients.

Lemma 5.2. *Let X be a central irreducible algebraic set. Assume $f \in \mathcal{K}(X)$ satisfies an integral equation with coefficients in $\mathcal{R}_0(X)$. Then f admits finitely many limit points at its indeterminacy points.*

Proof. The rational function f satisfies an integral equation of the form

$$f^d + a_{d-1}f^{d-1} + \dots + a_0 = 0$$

with $d \in \mathbb{N}^*$ and $a_i \in \mathcal{R}_0(X)$, for $i = 0, \dots, d-1$. Let $\pi : Y \rightarrow X$ be a resolution of the singularities of X . By Remark 3.11, the functions $a_i \circ \pi$ are regulous on Y . Then $f \circ \pi$ is a rational function on Y satisfying an integral equation with regulous coefficients. By Theorem 5.1, $f \circ \pi$ can be extended to Y as a regulous function \tilde{f} on Y . Remark that the fibers of π are non-empty because X is central (see the proof of [26, Prop. 2.29]). Let $x \in X$. By the arguments used in the proof of Theorem 5.1 then \tilde{f} is constant on the connected components of $\pi^{-1}(x)$. Since the restriction of a regulous function is regulous then $\tilde{f}|_{\pi^{-1}(x)} \in \mathcal{R}^0(\pi^{-1}(x))$. It follows that \tilde{f} is constant on each \mathcal{C} -irreducible component of $\pi^{-1}(x)$, and these components are of finite numbers (see [11]). As a consequence \tilde{f} takes a finite numbers of values on $\pi^{-1}(x)$, and therefore f admits finitely many limit points at x . \square

Definition 5.3. Let X be an algebraic set. Let Y_1, \dots, Y_k be the \mathcal{C} -irreducible components of X . We say that X is regulously connected by irreducible components or simply that X is regulously connected if $k = 1$ or else if $\forall i \neq j$ in $\{1, \dots, k\}$ there exists a sequence (i_1, \dots, i_l) , $l \geq 2$, of two by two distinct numbers in $\{1, \dots, k\}$ such that $i_1 = i$, $i_l = j$, and for $t = 1, \dots, l - 1$ then $Y_{i_t} \cap Y_{i_{t+1}} \neq \emptyset$.

Remark 5.4. For example, an algebraic set X is regulously connected when X is regulously irreducible, or also when X is connected.

We now extend Theorem 5.1 to some singular cases.

Theorem 5.5. *Let X be a central irreducible algebraic set such that there exists a resolution of singularities $\pi : \tilde{X} \rightarrow X$ such that $\forall x \in X$ the fiber $\pi^{-1}(x)$ is regulously connected. Then $\mathcal{R}_0(X)$ is integrally closed in $\mathcal{K}(X)$.*

Proof. Assume $f \in \mathcal{K}(X)^*$ there exist $d \in \mathbb{N}^*$ and $a_i \in \mathcal{R}_0(X)$, $i = 0, \dots, d - 1$ such that

$$f^d + a_{d-1}f^{d-1} + \dots + a_0 = 0$$

in $\mathcal{K}(X)$. Let $\pi : \tilde{X} \rightarrow X$ be a resolution of singularities such that $\forall x \in X$ the fiber $\pi^{-1}(x)$ is regulously connected. As we have already explained in the proof of Lemma 5.2, the rational function $f \circ \pi$ can be extended regulously to \tilde{X} . Let $\tilde{f} \in \mathcal{R}^0(\tilde{X})$ denote the extension. Let $x \in X$. We know that \tilde{f} is constant on the connected components of $\pi^{-1}(x)$. Let Y_1, \dots, Y_k be the \mathcal{C} -irreducible components of $\pi^{-1}(x)$. Since for $i = 1, \dots, k$ $\tilde{f}|_{Y_i} \in \mathcal{R}^0(Y_i)$ (see [11, Cor. 5.38]) then \tilde{f} is constant on Y_i (see [11, Cor. 6.6]). Since $\pi^{-1}(x)$ is regulously connected then \tilde{f} is constant on $\pi^{-1}(x)$. We conclude the proof in the same way we did in the proof of Theorem 5.1. \square

Example 5.6. Let X be the cuspidal plane curve given by $y^2 - x^3 = 0$. By Theorem 5.5 we know that $\mathcal{R}^0(X) = \mathcal{R}_0(X)$ is integrally closed.

Example 5.7. Let X be the algebraic surface in \mathbb{R}^3 defined as by $y^2 = (x^2 - z^2)(x^2 - 2z^2)$. It can be view as the cone over the smooth curve defined in the plane $z = 1$ by the irreducible curve with two connected components $y^2 = (x^2 - 1)(x^2 - 2)$. The origin is the only singular point of $X_{\mathbb{C}}$ and thus X is normal. Moreover the blowing-up of the origin gives a resolution of the singularities of X , with exceptional divisor a smooth irreducible curve with two connected components. It is in particular regulously irreducible, therefore it follows again from Theorem 5.5 that $\mathcal{R}^0(X) = \mathcal{R}_0(X)$ is integrally closed.

Example 5.8. Let X be the nodal plane curve given by $y^2 - (x + 1)x^2 = 0$. The rational function $f = \frac{y}{x}$ is integral over $\mathcal{O}(X)$ (and also over $\mathcal{R}_0(X)$) since $f^2 - (x + 1) = 0$ on $X \setminus \{(0, 0)\}$. It is easy to see that f cannot be extended continuously to whole X . Hence $\mathcal{R}_0(X) = \mathcal{R}^0(X)$ is not integrally closed. Of course the fiber over the node is never connected when we solve the node.

5.2. An algebraic characterisation of rational continuous functions via the normalization map. Let X be a central irreducible algebraic set. Let X' be its normalization and let $\pi : X' \rightarrow X$ be the normalization map. By Lemma 4.3 then X' is central and the normalization map is surjective. By Lemma 4.2, it induces a non-necessarily integral injective ring morphism

$$\pi_0 : \mathcal{R}_0(X) \hookrightarrow \mathcal{R}_0(X'), \quad f \mapsto f \circ \pi.$$

Remark that, if in addition X is a curve, then the situation is simpler because then $\mathcal{R}_0(X) = \mathcal{R}^0(X)$ and $\mathcal{R}_0(X') = \mathcal{R}^0(X') = \mathcal{O}(X)$ since X' is smooth.

Let A be a ring. We denote by $\text{Rad}(A)$ (resp. $\text{Rad}^{\mathbb{R}}(A)$) the (resp. real) Jacobson radical of A , namely the intersection of all the maximal (resp. and real) ideals of A .

We give an algebraic characterization of the ring of rational continuous functions through the normalization map.

Proposition 5.9. *Let X be a central irreducible algebraic set. Let X' be its normalization. We have*

$$\mathcal{R}_0(X) = \{f \in \mathcal{R}_0(X') \mid f_x \in \mathcal{O}_{X,x} + \text{Rad}(\mathcal{R}_0(X')_x)\}$$

where $\mathcal{R}_0(X)_x = \mathcal{R}_0(X)_{\mathcal{I}_{\mathcal{P}(X)}(\{x\})}$ and

$$\mathcal{R}_0(X')_x = (\mathcal{R}_0(X) \setminus \mathcal{I}_{\mathcal{R}_0(X)}(\{x\}))^{-1} \mathcal{R}_0(X') = \mathcal{R}_0(X)_x \otimes_{\mathcal{R}_0(X)} \mathcal{R}_0(X')$$

and f_x is the image of f in $\mathcal{R}_0(X')_x$.

Proof. Let $\pi : X' \rightarrow X$ be the normalization map. Before proving the proposition we need to demonstrate the following claim.

Let $x \in X$, we claim that $\text{Max } \mathcal{R}_0(X')_x = \{\mathcal{I}_{\mathcal{R}_0(X')_x}(\{x'\}) \mid x' \in \pi^{-1}(x)\}$: Since $\mathcal{R}_0(X)_x = (\mathcal{R}_0(X) \setminus \mathcal{I}_{\mathcal{R}_0(X)}(\{x\}))^{-1} \mathcal{R}_0(X)$ is a local ring with maximal ideal $\mathcal{I}_{\mathcal{R}_0(X)_x}(\{x\})$ then $\text{Max } \mathcal{R}_0(X')_x$ corresponds to the set of maximal ideals \mathbf{m}' of $\mathcal{R}_0(X')$ such that $\mathbf{m}' \cap \mathcal{R}_0(X) = \mathcal{I}_{\mathcal{R}_0(X)}(\{x\})$ or equivalently such that $\mathbf{m}' \cap \mathcal{P}(X) = \mathcal{I}_{\mathcal{P}(X)}(\{x\})$. By Corollary 3.17 such \mathbf{m}' is of the form $\mathcal{I}_{\mathcal{R}_0(X')_x}(\{x'\})$ for an $x' \in X'$ and this proves the claim.

We denote by A the ring $\{f \in \mathcal{R}_0(X') \mid f_x \in \mathcal{O}_{X,x} + \text{Rad}(\mathcal{R}_0(X')_x)\}$. Let $f \in A$. Since $f \in \mathcal{R}_0(X')$ then f is rational on X . By the claim the function f is constant on the fibers of π . By Proposition 4.3, we get $f \in \mathcal{R}_0(X)$.

Let $f \in \mathcal{R}_0(X)$ and let $x \in X$. The function $f \circ \pi$ takes a constant value $\alpha \in \mathbb{R}$ on the fiber $\pi^{-1}(x)$. Let $g \in \mathcal{O}(X)$ such that $g(x) = \alpha$ (we can take $g = \alpha$). By the claim we get $f \circ \pi - g \circ \pi \in \text{Rad}(\mathcal{R}_0(X')_x)$ and the proof is done. \square

If the normalization map is not injective, then we prove that the ring of rational continuous functions is not integrally closed.

Proposition 5.10. *Let X be a central irreducible algebraic set. Let $\pi : X' \rightarrow X$ be the normalization map. If π is not injective, then $\mathcal{R}_0(X)$ is not integrally closed in $\mathcal{K}(X)$ and moreover the injective map $\pi_0 : \mathcal{R}_0(X) \hookrightarrow \mathcal{R}_0(X')$ is not surjective.*

Proof. We know that the fibers of π are finite since π is a finite map. Assume there exists $x \in X$ such that we have $\{y_1, y_2\} \subset \pi^{-1}(x)$ and $y_1 \neq y_2$. There exists $p \in \mathcal{P}(X')$ such that $p(y_1) \neq p(y_2)$. Obviously p is integral over $\mathcal{R}_0(X)$ and moreover $p \notin \pi_0(\mathcal{R}_0(X))$ since p is not constant on the fibers of π (Proposition 4.3). \square

Remark 5.11. Note that even for normal central surfaces, the ring $\mathcal{R}^0(X) = \mathcal{R}_0(X)$ is not necessarily integrally closed. Consider for example the surface X given by $z^2 = (x^2 + y^2)^2 + x^6$ in \mathbb{R}^3 . The origin is the only singular point of $X_{\mathbb{C}}$ and thus X is normal (Proposition 1.11). The rational function

$f = \frac{z}{x^2 + y^2}$ satisfies the integral equation $f^2 = 1 + \frac{x^6}{(x^2 + y^2)^2}$ with coefficients in $\mathcal{R}^0(X)$. As a consequence f^2 converges to 1 at the origin, but f has different signs depending on the sign of z . Therefore f can not be continuous at the origin.

5.3. The case of curves.

Throughout this section X will be a central irreducible algebraic curve. Let $\pi : X' \rightarrow X$ be the normalization map. The goal of this section is to determine the integral closure $\overline{\mathcal{R}^0(X)}$ of $\mathcal{R}^0(X)$ in $\mathcal{K}(X)$. The central hypothesis is crucial here since, without it, $\mathcal{R}^0(X)$ is not included in $\mathcal{K}(X)$ [26, Ex. 2.4]. Recall also that the central hypothesis implies the surjectivity of π . By [12, Prop. 2.4], we have $\mathcal{R}_0(X') = \mathcal{R}^0(X') = \mathcal{O}(X')$ and it is an integrally closed ring (Theorem 5.1 or Proposition 1.9). Hence the following sequence of inclusions

$$\mathcal{O}(X) \subset \mathcal{R}_0(X) \subset \mathcal{O}(X')$$

induces

$$(1) \quad \overline{\mathcal{O}(X)} \subset \overline{\mathcal{R}_0(X)} \subset \mathcal{O}(X')$$

i.e. $\overline{\mathcal{R}_0(X)}$ is an intermediate ring between $\overline{\mathcal{O}(X)}$ and $\mathcal{O}(X')$. By Proposition 5.9, we know that

$$\mathcal{R}_0(X) = \{f \in \mathcal{O}(X') \mid f_x \in \mathcal{O}_{X,x} + \text{Rad}(\mathcal{O}(X')_x)\}.$$

The goal of this section is to determine the ring $\overline{\mathcal{R}_0(X)}$ in the case X is a curve, it is a non-trivial problem essentially because $\mathcal{O}(X')$ is not in general integral over $\mathcal{O}(X)$.

For curves, we know exactly when the ring of rational continuous functions is integrally closed (use Theorem 5.5 and Proposition 4.4).

Proposition 5.12. *Let X be a central irreducible algebraic curve. Let $\pi : X' \rightarrow X$ be the normalization map. The ring $\mathcal{R}_0(X)$ is integrally closed in $\mathcal{K}(X)$ if and only if π is a bijection.*

In the case the normalization map is a bijection we can easily determine the ring $\overline{\mathcal{R}_0(X)}$.

Corollary 5.13. *Let X be a central irreducible algebraic curve. Let $\pi : X' \rightarrow X$ be the normalization map. Then π is a bijection if and only if $\mathcal{R}_0(X) = \overline{\mathcal{R}_0(X)} = \mathcal{O}(X')$.*

Proof. If $\mathcal{R}_0(X) = \overline{\mathcal{R}_0(X)} = \mathcal{O}(X')$ the π is a bijection by Proposition 5.12. Assume now π is a bijection. By Propositions 5.12 and 4.4 we get $\mathcal{R}_0(X) = \overline{\mathcal{R}_0(X)} = \mathcal{O}(X')$. \square

Example 5.14. Let X' be a geometrically smooth algebraic curve. We can create, as it is explained in [30], an affine singular real algebraic curve X with a unique singular point by putting together two complex conjugated points of $X'_\mathbb{C}$ and a point of X' . Since the normalization map $\pi : X' \rightarrow X$ is a bijection then $\mathcal{R}_0(X) = \overline{\mathcal{R}_0(X)} = \mathcal{O}(X')$ (Corollary 5.13). Since the fiber of $\pi_\mathbb{C} : X'_\mathbb{C} \rightarrow X_\mathbb{C}$ over the singular point of X is not totally real then $\overline{\mathcal{R}_0(X)} = \mathcal{O}(X') \neq \overline{\mathcal{O}(X)}$ (Proposition 1.16). In this example, the first inclusion in (1) is strict and the second one is an equality.

We can also easily determine the ring $\overline{\mathcal{R}_0(X)}$ in case the fibers of the complex normalization map are totally real over X .

Proposition 5.15. *Let X be a central irreducible algebraic curve. Let $\pi : X' \rightarrow X$ be the normalization map. The fibers of $\pi_\mathbb{C} : X'_\mathbb{C} \rightarrow X_\mathbb{C}$ over the singular point of the points of X are totally real if and only if we have*

$$\overline{\mathcal{O}(X)} = \overline{\mathcal{R}_0(X)} = \mathcal{O}(X').$$

Proof. The proof follows from Proposition 1.16 and (1). \square

We finally determine the ring $\overline{\mathcal{R}_0(X)}$ in the general case.

Theorem 5.16. *Let X be a central irreducible algebraic curve. Let $\pi : X' \rightarrow X$ be the normalization map. Then we have $\overline{\mathcal{R}^0(X)} = \mathcal{O}(X')$.*

Proof. By Proposition 1.4 and Lemma 1.12, we only have to consider the local case. Let $x \in X$. We consider the fiber $\pi_\mathbb{C}^{-1}(x) = \{y_1, \dots, y_r, z_1, \sigma(z_1), \dots, z_t, \sigma(z_t)\}$ where r, t are integers, σ is the complex involution, y_1, \dots, y_r correspond to points of X' (or points of $X'_\mathbb{C}$ fixed by σ) and $z_1, \sigma(z_1), \dots, z_t, \sigma(z_t)$ is a set of distinct two-by-two points of $X'_\mathbb{C}$. By Proposition 4.3, we have $r \geq 1$. By Proposition 4.3, the regulous functions on X correspond to the regular functions on X' which are constant on the fibers of π . It follows that (see also the proof of Proposition 5.9)

$$\mathcal{R}^0(X)_{\mathcal{I}_{\mathcal{R}^0(X)}(\{x\})} = \mathcal{O}_{X,x} + \mathcal{I}_{\mathcal{O}(X')}(\{y_1\}) \cap \dots \cap \mathcal{I}_{\mathcal{O}(X')}(\{y_r\}).$$

By Proposition 1.4, it suffices to prove

$$\overline{\mathcal{R}^0(X)_{\mathcal{I}_{\mathcal{R}^0(X)}(\{x\})}} = \mathcal{O}_{X',y_1} \cap \dots \cap \mathcal{O}_{X',y_r}.$$

Let $f \in \mathcal{O}_{X',y_1} \cap \dots \cap \mathcal{O}_{X',y_r}$. For $i = 1, \dots, r$, there exists $\alpha_i \in \mathbb{R}$ such that $f - \alpha_i \in \mathcal{I}_{\mathcal{O}(X')}(\{y_i\})$. Consequently $\prod_{i=1}^r (f - \alpha_i) \in \mathcal{R}^0(X)_{\mathcal{I}_{\mathcal{R}^0(X)}(\{x\})}$ and it proves $f \in \overline{\mathcal{R}^0(X)_{\mathcal{I}_{\mathcal{R}^0(X)}(\{x\})}}$. \square

6. WEAK NORMALIZATION AND SEMINORMALIZATION

In [24], Marinari and Raimondo have shown that the classical seminormalization of real algebraic varieties does not have natural universal properties contrary to the seminormalization of complex algebraic varieties (see also [28]). The goal of this section is to define the concepts of (real) weak normalization and (real) seminormalization of real algebraic varieties and real rings. We prove that these concepts are satisfactory since they have natural universal properties. Note that in the real analytic case, such an approach has been already investigated in [1]. We show in particular that the notions of real weak normalization and real seminormalization are not equivalent whereas for complex algebraic varieties weak normalization and seminormalization coincide.

6.1. Several normalizations of a real algebraic set. In this section we will introduce the notions of real weak normalisation and seminormalization of an algebraic set in parallel with the notion of normalization.

We need some preliminary results. Let A , A_1 and A_2 be rings. We say that A is an intermediate ring between A_1 and A_2 (in this order) if $A_1 \subset A \subset A_2$.

Proposition 6.1. 1) *Let X be an irreducible algebraic set. Let A be an intermediate ring between $\mathcal{P}(X)$ and $\mathcal{K}(X)$ such that A is integral over $\mathcal{P}(X)$. There exists a unique irreducible algebraic set Z such that $A = \mathcal{P}(Z)$. Moreover the induced map $Z \rightarrow X$ is birational and finite.*
 2) *Let $\pi : Y \rightarrow X$ be a birational finite map between irreducible algebraic sets. Let A be an intermediate ring between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$. There exists a unique irreducible algebraic set Z such that $A = \mathcal{P}(Z)$. Moreover the induced maps $Y \rightarrow Z$ and $Z \rightarrow X$ are birational and finite.*

When the conditions of 2) are satisfied, we say that Z is an intermediate algebraic set between X and Y .

Proof. Assume A is an intermediate ring between $\mathcal{P}(X)$ and $\mathcal{K}(X)$ such that A is integral over $\mathcal{P}(X)$. Since $\mathcal{P}(X) \subset A \subset \mathcal{K}(X)$ then A is a domain with $\mathcal{K}(X)$ as fraction field. Since A is integral over $\mathcal{P}(X)$ then A is a finite $\mathcal{P}(X)$ -module, thus A is a finitely generated \mathbb{R} -algebra and therefore it is the ring of polynomial functions of an irreducible algebraic set Z . The induced map $Z \rightarrow X$ is finite (A is integral over $\mathcal{P}(X)$) and birational ($\mathcal{K}(Z) \simeq \mathcal{K}(X)$). We have proved 1).

Let $\pi : Y \rightarrow X$ be a birational finite map between irreducible algebraic sets and let A be an intermediate ring between $\mathcal{P}(X)$ and $\mathcal{P}(Y)$. Since $\mathcal{P}(X) \hookrightarrow \mathcal{P}(Y)$ is an integral morphism (Lemma 4.1) then A is integral over $\mathcal{P}(X)$. By 1), A is the ring of polynomial functions of an irreducible algebraic set Z . The rest of the proof follows easily from 1) since $\mathcal{P}(X) \hookrightarrow \mathcal{P}(Z)$ and $\mathcal{P}(Z) \hookrightarrow \mathcal{P}(Y)$ are both integral morphisms. \square

Using Proposition 6.1, we can define several kinds of normalization of an algebraic set.

Definition 6.2. Let X be a central irreducible algebraic set. Recall that we have the following sequence of inclusions

$$\mathcal{P}(X) \hookrightarrow \mathcal{R}^0(X) \hookrightarrow \mathcal{R}_0(X) \hookrightarrow \mathcal{K}(X).$$

- 1) The (classical) normalization X' of X is the algebraic set whose ring of polynomial functions is the integral closure of $\mathcal{P}(X)$ in $\mathcal{K}(X)$.
- 2) The real weak normalization X_* of X is the algebraic set whose ring of polynomial functions is the integral closure of $\mathcal{P}(X)$ in $\mathcal{R}_0(X)$.
- 3) The real seminormalization X^* of X is the algebraic set whose ring of polynomial functions is the integral closure of $\mathcal{P}(X)$ in $\mathcal{R}^0(X)$.

The finite birational maps $\pi : X' \rightarrow X$, $\pi_* : X_* \rightarrow X$, $\pi^* : X^* \rightarrow X$ are respectively called the normalization map, the real weak normalization map and the real seminormalization map. The algebraic set X is called normal if $X = X'$, real weakly normal if $X = X_*$ and real seminormal if $X = X^*$.

Remark 6.3. 1) We do not need that X is central in order to define the normalization of X . On the contrary, it is a necessary hypothesis in order to define the real weak normalization and the real seminormalization. Indeed, if X is not central then we have already seen that $\mathcal{R}_0(X)$ and $\mathcal{R}^0(X)$ are not necessarily included in $\mathcal{K}(X)$.

2) The sets X_* and X^* are intermediate algebraic sets between X and X' . Moreover X^* is an intermediate algebraic set between X and X_* .

3) The polynomial functions on X' are the rational functions on X that are integral over $\mathcal{P}(X)$. The polynomial functions on X_* are the rational continuous functions on X that are integral over $\mathcal{P}(X)$. The polynomial functions on X^* are the regulous functions on X that are integral over $\mathcal{P}(X)$. Namely, we have

$$\mathcal{P}(X_*) = \mathcal{P}(X') \cap \mathcal{R}_0(X)$$

and

$$\mathcal{P}(X^*) = \mathcal{P}(X') \cap \mathcal{R}^0(X).$$

Remark 6.4. Let X be a central irreducible algebraic set. In the spirit of [24], one can mimic Definition 6.2 replacing $\mathcal{P}(X)$ by $\mathcal{O}(X)$. However in general, the integral closure of $\mathcal{O}(X)$ in $\mathcal{K}(X)$ or $\mathcal{R}_0(X)$ or $\mathcal{R}^0(X)$ is not the ring of regular functions of an intermediate algebraic set between X and X' (see Proposition 1.17). This justifies why we have decided to work with $\mathcal{P}(X)$ rather than $\mathcal{O}(X)$ and why we consider polynomial maps rather than regular maps.

Example 6.5. (1) Let $X \subset \mathbb{R}^2$ be the cuspidal cubic with equation $y^2 - x^3 = 0$. Then $\mathcal{P}(X') = \mathcal{P}(X)[y/x]$, where the rational function y/x satisfies $(y/x)^2 = y$ on X . Note that y/x can be extended continuously at the origin, so that X' coincides with the real weak normalisation of X . Since X is a curve then $X^* = X_*$ (see Remark 6.23).

(2) Consider the nodal curve $X \subset \mathbb{R}^2$ with equation $y^2 - x^2(x+1) = 0$. Then $\mathcal{P}(X') = \mathcal{P}(X)[y/x]$, where the rational function y/x satisfies $(y/x)^2 = x+1$ on X . Here the rational function y/x cannot be extended continuously at the origin (the limit is ± 1 depending on the local branch one considers), so that $X_* = X^*$ is equal to X .

(3) Consider the curve $X \subset \mathbb{R}^2$ with equation $y^2 - x^4(x+1) = 0$. The origin is the unique singular point of X , where two distinct branches intersect with tangency. Note that the rational function y/x satisfies $(y/x)^2 = x^2(x+1)$, so that it can be extended continuously at the origin with the value 0. A contrario, the rational function y/x^2 , which satisfies $(y/x^2)^2 = x+1$, cannot be extended continuously at the origin. In this case $\mathcal{P}(X') = \mathcal{P}(X)[y/x^2]$ whereas $\mathcal{P}(X_*) = \mathcal{P}(X)[x/y]$, so that $X_* = X^*$ is given by the nodal curve $y^2 - x^2(x+1) = 0$.

(4) Consider the surface $X = \mathcal{Z}(x^6 - (x^2 + y^2)^2 yz)$. The singular locus of X is the z -axis and X is central (look at the intersection of X with the planes $H_a = \mathcal{Z}(z - a)$, $a \in \mathbb{R}$). Then $\mathcal{P}(X') = \mathcal{P}(X)[\frac{x^3}{x^2 + y^2}] \simeq \mathbb{R}[t, y, z]/(t^2 - yz)$, setting $t = \frac{x^3}{x^2 + y^2}$. Since $\frac{x^3}{x^2 + y^2}$ is regulous on X then X' is the real weak normalization and the real seminormalization of X .

6.2. Normalization. In this section, we investigate the universal properties that characterize the normalization of an algebraic set.

Theorem 6.6. *Let X be an irreducible algebraic set and let $\pi : X' \rightarrow X$ be the normalization map. Let Y be an irreducible algebraic set and $\varphi : Y \rightarrow X$ be a finite birational map. Then there exists a unique map $\pi' : X' \rightarrow Y$ such that $\pi = \varphi \circ \pi'$ i.e. Y is an intermediate algebraic set between X and X' . If we consider all pairs (Y, φ) consisting of an irreducible algebraic set Y and a finite birational map $\varphi : Y \rightarrow X$ and we declare $(Y, \varphi) \leq (Z, \psi)$ if and only if there exists a map $\phi : Z \rightarrow Y$ such that $\psi = \varphi \circ \phi$, then (X', π) is the maximal pair.*

Proof. By Lemma 4.1, φ is induced by the integral inclusion $\mathcal{P}(X) \hookrightarrow \mathcal{P}(Y)$. Since φ is birational then we have moreover $\mathcal{P}(Y) \hookrightarrow \mathcal{K}(X)$. By definition of the normalization, we get $\mathcal{P}(X) \hookrightarrow \mathcal{P}(Y) \hookrightarrow \mathcal{P}(X')$ and thus $\pi : X' \rightarrow X$ uniquely factors through $\varphi : Y \rightarrow X$. The rest of the proof follows easily. \square

Remark 6.7. The previous theorem says that X' is the biggest algebraic set (in the sense that its ring of polynomial functions is the biggest intermediate ring between $\mathcal{P}(X)$ and $\mathcal{K}(X)$) among the irreducible algebraic sets Y with a finite birational map $Y \rightarrow X$.

6.3. Real weak normalisation. In this section we will introduce the notion of real weak normalisation of a ring. We prove afterwards that the two notions of real weak normalizations of algebraic sets and rings are the same through the equivalence “algebraic sets” with “rings of polynomial functions”. We begin by giving a characterization of the real weak normalisation by some universal properties.

6.3.1. Universal properties for the real weak normalization of a real algebraic set. By Theorem 4.4, Proposition 6.1 and since $\mathcal{P}(X_*) = \mathcal{P}(X') \cap \mathcal{R}_0(X)$, we may characterize the real weak normalisation of a central irreducible algebraic set X as the biggest intermediate algebraic set between X and its normalization that satisfies several different but equivalent universal properties.

Theorem 6.8. *Let X be a central irreducible algebraic set and let X' be its normalization. The real weak normalization X_* of X is the biggest algebraic set among the intermediate algebraic sets Y between X and X' (in the sense that its ring of polynomial functions is the biggest intermediate ring between $\mathcal{P}(X)$ and $\mathcal{P}(X')$) satisfying one of the following properties:*

- 1) *The polynomial functions on Y are rational continuous on X i.e. $\mathcal{P}(Y) \subset \mathcal{R}_0(X)$.*
- 2) *The map $Y \rightarrow X$ is a bijection.*
- 3) *X and Y have the same rational continuous functions, namely the ring morphism $\mathcal{R}_0(X) \rightarrow \mathcal{R}_0(Y)$ is an isomorphism.*
- 4) *The map $Y \rightarrow X$ is an homeomorphism for the Euclidean topology.*
- 5) *The map $Y \rightarrow X$ is an homeomorphism for the regulous topology.*
- 6) *$\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ satisfies the strong real lying over property i.e. $\psi_R : \text{R-Spec } \mathcal{P}(Y) \rightarrow \text{R-Spec } \mathcal{P}(X)$ is a bijection.*

Remark 6.9. Let Y be an intermediate algebraic set between an irreducible central algebraic set X and its real weak normalization X_* . Let π denote the map $Y \rightarrow X$. Then the map $\pi^{-1} : X \rightarrow Y$ is a rational continuous map in the sense that its components are rational continuous functions on X . Actually, assume $Y \subset \mathbb{R}^n$ and consider a coordinate function y_i on Y for $i \in \{1, \dots, n\}$. We want to prove that the rational function $f_i = y_i \circ \pi^{-1}$ is continuous on X . However $f_i \circ \pi_*$ is polynomial on Y , so that, by Theorem 6.8, f_i belongs to $\mathcal{R}_0(X)$ as required. In particular, $(\pi_*)^{-1} : X \rightarrow X_*$ is a rational continuous map.

We can rephrase Theorem 6.8 in more geometric terms. The reader is invited to compare with Theorem 6.6.

Theorem 6.10. *Let X be an irreducible algebraic set. Let Y be an irreducible algebraic set and $\varphi : Y \rightarrow X$ be a bijective finite birational map. Then there exists a unique map $\pi' : X_* \rightarrow Y$ such that $\pi_* = \varphi \circ \pi'$ i.e Y is an intermediate algebraic set between X and X_* . If we consider all pairs (Y, φ) consisting of an irreducible algebraic set Y and a bijective finite birational map $\varphi : Y \rightarrow X$ and we declare $(Y, \varphi) \leq (Z, \psi)$ if and only if there exists a map $\phi : Z \rightarrow Y$ such that $\psi = \varphi \circ \phi$, then (X_*, π_*) is the maximal pair.*

Proof. By Lemma 4.1, φ is induced by the integral inclusion $\mathcal{P}(X) \hookrightarrow \mathcal{P}(Y)$. By Theorem 6.8, we get $\mathcal{P}(X) \hookrightarrow \mathcal{P}(Y) \hookrightarrow \mathcal{P}(X_*)$ and thus $\pi_* : X_* \rightarrow X$ uniquely factors through $\varphi : Y \rightarrow X$. The rest of the proof follows easily. \square

An immediate application of Theorem 6.10 gives a real version of [23, Cor. 2.8].

Proposition 6.11. *Let X be a central irreducible algebraic set. Suppose that X is real weakly normal and that $\varphi : Y \rightarrow X$ is a finite birational polynomial map with Y an irreducible algebraic set. Then φ is a bijection if and only if φ is an isomorphism.*

Proof. If φ is a bijection, Theorem 6.10 gives us an inverse polynomial mapping for φ . \square

Remark 6.12. We can not remove the “finite” hypothesis in the previous proposition. For instance, let X be the nodal curve given by $y^2 = x^2(x+1)$ in \mathbb{R}^2 , and Y be the hyperbola given by $xy = 1$ in \mathbb{R}^2 . The curve X is real weakly normal (see Proposition 6.38). They are both in bijection with the punctured line $\mathbb{R} \setminus \{1\}$ via the regular maps

$$\mathbb{R} \setminus \{1\} \rightarrow X, t \mapsto (t^2 - 1, t(t^2 - 1))$$

and

$$\mathbb{R} \setminus \{1\} \rightarrow Y, t \mapsto (t - 1, 1/(t - 1)).$$

As a consequence, the polynomial map

$$Y \rightarrow X, (x, y) \mapsto (x(x+2), x(x+1)(x+2))$$

is birational and bijective. It is even a homeomorphism with respect to the Zariski topology. However they are not isomorphic curves since X is singular whereas Y is smooth. We do not contradict Proposition 6.11 since the map $Y \rightarrow X$ is not finite. Indeed, we can see Y as the normalization X' of X where we have deleted one of the real points over the real node of X . Hence $\mathcal{P}(Y)$ is bigger than $\mathcal{P}(X')$ and thus there exists an element of $\mathcal{P}(Y)$ that is not integral over $\mathcal{P}(X)$.

Weakly normal sets are stable under the product of varieties. It is a real version of [23, Cor. 2.13].

Corollary 6.13. *Let X and Y be real weakly normal algebraic sets. Then $X \times Y$ is real weakly normal.*

Proof. We use the same strategy as in [23]. Let f be a rational continuous function on $X \times Y$ which is integral over $\mathcal{P}(X \times Y)$. Then, for any $x \in X$, the restriction f_x of f to $\{x\} \times Y$ satisfied an integral equation over $\mathcal{P}(Y)$. Note however that, if f_x is not necessarily a rational function on Y , there exists a Zariski dense subset U in X such that f_x is rational for any $x \in U$. By real weak normality of Y , it follows that f_x belongs to $\mathcal{P}(Y)$ for any $x \in U$. Similarly, there exists a Zariski dense subset V in Y such that f_y belongs to $\mathcal{P}(X)$ for any $y \in V$.

We want to conclude that f is a polynomial function on $X \times Y$. We know by Palais [27] that f is polynomial on $U \times V$, so that there exists a polynomial function $p \in \mathcal{P}(X \times Y)$ such that $f = p$ on $U \times V$. Since f is continuous and X and Y are central, it implies that $f = p$ on $X \times Y$. As a consequence $X \times Y$ is real weakly normal. \square

6.3.2. *Real weak normalization of a ring.* We define the real weak normalization of a ring.

Definition 6.14. Let A be an integral domain with integral closure denoted by A' . We assume $\text{R-Spec } A \neq \emptyset$. The ring

$$A_* = \{f \in A' \mid \forall \mathfrak{m} \in \text{R-Spec } A \cap \text{Max } A, f_{\mathfrak{m}} \in A_{\mathfrak{m}} + \text{Rad}^{\text{R}}(A'_{\mathfrak{m}})\}$$

is called the real weak normalization of A . In case $A = A_*$, we say that A is real weakly normal.

We prove that in the geometric setting we recover the real weak normalization defined in 6.2.

Theorem 6.15. *Let X be a central irreducible algebraic set and let X' be its normalization. We have*

$$\mathcal{P}(X_*) = \mathcal{P}(X)_*$$

i.e.

$$\mathcal{P}(X_*) = \{f \in \mathcal{P}(X') \mid \forall x \in X, f_x \in \mathcal{O}_{X,x} + \text{Rad}^{\text{R}}(\mathcal{P}(X')_x)\}$$

where f_x is the class of f in $\mathcal{P}(X')_x = (\mathcal{P}(X) \setminus \mathcal{I}_{\mathcal{P}(X)}(\{x\}))^{-1} \mathcal{P}(X') = \mathcal{P}(X') \otimes_{\mathcal{P}(X)} \mathcal{P}(X)_{\mathcal{I}_{\mathcal{P}(X)}(\{x\})}$.

Proof. Let $\pi : X' \rightarrow X$ be the normalization map. By definition of $\mathcal{P}(X)_*$ it is clear that $\mathcal{P}(X)_* \subset \mathcal{R}_0(X)$. Indeed, the functions in $\mathcal{P}(X)_*$ are rational on X and are clearly constant on the fibers of π and we can use Proposition 4.3. It follows that

$$\mathcal{P}(X)_* \subset \mathcal{P}(X_*).$$

The converse inclusion is a consequence of Proposition 5.9 but we still give a short proof it. Let $f \in \mathcal{P}(X_*)$ and let $x \in X$. Since f is a rational continuous function on X then f takes a constant value α on the fiber $\pi^{-1}(x)$. Consequently $f_x - \alpha \in \text{Rad}^{\mathbb{R}}(\mathcal{P}(X')_x)$ since the maximal and real ideals of $\mathcal{P}(X')_x$ correspond to the points in $\pi^{-1}(x)$. \square

6.4. Real seminormalization. Closely related to the notion of “Weak normalization”, Traverso [33] has introduced the notions of “seminormalization” and “seminormal rings”. Precisely, for an integral domain A with integral closure denoted by A' , the ring

$${}^+A = \{f \in A' \mid \forall \mathfrak{p} \in \text{Spec } A, f_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}(A'_{\mathfrak{p}})\}$$

is called the seminormalization of A . The ring A is called seminormal if $A = {}^+A$. We indicate here the principal references concerning the seminormality: [33],[35], [14], [32], [17]. The goal of this section is to introduce the concept of real seminormalization of a ring. We prove that, in the geometric setting, real seminormalizations of algebraic sets and rings of polynomial functions give the same.

6.4.1. Universal properties for the real seminormalization of a real algebraic set. We adapt the work we have done for the real weak normalization of an algebraic set, replacing the ring of rational continuous functions by the ring of regulous functions.

By Theorems 4.7, Proposition 6.1 and since $\mathcal{P}(X^*) = \mathcal{P}(X') \cap \mathcal{R}^0(X)$, we may characterize the real seminormalization of a central irreducible algebraic set X as the biggest intermediate algebraic set between X and its normalization that satisfies the following universal properties.

Theorem 6.16. *Let X be a central irreducible algebraic set and let X' be its normalization. The real seminormalization X^* of X is the biggest algebraic set among the intermediate algebraic sets Y between X and X' satisfying one of the following properties:*

- 1) *The polynomial functions on Y are regulous on X i.e. $\mathcal{P}(Y) \subset \mathcal{R}^0(X)$.*
- 2) *$\mathcal{P}(X) \rightarrow \mathcal{P}(Y)$ satisfies the very strong real lying over property.*

Corollary 6.17. *Let X be a central irreducible algebraic set and $\pi^* : X^* \rightarrow X$ be the real seminormalization map. Then $(\pi^*)^{-1}$ is a regulous map.*

Proof. We proceed as in Remark 6.9. Assume $X^* \subset \mathbb{R}^n$ and consider a coordinate function y_i on X^* for $i \in \{1, \dots, n\}$. We want to prove that the rational function $f_i = y_i \circ (\pi^*)^{-1}$ is regulous on X . However $f_i \circ \pi^*$ is polynomial on X^* , so that, by Theorem 6.16, f_i belongs to $\mathcal{R}^0(X)$ as required. \square

Using Proposition 4.10, we can rephrase Theorem 6.16 in more geometric terms. The reader is invited to compare with Theorems 6.6 and 6.10.

Theorem 6.18. *Let X be an irreducible algebraic set. Let Y be an irreducible algebraic set and $\varphi : Y \rightarrow X$ be a bijective finite birational map that is hereditarily birational. Then there exists a unique map $\pi' : X^* \rightarrow Y$ such that $\pi^* = \varphi \circ \pi'$ i.e Y is an intermediate algebraic set between X and X^* . If we consider all pairs (Y, φ) consisting of an irreducible algebraic set Y and a bijective finite birational map $\varphi : Y \rightarrow X$ that is hereditarily birational and we declare $(Y, \varphi) \leq (Z, \psi)$ if and only if there exists a map $\phi : Z \rightarrow Y$ such that $\psi = \varphi \circ \phi$ then (X^*, π^*) is the maximal pair.*

Proof. By Lemma 4.1, φ is induced by the integral inclusion $\mathcal{P}(X) \hookrightarrow \mathcal{P}(Y)$. By Theorem 6.16 and Proposition 4.10, we get $\mathcal{P}(X) \hookrightarrow \mathcal{P}(Y) \hookrightarrow \mathcal{P}(X^*)$ and thus $\pi^* : X^* \rightarrow X$ uniquely factors through $\varphi : Y \rightarrow X$. The rest of the proof follows easily. \square

An immediate application of Theorem 6.18 gives a seminormal version of Proposition 6.11.

Proposition 6.19. *Let X be a central irreducible real algebraic set. Suppose that X is real seminormal and that $\varphi : Y \rightarrow X$ is a finite birational polynomial map with Y an irreducible algebraic set. Then φ is bijective and hereditarily birational if and only if φ is an isomorphism.*

Proof. If φ is a bijection that is hereditarily birational, Theorem 6.18 gives us an inverse polynomial mapping for φ . \square

Corollary 6.20. *Let X and Y be real seminormal algebraic sets. Then $X \times Y$ is real seminormal.*

Proof. We use the same proof as in Corollary 6.13. Note that the proof is even simpler since the restriction of a regular function is rational, so that we can choose $U = X$ and $Y = V$. \square

6.4.2. *Real weak normalization versus real seminormalization.* Over the complex number, the notions of weak normalization and seminormalization coincide [35]. In this section, we investigate the differences between the notions of real weak normalization and real seminormalization.

Remark 6.21. Let X be a central irreducible algebraic set. Let $\pi : X' \rightarrow X$ be the normalization map. We clearly get the following sequence of integral inclusions

$$(2) \quad \mathcal{P}(X) \subset \mathcal{P}(X^*) \subset \mathcal{P}(X_*) \subset \mathcal{P}(X')$$

that induces the following decomposition of the normalization map by finite birational mappings between central (Proposition 4.3) irreducible algebraic sets

$$(3) \quad X' \rightarrow X_* \xrightarrow{\pi_*^*} X^* \xrightarrow{\pi^*} X,$$

where π_*^* and π^* are homeomorphisms with respect to the Euclidean and regulous topologies.

From the definitions of real weak normalization and real seminormalization, we get:

Proposition 6.22. *Let X be a central irreducible algebraic set. If $\mathcal{R}_0(X) = \mathcal{R}^0(X)$ then $X^* = X_*$.*

Remark 6.23. It follows that for irreducible algebraic curves, the real weak normalization and real normalization coincide.

We exhibit now an example of a surface for which the real seminormalization and real weak normalization differ.

Example 6.24. Let $X = \mathcal{Z}(x^3 - y^3(1 + z^2))$ be the surface considered in [19, Ex. 2] and in Example 1.15. Then $\mathcal{P}(X') = \mathcal{P}(X)[x/y] \simeq \mathbb{R}[t, y, z]/(t^3 - (1 + z^2))$, setting $t = x/y$. Moreover the rational function $f = x/y$ can be extended to a continuous function on X , equals to $\sqrt[3]{1 + z^2}$. As a consequence X' is the real weak normalisation of X (the normalization map is a bijection). However the restriction of f to the z -axis is no longer rational, so $f \notin \mathcal{R}^0(X)$. In particular $X = X^* \neq X_* = X'$. By Theorems 6.8 and 6.16, we know that $\mathcal{P}(X) \hookrightarrow \mathcal{P}(X_*)$ satisfies the strong real lying over property but doesn't satisfy the very strong real lying over property. We give an explicit example of that. Let $\mathfrak{p} = (x, y) \in \mathbf{R}\text{-Spec } \mathcal{P}(X)$ and let $\mathfrak{q} \in \mathbf{R}\text{-Spec } \mathcal{P}(X')$ be the unique real prime ideal of $\mathcal{P}(X')$ such that $\mathfrak{q} \cap \mathcal{P}(X) = \mathfrak{p}$. We have $k(\mathfrak{p}) = \mathbb{R}(z)$ and $k(\mathfrak{q}) \not\cong k(\mathfrak{p})$ since the class of f in $k(\mathfrak{q})$ is not contained in $k(\mathfrak{p})$ because f is not rational by restriction to $\mathcal{Z}(\mathfrak{p})$. In fact we have $k(\mathfrak{q}) = \mathbb{R}(z)(\sqrt[3]{1 + z^2})$. Here, the normalization map $X' \rightarrow X$ is a bijective finite birational map which is not hereditarily birational.

Remark 6.25. In the previous example the real seminormalization and real weak normalization differ essentially because the rings of rational continuous and regulous functions are different. However it may happen that $\mathcal{R}_0(X) \neq \mathcal{R}^0(X)$ but still X^* coincides with X_* . Consider for instance the normal algebraic hypersurface of \mathbb{R}^4 defined by $X = \mathcal{Z}((x^3 - (1 + t^2)y^3)^2 + z^6 + y^7)$ (cf. [19, Ex. 3]; it is a perturbation of Example 6.24 so that X becomes normal). Then again there exist rational continuous functions which are not regulous, nevertheless $X = X^* = X_* = X'$ since X is normal.

Remark 6.26. Let X be an irreducible and central algebraic curve. Let $\pi : X' \rightarrow X$ be the normalization map. Real seminormalization of curves is the subject of the latest section of the paper, however in the following cases it is easy to determine the seminormalization of X by looking at the possible intermediate curves between X and X' .

- If $X = \mathcal{Z}(y^2 - x^3)$ is the cuspidal curve then $X_* = X'$ since π is injective.
- If $X = \mathcal{Z}(y^2 - x^2(x + 1))$ is the nodal curve then $X_* = X$.
- If the singularities of $X_{\mathbb{C}}$ are only ordinary nodes and cusps which belong to X , then X_* is the curve we obtain when we blow-up all the cusps of X and we leave intact the nodal singularities.

We end this section with some trivial, but important, consequences of Theorems 6.8, and 6.16 concerning the idempotency and the composition of the notions of real weak normalization and real seminormalization.

Theorem 6.27. *Let X be a central irreducible algebraic set. Then we have:*

- 1) $(X_*)_* = X_*$.
- 2) $(X^*)^* = X^*$.
- 3) $(X_*)^* = X_*$.
- 4) $(X^*)_* = X_*$.

6.4.3. *Real seminormalization of rings.* Inspired by Traverso's definition of a seminormal ring given previously, we propose the following definitions of real seminormalization of a ring and real seminormal ring.

Definition 6.28. Let A be an integral domain with integral closure denoted by A' . We assume $\text{R-Spec } A \neq \emptyset$. The ring

$$A^* = \{f \in A' \mid \forall \mathfrak{p} \in \text{R-Spec } A, f_{\mathfrak{p}} \in A_{\mathfrak{p}} + \text{Rad}^{\text{R}}(A'_{\mathfrak{p}})\}$$

is called the real seminormalization of A . In case $A = A^*$, we say that A is real seminormal.

Remark 6.29. We have a sequence of inclusions

$$A \subset A^* \subset A_* \subset A'$$

It follows that a real weakly normal ring is automatically real seminormal. The polynomial ring of the surface $X = \mathcal{Z}(x^3 - y^3(1 + z^2))$ of Example 6.24 is a real seminormal ring that is not real weakly normal.

We prove now that in the geometric setting the real seminormalization of a central irreducible algebraic set is the algebraic set corresponding to the real seminormalization of its ring of polynomial functions.

Theorem 6.30. *Let X be a central irreducible algebraic set and let X' be its normalization. We have*

$$\mathcal{P}(X^*) = \mathcal{P}(X)^*$$

i.e.

$$\mathcal{P}(X^*) = \{f \in \mathcal{P}(X') \mid \forall \mathfrak{p} \in \text{R-Spec } \mathcal{P}(X), f_{\mathfrak{p}} \in \mathcal{P}(X)_{\mathfrak{p}} + \text{Rad}^{\text{R}}(\mathcal{P}(X')_{\mathfrak{p}})\}$$

where $f_{\mathfrak{p}}$ is the class of f in $\mathcal{P}(X')_{\mathfrak{p}} = (\mathcal{P}(X) \setminus \mathfrak{p})^{-1} \mathcal{P}(X') = \mathcal{P}(X') \otimes_{\mathcal{P}(X)} \mathcal{P}(X)_{\mathfrak{p}}$.

Proof. By Theorem 6.16, we have to prove that $\mathcal{P}(X)^*$ is the biggest intermediate ring B between $\mathcal{P}(X)$ and $\mathcal{P}(X')$ such that $\mathcal{P}(X) \rightarrow B$ satisfies the very strong real lying over property.

We begin by proving $\mathcal{P}(X)^*$ satisfies the very strong real lying over property.

Let $\mathfrak{p} \in \text{R-Spec } \mathcal{P}(X)$. Since $\mathcal{P}(X)^* \subset \mathcal{P}(X)_*$, it follows from Theorems 6.15 and 6.8 that $\mathcal{P}(X) \rightarrow \mathcal{P}(X)^*$ satisfies the strong real lying over property. Let \mathfrak{p}' be the unique real prime ideal of $\mathcal{P}(X)^*$ lying over \mathfrak{p} . We have to show that $k(\mathfrak{p}) \simeq k(\mathfrak{p}')$. Since the contraction of a real ideal is a real ideal, since the following sequence of injective maps between semi-local rings $\mathcal{P}(X)_{\mathfrak{p}} \hookrightarrow \mathcal{P}(X)_{\mathfrak{p}}^* \hookrightarrow \mathcal{P}(X')_{\mathfrak{p}}$ is

integral, it follows from Proposition 2.3 that $\mathbf{p}' \mathcal{P}(X)_{\mathbf{p}}^* = \text{Rad}^R((\mathcal{P}(X)_{\mathbf{p}})^*) = \text{Rad}^R(\mathcal{P}(X')_{\mathbf{p}}) \cap \mathcal{P}(X)_{\mathbf{p}}^*$. Let $b \in \mathcal{P}(X)^*$. By definition of $\mathcal{P}(X)^*$, there exist $\alpha \in \mathcal{P}(X)_{\mathbf{p}}$ and $\beta \in \text{Rad}^R(\mathcal{P}(X')_{\mathbf{p}})$ such that $b_{\mathbf{p}} = \alpha + \beta$. By Proposition 2.16, there exists $\mathbf{q}' \in \text{R-Spec } \mathcal{P}(X')$ such that $\mathbf{q}' \cap \mathcal{P}(X)^* = \mathbf{p}'$. From the following sequence of injective integral maps between integral domains

$$\frac{\mathcal{P}(X)}{\mathbf{p}} \hookrightarrow \frac{\mathcal{P}(X)^*}{\mathbf{p}'} \hookrightarrow \frac{\mathcal{P}(X')}{\mathbf{q}'}$$

we get the following sequence of algebraic extensions of residue fields $k(\mathbf{p}) \hookrightarrow k(\mathbf{p}') \hookrightarrow k(\mathbf{q}')$. It follows that the class $b(\mathbf{p}')$ of b in $k(\mathbf{p}')$ is also the class $b_{\mathbf{p}}(\mathbf{q}')$ of $b_{\mathbf{p}}$ in $k(\mathbf{q}')$. Therefore

$$b(\mathbf{p}') = b_{\mathbf{p}}(\mathbf{q}') = (\alpha + \beta)(\mathbf{q}') = \alpha(\mathbf{p}) + \beta(\mathbf{q}').$$

Since $\text{Rad}^R(\mathcal{P}(X')_{\mathbf{p}}) \subset \mathbf{q}' \mathcal{P}(X')_{\mathbf{p}}$ then $\beta(\mathbf{q}') = 0$ and thus $b(\mathbf{p}') = \alpha(\mathbf{p}) \in k(\mathbf{p})$.

It remains to prove that if B is an intermediate ring between $\mathcal{P}(X)$ and $\mathcal{P}(X')$ such that $\mathcal{P}(X) \rightarrow B$ satisfies the very strong real lying-over property then $B \subset \mathcal{P}(X)^*$. Take such a ring B . Assume there exists $b \in B$ such that $b \notin \mathcal{P}(X)^*$. It means there exists $\mathbf{p} \in \text{R-Spec } \mathcal{P}(X)$ such that $b_{\mathbf{p}} \notin \mathcal{P}(X)_{\mathbf{p}} + \text{Rad}^R(\mathcal{P}(X')_{\mathbf{p}})$. Let \mathbf{q} be the unique real prime ideal of B lying over \mathbf{p} . We have the following sequence of injective maps between semi-local rings

$$\mathcal{P}(X)_{\mathbf{p}} \hookrightarrow B_{\mathbf{p}} \hookrightarrow \mathcal{P}(X')_{\mathbf{p}}.$$

We clearly have $\text{Rad}^R(B_{\mathbf{p}}) = \mathbf{q}B_{\mathbf{p}}$. By Proposition 2.16 we have $\text{Rad}^R(\mathcal{P}(X')_{\mathbf{p}}) \neq \emptyset$. Since $B_{\mathbf{p}} \hookrightarrow \mathcal{P}(X')_{\mathbf{p}}$ is integral, it follows from Proposition 2.3 that $\text{Rad}^R(\mathcal{P}(X')_{\mathbf{p}}) \cap B_{\mathbf{p}} = \mathbf{q}B_{\mathbf{p}}$. Hence $b_{\mathbf{p}} \notin \mathcal{P}(X)_{\mathbf{p}} + \mathbf{q}B_{\mathbf{p}}$. Since $\mathbf{q}B_{\mathbf{p}}$ is the kernel of $B_{\mathbf{p}} \rightarrow k(\mathbf{q})$ then $b(\mathbf{q}) = b_{\mathbf{p}}(\mathbf{q}) \in k(\mathbf{q})$ is not in $k(\mathbf{p}) = \frac{(\mathcal{P}(X)_{\mathbf{p}} + \mathbf{q}B_{\mathbf{p}}) \cap \mathcal{P}(X)_{\mathbf{p}}}{\mathbf{q}B_{\mathbf{p}} \cap \mathcal{P}(X)_{\mathbf{p}}}$, a contradiction. \square

We give now some consequences of Theorems 6.27 and 6.30.

Proposition 6.31. *Let X be a central irreducible algebraic set. Let $\pi : X' \rightarrow X$ be the normalization map.*

(1) *The ring $\mathcal{P}(X^*)$ is real seminormal, namely*

$$\mathcal{P}(X^*) = \{f \in \mathcal{P}(X') \mid \forall \mathbf{p} \in \text{R-Spec } \mathcal{P}(X^*), f_{\mathbf{p}} \in \mathcal{P}(X^*)_{\mathbf{p}} + \text{Rad}^R(\mathcal{P}(X')_{\mathbf{p}})\}.$$

(2) *Let X be a central irreducible algebraic set. The ring $\mathcal{P}(X_*)$ is real weakly normal, namely*

$$\mathcal{P}(X_*) = \{f \in \mathcal{P}(X') \mid \forall x \in X_*, f_{\mathcal{I}_{\mathcal{P}(X_*)}(\{x\})} \in \mathcal{P}(X_*)_{\mathcal{I}_{\mathcal{P}(X_*)}(\{x\})} + \text{Rad}^R(\mathcal{P}(X')_{\mathcal{I}_{\mathcal{P}(X_*)}(\{x\})})\}.$$

(3) *Let X be a central irreducible algebraic set. The ring $\mathcal{P}(X_*)$ is real seminormal, namely*

$$\mathcal{P}(X_*) = \{f \in \mathcal{P}(X') \mid \forall \mathbf{p} \in \text{R-Spec } \mathcal{P}(X_*), f_{\mathbf{p}} \in \mathcal{P}(X_*)_{\mathbf{p}} + \text{Rad}^R(\mathcal{P}(X')_{\mathbf{p}})\}.$$

Proof. Concerning the first point, it is a consequence of the equality $(X^*)^* = X^*$, which is part of the conclusion of Theorem 6.27. The second point comes from the equality $(X_*)_* = X_*$ in Theorem 6.27 also, whereas the third comes from the equality $(X_*)^* = X_*$ *ibid.* \square

6.5. A characterization of real seminormal rings. We focus our interest in this section on general properties related to real seminormalization and real seminormal rings. We begin with a remark on the relationships between seminormalization and real seminormalization.

Remark 6.32. Let A be an integral domain with integral closure denoted by A' . We assume $\text{R-Spec } A \neq \emptyset$. If we assume moreover that $A \hookrightarrow A'$ satisfies the real lying over property then we have

$${}^+A \subset A^*.$$

Indeed, let $\mathbf{p} \in \text{R-Spec } A$, we have $\text{Rad}(A'_{\mathbf{p}}) \subset \text{Rad}^R(A'_{\mathbf{p}})$ if and only if $\text{Rad}^R(A'_{\mathbf{p}}) \neq \emptyset$. Under the above hypotheses, a real seminormal ring is seminormal.

Notice that, proceeding analogously to the proof of Theorem 4.7, we can state an abstract form of it. It gives a real version of Traverso's theorem [33, 1.1].

Theorem 6.33. *Let A be an integral domain with integral closure denoted by A' . We assume $\text{R-Spec } A \neq \emptyset$ and moreover that $A \hookrightarrow A'$ satisfies the real lying over property. Then, the real seminormalization A^* of A is the biggest ring among the rings B between A and A' such that $A \hookrightarrow B$ satisfies the very strong real lying-up property.*

We are interested in giving a simple characterization of real seminormal rings. Let us recall the characterization of seminormal rings by Traverso.

Theorem 6.34. [14, Thm. 1.1]

Let A be an integral domain with integral closure denoted by A' . The following conditions are equivalent:

- 1) A is seminormal.
- 2) For each $f \in A'$, the conductor of A in $A[f]$ is a radical ideal of $A[f]$.
- 3) A contains each element f of A' such that $f^n \in A$ for all sufficiently large n .

We state a real version of this theorem, assuming now that $A \hookrightarrow A'$ satisfies moreover the real lying over property. Notice that conditions 2) and 3) must be strengthened since a real seminormal ring is necessarily seminormal.

Theorem 6.35. *Let A be an integral Noetherian domain with integral closure denoted by A' . We assume $\text{R-Spec } A \neq \emptyset$ and moreover that $A \hookrightarrow A'$ satisfies the real lying over property. The following conditions are equivalent:*

- 1) A is real seminormal.
- 2) For each $f \in A'$, the conductor of A in $A[f]$ is a real ideal of $A[f]$.
- 3) A contains each element f of A' such that there exist $m \in \mathbb{N} \setminus \{0\}$ and $f_1, \dots, f_l \in A[f]$ such that $f^{2m} + f_1^2 + \dots + f_l^2 \in A$.

Remark 6.36. The condition 2) of Theorem 6.35 is stronger than the condition 2) of Theorem 6.34 since a real ideal is radical [7, Lem. 4.1.5].

Proof. The equivalence between 2) and 3) is given by [7, Prop. 4.1.7].

We prove that 1) implies 2) by adapting the proof of [33, Lem. 1.3] to the real case. Let $f \in A'$. Let \mathcal{C} be the conductor of A in $A[f]$. Since \mathcal{C} is the greatest ideal of $A[f]$ contained in A , we have to prove that the real radical of \mathcal{C} in $A[f]$ is contained in A . Let $g \in A[f]$ such that there exist $m \in \mathbb{N} \setminus \{0\}$ and $f_1, \dots, f_l \in A[f]$ such that $g^{2m} + f_1^2 + \dots + f_l^2 \in \mathcal{C}$. Let $\mathfrak{p} \in \text{R-Spec } A$. If $\mathcal{C} \subset \mathfrak{p}$ then $(g^{2m} + f_1^2 + \dots + f_l^2)_{\mathfrak{p}} \in \mathcal{C}_{\mathfrak{p}} \subset \mathfrak{p}A_{\mathfrak{p}}$. The extension $A_{\mathfrak{p}} \hookrightarrow A'_{\mathfrak{p}}$ is integral and clearly satisfies the real lying-over property since $A \hookrightarrow A'$ does. Consequently, we have $\text{Rad}^{\text{R}}(A'_{\mathfrak{p}}) \neq \emptyset$ and thus it follows from Proposition 2.3 that $\mathfrak{p}A_{\mathfrak{p}} \subset \text{Rad}^{\text{R}}(A'_{\mathfrak{p}})$. We have proved that if $\mathcal{C} \subset \mathfrak{p}$ then $(g^{2m} + f_1^2 + \dots + f_l^2)_{\mathfrak{p}} \in \text{Rad}^{\text{R}}(A'_{\mathfrak{p}})$ and thus $g_{\mathfrak{p}} \in \text{Rad}^{\text{R}}(A'_{\mathfrak{p}})$ since $\text{Rad}^{\text{R}}(A'_{\mathfrak{p}})$ is a real ideal. If $\mathcal{C} \not\subset \mathfrak{p}$ then clearly $A_{\mathfrak{p}} = A[f]_{\mathfrak{p}}$ and thus $g_{\mathfrak{p}} \in A_{\mathfrak{p}}$. We have proved that $g \in A^*$ and thus $g \in A$ since A is real seminormal.

To prove that 2) implies 1), we adapt the proof of [14, Thm. 1.1] to our case. We assume that A satisfies the condition 2) and A is not real seminormal. Let $f \in A^* \setminus A$. Let \mathcal{C} be the conductor of A in $A[f]$. Since A satisfies 2) then we know that \mathcal{C} is a real ideal. Since $\mathcal{C} \subset A$ then \mathcal{C} is also a real ideal of A . Let \mathfrak{p} be a minimal prime ideal of A that contains \mathcal{C} . Since A is Noetherian then \mathfrak{p} is a real ideal. We have $f_{\mathfrak{p}} \in (A^*)_{\mathfrak{p}} \setminus A_{\mathfrak{p}}$ and $\text{Rad}(\mathcal{C}A_{\mathfrak{p}}) = \text{Rad}^{\text{R}}(\mathcal{C}A_{\mathfrak{p}}) = \mathfrak{p}A_{\mathfrak{p}}$. It follows that $\mathfrak{p}A_{\mathfrak{p}}$ is the conductor of $A_{\mathfrak{p}}$ in $A_{\mathfrak{p}}[f_{\mathfrak{p}}]$ and by hypothesis it is thus a real ideal of $A_{\mathfrak{p}}[f_{\mathfrak{p}}]$. We clearly get that $A_{\mathfrak{p}} \hookrightarrow (A^*)_{\mathfrak{p}}$ satisfies the real lying over property since $A \hookrightarrow A^*$ (or $A \hookrightarrow A'$) does. Since $f \in A^*$ then $A[f] \hookrightarrow A^*$. It follows from Theorem 6.33 that $A \hookrightarrow A[f]$ satisfies the very strong real lying over property. It is thus clear that $A_{\mathfrak{p}} \hookrightarrow A_{\mathfrak{p}}[f_{\mathfrak{p}}]$ satisfies the very strong real lying over property. Therefore there exists a

unique prime and real ideal of $A_{\mathbf{p}}[f_{\mathbf{p}}]$ lying over $\mathbf{p}A_{\mathbf{p}}$. This ideal is $\mathbf{p}A_{\mathbf{p}}$ since we have already noticed that $\mathbf{p}A_{\mathbf{p}}$ is a real ideal of $A_{\mathbf{p}}[f_{\mathbf{p}}]$. By the very strong real lying over property, the canonical injection $\frac{A_{\mathbf{p}}}{\mathbf{p}A_{\mathbf{p}}} \hookrightarrow \frac{A_{\mathbf{p}}[f_{\mathbf{p}}]}{\mathbf{p}A_{\mathbf{p}}}$ is an isomorphism. It follows that $A_{\mathbf{p}}[f_{\mathbf{p}}] = A_{\mathbf{p}}$ and we get a contradiction. \square

6.6. Real seminormal algebraic curves. Let X be a central irreducible algebraic curve. The goal of this section is to determine what kind of singularities can occur when X is real seminormal. Notice that since $\mathcal{R}^0(X) = \mathcal{R}_0(X)$ then X is real seminormal if and only if X is real weakly normal.

Bombieri [8] determined the singularities of seminormal complex algebraic curves. It was a generalization of a result of Salmon [29] concerning plane curves. In [10], there is a geometric characterisation of seminormal local rings of dimension one. Let y be a point of a complex algebraic curve Y . We say that y is an ordinary k -fold point if y is a point of multiplicity k with k linearly independent tangents (if $Y \subset \mathbb{C}^n$ then $k \leq n$). It means that the singularity at y is analytically isomorphic to the union of the k coordinate axes in \mathbb{C}^k (see [17]).

Theorem 6.37. [8]

Let Y be an irreducible algebraic curve. Then Y is seminormal (i.e. $\mathcal{P}(Y)$ is a seminormal ring) if and only if the singularities of Y are ordinary k -fold points.

The following proposition is largely a consequence of the fact that a real seminormal ring is seminormal, cf Remark 6.31.

Proposition 6.38. *Let X be a central irreducible algebraic curve. Let $\pi : X' \rightarrow X$ be the normalization map. The curve X is real seminormal if and only if the following properties are satisfied:*

- (1) *the singularities of $X_{\mathbb{C}}$ are ordinary k -fold points.*
- (2) *$\text{Sing}(X_{\mathbb{C}}) \subset X$.*
- (3) *$\forall x \in \text{Sing}(X)$ then $\pi_{\mathbb{C}}^{-1}(x)$ is totally real i.e. $\#(\pi_{\mathbb{C}}^{-1}(x)) = \#(\pi^{-1}(x))$.*

Proof. Assume X is real seminormal. Thus $\mathcal{P}(X)$ is real seminormal (Theorem 6.30) and it follows that $\mathcal{P}(X)$ is a seminormal ring. From the definitions or by [15, Cor. 5.7], it follows that $\mathcal{P}(X)$ is seminormal if and only if $\mathcal{P}(X_{\mathbb{C}})$ is seminormal. By Theorem 6.37, we get (1). Assume $X_{\mathbb{C}}$ admit as singularities two complex conjugated points (which are then non-real). We resolve the singularity at these two points (by performing a sequence of blowings-up along points) and we get a central irreducible algebraic curve Y (Proposition 4.3) such that the map $Y \rightarrow X$ is birational, finite, bijective. By Proposition 1.5 and since $Y_{\mathbb{C}}$ and $X_{\mathbb{C}}$ are not isomorphic (the map $Y_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ is not bijective) then the map $Y \rightarrow X$ is not an isomorphism. Since X is real weakly normal then we contradict Proposition 6.11. We have proved (2). Assume now there exists $x \in \text{Sing}(X)$ such that $\pi_{\mathbb{C}}^{-1}(x)$ is not totally real. By Proposition 4.3, we have $\#(\pi^{-1}(x)) \geq 1$. We resolve the singularity at the point x and then we glue together the real points over x . We get a real central irreducible real algebraic curve Y (Proposition 4.3) such that the map $Y \rightarrow X$ is birational, finite, bijective but not an isomorphism since $Y_{\mathbb{C}} \rightarrow X_{\mathbb{C}}$ is not bijective. By Proposition 6.11 we get a contradiction and it gives (3).

Assume now the curve X satisfies the three properties of the proposition. From (2), (3) and the definitions, it follows that X is real seminormal if and only if X is seminormal. By (1) and Theorem 6.37, it follows that $X_{\mathbb{C}}$ is a weakly normal algebraic curve i.e. $\mathcal{P}(X_{\mathbb{C}})$ is seminormal. By [15, Cor. 5.7], $\mathcal{P}(X)$ is seminormal if and only if $\mathcal{P}(X_{\mathbb{C}})$ is seminormal and the proof is done. \square

Remark 6.39. By Proposition 1.16, condition (3) of Proposition 6.38 is equivalent to the equality $\overline{\mathcal{O}(X)} = \mathcal{O}(X')$.

From Propositions 6.37 and 6.38, we get:

Example 6.40. (1) The cuspidal curve $\mathcal{Z}(y^2 - x^3)$ is not seminormal, since the origin is a singular point of multiplicity 2 which is not an ordinary 2-fold point.
 (2) The nodal curve $\mathcal{Z}(y^2 - x^2(x + 1))$ is real seminormal.

- (3) The curve $\mathcal{Z}(y^3 - x^3(x + 1))$ is not seminormal since the origin, which is a singular point of multiplicity 3, is not an ordinary 3-fold point.
- (4) The curve $\mathcal{Z}(y^2 - (x^2 + 1)^2x)$ is seminormal but not real seminormal, since the singular points are not all real.

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