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On the Ball–Marsden–Slemrod obstruction in bilinear control systems

Nabile Boussaïd

Laboratoire de Mathématiques de Besançon, UMR 6623
Université de Franche Comté, Besançon, France
Nabile.Boussaïd@univ-fcomte.fr

Marco Caponigro

Équipe M2N
Conservatoire National des Arts et Métiers, Paris, France
Marco.Caponigro@cnam.fr

Thomas Chambrion

Université de Lorraine, IECL, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France
CNRS, IECL, UMR 7502, Vandoeuvre-lès-Nancy, F-54506, France
Inria, Villers-lès-Nancy, F-54600, France
Thomas.Chambrion@univ-lorraine.fr

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Abstract

This note presents an extension to the case of L^1 controls of a well-known obstruction to the controllability of bilinear control systems in infinite dimensional spaces.

1 Introduction

1.1 Bilinear control systems

Let X be a Banach space, $A : D(A) \rightarrow X$ a linear operator in X with domain $D(A)$, $B : X \rightarrow X$ a linear bounded operator and ψ_0 an element in X . We consider the following bilinear control system

$$\begin{cases} \dot{\psi}(t) &= A\psi(t) + u(t)B\psi(t), \\ \psi(0) &= \psi_0, \end{cases} \quad (1)$$

where $u : [0, +\infty) \rightarrow \mathbf{R}$ is a scalar function representing the control.

Assumption 1. *The pair (A, B) of linear operators in X satisfies*

- 1) *the operator A generates a C^0 -semigroup of linear bounded operators on X .*
- 2) *the operator B is bounded.*

Definition 1. Assume that (A, B) satisfies Assumption 1. Let $T > 0$. A function $\psi : [0, T] \rightarrow X$ is a mild solution of (1) if for every t in $[0, T]$,

$$\psi(t) = e^{tA}\psi_0 + \int_0^t e^{(t-s)A}B\psi(s)u(s)ds \quad (2)$$

Equation (2) is often called Duhamel formula.

Existence and uniqueness for equation (1) is given by the following result (see, for instance, Proposition 2.1 and Remark 2.7 in [BMS82]).

Proposition 1. Assume that (A, B) satisfies Assumption 1. Then, for every ψ_0 in X , for every u in $L^1_{loc}([0, +\infty), \mathbf{R})$, there exists a unique mild solution $t \mapsto \Upsilon_{t,0}^u\psi_0$ to the Cauchy problem (1). Moreover, for every ψ_0 in X , the end-point mapping $\Upsilon_{\cdot,0}\psi_0 : [0, +\infty) \times L^1_{loc}([0, +\infty), \mathbf{R}) \rightarrow X$ is continuous.

Definition 2. Assume that (A, B) satisfies Assumption 1 and let \mathcal{U} be a subset of $L^1_{loc}([0, +\infty), \mathbf{R})$. For every ψ_0 in X , the attainable set from ψ_0 with controls in \mathcal{U} is defined as

$$\mathcal{A}(\psi_0, \mathcal{U}) = \bigcup_{T \geq 0} \bigcup_{u \in \mathcal{U}} \{\Upsilon_{T,0}^u\psi_0\}.$$

Our main result is the following property of the attainable set of system (1) with L^1 controls.

Theorem 2. Assume that (A, B) satisfies Assumption 1. Then, for every ψ_0 in X , the attainable set $\mathcal{A}(\psi_0, L^1_{loc}([0, +\infty), \mathbf{R}))$ from ψ_0 with L^1_{loc} controls is contained in a countable union of compact subsets of X .

1.2 The Ball–Marsden–Slemrod obstruction

Our main result Theorem 2 is an extension of the well-known Ball–Marsden–Slemrod obstruction to controllability.

Theorem 3 (Theorem 3.6 in [BMS82]). Assume that (A, B) satisfies Assumption 1. Then, for every ψ_0 in X , the attainable set $\mathcal{A}(\psi_0, \cup_{r>1} L^r_{loc}([0, +\infty), \mathbf{R}))$ from ψ_0 with L^r_{loc} controls, $r > 1$, is contained in a countable union of compact subsets of X .

A consequence of Theorem 3 to the framework of the conservative bilinear Schrödinger equation is given by Turinici.

Theorem 4 (Theorem 1 in [Tur00]). Assume that (A, B) satisfies Assumption 1. Then, for every ψ_0 in X , the set $\cup_{\alpha>0} \alpha \mathcal{A}(\psi_0, \cup_{r>1} L^r_{loc}([0, +\infty), \mathbf{R}))$ is contained in a countable union of compact subsets of X .

Theorems 2 and 3 are basically empty in the case in which X is finite dimensional, indeed, in this case, X itself is a countable union of compact sets. On the other hand, when X is infinite dimensional, these results represent a strong topological obstruction to the exact controllability. Indeed, compact subsets of an infinite dimensional Banach space have empty interiors and so is a countable union of closed subsets with empty interiors (as a consequence of Baire Theorem).

Let us recall briefly the method used in [BMS82] for the proof of Theorem 3. The first step is to write

$$\begin{aligned} \mathcal{A}(\psi_0, \cup_{r>1} L^r_{loc}([0, +\infty), \mathbf{R})) &= \bigcup_{T \geq 0} \bigcup_{r > 1} \bigcup_{u \in L^r([0, T], \mathbf{R})} \{\Upsilon_{T,0}^u \psi_0\} \\ &= \bigcup_{l \in \mathbf{N}} \bigcup_{m \in \mathbf{N}} \bigcup_{k \in \mathbf{N}} \bigcup_{0 \leq t \leq l} \left(\bigcup_{\|u\|_{L^{1+1/m} \leq k}} \{\Upsilon_{t,0}^u \psi_0\} \right). \end{aligned}$$

Hence it is sufficient to prove that, for every (l, m, k) in \mathbf{N}^3 , the set

$$\mathcal{A}^{l,m,k} = \bigcup_{0 \leq t \leq l} \left(\bigcup_{\|u\|_{L^{1+1/m} \leq k}} \{\Upsilon_{t,0}^u \psi_0\} \right)$$

has compact closure in X . To this end, one considers a sequence $(\psi_n)_{n \in \mathbf{N}}$ in $\mathcal{A}^{l,m,k}$, associated with a sequence of times $(t_n)_{n \in \mathbf{N}}$ in $[0, l]$ and a sequence of controls $(u_n)_{n \in \mathbf{N}}$ in the ball of radius k of $L^{1+1/m}([0, +\infty), \mathbf{R})$. By compactness of $[0, l]$, up to extraction, one can assume that $(t_n)_{n \in \mathbf{N}}$ tends to t_∞ in $[0, l]$. By Banach–Alaoglu–Bourbaki Theorem, the balls of $L^{1+1/m}([0, +\infty), \mathbf{R})$ are weakly (sequentially) compact and, hence, up to extraction, one can assume that $(u_n)_{n \in \mathbf{N}}$ converges weakly in $L^{1+1/m}([0, +\infty), \mathbf{R})$ to some u_∞ . The hard point of the proof (Lemma 3.7 in [BMS82]) is then to show that $\Upsilon_{t_n,0}^{u_n} \psi_0$ tends to $\Upsilon_{t_\infty,0}^{u_\infty}$ as n tends to infinity.

A crucial point in the proof of Theorem 3 given in [BMS82] is the fact that the closed balls of L^p , $p > 1$ are weakly sequentially compact. This is no longer true for the balls of L^1 , and this prevents a direct extension of the proof of Theorem 3 to the proof of Theorem 2.

1.3 Content

In this note we present a simple and short proof of Theorem 2. However, historical reasons have made different communities use incompatible terminologies and, in order to avoid ambiguities, we present in Section 2 a quick reminder of basic facts in Banach topologies. Section 3 gives a short introduction to the classical Dyson expansion (Section 3.1) and the proof of an instrumental compactness property (Section 3.2). We conclude in Section 4 with the proof of Theorem 2.

2 Basic facts on the topology in Banach spaces

2.1 Notations

The Banach space X is endowed with norm $\|\cdot\|$. For every ψ_c in X and every $r > 0$, $B_X(\psi_c, r)$ denotes the ball of center ψ_c and of radius r :

$$B_X(\psi_c, r) = \{\psi \in X \mid \|\psi - \psi_c\| < r\}.$$

In the following, all we need to know about generators of C^0 -semigroup is the classical result stated in Proposition 5 (see Chapter VII of [HP57]).

Proposition 5. *Assume that A generates a C^0 -semigroup. Then there exist $M, \omega > 0$ such that $\|e^{At}\| \leq Me^{\omega t}$ for every $t \geq 0$.*

2.2 Compact subset of Banach spaces

Definition 3. Let X be a Banach space and Y be a subset of X . A family $(O_i)_{i \in I}$ is an open cover of Y if O_i is open in X for every i in I and $Y \subset \cup_{i \in I} O_i$.

Definition 4. Let X be a Banach space. A subset Y of X is said to be compact if from any open cover of Y , it is possible to extract a finite cover of Y .

Definition 5. Let X be a Banach space. A subset Y of X is said to be sequentially compact if from any sequence $(\psi_n)_{n \in \mathbf{N}}$ taking value in Y , it is possible to extract a subsequence $(\psi_{\phi(n)})_{n \in \mathbf{N}}$ converging in Y .

Definition 6. Let X be a Banach space. A subset Y of X is said to be totally bounded if for every $\varepsilon > 0$, there exist $N \in \mathbf{N}$ and a finite family $(x_i)_{1 \leq i \leq N}$ in X such that

$$Y \subset \bigcup_{i=1}^N B_X(x_i, \varepsilon).$$

Proposition 6. Let X be a Banach space. For every subset Y of X , the following assertions are equivalent:

1. Y is compact.
2. Y is sequentially compact.
3. Y is complete and totally bounded.
4. Y is closed and totally bounded.

Proposition 7. Let X be a Banach space, N in \mathbf{N} and $(Y_i)_{1 \leq i \leq N}$ a finite family of compact subsets of X . Then, the finite sum

$$\sum_{i=1}^N Y_i = \{y_1 + y_2 + \dots + y_N \mid y_i \in Y_i, i = 1, \dots, N\}$$

is compact as well.

Proposition 8. Let X be a Banach space, N in \mathbf{N} and $(Y_i)_{1 \leq i \leq N}$ a finite family of totally bounded subsets of X . Then, the finite sum

$$\sum_{i=1}^N Y_i = \{y_1 + y_2 + \dots + y_N \mid y_i \in Y_i, i = 1, \dots, N\}$$

is totally bounded as well.

Proposition 9. Let X be a Banach space, $T > 0$ and (A, B) satisfies Assumption 1. Define the mapping

$$F : \begin{array}{ccc} [0, T] \times [0, T] \times X & \rightarrow & X \\ (s, t, \psi) & \mapsto & e^{|t-s|A} B\psi \end{array}$$

Then, for every totally bounded subset Y of X , the set $F([0, T] \times [0, T] \times Y)$ is totally bounded as well.

Proof. We claim that $G : (t, \psi) \mapsto e^{tA}\psi$ is jointly continuous in its two variables. Indeed, for every ψ, ψ_0 in X , for every $t, t_0 \geq 0$,

$$\begin{aligned} \|e^{tA}\psi - e^{t_0A}\psi_0\| &\leq \|e^{tA}(\psi - \psi_0)\| + \|(e^{tA} - e^{t_0A})\psi_0\| \\ &\leq Me^{\omega t}\|\psi - \psi_0\| + \|(e^{tA} - e^{t_0A})\psi_0\|. \end{aligned}$$

This last quantity tends to zero as (t, ψ) tends to (t_0, ψ_0) . As a consequence, F is continuous (as composition of continuous functions).

If Y is totally bounded, the topological closure \bar{Y} of Y is compact (because the ambient space X is complete). Hence $[0, T] \times [0, T] \times \bar{Y}$ is compact. By continuity, $F([0, T] \times [0, T] \times \bar{Y})$ is compact, hence is totally bounded. The set $F([0, T] \times [0, T] \times Y)$, which is contained in $F([0, T] \times [0, T] \times \bar{Y})$, is, therefore, totally bounded as well. \square

2.3 Partition of unity in Banach spaces

Definition 7. Let X a Banach space. A family $(x_i)_{i \in I}$ of points of X is locally finite if for every x in X and every $R > 0$, the cardinality of the set

$$\left(\bigcup_{i \in I} \{x_i\} \right) \cap B_X(x, R)$$

is finite.

Definition 8. Let X be a Banach space, Y be a subset of X , and $(O_i)_{i \in I}$ be an open cover of Y . A family $(\phi_i)_{i \in I}$ of continuous functions from Y to $[0, 1]$ is called a partition of the unity of Y adapted to the cover $(O_i)_{i \in I}$ if

- (i) for every $i \in I$, $\phi_i(x) = 0$ for every $x \notin O_i$;
- (ii) $\sum_{i \in I} \phi_i(x) = 1$ for every $x \in Y$.

Proposition 10. Let X be a Banach space, Y a subset of X , $\delta > 0$, $(x_j)_{j \in J}$ a locally finite family of points in Y such that $Y \subset \cup_{j \in J} B_X(x_j, \delta)$. Then, there exists $(\phi_j)_{j \in J}$ a partition of the unity adapted to the open cover $(B(x_j, 2\delta))_{j \in J}$ of Y .

Moreover, if a family $(\phi_j)_{j \in J}$ is a partition of the unity adapted to the open cover $(B_X(x_j, 2\delta))_{j \in J}$, then for every x in Y , $\|x - \sum_{j \in J} \phi_j(x)x_j\| \leq 2\delta$.

Proof. We first prove the existence of a partition of the unity adapted to the open cover $(B_X(x_j, 2\delta))_{j \in J}$ of Y . To this end, we define, for every j in J , the continuous functions $\varphi_j : X \rightarrow [0, 1]$ by

$$\begin{cases} \varphi_j(x) = 1, & \text{if } \|x - x_j\| < \delta, \\ \varphi_j(x) = 2 - \|x - x_j\|/\delta, & \text{if } \delta \leq \|x - x_j\| < 2\delta, \\ \varphi_j(x) = 0, & \text{if } 2\delta \leq \|x - x_j\|. \end{cases}$$

Since the family $(x_j)_{j \in J}$ is locally finite, the sum $\sum_{j \in J} \varphi_j(x)$ converges for every x in Y . Moreover, since $Y \subset \cup_{j \in J} B(x_j, \delta)$, the function $x \mapsto \sum_{j \in J} \varphi_j(x)$ does not vanish on Y . For every j_0 in J , we define ϕ_{j_0} by

$$\phi_{j_0}(x) = \varphi_{j_0}(x) \frac{1}{\sum_{j \in J} \varphi_j(x)},$$

and the family $(\phi_j)_{j \in J}$ is a partition of the unity adapted to the open cover $(B_X(x_j, 2\delta))_{j \in J}$ of Y .

We now prove the second point of Proposition 10. Let $(\phi_j)_{j \in J}$ be a partition of unity of Y adapted to the cover $(B_X(x_j, 2\delta))_{j \in J}$. Then, for every x in Y ,

$$\begin{aligned} \left\| x - \sum_{j \in J} \phi_j(x) x_j \right\| &= \left\| \sum_{j \in J} \phi_j(x) x - \sum_{j \in J} \phi_j(x) x_j \right\| \\ &= \left\| \sum_{j \in J} \phi_j(x) (x - x_j) \right\| \\ &\leq \sum_{j \in J} \phi_j(x) \|x - x_j\|. \end{aligned}$$

By construction, $\phi_j(x) = 0$ as soon as $\|x - x_j\| \geq 2\delta$. Hence,

$$\left\| x - \sum_{j \in J} \phi_j(x) x_j \right\| \leq 2\delta \sum_{j \in J} \phi_j(x) \leq 2\delta,$$

which concludes the proof. \square

3 The Dyson expansion

3.1 The Dyson Operators

For every u in $L^1_{loc}([0, +\infty), \mathbf{R})$, $p \in \mathbf{N}$, and $t \geq 0$ we define the linear bounded operator $W_p(t, u) : X \rightarrow X$ recursively by

$$\begin{aligned} W_0(t, u)\psi &= e^{(t-s)A}\psi \\ W_p(t, u)\psi &= \int_0^t e^{(t-s)A} B W_{p-1}(s, u)\psi u(s) ds, \quad \text{for } p \geq 1, \end{aligned}$$

for every ψ in X . We have the following estimate on the norm of the operator.

Proposition 11. *For every u in $L^1_{loc}([0, +\infty), \mathbf{R})$, $p \in \mathbf{N}$, and $t \geq 0$*

$$\|W_p(t, u)\| \leq \frac{M e^{\omega t} \|B\|^p (\int_0^t |u(s)| ds)^p}{p!}.$$

Proof. We prove the result by induction on p in \mathbf{N} . For $p = 0$ the result clearly follows from Proposition 5. Assume that the result holds for $p \geq 0$. Then, for every ψ in X ,

$$\begin{aligned} \|W_{p+1}(t, u)\psi\| &\leq \int_0^t e^{(t-s)A} B W_p(s, u)\psi u(s) ds \\ &\leq M \int_0^t e^{(t-s)\omega} \|B\| \frac{e^{\omega s} \|B\|^p (\int_0^s |u(\tau)| d\tau)^p}{p!} u(s) ds \\ &\leq M e^{\omega t} \|B\|^{p+1} \frac{(\int_0^t |u(\tau)| d\tau)^{p+1}}{(p+1)!}. \end{aligned}$$

The last inequality follows from Proposition 5. We conclude the proof by induction on p . \square

3.2 A compactness property

Lemma 12. *For every j in \mathbf{N} , $T \geq 0$ and $K \geq 0$, and ψ in X the set*

$$\mathcal{W}_j^{T,K} = \{W_j(t, u)\psi \mid 0 \leq t \leq T, \|u\|_{L^1} \leq K\}, \quad (3)$$

is totally bounded

Proof. We prove the result by induction on j in \mathbf{N} . For $j = 0$, consider $\mathcal{W}_0^{T,K} = \{e^{tA}\psi, 0 \leq t \leq T\}$ and let $(w_n)_{n \in \mathbf{N}}$ be a sequence in $\mathcal{W}_0^{T,K}$. Then there exists a sequence $(t_n)_{n \in \mathbf{N}}$ such that $w_n = e^{t_n A}\psi$ for every n . Up to extraction $\lim_{n \rightarrow \infty} t_n = t \in [0, T]$ since $[0, T]$ is compact. By definition of C^0 -semigroup, $\lim_{n \rightarrow \infty} e^{t_n A}\psi = e^{tA}\psi$. This proves that $\mathcal{W}_0^{T,K}$ is sequentially compact, hence compact and, in particular, totally bounded (Proposition 6).

Assume that, for $j \geq 0$, $\mathcal{W}_j^{T,K}$ is totally bounded. By Proposition 9, the set

$$\begin{aligned} Z_j^{T,K} &:= \{e^{(t-s)A}B\psi, \psi \in W_j^{T,K}, 0 \leq s \leq t \leq T\} \\ &\subset F([0, T]^2 \times \mathcal{W}_j^{T,K}) \end{aligned}$$

is totally bounded as well.

Let $\varepsilon > 0$ be given and define $\delta = \frac{\varepsilon}{2K+1} > 0$. Since $Z_j^{T,K}$ is totally bounded, there exists a finite family $(x_i)_{1 \leq i \leq N_\delta}$ in $Z_j^{T,K}$ such that

$$Z_j^{T,K} \subset \bigcup_{i=1}^{N_\delta} B_X(x_i, \delta).$$

Let $(\phi_i)_{1 \leq i \leq N_\delta}$ be a partition of the unity adapted to the cover $\cup_{i=1}^{N_\delta} B(x_i, 2\delta)$ of $Z_j^{T,K}$. Such a partition of the unity exists by Proposition 10, and moreover, for every x in $Z_j^{T,K}$, we have

$$\left\| x - \sum_{i=1}^{N_\delta} \phi_i(x)x_i \right\| \leq 2\delta. \quad (4)$$

Applying the inequality (4) with $x = e^{(t-s)A}BW_j(s, u)\psi_0$, we get, for every u in L^1 and every (s, t) such that $0 \leq s \leq t$,

$$\left\| e^{(t-s)A}BW_j(s, u)\psi_0 - \sum_{i=1}^{N_\delta} \phi_i(e^{(t-s)A}BW_j(s, u)\psi_0)x_i \right\| \leq 2\delta.$$

Multiplying by $u(s)$ and integrating for s in $[0, t]$, one gets for $\|u\|_{L^1} \leq K$

$$\left\| \int_0^t e^{(t-s)A}BW_j(s, u)\psi_0 u(s) ds - \sum_{i=1}^{N_\delta} \int_0^t \phi_i(e^{(t-s)A}BW_j(s, u)\psi_0) u(s) ds x_i \right\| \leq 2\delta K. \quad (5)$$

The set $\sum_{i=1}^{N_\delta} [0, K]x_i$ is compact by Proposition 7 and, hence, totally bounded. Then there exists a finite family $(y_i)_{1 \leq i \leq N'_\delta}$ such that

$$\sum_{i=1}^{N_\delta} [0, K]x_i \subset \bigcup_{i=1}^{N'_\delta} B_X(y_i, \delta). \quad (6)$$

From (5) and (6), one deduces that

$$\mathcal{W}_{j+1}^{T,K} \subset \bigcup_{i=1}^{N'_\delta} B_X(y_i, (2K+1)\delta) = \bigcup_{i=1}^{N'_\delta} B_X(y_i, \varepsilon).$$

This proves that $\mathcal{W}_{j+1}^{T,K}$ is totally bounded and concludes the proof. \square

3.3 Convergence of the Dyson expansion

Proposition 13. *For every u in $L^1([0, +\infty), \mathbf{R})$, $t \geq 0$, and $\psi_0 \in X$*

$$\|\Upsilon_{t,0}^u \psi_0\| \leq M e^{\omega t} \|\psi_0\| \exp\left(M e^{\omega t} \|B\| \int_0^t |u(s)| ds\right)$$

Proof. The proof follows the proof of [BMS82, Theorem 2.5]. By Duhamel formula (2) and Proposition 5,

$$\begin{aligned} \|\Upsilon_{t,0}^u \psi_0\| &= \left\| e^{tA} \psi_0 + \int_0^t e^{(t-s)A} B \Upsilon_{s,0}^u \psi_0 u(s) ds \right\| e^{t\omega} \|\psi_0\| \\ &\leq M e^{t\omega} \|\psi_0\| + \int_0^t M e^{\omega(t-s)} \|B\| \|\Upsilon_{s,0}^u \psi_0\| |u(s)| ds, \end{aligned}$$

and the conclusion follows by Gronwall lemma (see [BMS82, Lemma 2.6]). \square

Proposition 14. *For every u in $L^1([0, +\infty), \mathbf{R})$, p in \mathbf{N} , $t \geq 0$, and ψ_0 in X*

$$\lim_{p \rightarrow \infty} \left\| \int_0^t e^{(t-s)A} B W_p(s, t) \Upsilon_{s,0}^u \psi_0 u(s) ds \right\| = 0.$$

Proof. Consider

$$\left\| \int_0^t e^{(t-s)A} B W_p(s, t) \Upsilon_{s,0}^u \psi_0 u(s) ds \right\| \leq \int_0^t \|e^{(t-s)A}\| \|B\| \|W_p(s, t)\| \|\Upsilon_{s,0}^u \psi_0\| |u(s)| ds$$

and recall that $\|W_p(s, t)\|$ tends to zero as p tends to infinity (Proposition 11). \square

Proposition 15. *For every u in $L^1_{loc}([0, +\infty), \mathbf{R})$, p in \mathbf{N} , $t \geq 0$, and ψ_0 in X*

$$\Upsilon_{t,0}^u \psi_0 = \sum_{p=0}^{\infty} W_p(t, 0, u) \psi_0.$$

Proof. Applying iteratively p -times Duhamel formula (2), one gets

$$\begin{aligned} \Upsilon_{t,0}^u \psi_0 &= e^{tA} \psi_0 + \int_0^t e^{(t-s)A} B u(s) \Upsilon_{s,0}^u \psi_0 ds \\ &= e^{tA} \psi_0 + \int_0^t e^{(t-s)A} B u(s) e^{sA} \psi_0 ds + \int_0^t e^{(t-s)A} B W_1(s, t) \Upsilon_{s,0}^u \psi_0 u(s) ds \\ &= \sum_{j=1}^p W_p(t, 0) \psi_0 + \int_0^t e^{(t-s)A} B W_p(s, t) \Upsilon_{s,0}^u \psi_0 u(s) ds. \end{aligned}$$

Hence, for every $p \geq 1$,

$$\Upsilon_{t,0}^u \psi_0 - \sum_{j=0}^p W_j(t,0) \psi_0 = \int_0^t e^{(t-s)A} B W_p(s,t) \Upsilon_{s,0}^u \psi_0 u(s) ds$$

and the result follow from Proposition 14 as p tends to ∞ . \square

4 Proof of Theorem 2

We proceed now to the proof of Theorem 2. First of all, notice that, for every ψ_0 in X ,

$$\begin{aligned} \mathcal{A}(\psi_0, L_{loc}^1([0, +\infty), \mathbf{R})) \\ = \bigcup_{l \in \mathbf{N}} \bigcup_{m \in \mathbf{N}} \{ \Upsilon_{t,0}^u \psi_0, \|u\|_{L^1} \leq l, 0 \leq t \leq m \}, \end{aligned}$$

and it is enough to prove that, for every l and m in \mathbf{N} , the set

$$\{ \Upsilon_{t,0}^u \psi_0, \|u\|_{L^1} \leq l, 0 \leq t \leq m \}$$

is totally bounded.

Let $\varepsilon > 0$. From the convergence of the Dyson expansion (Proposition 15) and the bound on the operators W_j (Proposition 11), there exists a integer N_ε such that

$$\left\| \sum_{p \geq N_\varepsilon} W_p(t, u) \psi_0 \right\| \leq \frac{\varepsilon}{2}, \quad (7)$$

for every t in $[0, m]$ and every u such that $\|u\|_{L^1} \leq l$. For each $j = 1, \dots, N_\varepsilon$ the sets $\mathcal{W}_j^{m,l}$, defined by (3), are totally bounded (Lemma 12), hence their sum

$$\sum_{j=0}^{N_\varepsilon} \mathcal{W}_j^{m,l}$$

is totally bounded as well (Proposition 8). Hence there exists a family $(x_i)_{1 \leq i \leq N_1}$ of points of $\sum_{j=0}^{N_\varepsilon} \mathcal{W}_j^{m,l}$ such that

$$\sum_{j=0}^{N_\varepsilon} \mathcal{W}_j^{m,l} \subset \bigcup_{i=1}^{N_1} B_X \left(x_i, \frac{\varepsilon}{2} \right). \quad (8)$$

Gathering (7) and (8), one gets

$$\sum_{j=0}^{\infty} \mathcal{W}_j^{m,l} \subset \bigcup_{i=1}^{N_1} B_X(x_i, \varepsilon),$$

which concludes the proof of Theorem 2.

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