On the Ball-Marsden-Slemrod obstruction for bilinear control systems
Nabile Boussaid, Marco Caponigro, Thomas Chambrion

To cite this version:
Nabile Boussaid, Marco Caponigro, Thomas Chambrion. On the Ball-Marsden-Slemrod obstruction for bilinear control systems. CDC 2019 - 58th IEEE Conference on Decision and Control, Dec 2019, Nice, France. 10.1109/CDC40024.2019.9029511 . hal-01537743v3

HAL Id: hal-01537743
https://hal.science/hal-01537743v3
Submitted on 29 Nov 2019

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L’archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d’enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.
On the Ball-Marsden-Slemrod obstruction for bilinear control systems

Nabile Boussaïd
Laboratoire de Mathématiques de Besançon, UMR 6623
Université de Bourgogne Franche-Comté, Besançon, France
Nabile.Boussaid@univ-fcomte.fr

Marco Caponigro
Équipe M2N
Conservatoire National des Arts et Métiers, Paris, France
Marco.Caponigro@cnam.fr

Thomas Chambrion
Université de Lorraine, CNRS, Inria, IECL, Nancy, France
Thomas.Chambrion@univ-lorraine.fr

Abstract—In this paper we present an extension to the case of \( L^1 \)-controls of a famous result by Ball–Marsden–Slemrod on the obstruction to the controllability of bilinear control systems in infinite dimensional spaces.

I. INTRODUCTION

A. Bilinear control systems

Let \( X \) be a Banach space, \( A : D(A) \to X \) a linear operator in \( X \) with domain \( D(A) \), \( B : X \to X \) a linear bounded operator and \( \psi_0 \) an element in \( X \).

We consider a following bilinear control system on \( X \)
\[
\begin{align*}
\dot{\psi}(t) &= A\psi(t) + u(t)B\psi(t), \\
\psi(0) &= \psi_0,
\end{align*}
\]
where \( u : [0, +\infty) \to \mathbb{R} \) is a scalar function representing the control.

Assumption 1: The pair \((A, B)\) of linear operators in \( X \) satisfies
1) the operator \( A \) generates a \( C^0 \)-semigroup of linear bounded operators on \( X \).
2) the operator \( B \) is bounded.

Definition 1: Let \((A, B)\) satisfy Assumption 1. A function \( \psi : [0, +\infty) \to X \) is a mild solution of (1) if \( \psi \) is continuous from \([0, +\infty)\) to \( X \) and, for every \( t \in [0, T] \),
\[
\psi(t) = e^{tA}\psi_0 + \int_0^t e^{(t-s)A}B\psi(s)u(s)ds.
\]
Equation (2) is often called Duhamel formula. Existence and uniqueness of mild solutions for equation (1) is given by the following result (see, for instance, Proposition 2.1 and Remark 2.7 in [BMS82]).

Proposition 1: Assume that \((A, B)\) satisfies Assumption 1. Then, for every \( \psi_0 \) in \( X \), for every \( u \) in \( L^1_{loc}([0, +\infty), \mathbb{R}) \), there exists a unique mild solution \( t \to \Upsilon_{t,0,\psi_0} \) to the Cauchy problem (1). Moreover, for every \( \psi_0 \) in \( X \), the end-point mapping \( \Upsilon_{-t,0,\psi_0} : [0, +\infty) \times L^1_{loc}([0, +\infty), \mathbb{R}) \to X \) is continuous.

For the sake of completeness, we will prove a weak version of Proposition 1 (without the continuity statement), sufficient for our purpose, in Section III-C.

Definition 2: Assume that \((A, B)\) satisfies Assumption 1 and let \( U \) be a subset of \( L^1_{loc}([0, +\infty), \mathbb{R}) \). For every \( \psi_0 \) in \( X \), the attainable set from \( \psi_0 \) with controls in \( U \) is defined as
\[
\mathcal{A}(\psi_0, U) = \bigcup_{T>0} \bigcup_{u \in U} \{ \Upsilon_{T,0,\psi_0} \}.
\]

Our main result is the following property of the attainable set of system (1) with \( L^1 \) controls.

Theorem 2: Assume that \((A, B)\) satisfies Assumption 1. Then, for every \( \psi_0 \) in \( X \), the attainable set \( \mathcal{A}(\psi_0, L^1_{loc}([0, +\infty), \mathbb{R})) \) from \( \psi_0 \) with \( L^1_{loc} \) controls is contained in a countable union of compact subsets of \( X \).

B. The Ball–Marsden–Slemrod obstruction

Our main result, Theorem 2 is an extension of the well-known Ball–Marsden–Slemrod obstruction to controllability (see also [ILT06]) which is as follows.

Theorem 3 (Theorem 3.6 in [BMS82]): Assume that \((A, B)\) satisfies Assumption 1. Then, for every \( \psi_0 \) in \( X \), the attainable set \( \mathcal{A}(\psi_0, L^1_{loc}([0, +\infty), \mathbb{R})) \) from \( \psi_0 \) with \( L^1_{loc} \) controls, \( r > 1 \), is contained in a countable union of compact subsets of \( X \).

A consequence of Theorem 3 to the framework of the conservative bilinear Schrödinger equation is given by Turinici.

Theorem 4 (Theorem 1 in [Tur00]): Assume that \((A, B)\) satisfies Assumption 1. Then, for every \( \psi_0 \) in \( X \), the set \( \bigcup_{r>1} \mathcal{A}(\psi_0, L^r_{loc}([0, +\infty), \mathbb{R})) \) is contained in a countable union of compact subsets of \( X \).

Theorems 2 and 3 are basically empty in the case in which \( X \) is finite dimensional, since, in this case, \( X \) itself is a countable union of compact sets. On the other hand, when \( X \) is infinite dimensional, these results represent a strong topological obstruction to the exact controllability. Indeed, compact subsets of an infinite dimensional Banach space have empty interiors and so is a countable union of closed subsets with empty interiors (as a consequence of Baire Theorem).

Whether the non-controllability result of Ball, Marsden, and Slemrod, Theorem 3 holds for \( L^1 \) control has been an
open question for decades. Indeed, the proof of Theorem 3 does not apply directly to the $L^1$ case. To see what fails let us briefly recall the method used in [BMS82] for the proof of Theorem 3. The first step is to write

$$\mathcal{A}(\psi_0, \cup_{r>1} L^r_{loc}([0, +\infty), \mathbb{R}))$$

$$= \bigcup_{T \geq 0} \bigcup_{r>1} \{ \mathcal{Y}_{T,0}^r \psi_0 \}$$

$$= \bigcup_{i \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \bigcup_{k \in \mathbb{N}} \left( \bigcup_{0 \leq t \leq l} \bigcup_{\|u\|_{L^{1+1/m}} \leq k} \{ \mathcal{Y}_{T,0}^r \psi_0 \} \right).$$

Hence it is sufficient to prove that, for every $(l, m, k)$ in $\mathbb{N}^3$, the set

$$\mathcal{A}^{l,m,k} = \bigcup_{0 \leq t \leq l} \left( \bigcup_{\|u\|_{L^{1+1/m}} \leq k} \{ \mathcal{Y}_{T,0}^r \psi_0 \} \right)$$

has compact closure in $X$. To this end, one considers a sequence $(\psi_n)_{n \in \mathbb{N}}$ in $\mathcal{A}^{l,m,k}$, associated with a sequence of times $(t_n)_{n \in \mathbb{N}}$ in $[0, l]$ and a sequence of controls $(u_n)_{n \in \mathbb{N}}$ in the ball of radius $k$ of $L^{1+1/m}([0, +\infty), \mathbb{R})$. By compactness of $[0, l]$, up to extraction, one can assume that $(t_n)_{n \in \mathbb{N}}$ tends to $t_\infty$ in $[0, l]$. By Banach–Alaoglu–Bourbaki Theorem, the balls of $L^{1+1/m}([0, +\infty), \mathbb{R})$ are weakly (sequentially) compact and, hence, up to extraction, one can assume that $(u_n)_{n \in \mathbb{N}}$ converges weakly in $L^{1+1/m}([0, +\infty), \mathbb{R})$ to some $u_\infty$. The hard step of the proof (Lemma 3.7 in [BMS82]) is then to show that $\mathcal{Y}_{t_\infty,0}^n \psi_0$ tends to $\mathcal{Y}_{t_\infty,0}^\infty \psi_0$ as $n$ tends to infinity.

A crucial point in the proof of Theorem 3 given in [BMS82] is the fact that the closed balls of $L^p$, $p > 1$ are weakly sequentially compact. This is no longer true for the balls of $L^1$, and this prevents a direct extension of the proof of Theorem 3 to Theorem 2. Here we present a brief and self-contained proof of Theorem 2 mainly based on Dyson expansions and basics compactness properties on Banach spaces.

An alternative proof of Theorem 3, not relying on the reflectiveness of the set of admissible controls, has been recently given in [BCC17]. The proof applies to a very large class of controls (namely, Radon measures) which contains locally integrable functions, and for its generality it is technically quite involved, in contrast with the simplicity of the underlying ideas. It applies also, with minor modifications, to nonlinear problems [CT19].

C. Content

In this note we present a simple proof of Theorem 2. However, historical reasons have made different communities use incompatible terminologies and, in order to avoid ambiguities, we present in Section II a quick reminder of basic facts in Banach topologies. Section III gives a short introduction to the classical Dyson expansion (Section III-A), the proof of an instrumental compactness property (Section III-B) and a proof of the well-posedness of the Cauchy problem (1) (Section III-C). We conclude in Section IV with the proof of Theorem 2.

II. BASIC FACTS ABOUT TOPOLOGY IN BANACH SPACES

A. Notations

The Banach space $X$ is endowed with norm $\| \cdot \|$. For every $\psi_c$ in $X$ and every $r > 0$, $B_X(\psi_c, r)$ denotes the ball of center $\psi_c$ and of radius $r$:

$$B_X(\psi_c, r) = \{ \psi \in X : \| \psi - \psi_c \| < r \}.$$

In the following, all we need to know about generators of $C^0$-semigroup is the classical result stated in Proposition 5 (see Chapter VII of [HP57]).

Proposition 5: Assume that $A$ generates a $C^0$-semigroup. Then there exist $M, \omega > 0$ such that $\| e^{At} \| \leq M e^{\omega t}$ for every $t \geq 0$.

B. Compact subset of Banach spaces

Definition 3: Let $X$ be a Banach space and $Y$ be a subset of $X$. A family $(O_i)_{i \in I}$, indexed by an arbitrary set $I$, is an open cover of $Y$. If $O_i$ is open in $X$ for every $i \in I$ and $Y \subset \bigcup_{i \in I} O_i$.

Definition 4: Let $X$ be a Banach space. A subset $Y$ of $X$ is said to be compact if from any open cover of $Y$, it is possible to extract a finite cover of $Y$.

Definition 5: Let $X$ be a Banach space. A subset $Y$ of $X$ is said to be sequentially compact if from any sequence $(\psi_n)_{n \in \mathbb{N}}$ taking value in $Y$, it is possible to extract a subsequence $(\psi_{n_{(i)}})_{i \in \mathbb{N}}$ converging in $Y$.

Definition 6: Let $X$ be a Banach space. A subset $Y$ of $X$ is said to be totally bounded if for every $\varepsilon > 0$, there exist $N \in \mathbb{N}$ and a finite family $(x_i)_{1 \leq i \leq N}$ in $X$ such that

$$Y \subset \bigcup_{i=1}^{N} B_X(x_i, \varepsilon).$$

Proposition 6: Let $X$ be a Banach space. For every subset $Y$ of $X$, the following assertions are equivalent:

1) $Y$ is compact.
2) $Y$ is sequentially compact.
3) $Y$ is complete and totally bounded.
4) $Y$ is closed and totally bounded.

Proposition 7: Let $X$ be a Banach space, $N \in \mathbb{N}$ and $(Y_i)_{1 \leq i \leq N}$ be a finite family of compact subsets of $X$. Then, the finite sum

$$\sum_{i=1}^{N} Y_i = \{ y_1 + y_2 + \ldots + y_N \mid y_i \in Y_i, i = 1, \ldots, N \}$$

is compact as well.

Proposition 8: Let $X$ be a Banach space, $N \in \mathbb{N}$ and $(Y_i)_{1 \leq i \leq N}$ be a finite family of totally bounded subsets of $X$. Then, the finite sum

$$\sum_{i=1}^{N} Y_i = \{ y_1 + y_2 + \ldots + y_N \mid y_i \in Y_i, i = 1, \ldots, N \}$$

is totally bounded as well.
Proposition 9: Let $X$ be a Banach space, $T > 0$ and $(A, B)$ satisfy Assumption 1. Define the mapping

$$F : [0, T] \times [0, T] \times X \to X$$

$$(s, t, \psi) \mapsto e^{t-s}A B \psi$$

Then, for every totally bounded subset $Y$ of $X$, the set $F([0, T] \times [0, T] \times Y)$ is totally bounded as well.

Proof: We claim that $G : (t, \psi) \mapsto e^{tA} \psi$ is jointly continuous in its two variables. Indeed, for every $\psi, \psi_0$ in $X$, for every $t, t_0 \geq 0$,

$$||e^{tA} \psi - e^{t_0A} \psi_0|| \leq ||e^{tA} (\psi - \psi_0)|| + ||e^{tA} - e^{t_0A}|| \psi_0||$$

$$\leq M e^{t_0 ||\psi_0||} + ||e^{tA} - e^{t_0A}|| \psi_0||$$

This last quantity tends to zero as $(t, \psi)$ tends to $(t_0, \psi_0)$. As a consequence, $F$ is continuous (as composition of continuous functions).

If $Y$ is totally bounded, the topological closure $\bar{Y}$ of $Y$ is compact (because the ambient space $X$ is complete). Hence, $F([0, T] \times [0, T] \times \bar{Y})$ is compact. By compactness, $F([0, T] \times [0, T] \times \bar{Y})$, which is contained in $F([0, T] \times [0, T] \times \bar{Y})$, is, therefore, totally bounded as well.

C. Partition of unity in Banach spaces

Definition 7: Let $X$ be a Banach space. A family $(x_i)_{i \in I}$ of points of $X$ is locally finite if for every $x \in X$ and every $R > 0$, the cardinality of the set

$$\bigcup_{i \in I} \{x_i\} \cap B_X(x, R)$$

is finite.

Definition 8: Let $X$ be a Banach space, $Y$ be a subset of $X$, and $(O_i)_{i \in I}$ be an open cover of $Y$. A family $(\phi_i)_{i \in I}$ of continuous functions from $Y$ to $[0, 1]$ is called a partition of the unity of $Y$ adapted to the cover $(O_i)_{i \in I}$ if

(i) for every $i \in I$, $\phi_i(x) = 0$ for every $x \not\in O_i$;

(ii) $\sum_{i \in I} \phi_i(x) = 1$ for every $x \in Y$.

Proposition 10: Let $X$ be a Banach space, $Y$ be a subset of $X$, $\delta > 0$, $(x_j)_{j \in J}$ be a locally finite family of points in $Y$ such that $Y \subset \bigcup_{j \in J} B_X(x_j, \delta)$. Then, there exists a partition of the unity, $(\phi_j)_{j \in J}$, adapted to the open cover $(B_X(x_j, 2\delta))_{j \in J}$ of $Y$.

Moreover, if a family $(\phi_j)_{j \in J}$ is a partition of the unity adapted to the open cover $(B_X(x_j, \delta))_{j \in J}$, then for every $x \in Y$, $||x - \sum_{j \in J} \phi_j(x) x_j|| \leq 2\delta$.

Proof: We first prove the existence of a partition of the unity adapted to the open covering $(B_X(x_j, 2\delta))_{j \in J}$ of $Y$. To this end, we define, for every $j \in J$, the continuous functions $\varphi_j : X \to [0, 1]$ by

$$\varphi_j(x) = 1,$$

$$\varphi_j(x) = 2 - ||x - x_j||/\delta, \quad \text{if } \delta \leq ||x - x_j|| < 2\delta,$$

$$\varphi_j(x) = 0,$$

$$\text{if } 2\delta \leq ||x - x_j||.$$

Since the family $(x_j)_{j \in J}$ is locally finite, the sum $\sum_{j \in J} \varphi_j(x)$ converges for every $x \in Y$. Moreover, since $Y \subset \bigcup_{j \in J} B(x_j, \delta)$, the function $x \mapsto \sum_{j \in J} \varphi_j(x)$ does not vanish on $Y$. For every $j_0 \in J$, we define $\phi_{j_0}$ by

$$\phi_{j_0}(x) = \varphi_{j_0}(x) \frac{1}{\sum_{j \in J} \varphi_j(x)},$$

and the family $(\phi_j)_{j \in J}$ is a partition of the unity adapted to the open cover $(B_X(x_j, 2\delta))_{j \in J}$ of $Y$.

We now prove the second point of Proposition 10. Let $(\phi_j)_{j \in J}$ be a partition of unity of $Y$ adapted to the cover $(B_X(x_j, 2\delta))_{j \in J}$. Then, for every $x \in Y$,

$$\left| x - \sum_{j \in J} \phi_j(x) x_j \right| = \left| \sum_{j \in J} \phi_j(x) x - \sum_{j \in J} \phi_j(x) x_j \right|$$

$$= \left| \sum_{j \in J} \phi_j(x) (x - x_j) \right| \leq \sum_{j \in J} \phi_j(x) ||x - x_j||$$

By construction, $\phi_j(x) = 0$ as soon as $||x - x_j|| \geq 2\delta$. Hence,

$$\left| x - \sum_{j \in J} \phi_j(x) x_j \right| \leq 2\delta \sum_{j \in J} \phi_j(x) \leq 2\delta,$$

which concludes the proof.

III. DYSON EXPANSION

A. The Dyson Operators

Let $(A, B)$ satisfy Assumption 1. For every $u \in L^1_{loc}([0, +\infty), \mathbb{R})$, $p \in \mathbb{N}$, and $t \geq 0$ we define the linear bounded operator $W_p(t, u) : X \to X$ recursively by

$$W_0(t, u) = e^{tA} \psi$$

$$W_p(t, u) = \int_0^t e^{(t-s)A} BW_{p-1}(s, u) \psi(s) ds$$

for $p \geq 1$, for every $u$ in $X$. We have the following estimate for the norm of the operator.

Proposition 11: For every $u \in L^1_{loc}([0, +\infty), \mathbb{R})$, $p \in \mathbb{N}$, and $t \geq 0$

$$||W_p(t, u)|| \leq \frac{M^{p+1} e^{(p+1)\omega t} ||B||^{p} ||u(s)|| ds ||}{p!}.$$

Proof: We prove the result by induction on $p$ in $\mathbb{N}$. For $p = 0$ the result clearly follows from Proposition 5. Assume that the result holds for $p \geq 0$. Then, for every $\psi$ in $X$

$$||W_{p+1}(t, u)\psi||$$

$$\leq \left| \int_0^t e^{(t-s)A} BW_{p}(s, u) \psi(s) ds \right|$$

$$\leq \int_0^t \left| e^{(t-s)A} \right| ||B|| ||W_p(s, u)|| ||\psi|| ||u(s)|| ds$$

$$\leq M \int_0^t e^{(t-s)\omega} ||B||^{p+1} e^{(p+1)\omega s} ||u(s)|| ds$$

$$\leq M^{p+2} e^{(p+2)\omega t} ||B||^{p+1} \frac{\int_0^t ||u(s)|| ds ||\psi||}{(p+1)!}.$$
The third inequality follows from Proposition 5 and the induction assumption at rank \( p \). We conclude the proof by induction on \( p \). 

**B. A compactness property**

**Lemma 12:** For every \( j \) in \( \mathbb{N} \), \( T \geq 0 \) and \( K \geq 0 \), and \( \psi_0 \in X \) in the set

\[
W_j^{T,K} = \{ W_j(t,u)\psi_0 \mid 0 \leq t \leq T, \| u \|_{L^1} \leq K \},
\]

(3)

is totally bounded.

**Proof:** We prove the result by induction on \( j \) in \( \mathbb{N} \). For \( j = 0 \), consider \( W_0^{T,K} = \{ e^{tA} \psi_0, 0 \leq t \leq T \} \) and let \( (w_n)_{n \in \mathbb{N}} \) be a sequence in \( W_0^{T,K} \). Then there exists a sequence \( (t_n)_{n \in \mathbb{N}} \) such that \( w_n = e^{t_n A} \psi \) for every \( n \). Up to extraction, \( \lim_{n \to \infty} t_n = t \in [0,T] \), since \( [0,T] \) is compact. By definition of \( C^0 \)-semigroup, \( \lim_{n \to \infty} e^{t_n A} \psi_0 = e^{tA} \psi_0 \).

This proves that \( W_0^{T,K} \) is sequentially compact, hence compact and, in particular, totally bounded (Proposition 6).

Assume that, for \( j \geq 0 \), \( W_j^{T,K} \) is totally bounded. By Proposition 9, the set

\[
Z_j^{T,K} := \{ e^{(t-s)A} B \psi_0, \psi \in W_j^{T,K}, 0 \leq s \leq t \leq T \}
\]

is totally bounded as well.

Let \( \varepsilon > 0 \) be given and define \( \delta = \frac{\varepsilon}{2K+1} > 0 \). Since \( Z_j^{T,K} \) is totally bounded, there exists a finite family \( (x_i)_{1 \leq i \leq N_j} \) in \( Z_j^{T,K} \) such that

\[
Z_j^{T,K} \subset \bigcup_{i=1}^{N_j} B_X(x_i, \delta).
\]

Let \( (\phi_i)_{1 \leq i \leq N_j} \) be a partition of the unity adapted to the cover \( \bigcup_{i=1}^{N_j} B(x_i, 2\delta) \) of \( Z_j^{T,K} \). Such a partition of the unity exists by Proposition 10, and moreover, for every \( x \) in \( Z_j^{T,K} \), we have

\[
\| x - \sum_{i=1}^{N_j} \phi_i(x) x_i \| \leq 2\delta.
\]

(4)

Applying the inequality (4) with \( x = e^{(t-s)A} BW_j(s,u)\psi_0 \), we get, for every \( u \) in \( L^1_{loc}([0, +\infty), \mathbb{R}) \) and every \( (s,t) \) such that \( 0 \leq s \leq t \),

\[
\left\| e^{(t-s)A} BW_j(s,u)\psi_0 \right\|
\]

\[
- \sum_{i=1}^{N_j} \phi_i e^{(t-s)A} BW_j(s,u)\psi_0 x_i \leq 2\delta.
\]

Multiplying by \( u(s) \) and integrating for \( s \) in \( [0,t] \), one gets, for every \( u \) such that \( \int_0^t |u| \leq K \),

\[
\left\| \int_0^t e^{(t-s)A} BW_j(s,u)\psi_0 u(s)ds - \sum_{i=1}^{N_j} \phi_i e^{(t-s)A} BW_j(s,u)\psi_0 u(s)ds \right\| \leq 2\delta \int_0^t |u(s)|ds,
\]

that is

\[
\left\| \int_0^t e^{(t-s)A} BW_j(s,u)\psi_0 u(s)ds - \sum_{i=1}^{N_j} \phi_i e^{(t-s)A} BW_j(s,u)\psi_0 u(s)ds \right\| \leq 2\delta \int_0^t |u(s)|ds,
\]

(5)

The set \( \sum_{i=1}^{N_j} [0,K] x_i \) is compact by Proposition 7 and, hence, totally bounded. Then, there exists a finite family \( (y_i)_{1 \leq i \leq N_j} \) such that

\[
\sum_{i=1}^{N_j} [0,K] x_i \subset \bigcup_{i=1}^{N_j} B_X(y_i, \delta).
\]

(6)

From (5) and (6), one deduces that

\[
W_j^{T,K} \subset \bigcup_{i=1}^{N_j} B_X(y_i, (2K+1)\delta) = \bigcup_{i=1}^{N_j} B_X(y_i, \varepsilon).
\]

This proves that \( W_{j+1}^{T,K} \) is totally bounded and concludes the proof.

**C. Convergence of the Dyson expansion**

Let \( T > 0 \), \( \psi_0 \) in \( X \) and \( u \) in \( L^1([0,T], \mathbb{R}) \) be fixed. We define \( C([0,T], X) \) the set of continuous functions from \( [0,T] \) to \( X \). When endowed with the norm

\[
\| \psi \|_{C([0,T], X)} = \sup_{t \in [0,T]} \| \psi(t) \|
\]

the set \( C([0,T], X) \) turns into a Banach space. We define also the mapping

\[
H_T : C([0,T], X) \to C([0,T], X)
\]

by

\[
H_T(\psi)(t) = e^{tA} \psi_0 + \int_0^t e^{(t-s)A} u(s) B \psi(s)ds
\]

for every \( t \) in \( [0,T] \) and \( \psi \) in \( C([0,T], X) \).

**Proposition 13:** A function \( \psi \) in \( C([0, +\infty), X) \) is a mild solution of (1) if and only if, for every \( T > 0 \), the restriction of \( \psi \) to \( [0,T] \) is a fixed point of \( H_T \).

**Proof:** This is a rewriting of the Definition 1.

**Proposition 14:** For every \( k \) in \( \mathbb{N} \), \( H_T^k \) is Lipschitz continuous from \( C([0,T], X) \) to itself.

**Proof:** Let \( \psi_1, \psi_2 \) in \( C([0,T], X) \). We prove by induction on \( k \) that, for every \( k \) in \( \mathbb{N} \), for every \( t \) in \( [0,T] \),

\[
\| H_T^k(\psi_1)(t) - H_T^k(\psi_2)(t) \| \leq M^k\epsilon^{kT}\sup_{0 \leq s \leq t} \| \psi_1(s) - \psi_2(s) \|
\]

(7)
For the step $k = 1$, we write, for every $t$ in $[0, T]$,

\[
\|H_T(\psi_1)(t) - H_T(\psi_2)(t)\| = \left\| \int_0^t e^{(t-s)A}u(s)B(\psi_1(s) - \psi_2(s))ds \right\|
\]

\[
\leq \int_0^t \left\| e^{(t-s)A}u(s)B(\psi_1(s) - \psi_2(s)) \right\| ds
\]

\[
\leq M e^{\omega t} \|B\| \left( \int_0^t |u(s)| ds \right) \sup_{0 \leq \tau \leq t} \|\psi_1(\tau) - \psi_2(\tau)\|
\]

and (7) is proved for $k = 1$.

Assume that (7) is proved for every $k$ in $N$. Then, for every $t$ in $[0, T]$,

\[
\|H_T^{k+1}(\psi_1)(t) - H_T^{k+1}(\psi_2)(t)\| = \left\| \int_0^t e^{(t-s)A}u(s)BH_T^k(\psi_1(s) - \psi_2(s))ds \right\|
\]

\[
\leq \int_0^t \left\| e^{(t-s)A}u(s)BH_T^k(\psi_1(s) - \psi_2(s)) \right\| ds
\]

\[
\leq M e^{\omega t} \|B\| \int_0^t |u(s)| ds \sup_{0 \leq \tau \leq t} \|H_T^k(\psi_1(\tau) - \psi_2(\tau))\|ds
\]

\[
\leq M e^{\omega t} \|B\| \int_0^t |u(s)| ds M^k e^{\omega s} \left( \int_0^s |u(\tau)| d\tau \right)^k
\]

\[
\leq \frac{M^k+1 e^{(k+1)\omega t}}{(k+1)!} \|B\| \sup_{0 \leq \tau \leq t} |\psi_1(\tau) - \psi_2(\tau)| ds
\]

This finishes the proof of (7) for $k + 1$. Proposition 14 follows by noting that $\sup_{0 \leq \tau \leq t} |\psi_1(\tau) - \psi_2(\tau)| \leq |\psi_1 - \psi_2|_{C([0,T],X)}$.

**Proposition 15:** For every $u$ in $L_{\text{loc}}^1([0, +\infty), \mathbb{R})$, $T \geq 0$, $\psi_0$ in $X$ and $p$ in $N \cap \{0\}$,

\[
H_T \left( t \mapsto \sum_{j=0}^p W_j(t, 0)\psi_0 \right) = \left( t \mapsto \sum_{j=0}^{p+1} W_j(t, 0)\psi_0 \right).
\]

**Proof:** The proof goes by induction on $p$.

For $p = 1$, we write, for every $t$ in $[0, T]$,

\[
H_T(t \mapsto e^{tA}) = e^{tA}\psi_0 + \int_0^t e^{(t-s)A}u(s)Be^{sA}\psi_0 ds
\]

\[
= W_0(t, u)\psi_0 + W_1(t, u)\psi_0.
\]

Assume the result is true for some integer $p$. Then, we write, for every $t$ in $[0, T]$,

\[
H_T \left( t \mapsto \sum_{j=0}^p W_j(t, 0)\psi_0 \right) = e^{tA}\psi_0 + \int_0^t e^{(t-s)A}u(s)B\sum_{j=0}^p W_j(t, 0)\psi_0 ds
\]

\[
= W_0(t, u)\psi_0 + \sum_{j=0}^p W_{j+1}(t, u)\psi_0
\]

and the result is true also at rank $p + 1$.

**Proposition 16:** For every $u$ in $L_{\text{loc}}^1([0, +\infty), \mathbb{R})$, $T \geq 0$, and $\psi_0$ in $X$,

\[
\Upsilon_{\cdot,0}^u \psi_0 \colon [0, +\infty) \to X \quad t \mapsto \sum_{p=0}^\infty W_p(t, u)\psi_0,
\]

is the unique mild solution of (1).

**Proof:** Let $T > 0$. From Proposition 13, the restriction of any mild solution of (1) to $[0, T]$ is a fixed point of $H_T$. From Proposition 14, there exists $k$ in $N$ large enough such that $H_T^k$ is a contraction from $C([0,T],X)$ to itself. Hence $H_T^k$ admits one and only one fixed point in $C([0,T],X)$. Since every fixed point of $H_T$ is also a fixed point of $H_T^k$, the mapping $H_T$ has at most one fixed point in $C([0,T],X)$. Letting $T$ vary from 0 to $\infty$, this means that (1) admits at most one mild solution.

From Proposition 11, for every $t$ in $[0, T]$, the sum $\sum_{j=0}^\infty W_j(t, 0)\psi_0$ converges in $X$ to a limit we note $\Upsilon_{t,0}^u\psi_0$. The aim of the rest of the proof is to show that $t \mapsto \Upsilon_{t,0}^u\psi_0$ is a mild solution of (1).

Let $T > 0$. The convergence of the sum $\sum_{j=0}^\infty W_j(t, 0)\psi_0$ being uniform with respect to $t$ in $[0, T]$, the function $t \mapsto \Upsilon_{t,0}^u\psi_0$ is continuous. From Proposition 15, for every $t$ in $[0, T]$, for every $p$ in $N$,

\[
H_T \left( t \mapsto \sum_{j=0}^p W_j(t, u)\psi_0 \right) = \left( t \mapsto \sum_{j=0}^{p+1} W_j(t, u)\psi_0 \right).
\]

As $p$ tends to infinity, $\sum_{j=0}^p W_j(t, u)\psi_0$ tends to $\Upsilon_{t,0}^u\psi_0$. Since $H_T$ is Lipschitz continuous (Proposition 15 with $k = 1$), $H_T(t \mapsto \sum_{j=0}^p W_j(t, u)\psi_0)$ tends to $H_T(t \mapsto \Upsilon_{t,0}^u\psi_0)$ as $p$ tends to infinity. In other words, the function $t \mapsto \sum_{j=0}^\infty W_j(t, 0)\psi_0$ is a fixed point of $H_T$ (the only one from what precedes). In conclusion, $t \mapsto \Upsilon_{t,0}^u\psi_0$ is the unique mild solution of (1).

**IV. PROOF OF THEOREM 2**

We proceed now to the proof of Theorem 2. First of all, notice that, for every $\psi_0$ in $X$,

\[
A(\psi_0, L^1_{\text{loc}}([0, +\infty), \mathbb{R})) = \bigcup_{l \in \mathbb{N}} \bigcup_{m \in \mathbb{N}} \{\Upsilon_{l,m}^u\psi_0, \|u\|_{L^1} \leq l, 0 \leq t \leq m\},
\]
and it is enough to prove that, for every $l$ and $m$ in $\mathbb{N}$, the set
\[ \{ \Upsilon_t^u \psi_0, \|u\|_{L^1} \leq l, 0 \leq t \leq m \} \]
is totally bounded.

Let $\varepsilon > 0$. From the convergence of the Dyson expansion (Proposition 16) and the bound on the operators $W_j$ (Proposition 11), there exists an integer $N_\varepsilon$ such that
\[ \left\| \sum_{p \geq N_\varepsilon} W_p(t, u) \psi_0 \right\| \leq \frac{\varepsilon}{2}, \tag{8} \]
for every $t$ in $[0, m]$ and every $u$ such that $\|u\|_{L^1} \leq l$. For each $j = 1, \ldots, N_\varepsilon$ the sets $W_j^{m,l}$, defined by (3), are totally bounded (Lemma 12), hence their sum
\[ \sum_{j=0}^{N_\varepsilon} W_j^{m,l} \]
is totally bounded as well (Proposition 8). Hence there exists a family $(x_i)_{1 \leq i \leq N_1}$ of points of $\sum_{j=0}^{N_\varepsilon} W_j^{m,l}$ such that
\[ \sum_{j=0}^{N_\varepsilon} W_j^{m,l} \subset \bigcup_{i=1}^{N_1} B_X(x_i, \frac{\varepsilon}{2}). \tag{9} \]
Gathering (8) and (9), one gets
\[ \sum_{j=0}^{\infty} W_j^{m,l} \subset \bigcup_{i=1}^{N_1} B_X(x_i, \varepsilon), \]
which concludes the proof of Theorem 2.

V. ACKNOWLEDGMENTS

This work has been supported by the project DISQUO of the DEFI InFlniTI 2017 by CNRS and the QUACO project by French Agence Nationale de la Recherche, contract 17-CE40-0007-01.

REFERENCES