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A new lifetime model by mixing gamma and geometric distributions with fitting hydrologic data

Hassan S. Bakouch^{a,*}, Abdus Saboor^b, Christophe Chesneau^c, Anwaar Saeed^b

^a*Department of Mathematics, Faculty of Science, Tanta University, Tanta, Egypt*

^b*Department of Mathematics, Kohat University of Science & Technology, Kohat, Pakistan*

^c*LMNO, University of Caen, France*

Abstract

In this paper, we propose a new distribution obtained by mixing gamma and geometric distributions. We discuss different shapes of the probability density function and the hazard rate functions. We study several statistical properties. The maximum likelihood estimation method is performed for estimating the parameters. We determine the observed information matrix and discuss inference. Illustrative three hydrologic data sets are given to show the flexibility and potentiality of the proposed distribution.

Keywords: Gamma distribution, Geometric distribution, Order statistics, Record values, Estimation.

Mathematics Subject Classification: MSC 2010: 2E10, 62E15, 62F10.

[☆]Corresponding author

Email addresses: hnbakouch@yahoo.com (Hassan S. Bakouch), saboorhangu@gmail.com;
dr.abdussaboor@kust.edu.pk (Abdus Saboor), christophe.chesneau@unicaen.fr (Christophe Chesneau),
Anwaarsaeedqureshi@gmail.com (Anwaar Saeed)

1. Introduction

An important aspect of statistics is the determination of flexible distributions to elaborate useful models for lifetime data. Among the existing approaches, new distributions can be obtained by mixing discrete and continuous distributions. Those using geometric distributions include the exponential geometric distribution [1], the exponential-power series distribution [9], the extended exponential geometric distribution [2], the complementary exponential geometric distribution [14], the Weibull-geometric distribution [6], the Weibull-power series distribution [18], the generalized exponential-power series distribution [16], the complementary exponentiated exponential geometric distribution [15], the extended Weibull-power series distribution [21], the complementary extended Weibull-power series [11], the exponentiated extended Weibull-power series distribution [23], the G-geometric distribution [3], the alternative G-geometric distribution [8], the generalized linear failure rate-geometric distribution [12] and the linear failure rate-power series distribution [17]. We also refer to the review of [22], and the references therein.

On the other hand, among the continuous distributions, the gamma distribution is one of the most commonly used in modeling life-time data. In practice, it has been shown to be very flexible in modeling various types of lifetime distributions. To the best of our knowledge, the mixing of the geometric distribution with the Gamma distribution (not reduced to the exponential one) has not ever been considered in the literature. Based on such a mixing, this paper offers a new distribution with two parameters, called the gamma-geometric (GG) distribution. The formulation and motivations of such distribution are as follows.

Let $\lambda > 0$, $\theta \in (0, 1)$ and $\bar{\theta} = 1 - \theta$. We say that a random variable X follows the GG distribution with parameters (θ, λ) , denoted by $GG(\theta, \lambda)$, if it has the probability density function (pdf) given by

$$f(x) = \theta \lambda^2 x e^{-\lambda x} \frac{1 + \bar{\theta} e^{-\lambda x}}{(1 - \bar{\theta} e^{-\lambda x})^3}, \quad x > 0. \quad (1)$$

The corresponding cumulative distribution function (cdf) is given by

$$F(x) = 1 - \theta e^{-\lambda x} \frac{1 + \lambda x - \bar{\theta} e^{-\lambda x}}{(1 - \bar{\theta} e^{-\lambda x})^2}, \quad x > 0. \quad (2)$$

The GG distribution arises from the following stochastic representation. Let X be a random variable having the following stochastic representation:

$$X \mid \{N = n\} \sim \mathcal{G}_{am}(2, \lambda n), \quad N \mid \theta \sim G_{trunc}(\theta), \quad (3)$$

that is N is a random variable having the truncated geometric distribution with parameter θ : $P(N = n) = \theta \bar{\theta}^{n-1}$, $n = 1, 2, \dots$ and the distribution of X conditionally to $\{N = n\}$ is the gamma distribution $\mathcal{G}_{am}(2, \lambda n)$, with a conditional pdf given by $f_{X|\{N=n\}}(x) = \lambda^2 n^2 x e^{-\lambda n x}$, $x > 0$. Then X follows the $GG(\theta, \lambda)$ distribution; using the geometric series expansions: $\sum_{n=1}^{+\infty} n^2 x^n = \frac{x(1+x)}{(1-x)^3}$, $|x| < 1$, the pdf of X is given by

$$f_X(x) = \sum_{n=1}^{+\infty} f_{X|\{N=n\}}(x) P(N = n) = \frac{\theta}{\bar{\theta}} \lambda^2 x \sum_{n=1}^{+\infty} n^2 (\bar{\theta} e^{-\lambda x})^n = \theta \lambda^2 x e^{-\lambda x} \frac{1 + \bar{\theta} e^{-\lambda x}}{(1 - \bar{\theta} e^{-\lambda x})^3}. \quad (4)$$

The stochastic representation (3) can be viewed as a natural extension of the stochastic representation $X \mid \{N = n\} \sim \mathcal{E}_{xp}(\lambda n) = \mathcal{G}_{am}(1, \lambda n)$, with pdf corresponding to the one of the G-geometric class proposed by [3] (applied with the exponential distribution).

Ratio of two independent variables. An example of simple model using the GG distribution is given by the ratio of two independent variables as described as follows. Let $Y \sim \mathcal{G}_{am}(2, \lambda)$ and $N \sim G_{trunc}(\theta)$. Suppose that Y and N are independent. Then

$$X = \frac{Y}{N},$$

follows the $GG(\theta, \lambda)$ distribution. It is enough to note that $X | \{N = n\} = Y/n \sim \mathcal{G}_{am}(2, \lambda n)$. This ratio representation will be useful to determine statistical properties of the GG distribution.

Note that, from the ratio representation, the cdf of X given by (2) can be expressed directly: using the cdf of Y : $F_Y(x) = 1 - e^{-\lambda x} - \lambda x e^{-\lambda x}$, $x > 0$, and the geometric series expansions: $\sum_{n=0}^{+\infty} x^n = \frac{1}{1-x}$ and $\sum_{n=1}^{+\infty} n x^{n-1} = \frac{1}{(1-x)^2}$, $|x| < 1$, the cdf of X is given by

$$\begin{aligned} F_X(x) &= P(Y \leq xN) = \sum_{n=1}^{\infty} F_Y(xn)P(N = n) \\ &= 1 - \theta e^{-\lambda x} \sum_{n=0}^{\infty} (\bar{\theta} e^{-\lambda x})^n - \lambda x \theta e^{-\lambda x} \sum_{n=1}^{\infty} n (\bar{\theta} e^{-\lambda x})^{n-1} \\ &= 1 - \theta e^{-\lambda x} \frac{1 + \lambda x - \bar{\theta} e^{-\lambda x}}{(1 - \bar{\theta} e^{-\lambda x})^2}. \end{aligned}$$

Some limit properties for $f(x)$ are given as:

$$f(x) \sim \lambda^2 \frac{2-\theta}{\theta^2} x \rightarrow 0, \quad x \rightarrow 0, \quad f(x) \sim \theta \lambda^2 x e^{-\lambda x} \rightarrow 0, \quad x \rightarrow +\infty.$$

Moreover, one can show that $f(x)$ has a unique maximum on $(0, +\infty)$ given by $f(x_*)$ where x_* satisfies the equation: $4\theta \lambda x_* e^{\lambda x_*} + (\lambda x_* - 1)e^{2\lambda x_*} + \theta^2(\lambda x_* + 1) = 0$. Some plots of $f(x)$ are given in Figure 1 for several values of (θ, λ) . The rest of the paper is organized as follows. In Section 2, we give some

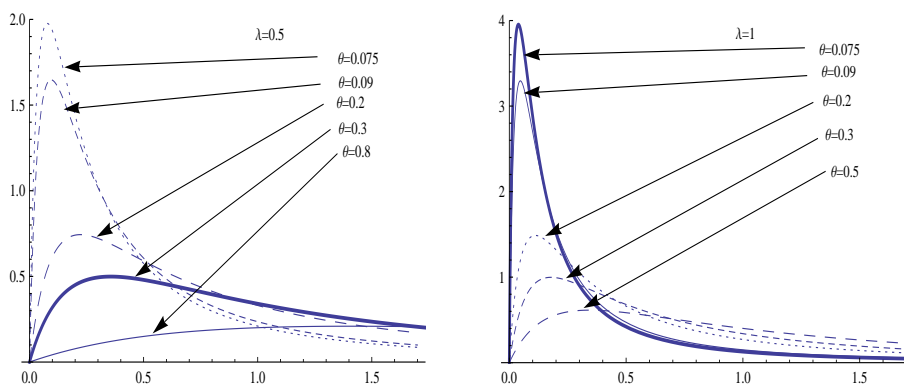


Figure 1: Plots of the GG density function

properties of the GG distribution. The estimation by maximum likelihood is discussed in Section 3 with illustrative real data examples.

2. Properties of the GG distribution

In this section, we propose many features and statistical properties of the GG distribution.

2.1. The survival function and hazard rate functions

The survival function (sf) of X is given by

$$S(x) = 1 - F(x) = \theta e^{-\lambda x} \frac{1 + \lambda x - \bar{\theta} e^{-\lambda x}}{(1 - \bar{\theta} e^{-\lambda x})^2}, \quad x > 0, \quad (5)$$

and the associated hazard rate function (hrf) of X is

$$h(x) = \frac{f(x)}{S(x)} = \lambda^2 x \frac{1 + \bar{\theta}e^{-\lambda x}}{(1 - \bar{\theta}e^{-\lambda x})(1 + \lambda x - \bar{\theta}e^{-\lambda x})}, \quad x > 0. \quad (6)$$

Observe that

$$h(x) \sim \lambda^2 \frac{2 - \theta}{\theta^2} x \rightarrow 0, \quad x \rightarrow 0, \quad h(x) \sim \lambda, \quad x \rightarrow +\infty.$$

Some plots of $h(x)$ are given in Figure (2) for several values of (θ, λ) .

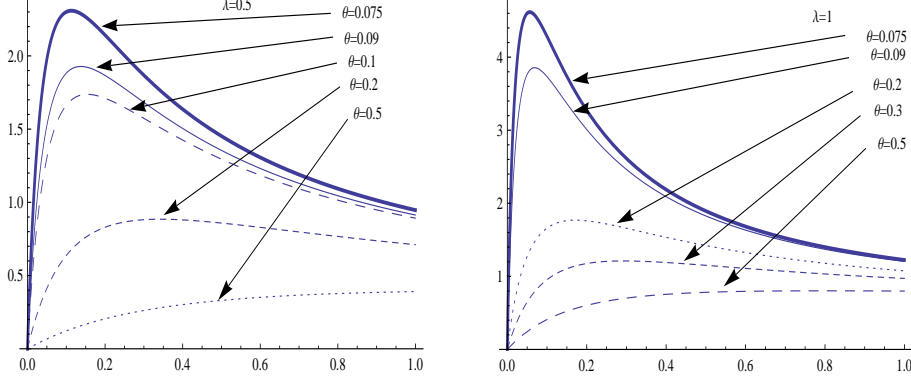


Figure 2: Plots of the GG hazard rate function

2.2. Quantile function

The quantile function of X is determined by inverting the cdf $F(x)$. The p -th quantile x_p of X is the real solution of the nonlinear equation:

$$F(x_p) = p \Leftrightarrow \theta e^{-\lambda x_p} (1 + \lambda x_p - \bar{\theta} e^{-\lambda x_p}) = (1 - p)(1 - \bar{\theta} e^{-\lambda x_p})^2.$$

2.3. Moments

Some key features of a distribution, like mean and variance, can be investigated through its r -th moments $E(X^r)$. For finding $E(X^r)$, we can use the ratio representation of X : X has the same distribution of the ratio of 2 random variables: Y/N with $Y \sim \mathcal{G}_{am}(2, \lambda)$ and $N \sim G_{trunc}(\theta)$, Y and N independent. Therefore, considering the Gamma function: $\Gamma(\nu) = \int_0^{+\infty} x^{\nu-1} e^{-x} dx$, $\nu > 0$, $\mathbb{E}(Y^r) = \frac{\Gamma(2+r)}{\lambda^r}$ and the polylogarithm function: $\text{Li}_r(x) = \sum_{n=1}^{+\infty} \frac{x^n}{n^r}$, $r > 0$, $|x| < 1$, we have

$$E(X^r) = \mathbb{E}\left(\frac{Y^r}{N^r}\right) = \mathbb{E}(Y^r) \mathbb{E}\left(\frac{1}{N^r}\right) = \frac{\Gamma(2+r)}{\lambda^r} \sum_{n=1}^{+\infty} \frac{1}{n^r} P(N=n) = \frac{\Gamma(2+r)}{\lambda^r} \frac{\theta}{\bar{\theta}} \text{Li}_r(\bar{\theta}).$$

In particular, by taking $r = 1$, since $\text{Li}_1(x) = -\log(1-x)$, we obtain

$$E(X) = -\frac{2\theta}{\lambda\bar{\theta}} \log(\theta).$$

The variance of X can be explicit in some cases. For instance, if $\theta = \frac{1}{2}$, since $\text{Li}_1(\frac{1}{2}) = \log(2)$ and $\text{Li}_2(\frac{1}{2}) = \frac{1}{12}[\pi^2 - 6(\log(2))^2]$, we have

$$\text{Var}(X) = E(X^2) - [E(X)]^2 = \frac{\Gamma(4)}{\lambda^2} \text{Li}_2\left(\frac{1}{2}\right) - \left[\frac{\Gamma(3)}{\lambda} \text{Li}_1\left(\frac{1}{2}\right)\right]^2 = \frac{1}{\lambda^2} \left(\frac{\pi^2}{2} - 7(\log(2))^2\right).$$

2.4. Moment generating function

As for the moments, the moment generating function of X can be obtain via the ratio representation Y/N . Using $E(e^{tY}) = \frac{\lambda^2}{(\lambda - t)^2}$, $t < \lambda$, and the conditional expectation, we get

$$\begin{aligned} M(t) &= E(e^{tX}) = E(e^{t\frac{Y}{N}}) = E\left(E\left(e^{t\frac{Y}{N}} \mid N\right)\right) = E\left(\frac{\lambda^2}{\left(\lambda - \frac{t}{N}\right)^2}\right) = \lambda^2 E\left(\frac{N^2}{(\lambda N - t)^2}\right) \\ &= \lambda^2 \sum_{n=1}^{+\infty} \frac{n^2}{(\lambda n - t)^2} P(N = n) = \lambda^2 \theta \sum_{n=1}^{+\infty} \frac{n^2}{(\lambda n - t)^2} \bar{\theta}^{n-1}. \end{aligned}$$

2.5. Conditional and reversed moments

The r -th conditional moments of X is given by

$$E(X^r \mid X > t) = \frac{1}{S(t)} \left(E(X^r) - \int_0^t x^r f(x) dx \right), \quad t > 0,$$

and the r -th reversed moments of X is given by

$$E(X^r \mid X \leq t) = \frac{1}{F(t)} \int_0^t x^r f(x) dx, \quad t > 0.$$

The integral term can be expressed using the expansion (4). Indeed, introducing the lower incomplete gamma function $\Gamma(t, \nu) = \int_0^t x^{\nu-1} e^{-x} dx$, $\nu > 0$, $t > 0$, we have

$$\int_0^t x^r f(x) dx = \frac{\theta}{\bar{\theta}} \lambda^2 \sum_{n=1}^{+\infty} n^2 \bar{\theta}^n \int_0^t x^{r+1} e^{-n\lambda x} dx = \frac{\theta}{\bar{\theta}} \frac{1}{\lambda^r} \sum_{n=1}^{+\infty} \frac{\bar{\theta}^n}{n^r} \Gamma(n\lambda t, r+2). \quad (7)$$

2.6. Rényi entropy

An entropy plays a central role in information theory. It provides a suitable measure of randomness or uncertainty of X . For continuous distributions, Rényi entropy (see [20]) can be determined as follows:

$$I_R(\gamma) = \frac{1}{1-\gamma} \log \left(\int_{-\infty}^{+\infty} [f(x)]^\gamma dx \right), \quad \gamma > 0, \gamma \neq 1.$$

We have

$$[f(x)]^\gamma = \theta^\gamma \lambda^{2\gamma} x^\gamma e^{-\lambda\gamma x} \frac{(1 + \bar{\theta}e^{-\lambda x})^\gamma}{(1 - \bar{\theta}e^{-\lambda x})^{3\gamma}}.$$

Using the generalized binomial series: $(1+x)^\alpha = \sum_{k=0}^{+\infty} \binom{\alpha}{k} x^k$, $\alpha \in \mathbb{C}$, $|x| < 1$, $\binom{\alpha}{k} = \frac{\alpha(\alpha-1)(\alpha-2)\dots(\alpha-k+1)}{k!}$, we have

$$(1 + \bar{\theta}e^{-\lambda x})^\gamma = \sum_{k=0}^{+\infty} \binom{\gamma}{k} \bar{\theta}^k e^{-\lambda k x}, \quad \frac{1}{(1 - \bar{\theta}e^{-\lambda x})^{3\gamma}} = \sum_{\ell=0}^{+\infty} \binom{-3\gamma}{\ell} (-1)^\ell \bar{\theta}^\ell e^{-\lambda \ell x}.$$

Hence,

$$[f(x)]^\gamma = \theta^\gamma \lambda^{2\gamma} \sum_{\ell=0}^{+\infty} \sum_{k=0}^{+\infty} \binom{-3\gamma}{\ell} \binom{\gamma}{k} (-1)^\ell \bar{\theta}^{\ell+k} x^\gamma e^{-\lambda(\ell+k+\gamma)x}.$$

Therefore,

$$\begin{aligned} I_R(\gamma) &= \frac{1}{1-\gamma} \left[\gamma \log(\theta) + 2\gamma \log(\lambda) + \log \left(\sum_{\ell=0}^{+\infty} \sum_{k=0}^{+\infty} \binom{-3\gamma}{\ell} \binom{\gamma}{k} (-1)^\ell \bar{\theta}^{\ell+k} \int_0^{+\infty} x^\gamma e^{-\lambda(\ell+k+\gamma)x} dx \right) \right] \\ &= \frac{1}{1-\gamma} \left[\gamma \log(\theta) + 2\gamma \log(\lambda) + \log \left(\sum_{\ell=0}^{+\infty} \sum_{k=0}^{+\infty} \binom{-3\gamma}{\ell} \binom{\gamma}{k} (-1)^\ell \bar{\theta}^{\ell+k} \frac{\Gamma(\gamma+1)}{\lambda^{\gamma+1}(\ell+k+\gamma)^{\gamma+1}} \right) \right]. \end{aligned}$$

2.7. Order statistics distributions

The order statistics are central tools in non-parametric statistics and inference. Let us now present the distributions of some fundamental order statistics related to the $GG(\theta, \lambda)$ distribution. Let a sample X_1, X_2, \dots, X_n is randomly chosen from the $GG(\theta, \lambda)$ distribution and $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ are its corresponding order statistics. A pdf of $X_{i:n}$ is given by

$$\begin{aligned} f_{X_{i:n}}(x) &= \frac{n!}{(i-1)!(n-i)!} f(x) \sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^l [F(x)]^{i-1+l} \\ &= \frac{n!}{(i-1)!(n-i)!} \theta \lambda^2 x e^{-\lambda x} \frac{1 + \bar{\theta} e^{-\lambda x}}{(1 - \bar{\theta} e^{-\lambda x})^3} \sum_{l=0}^{n-i} \binom{n-i}{l} (-1)^l \left[1 - \theta e^{-\lambda x} \frac{1 + \lambda x - \bar{\theta} e^{-\lambda x}}{(1 - \bar{\theta} e^{-\lambda x})^2} \right]^{i-1+l}, \\ & \quad x > 0 \end{aligned}$$

The cdf of $X_{i:n}$ is given by

$$\begin{aligned} F_{X_{i:n}}(x) &= \frac{n!}{(i-1)!(n-i)!} \sum_{l=0}^{n-i} \binom{n-i}{l} \frac{(-1)^l}{i+l} [F(x)]^{i+l} \\ &= \frac{n!}{(i-1)!(n-i)!} \sum_{l=0}^{n-i} \binom{n-i}{l} \frac{(-1)^l}{i+l} \left[1 - \theta e^{-\lambda x} \frac{1 + \lambda x - \bar{\theta} e^{-\lambda x}}{(1 - \bar{\theta} e^{-\lambda x})^2} \right]^{i+l}, \quad x > 0. \end{aligned}$$

A joint pdf of $(X_{1:n}, \dots, X_{n:n})$ is given by

$$\begin{aligned} f_{(X_{1:n}, \dots, X_{n:n})}(x_1, \dots, x_n) &= n! \prod_{k=1}^n f(x_k) = n! \theta^n \lambda^{2n} \left(\prod_{k=1}^n x_k \right) e^{-\lambda \sum_{k=1}^n x_k} \frac{\prod_{k=1}^n (1 + \bar{\theta} e^{-\lambda x_k})}{\prod_{k=1}^n (1 - \bar{\theta} e^{-\lambda x_k})^3}, \\ & \quad 0 < x_1 < \dots < x_n. \end{aligned}$$

A joint pdf of $(X_{i:n}, X_{j:n})$, $i < j$, is given by

$$\begin{aligned} f_{(X_{i:n}, X_{j:n})}(x_i, x_j) &= \frac{n!}{(i-1)!(n-j)!(j-i-1)!} [F(x_i)]^{i-1} [F(x_j) - F(x_i)]^{j-i-1} [S(x_j)]^{n-j} f(x_i) f(x_j) \\ &= \frac{n!}{(i-1)!(n-j)!(j-i-1)!} \left[1 - \theta e^{-\lambda x_i} \frac{1 + \lambda x_i - \bar{\theta} e^{-\lambda x_i}}{(1 - \bar{\theta} e^{-\lambda x_i})^2} \right]^{i-1} \\ &\quad \times \left[\theta e^{-\lambda x_i} \frac{1 + \lambda x_i - \bar{\theta} e^{-\lambda x_i}}{(1 - \bar{\theta} e^{-\lambda x_i})^2} - \theta e^{-\lambda x_j} \frac{1 + \lambda x_j - \bar{\theta} e^{-\lambda x_j}}{(1 - \bar{\theta} e^{-\lambda x_j})^2} \right]^{j-i-1} \\ &\quad \times \left[\theta e^{-\lambda x_j} \frac{1 + \lambda x_j - \bar{\theta} e^{-\lambda x_j}}{(1 - \bar{\theta} e^{-\lambda x_j})^2} \right]^{n-j} \theta^2 \lambda^4 x_i x_j e^{-\lambda(x_i+x_j)} \frac{1 + \bar{\theta} e^{-\lambda x_i}}{(1 - \bar{\theta} e^{-\lambda x_i})^3} \frac{1 + \bar{\theta} e^{-\lambda x_j}}{(1 - \bar{\theta} e^{-\lambda x_j})^3}, \\ & \quad 0 < x_i < x_j. \end{aligned}$$

2.8. Record values distributions

Record values arise in a wide variety of real-life applications as hydrology, industry, lifetesting, economics, among the others. See, for instance, [4], [5] and [10]. We now present important distributions related to record values using the $GG(\theta, \lambda)$ distribution as baseline. Let X_1, X_2, \dots , be a sequence of i.i.d. random variables having the $GG(\theta, \lambda)$ distribution. We define a sequence of record times $U(n)$ as follows: $U(1) = 1$, $U(n) = \min\{j; j > U(n-1), X_j > X_{U(n-1)}\}$ for $n \geq 2$. We define the i -th upper record value by $R_i = X_{U(i)}$, with $R_1 = X_1$. A pdf of R_i is given by

$$\begin{aligned} f_{R_i}(x) &= \frac{1}{(i-1)!} [-\log(S(x))]^{i-1} f(x) \\ &= \frac{1}{(i-1)!} \left[-\log \left(\theta e^{-\lambda x} \frac{1 + \lambda x - \bar{\theta} e^{-\lambda x}}{(1 - \bar{\theta} e^{-\lambda x})^2} \right) \right]^{i-1} \theta \lambda^2 x e^{-\lambda x} \frac{1 + \bar{\theta} e^{-\lambda x}}{(1 - \bar{\theta} e^{-\lambda x})^3}, \quad x > 0. \end{aligned}$$

A joint pdf of (R_1, \dots, R_n) is given by

$$f_{(R_1, \dots, R_n)}(x_1, \dots, x_n) = f(x_n) \prod_{k=1}^{n-1} h(x_k) = \theta \lambda^{2n} x_n e^{-\lambda x_n} \frac{1 + \bar{\theta} e^{-\lambda x_n}}{(1 - \bar{\theta} e^{-\lambda x_n})^3} \\ \times \left(\prod_{k=1}^{n-1} x_k \right) \frac{\prod_{k=1}^{n-1} (1 + \bar{\theta} e^{-\lambda x_k})}{\prod_{k=1}^{n-1} (1 - \bar{\theta} e^{-\lambda x_k})(1 + \lambda x_k - \bar{\theta} e^{-\lambda x_k})}, \quad 0 < x_1 < \dots < x_n.$$

A joint pdf of (R_i, R_j) , $i < j$, is given by

$$f_{(R_i, R_j)}(x_i, x_j) = \frac{1}{(i-1)!(j-i-1)!} [-\log(S(x_i))]^{i-1} [\log(S(x_i)) - \log(S(x_j))]^{j-i-1} h(x_i) f(x_j) \\ = \frac{1}{(i-1)!(j-i-1)!} \left[-\log \left(\theta e^{-\lambda x_i} \frac{1 + \lambda x_i - \bar{\theta} e^{-\lambda x_i}}{(1 - \bar{\theta} e^{-\lambda x_i})^2} \right) \right]^{i-1} \\ \times \left[\log \left(\theta e^{-\lambda x_i} \frac{1 + \lambda x_i - \bar{\theta} e^{-\lambda x_i}}{(1 - \bar{\theta} e^{-\lambda x_i})^2} \right) - \log \left(\theta e^{-\lambda x_j} \frac{1 + \lambda x_j - \bar{\theta} e^{-\lambda x_j}}{(1 - \bar{\theta} e^{-\lambda x_j})^2} \right) \right]^{j-i-1} \\ \times \lambda^2 x_i \frac{1 + \bar{\theta} e^{-\lambda x_i}}{(1 - \bar{\theta} e^{-\lambda x_i})(1 + \lambda x_i - \bar{\theta} e^{-\lambda x_i})} \times \theta \lambda^2 x_j e^{-\lambda x_j} \frac{1 + \bar{\theta} e^{-\lambda x_j}}{(1 - \bar{\theta} e^{-\lambda x_j})^3}, \quad 0 < x_i < x_j.$$

2.9. Residuals life functions

The residual life functions play a fundamental role in survival or reliability studies. See, for instance, [7], [13] and [19]. We now present some related mathematical objects with a potential of interest in the context of the GG(θ, λ) distribution.

The residual life is described by the conditional random variable $R_{(t)} = X - t \mid X > t$, $t \geq 0$. The sf of the residual lifetime $R_{(t)}$ is given by

$$S_{R_{(t)}}(x) = \frac{S(x+t)}{S(t)} = e^{-\lambda x} \frac{(1 + \lambda(x+t) - \bar{\theta} e^{-\lambda(x+t)})(1 - \bar{\theta} e^{-\lambda t})^2}{(1 - \bar{\theta} e^{-\lambda(x+t)})^2 (1 + \lambda t - \bar{\theta} e^{-\lambda t})}, \quad x > 0.$$

The associated cdf is given by

$$F_{R_{(t)}}(x) = 1 - e^{-\lambda x} \frac{(1 + \lambda(x+t) - \bar{\theta} e^{-\lambda(x+t)})(1 - \bar{\theta} e^{-\lambda t})^2}{(1 - \bar{\theta} e^{-\lambda(x+t)})^2 (1 + \lambda t - \bar{\theta} e^{-\lambda t})}, \quad x > 0.$$

Then, the corresponding pdf is given by

$$f_{R_{(t)}}(x) = \lambda^2 (x+t) e^{-\lambda x} \frac{(1 + \bar{\theta} e^{-\lambda(x+t)})(1 - \bar{\theta} e^{-\lambda t})^2}{(1 - \bar{\theta} e^{-\lambda(x+t)})^3 (1 + \lambda t - \bar{\theta} e^{-\lambda t})}, \quad x > 0.$$

The associated hrf is given by

$$h_{R_{(t)}}(x) = \lambda^2 (x+t) \frac{1 + \bar{\theta} e^{-\lambda(x+t)}}{(1 - \bar{\theta} e^{-\lambda(x+t)}) (1 + \lambda(x+t) - \bar{\theta} e^{-\lambda(x+t)})} \quad x > 0,$$

and the mean residual life is defined as

$$K(t) = E(R_{(t)}) = E(X - t \mid X > t) = \frac{1}{S(t)} \left(E(X) - \int_0^t x f(x) dx \right) - t.$$

The integral term can be expressed as (7) with $r = 1$.

On the other hand, the variance residual life is given by

$$V(t) = Var(R_{(t)}) = Var(X - t \mid X > t) = \frac{1}{S(t)} \left(E(X^2) - \int_0^t x^2 f(x) dx \right) - t^2 - 2tK(t) - [K(t)]^2.$$

Again, the integral term can be expressed as (7) with $r = 2$.

The reverse residual life is described by the conditional random variable $\bar{R}_{(t)} = t - X \mid X \leq t$, $t \geq 0$. The sf of the reversed residual lifetime $\bar{R}_{(t)}$ is given by

$$S_{\bar{R}_{(t)}}(x) = \frac{F(t-x)}{F(t)} = \frac{(1 - \bar{\theta}e^{-\lambda t})^2 [(1 - \bar{\theta}e^{-\lambda(t-x)})^2 - \theta e^{-\lambda(t-x)}(1 + \lambda(t-x) - \bar{\theta}e^{-\lambda(t-x)})]}{(1 - \bar{\theta}e^{-\lambda(t-x)})^2 [(1 - \bar{\theta}e^{-\lambda t})^2 - \theta e^{-\lambda t}(1 + \lambda t - \bar{\theta}e^{-\lambda t})]}, \quad 0 < x \leq t.$$

The associated cdf is given by

$$F_{\bar{R}_{(t)}}(x) = 1 - \frac{(1 - \bar{\theta}e^{-\lambda t})^2 [(1 - \bar{\theta}e^{-\lambda(t-x)})^2 - \theta e^{-\lambda(t-x)}(1 + \lambda(t-x) - \bar{\theta}e^{-\lambda(t-x)})]}{(1 - \bar{\theta}e^{-\lambda(t-x)})^2 [(1 - \bar{\theta}e^{-\lambda t})^2 - \theta e^{-\lambda t}(1 + \lambda t - \bar{\theta}e^{-\lambda t})]}, \quad 0 < x \leq t.$$

Therefore, the corresponding pdf is given by

$$f_{\bar{R}_{(t)}}(x) = \frac{\theta \lambda^2 (t-x) e^{-\lambda(t-x)} (1 + \bar{\theta}e^{-\lambda(t-x)}) (1 - \bar{\theta}e^{-\lambda t})^2}{(1 - \bar{\theta}e^{-\lambda(t-x)})^3 [(1 - \bar{\theta}e^{-\lambda t})^2 - \theta e^{-\lambda t}(1 + \lambda t - \bar{\theta}e^{-\lambda t})]}, \quad 0 < x \leq t,$$

and the associated hrf is given by

$$h_{\bar{R}_{(t)}}(x) = \frac{\theta \lambda^2 (t-x) e^{-\lambda(t-x)} (1 + \bar{\theta}e^{-\lambda(t-x)}) (1 - \bar{\theta}e^{-\lambda(t-x)})^2}{(1 - \bar{\theta}e^{-\lambda(t-x)})^3 [(1 - \bar{\theta}e^{-\lambda(t-x)})^2 - \theta e^{-\lambda(t-x)}(1 + \lambda(t-x) - \bar{\theta}e^{-\lambda(t-x)})]}, \quad 0 < x \leq t.$$

The mean reversed residual life is defined as

$$L(t) = E(\bar{R}_{(t)}) = E(t - X \mid X \leq t) = t - \frac{1}{F(t)} \int_0^t x f(x) dx.$$

The integral term can be expressed as (7) with $r = 1$.

The variance reversed residual life is given by

$$W(t) = Var(\bar{R}_{(t)}) = Var(t - X \mid X \leq t) = 2tL(t) - [L(t)]^2 - t^2 + \frac{1}{F(t)} \int_0^t x^2 f(x) dx,$$

Again, the last integral can be expressed as (7) with $r = 2$.

3. Estimation with hydrologic data examples

In this section, we estimate the unknown parameters of the GG distribution using the method of maximum likelihood. Moreover, three hydrologic data sets are given to show the flexibility and potentiality of the proposed distribution.

3.1. Maximum likelihood estimation

Let X_1, X_2, \dots, X_n be a random sample of size n from the $GG(\theta, \lambda)$ distribution with observed values x_1, x_2, \dots, x_n . Set $\Theta = \{\theta, \lambda\}$. The likelihood function associated to x_1, \dots, x_n is given by

$$L(\Theta) = \prod_{i=1}^n f(x_i) = \prod_{i=1}^n \left(\theta \lambda^2 x_i e^{-\lambda x_i} \frac{1 + \bar{\theta}e^{-\lambda x_i}}{(1 - \bar{\theta}e^{-\lambda x_i})^3} \right) = \theta^n \lambda^{2n} \left(\prod_{i=1}^n x_i \right) e^{-\lambda \sum_{i=1}^n x_i} \frac{\prod_{i=1}^n (e^{\lambda x_i} + 1 - \theta)}{\prod_{i=1}^n (e^{\lambda x_i} - 1 + \theta)^3}.$$

The maximum likelihood estimators (MLEs) of θ and λ are obtained by maximization of $L(\Theta)$, or alternatively, the log-likelihood defined by

$$\begin{aligned} \ell(\Theta) &= \log(L(\Theta)) = n \log(\theta) + 2n \log(\lambda) + \sum_{i=1}^n \log(x_i) + \lambda \sum_{i=1}^n x_i \\ &+ \sum_{i=1}^n \log(e^{\lambda x_i} + 1 - \theta) - 3 \sum_{i=1}^n \log(e^{\lambda x_i} - 1 + \theta). \end{aligned}$$

It follows that the MLEs are the simultaneous solutions of the equations according to (θ, λ) :

$$\frac{\partial \ell(\Theta)}{\partial \theta} = \frac{n}{\theta} - \sum_{i=1}^n \frac{1}{e^{\lambda x_i} + 1 - \theta} - 3 \sum_{i=1}^n \frac{1}{e^{\lambda x_i} - 1 + \theta} = 0$$

and

$$\frac{\partial \ell(\Theta)}{\partial \lambda} = \frac{2n}{\lambda} + \sum_{i=1}^n x_i + \sum_{i=1}^n \frac{e^{\lambda x_i} x_i}{e^{\lambda x_i} + 1 - \theta} - 3 \sum_{i=1}^n \frac{e^{\lambda x_i} x_i}{e^{\lambda x_i} - 1 + \theta} = 0.$$

Since we have no analytic forms, numerical methods, as the quasi-Newton algorithm, can be applied to determine the estimators. The observed information matrix is given by

$$J(\Theta) = \begin{pmatrix} J_{\theta\theta}(\Theta) & J_{\theta\lambda}(\Theta) \\ J_{\lambda\theta}(\Theta) & J_{\lambda\lambda}(\Theta) \end{pmatrix},$$

where

$$\begin{aligned} J_{\theta\theta}(\Theta) &= -\frac{\partial^2 \ell(\Theta)}{\partial \theta^2} = \frac{n}{\theta^2} + \sum_{i=1}^n \frac{1}{(e^{\lambda x_i} + 1 - \theta)^2} - 3 \sum_{i=1}^n \frac{1}{(e^{\lambda x_i} - 1 + \theta)^2}, \\ J_{\theta\lambda}(\Theta) &= -\frac{\partial^2 \ell(\Theta)}{\partial \theta \partial \lambda} = -\sum_{i=1}^n \frac{e^{\lambda x_i} x_i}{(e^{\lambda x_i} + 1 - \theta)^2} - 3 \sum_{i=1}^n \frac{e^{\lambda x_i} x_i}{(e^{\lambda x_i} - 1 + \theta)^2}, \\ J_{\lambda\lambda}(\Theta) &= -\frac{\partial^2 \ell(\Theta)}{\partial \lambda^2} = \frac{2n}{\lambda^2} + \sum_{i=1}^n \left(\frac{e^{2\lambda x_i} x_i^2}{(e^{\lambda x_i} + 1 - \theta)^2} - \frac{e^{\lambda x_i} x_i^2}{e^{\lambda x_i} + 1 - \theta} \right) \\ &\quad - 3 \sum_{i=1}^n \left(\frac{e^{2\lambda x_i} x_i^2}{(e^{\lambda x_i} - 1 + \theta)^2} - \frac{e^{\lambda x_i} x_i^2}{(e^{\lambda x_i} - 1 + \theta)} \right). \end{aligned}$$

This matrix is a key mathematical tool to obtain approximate confidence intervals or Wald tests for θ and λ in the case of a large sample.

3.2. Illustrative hydrologic data examples

In this section, we fit the GG distribution to three hydrologic data sets and compare with the Weibull, Gumbel, Exponentiated Exponential, Generalized Gumbel, Kappa and Weibull Geometric distributions for three data sets. Most of those distributions have received great attention for fitting hydrology data, like rainfall data, precipitation data and flood data. More precisely, the densities of the compared distributions are given as follows.:

- Weibull distribution with pdf:

$$f(x) = \frac{k}{\lambda} \left(\frac{x}{\lambda} \right)^{k-1} e^{-\left(\frac{x}{\lambda}\right)^k}, \quad \lambda > 0, k > 0, x > 0.$$

- Gumbel distribution with pdf:

$$f(x) = \frac{e^{-e^{-\frac{x-\mu}{\sigma}} - \frac{x-\mu}{\sigma}}}{\sigma}, \quad \sigma > 0, x, \mu \in \mathbb{R}.$$

- Exponentiated Exponential (EE) distribution [24] with pdf:

$$f(x) = \alpha \lambda (1 - e^{-\lambda x})^{(\alpha-1)} e^{-\lambda x}, \quad \alpha, \lambda, x > 0.$$

- Generalized Gumbel (GGu) distribution [25] with pdf:

$$f(x) = \frac{\alpha \left(1 - e^{-e^{-\frac{x-\mu}{\sigma}}} \right)^{\alpha-1} \left(e^{-\frac{x-\mu}{\sigma}} \right) e^{-e^{-\frac{x-\mu}{\sigma}}}}{\sigma}, \quad \alpha, \sigma > 0, \mu, x \in \mathbb{R}.$$

- Kappa distribution [26] with pdf:

$$f(x) = \frac{\alpha\theta}{\beta} \left(\frac{x}{\beta}\right)^{\theta-1} \left(\alpha + \left(\frac{x}{\beta}\right)^{\alpha\theta}\right)^{-\frac{(\alpha+1)}{\alpha}}, \quad \alpha, \theta, \beta, x > 0.$$

- Weibull geometric (WG) distribution [6] with pdf:

$$f(x) = \alpha\beta^\alpha(1-p)x^{\alpha-1}e^{-(\beta x)^\alpha} \left(1 - pe^{-(\beta x)^\alpha}\right)^{-2}, \quad p \in (0, 1), \alpha, \beta, x > 0.$$

For goodness-of-fit we have two main test statistics, i.e., information criterion and empirical distribution. The measures Akaike information criterion (AIC) [27], corrected Akaike information criterion (AICC) [28], Hannan–Quinn information criterion (HQIC) [29], and consistent Akaike information criterion (CAIC) [30] are widely used information criterion for selecting the appropriate model among different others models. The Anderson-Darling (A^*) due to Anderson and Darling [31], the Cramér–von Mises (W^*) due to Cramér and Mises [32] and the Kolmogorov Smirnov (KS) statistics due to Kolmogorov [33] with their p-values to compare the fitted models. These statistics are used to evaluate how a particular distribution with cdf, for a given data set, fits the corresponding empirical distribution. The distribution with better fit than the others will be the one having the smallest statistics and largest p-value.

Some information about the data sets are given below as:

The first data set is taken from engineering department and is presented by Linhart and Zucchini [34]. The data points are 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95

The second data contains of annual maximum stream amounts. The data is available in U.S. Geological Survey (USGS) website (<http://nwis.waterdata.usgs.gov>). The data points are 144, 179, 105, 280, 81, 35, 320, 248, 159, 570, 278, 315, 327, 182, 186, 102, 31, 350, 435, 520, 715, 1600, 660, 173, 239, 667, 44, 82, 70, 68, 69, 42, 16, 450, 333, 114, 121, 175, 299, 102, 93, 287, 64, 36, 438, 63, 146, 48, 37, 214, 25, 161, 104, 115, 32, 109, 128, 30

The third data describes the maximum rainfall of Pakistan from 1981 to 2010. The data points are

21.7, 172.9, 69.5, 96.5, 12.6, 265.5, 154, 28, 142.8, 14.2, 74.8, 32.5, 25, 28.5, 113.8, 25.7, 116.3, 28, 16.9, 6, 9, 17.6, 47.3, 55, 129, 72, 92, 28, 113, 194

For the first data set, Table 1 gives us estimates of the parameters of the considered models with their corresponding standard errors. Table 2 presents their goodness-of-fit statistics. Concerning the GG model, the MLEs corresponding to the data are given by $\hat{\theta} = 0.155401$ and $\hat{\lambda} = 0.0124595$, and the following information criterion are obtained: AIC = 306.205, AICC = 306.649, HQIC = 307.101 and CAIC = 306.649. These values are the smallest in comparison to those obtained for the other models. On the other hand, we have $A^* = 0.51583$, $W^* = 0.0892352$, KS = 0.114073 with $p = 0.829838$, which are also the best. The superiority of the GG model, in terms of goodness-of-fit statistics, in comparison to the others, is also observed for second data set (estimates are given in Table 3 and goodness of fit statistics in Table 4) and the third data set (estimates are given in Table 5 and goodness of fit statistics in Table 6).

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Table 1: Estimates of the parameters (standard errors in parentheses) for aeroplane data

Distributions	Estimates		
GG(θ, λ)	0.155401 (0.039411)	0.012459 (0.002647)	
Weibul(λ, k)	54.6134 (12.361500)	0.853587 (0.119402)	
Gumbel(μ, σ)	31.360100 (0.039411)	40.609000 (6.566440)	
EE(α, λ)	0.809287 (0.188512)	0.014543 (0.003720)	
GGu(α, σ, μ)	1.611330 (0.361676)	63.122100 (13.339200)	58.019100 (12.796100)
Kappa(α, θ, β)	1.035070 (0.827705)	1.271930 (0.695590)	29.742000 (12.688400)
WG(λ, a, b)	0.793447 (0.281986)	0.007379 (0.005622)	1.137020 (0.251924)

Table 2: Goodness of fit statistics for aeroplane data

Distributions	Log(L)	AIC	AICC	HQIC	CAIC
GG(θ, λ)	-151.102	306.205	306.649	307.101	306.649
Weibul(λ, k)	-151.937	307.874	308.318	308.770	308.318
Gumbel(μ, σ)	-161.982	327.964	328.408	328.860	328.408
EE(α, λ)	-152.201	308.401	308.846	309.298	308.846
GGu(α, σ, μ)	-164.116	334.233	335.156	335.577	335.156
Kappa(α, θ, β)	-152.183	310.367	311.290	311.712	311.290
WG(λ, a, b)	-151.278	308.557	309.480	309.902	309.480
Distributions	A*	W*	KS	p	
GG(θ, λ)	0.51583	0.0892352	0.114073	0.829838	
Weibul(λ, k)	0.567419	0.0990543	0.153363	0.480628	
Gumbel(μ, σ)	5.90727	0.931143	0.355422	0.00102163	
EE(α, λ)	0.691402	0.123059	0.171971	0.337505	
GGu(α, σ, μ)	8.98938	1.39876	0.39446	0.000176396	
Kappa(α, θ, β)	0.477954	0.0885657	0.121557	0.767156	
WG(λ, a, b)	0.476917	0.0869025	0.125281	0.734085	

Table 3: Estimates of the parameters (standard errors in parentheses) for carnero data

Distributions	Estimates		
GG(θ, λ)	0.336657 (0.0567315)	0.00480789 (0.000630488)	
Weibul(λ, k)	229.722 (30.3587)	1.05334 (0.100504)	
Gumbel(μ, σ)	133.735 (18.0217)	132.415 (15.0402)	
EE(α, λ)	1.27283 (0.234449)	0.00520083 (0.000861595)	
GGu(α, σ, μ)	70.6639 (5.77893)	1229.31 (73.4427)	2083.55 (55.7163)
Kappa(α, θ, β)	0.885474 (0.63983)	1.88622 (0.937705)	136.533 (35.2732)
WG(λ, a, b)	0.952363 (0.06844)	0.00108257 (0.00071915)	1.64806 (0.224096)

Table 4: Goodness of fit statistics for carnero data

Distributions	Log(L)	AIC	AICC	HQIC	CAIC
GG(θ, λ)	-369.200	742.400	742.618	744.005	742.618
Weibul(λ, k)	-371.839	747.678	747.896	749.283	747.896
Gumbel(μ, σ)	-381.104	766.207	766.426	767.813	766.426
EE(α, λ)	-371.153	746.305	746.523	747.910	746.523
GGu(α, σ, μ)	-410.482	826.963	827.408	829.371	827.408
Kappa(α, θ, β)	-369.537	745.075	745.519	747.483	745.519
WG(λ, a, b)	-369.040	744.080	744.524	746.488	744.524
Distributions	A*	W*	KS	p	
GG(θ, λ)	0.26411	0.0282419	0.0586334	0.988461	
Weibul(λ, k)	0.563524	0.0712539	0.0869865	0.772452	
Gumbel(μ, σ)	3.94994	0.727	0.197178	0.0219974	
EE(α, λ)	0.570699	0.0915234	0.0938397	0.686709	
GGu(α, σ, μ)	25.9378	4.87121	0.419101	2.8×10^{-9}	
Kappa(α, θ, β)	0.314206	0.0400859	0.0737592	0.910551	
WG(λ, a, b)	0.248297	0.0308173	0.067118	0.956358	

Table 5: Estimates of the parameters (standard errors in parentheses) for jiwani data

Distributions	Estimates		
GG(θ, λ)	0.453529 (0.105628)	0.0182027 (0.00303367)	
Weibul(λ, k)	77.5581 (12.8831)	1.16238 (0.165544)	
Gumbel(μ, σ)	45.4199 (8.35456)	43.7452 (6.80433)	
EE(α, λ)	1.34541 (0.340988)	0.0163824 (0.00365062)	
GGu(α, σ, μ)	10.4962 (1.5249)	156.021 (17.0742)	226.737 (16.1174)
Kappa(α, θ, β)	0.881944 (1.20898)	1.83423 (1.74096)	46.7766 (23.4308)
WG(λ, a, b)	0.648328 (0.454225)	0.00815663 (0.00451551)	1.40982 (0.333492)

Table 6: Goodness of fit statistics for jiwani data

Distributions	Log(L)	AIC	AICC	HQIC	CAIC
GG(θ, λ)	-157.521	319.042	319.487	319.939	319.487
Weibul(λ, k)	-158.359	320.717	321.162	321.614	321.162
Gumbel(μ, σ)	-162.542	329.084	329.529	329.981	329.529
EE(α, λ)	-158.223	320.446	320.891	321.343	320.891
GGu(α, σ, μ)	-166.937	339.874	340.797	341.219	340.797
Kappa(α, θ, β)	-159.344	324.688	325.611	326.033	325.611
WG(λ, a, b)	-158.038	322.077	323.000	323.422	323.000
Distributions	A*	W*	KS	p	
GG(θ, λ)	0.4193	0.0798668	0.141087	0.589015	
Weibul(λ, k)	0.517542	0.095075	0.165072	0.387045	
Gumbel(μ, σ)	2.10048	0.349666	0.265105	0.0294899	
EE(α, λ)	0.54145	0.102138	0.167958	0.365793	
GGu(α, σ, μ)	5.2192	0.858108	0.321454	0.00405898	
Kappa(α, θ, β)	0.567106	0.101475	0.141822	0.582326	
WG(λ, a, b)	0.503657	0.0945876	0.153904	0.476065	

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