

# Completed Iwahori-Hecke algebras and parahoric Hecke algebras for Kac-Moody groups over local fields

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## Abstract

Let  $G$  be a split Kac-Moody group over a non-archimedean local field. We define a completion of the Iwahori-Hecke algebra of  $G$ . We determine its center and prove that it is isomorphic to the spherical Hecke algebra of  $G$  using the Satake isomorphism. This is thus similar to the situation of reductive groups. Our main tool is the measure  $\mathcal{I}$  associated to this setting, which is the analogue of the Bruhat-Tits building for reductive groups. Then, for each type 0 spherical face  $F$ , we associate a Hecke algebra. In the Kac-Moody setting, this construction was known only for the spherical subgroup and for the Iwahori subgroup.

## 1 Introduction

Let  $\mathbf{G}_0$  be a split reductive group over a non archimedean local field  $\mathcal{K}$  and  $G_0 = \mathbf{G}_0(\mathcal{K})$ . An important tool in the study of the representation theory of  $G$  are Hecke algebras. They are attached to each compact open subgroup of  $G$ : if  $K$  is such a group, the Hecke algebra  $\mathcal{H}_K$  associated to  $K$  is the set of  $K$ -bi-invariant functions on  $G_0$ , with compact support; this set being equipped with some convolution product. Two choices of  $K$  are of particular interest. The first one is when  $K = K_s = \mathbf{G}(\mathcal{O})$ , where  $\mathcal{O}$  is the ring of integers of  $\mathcal{K}$ , which is a maximal compact open subgroup of  $G$ . Then  $\mathcal{H}_s = \mathcal{H}_{K_s}$  is commutative and is called the spherical Hecke algebra of  $G_0$ . This algebra can be described explicitly thanks to the Satake isomorphism: if  $W^v$  is the Weyl group of  $G$  and  $Q^\vee$  is the coweight lattice of  $G_0$ ,  $\mathcal{H}_s$  is isomorphic to  $\mathbb{C}[Q^\vee]^{W^v}$ , which is the sub-algebra of  $W^v$ -invariant elements of the algebra of the group  $(Q^\vee, +)$ . The second is when  $K = K_I$  is the Iwahori subgroup of  $G_0$ . Then  $\mathcal{H} := \mathcal{H}_{K_I}$  is called the Iwahori-Hecke algebra of  $G_0$ . It has a basis indexed by the affine Weyl group of  $G$ , the Bernstein-Lusztig basis, and the product of two elements of this basis can be expressed with the Bernstein-Lusztig presentation. It enables to determine the center of  $\mathcal{H}$  and one sees that it is isomorphic to the spherical Hecke algebra of  $G_0$ . We summarize these results as follows:

$$\mathcal{H}_s \xrightarrow[S]{\cong} \mathbb{C}[Q^\vee]^{W^v} \xrightarrow[g]{\hookrightarrow} \mathcal{H}, \text{ and } \text{Im}(g) = \mathcal{Z}(\mathcal{H})$$

where  $S$  is the Satake isomorphism and  $g$  comes from the Bernstein-Lusztig basis. The aim of this article is to study the extension of this theory to the Kac-Moody setting.

Kac-Moody groups are interesting generalizations of reductive groups. There are several definitions of Kac-Moody groups but we will study the groups defined by Tits in [Tit87]. Let  $\mathbf{G}$  be a split Kac-Moody group over  $\mathcal{K}$  and  $G = \mathbf{G}(\mathcal{K})$ . In order to study  $G$ , Gaussent and Rousseau constructed an object  $\mathcal{I} = \mathcal{I}(G)$  called “measure” (or affine ordered hovel) in [GR08], and it was generalized in [Rou16] and [Rou17]. This measure is a generalization of a Bruhat-Tits building (introduced in [BT72] and [BT84]) and when  $G$  is reductive,  $\mathcal{I}$  is the usual Bruhat-Tits building of  $G$ . The set  $\mathcal{I}$  is a union of apartments, all isomorphic to a standard one  $\mathbb{A}$  and  $G$  acts on  $\mathcal{I}$ . There is still an arrangement of hyperplanes, called walls, but it is no more locally finite in general. For this reason the faces in  $\mathbb{A}$  are no more sets but filters. An other important difference with buildings is that there can be two points which are not included in any apartment.

There is up to now no topology on  $G$  generalizing the usual topology of reductive groups. We can nevertheless define analogs of  $K_s$  and  $K_I$  in this setting. They are fixers of some faces in  $\mathcal{I}$ . However, as we shall see, there exists no topology of topological group on  $G$  such that  $K_s$  is open compact, and the same result holds for  $K_I$  (this is Proposition 3.7). Braverman, Kazhdan and Patnaik extended the definition of the spherical Hecke algebra and of the Iwahori-Hecke algebra of  $G$  when  $G$  is affine in [BK11] and [BKP16]. They obtained a Satake isomorphism and Bernstein-Lusztig relations. These definitions and results were extended to the general case ( $G$  no more assumed affine) by Bardy-Panse, Gaussent and Rousseau in [GR14] and [BPGR16]. In this framework, the Satake isomorphism is an isomorphism between  $\mathcal{H}_s := \mathcal{H}_{K_s}$  and  $\mathbb{C}[[Y]]^{W^v}$ , where  $Y$  is a lattice which can be thought of as the coroot lattice in a first approximation (but it can be different, notably when  $G$  is affine) and  $\mathbb{C}[[Y]]$  is the Looijenga’s algebra of  $Y$ , which is some completion of the group algebra  $\mathbb{C}[Y]$  of  $Y$ . Let  $\mathcal{H}$  be the Iwahori-Hecke algebra of  $G$ . As we shall see (Theorem 4.26), the center of  $\mathcal{H}$  is more or less trivial. Moreover,  $\mathbb{C}[[Y]]^{W^v}$  is a set of infinite formal series and there is no obvious injection from  $\mathbb{C}[[Y]]$  to  $\mathcal{H}$ . For these reasons, we define a “completion”  $\hat{\mathcal{H}}$  of  $\mathcal{H}$ . More precisely, let  $(Z^\lambda H_w)_{\lambda \in Y^+, w \in W^v}$ , where  $Y^+$  is a sub-monoid of  $Y$ , be the Bernstein-Lusztig basis of  $\mathcal{H}$ . Then  $\hat{\mathcal{H}}$  is the set of formal series  $\sum_{w \in W^v, \lambda \in Y^+} c_{w, \lambda} Z^\lambda H_w$ , whose support satisfies some conditions similar to what appears in the definition of  $\mathbb{C}[[Y]]$ . We equip it with a convolution compatible with the inclusion  $\mathcal{H} \subset \hat{\mathcal{H}}$ . The fact that this product is well defined is not obvious and this is our main result: Theorem 4.17. We then determine the center of  $\hat{\mathcal{H}}$  and we show that it is isomorphic to  $\mathbb{C}[[Y]]^{W^v}$  (Theorem 4.26), which is similar to the classical case. We thus get the following diagram:

$$\mathcal{H}_s \xrightarrow[S]{\cong} \mathbb{C}[[Y]]^{W^v} \xrightarrow[g]{\hookrightarrow} \hat{\mathcal{H}}, \text{ and } \text{Im}(g) = \mathcal{Z}(\hat{\mathcal{H}}),$$

where  $S$  is the Satake isomorphism (see Section 8 of [BK11] or Theorem 5.4 of [GR14]), and  $g$  comes from the Bernstein-Lusztig basis.

In a second part, we associate Hecke algebras to subgroups more general than  $K_I$ . The group  $K_s$  is the fixer of  $\{0\}$  and  $K_I$  is the fixer of the type 0 chamber  $C_0^+$ . When  $G$  is reductive, for all faces  $F$  between  $\{0\}$  and  $C_0^+$ , the fixer  $K_F$  (the parahoric group associated to  $F$ ) of  $F$  in  $G$  is compact open. Therefore it seems natural to try to associate a Hecke algebra to the fixer  $K_F$  of  $F$  in  $G$  for all face  $F$  between  $\{0\}$  and  $C_0^+$ . We succeed in defining such an algebra when  $F$  is spherical, which means that its fixer in the Weyl group is finite.

Our construction is very close to the construction of the Iwahori-Hecke algebra of [BGR16]. When  $F$  is no more spherical and different from  $\{0\}$  (this case does not occur when  $G$  is affine), we prove that this construction fails: the structure constants are infinite.

Actually, this article is written in a more general framework explained in Section 2: we ask  $\mathcal{I}$  to be an abstract measure and  $G$  to be a strongly transitive group of (positive, type-preserving) automorphisms of  $\mathcal{I}$ . This applies in particular to almost-split Kac-Moody groups over local fields.

In Section 2, we recall the definition of measures. The reader only interested in Section 4 can read only Subsection 2.1.

In Section 3 we prove that there exists no topology of topological group on  $G$  for which  $K_s$  or  $K_I$  are compact and open.

In Section 4, we define the completed Iwahori-Hecke algebra of  $G$  and determine its center.

In Section 5, we associate Hecke algebras to each spherical face between  $\{0\}$  and  $C_0^+$  and prove that this construction fails when  $F$  is not spherical and different from  $\{0\}$ .

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## 2 General framework, Measure

### 2.1 Standard apartment

#### 2.1.1 Root generating system

A reference for this section is [Rou11], Section 1 and 2.

A Kac-Moody matrix (or generalized Cartan matrix) is a square matrix  $A = (a_{i,j})_{i,j \in I}$  with integers coefficients, indexed by a finite set  $I$  and satisfying:

1.  $\forall i \in I, a_{i,i} = 2$
2.  $\forall (i,j) \in I^2 | i \neq j, a_{i,j} \leq 0$
3.  $\forall (i,j) \in I^2, a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0$ .

A root generating system is a 5-tuple  $\mathcal{S} = (A, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$  made of a Kac-Moody matrix  $A$  indexed by  $I$ , of two dual free  $\mathbb{Z}$ -modules  $X$  (of characters) and  $Y$  (of cocharacters) of finite rank  $\text{rk}(X)$ , a family  $(\alpha_i)_{i \in I}$  (of simple roots) in  $X$  and a family  $(\alpha_i^\vee)_{i \in I}$  (of simple coroots) in  $Y$ . They have to satisfy the following compatibility condition:  $a_{i,j} = \alpha_j(\alpha_i^\vee)$  for all  $i, j \in I$ . We also suppose that the family  $(\alpha_i)_{i \in I}$  is free in  $X$  and that the family  $(\alpha_i^\vee)_{i \in I}$  is free in  $Y$ .

We now fix a Kac-Moody matrix  $A$  and a root generating system with matrix  $A$ .

Let  $\mathbb{A} = Y \otimes \mathbb{R}$ . Every element of  $X$  induces a linear form on  $\mathbb{A}$ . We will consider  $X$  as a subset of the dual  $\mathbb{A}^*$  of  $\mathbb{A}$ : the  $\alpha_i$ 's,  $i \in I$  are viewed as linear forms on  $\mathbb{A}$ . For  $i \in I$ , we define an involution  $r_i$  of  $\mathbb{A}$  by  $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$  for all  $v \in \mathbb{A}$ . Its space of fixed points is  $\ker \alpha_i$ . The subgroup of  $\text{GL}(\mathbb{A})$  generated by the  $\alpha_i$ 's for  $i \in I$  is denoted by  $W^v$  and is called the Weyl group of  $\mathcal{S}$ . The system  $(W^v, \{r_i | i \in I\})$  is a Coxeter system. For  $w \in W^v$ , we denote by  $l(w)$  the length of  $w$  with respect to  $\{r_i | i \in I\}$ .

For  $x \in \mathbb{A}$  one sets  $\underline{\alpha}(x) = (\alpha_i(x))_{i \in I} \in \mathbb{R}^I$ .

Let  $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$  and  $P^\vee = \{v \in \mathbb{A} | \underline{\alpha}(v) \in \mathbb{Z}^I\}$ . We call  $Q^\vee$  the *coroot-lattice* and  $P^\vee$  the *co-weight-lattice* (but if  $\bigcap_{i \in I} \ker \alpha_i \neq \{0\}$ , this is not a lattice). Let  $Q_+^\vee = \bigoplus_{i \in I} \mathbb{N}\alpha_i^\vee$ ,  $Q_-^\vee = -Q_+^\vee$  and  $Q_{\mathbb{R}}^\vee = \bigoplus_{i \in I} \mathbb{R}\alpha_i^\vee$ . This enables us to define a pre-order  $\leq_{Q^\vee}$  on  $\mathbb{A}$  by the following way: for all  $x, y \in \mathbb{A}$ , one writes  $x \leq_{Q^\vee} y$  if  $y - x \in Q_+^\vee$ .

One defines an action of the group  $W^v$  on  $\mathbb{A}^*$  by the following way: if  $x \in \mathbb{A}$ ,  $w \in W^v$  and  $\alpha \in \mathbb{A}^*$  then  $(w.\alpha)(x) = \alpha(w^{-1}.x)$ . Let  $\Phi = \{w.\alpha_i | (w, i) \in W^v \times I\}$ ,  $\Phi$  is the set of *real roots*. Then  $\Phi \subset Q$ , where  $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ . Let  $W = Q^\vee \rtimes W^v \subset \text{GA}(\mathbb{A})$  the *affine Weyl group* of  $\mathcal{S}$ , where  $\text{GA}(\mathbb{A})$  is the group of affine isomorphisms of  $\mathbb{A}$ .

## 2.1.2 Vectorial faces and Tits cone

Define  $C_f^v = \{v \in \mathbb{A} \mid \alpha_i(v) > 0, \forall i \in I\}$ . We call it the *fundamental chamber*. For  $J \subset I$ , one sets  $F^v(J) = \{v \in \mathbb{A} \mid \alpha_i(v) = 0 \forall i \in J, \alpha_i(v) > 0 \forall i \in I \setminus J\}$ . Then the closure  $\overline{C_f^v}$  of  $C_f^v$  is the union of the  $F^v(J)$  for  $J \subset I$ . The *positive* (resp. *negative*) *vectorial faces* are the sets  $w.F^v(J)$  (resp.  $-w.F^v(J)$ ) for  $w \in W^v$  and  $J \subset I$ . A *vectorial face* is either a positive vectorial face or a negative vectorial face. We call *positive chamber* (resp. *negative*) every cone of the shape  $w.C_f^v$  for some  $w \in W^v$  (resp.  $-w.C_f^v$ ). For all  $x \in C_f^v$  and for all  $w \in W^v$ ,  $w.x = x$  implies that  $w = 1$ . In particular the action of  $w$  on the positive chambers is simply transitive. The *Tits cone*  $\mathcal{T}$  is defined by  $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$ . We also consider the negative cone  $-\mathcal{T}$ . We define a  $W^v$ -invariant relation  $\leq$  on  $\mathbb{A}$  by:  $\forall (x, y) \in \mathbb{A}^2, x \leq y \Leftrightarrow y - x \in \mathcal{T}$ .

We denote by  $Y^+$  the set  $Y \cap \mathcal{T}$  and by  $Y^{++}$  the set  $Y \cap \overline{C_f^v}$ .

We recall the simple but useful result (this is Lemma 2.4 a) of [GR14]):

**Lemma 2.1.** *Let  $\lambda \in Y^{++}$ . Then for all  $w \in W^v$ ,  $w.\lambda \leq_{Q^v} \lambda$ .*

In the next subsection, we define measures. The reader only interested in the completion of Iwahori-Hecke algebras can skip it and go directly to Section 4.

## 2.2 Measure

In this section, we define measures. They were introduced in [GR08] for symmetrizable split Kac-Moody groups over a valued field whose residue field contains  $\mathbb{C}$ , axiomatized in [Rou11] and developed and generalized to almost-split Kac-Moody groups over non-archimedean local fields in [Rou16] and [Rou17]. We consider semi-discrete measures which are thick of finite thickness.

### 2.2.1 Filters

**Definition 2.2.** *A filter in a set  $E$  is a nonempty set  $F$  of nonempty subsets of  $E$  such that, for all subsets  $S, S'$  of  $E$ , if  $S, S' \in F$  then  $S \cap S' \in F$  and, if  $S' \subset S$ , with  $S' \in F$  then  $S \in F$ .*

If  $F$  is a filter in a set  $E$ , and  $E'$  is a subset of  $E$ , one says that  $F$  contains  $E'$  if every element of  $F$  contains  $E'$ . If  $E'$  is nonempty, the set  $F_{E'}$  of subsets of  $E$  containing  $E'$  is a filter. By abuse of language, we will sometimes say that  $E'$  is a filter by identifying  $F_{E'}$  and  $E'$ . If  $F$  is a filter in  $E$ , its closure  $\overline{F}$  (resp. its convex hull) is the filter of subsets of  $E$  containing the closure (resp. the convex hull) of some element of  $F$ . A filter  $F$  is said to be contained in another filter  $F'$ :  $F \subset F'$  (resp. in a subset  $Z$  in  $E$ :  $F \subset Z$ ) if and only if any set in  $F'$  (resp. if  $Z$ ) is in  $F$ .

If  $x \in \mathbb{A}$  and  $\Omega$  is a subset of  $\mathbb{A}$  containing  $x$  in its closure, then the *germ* of  $\Omega$  in  $x$  is the filter  $germ_x(\Omega)$  of subsets of  $\mathbb{A}$  containing a neighborhood in  $\Omega$  of  $x$ .

A *sector* in  $\mathbb{A}$  is a set of the shape  $\mathfrak{s} = x + C^v$  with  $C^v = \pm w.C_f^v$  for some  $x \in \mathbb{A}$  and  $w \in W^v$ . The point  $x$  is its *base point* and  $C^v$  is its *direction*. The intersection of two sectors of the same direction is a sector of the same direction.

The *sector-germ* of a sector  $\mathfrak{s} = x + C^v$  is the filter  $\mathfrak{S}$  of subsets of  $\mathbb{A}$  containing an  $\mathbb{A}$ -translate of  $\mathfrak{s}$ . It only depends on the direction  $C^v$ . We denote by  $+\infty$  (resp.  $-\infty$ ) the sector-germ of  $C_f^v$  (resp. of  $-C_f^v$ ).

A ray  $\delta$  with base point  $x$  and containing  $y \neq x$  (or the interval  $[x, y] = [x, y] \setminus \{x\}$  or  $[x, y]$ ) is called *preordered* if  $x \leq y$  or  $y \leq x$  and *generic* if  $y - x \in \pm \mathcal{T}$ , the interior of  $\pm \mathcal{T}$ .

### 2.2.2 Definitions of enclosures, faces and related notions

Let  $\Delta = \Phi \cup \Delta_{im}^+ \cup \Delta_{im}^- \subset Q$  be the set of all roots (recall that  $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ ), defined in [Kac94]. The group  $W^v$  stabilizes  $\Delta$ . For  $\alpha \in \Delta$ , and  $k \in \mathbb{Z} \cup +\infty$ , let  $D(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k \geq 0\}$  (and  $D(\alpha, +\infty) = \mathbb{A}$  for all  $\alpha \in \Delta$ ) and  $D^\circ(\alpha, k) = \{v \in \mathbb{A} \mid \alpha(v) + k > 0\}$  (for  $\alpha \in \Delta$  and  $k \in \mathbb{Z} \cup \{+\infty\}$ ).

For  $\alpha \in \Phi$ , one sets  $\Lambda_\alpha = \mathbb{Z}$  and for  $\alpha \in \Delta_{im} = \Delta \setminus \Phi$ , one set  $\Lambda_\alpha = \mathbb{R}$ . Given a filter  $F$  of subsets of  $\mathbb{A}$ , its *enclosure*  $\text{cl}_{\mathbb{A}}(F)$  is the filter made of the subsets of  $\mathbb{A}$  containing an element of  $F$  of the shape  $\bigcap_{\alpha \in \Delta} D(\alpha, k_\alpha)$  where  $k_\alpha \in \Lambda_\alpha \cup \{+\infty\}$  for all  $\alpha \in \Delta$ . A *wall* (resp. *half-apartment*) is a set of the shape  $M(\alpha, k) = \{x \in \mathbb{A} \mid \alpha(x) + k = 0\}$  (resp.  $D(\alpha, k)$ ) for  $\alpha \in \Phi$  and  $k \in \mathbb{Z}$ .

**Faces** A *local face*  $F^l$  in  $\mathbb{A}$  is a filter associated to a point  $x \in \mathbb{A}$ , its *vertex* and a vectorial face  $F^v \subset \mathbb{A}$ , its *direction*. It is defined by  $F^l = \text{germ}_x(x + F^v)$  and we denote it by  $F^l(x, F^v)$ . A *face*  $F$  in  $\mathbb{A}$  is a filter associated to a point  $x \in \mathbb{A}$  and a vectorial face  $F^v \subset \mathbb{A}$ . More precisely, a subset  $S$  of  $\mathbb{A}$  is an element of the face  $F = F(x, F^v)$  if and only if, it contains an intersection of half-spaces  $D(\alpha, k_\alpha)$  or open half-spaces  $D^\circ(\alpha, k_\alpha)$ , with  $k_\alpha \in \mathbb{Z}$  for all  $\alpha \in \Delta$ , that contains  $F^l(x, F^v)$ .

There is an order on the (local) faces: if  $F \subset \overline{F'}$  we say that “ $F$  is a face of  $F'$ ” or “ $F'$  contains  $F$ ” or “ $F'$  dominates  $F$ ”. The dimension of a face  $F$  is the smallest dimension of an affine space generated by some  $S \in F$ . Such an affine space is unique and is called its support. A local face is said to be *spherical* if its direction is spherical; then its point-wise stabilizer  $W_F$  in  $W^v$  is finite.

We have  $W \subset P^\vee \rtimes W^v$ . As  $\alpha(P^\vee) \subset \mathbb{Z}$  for all  $\alpha$  in  $\bigoplus_{i \in I} \mathbb{Z}\alpha_i$ , if  $\tau$  is a translation of  $\mathbb{A}$  of a vector  $p \in P^\vee$ , then for all  $\alpha \in Q$ ,  $\tau$  permutes the sets of the shape  $D(\alpha, k)$  where  $k$  runs over  $\mathbb{Z}$ . As  $W^v$  stabilizes  $\Delta$ , any element of  $W^v$  permutes the sets of the shape  $D(\alpha, k)$  where  $\alpha$  runs over  $\Delta$ . Therefore,  $W$  permutes the sets  $D(\alpha, k)$ , where  $(\alpha, k)$  runs over  $\Delta \times \mathbb{Z}$  and thus  $W$  permutes the enclosures, faces, ... of  $\mathbb{A}$ .

A *local chamber* (or alcove) is a maximal local face, i.e a local face  $F^l(x, \pm w.C_f^v)$  for  $x \in \mathbb{A}$  and  $w \in W^v$ . The *fundamental local chamber* is  $C_0^+ = F^l(0, C_f^v)$ .

A *local panel* is a spherical local face maximal among faces that are not chambers or equivalently a spherical face of dimension  $\dim \mathbb{A} - 1$ . We have analog definitions of chambers and panels.

A *type 0 local face* is a local face whose vertex is in  $Y$ . Let  $\mathbb{A}_{in} = \bigcap_{i \in I} \ker(\alpha_i) = F^v(I)$ . We denote by  $F_0$  the local face  $F^l(0, \mathbb{A}_{in})$ . From now on, we will say “type 0 face” instead of “type 0 local face”.

**Chimneys** In [Rou11], Rousseau defines chimneys and uses it in its axiomatization of mesures. We do not define this notion, we only precise that each sector-germ is a splayed, solid chimney-germ, that each spherical face is included in a solid chimney and that  $W$  permutes the chimneys of  $\mathbb{A}$  (and preserve their properties: splayed, solid, ...).

### 2.2.3 Measure

An apartment of type  $\mathbb{A}$  is a set  $A$  with a nonempty set  $\text{Isom}(\mathbb{A}, A)$  of bijections (called Weyl-isomorphisms) such that if  $f_0 \in \text{Isom}(\mathbb{A}, A)$  then  $f \in \text{Isom}(\mathbb{A}, A)$  if and only if, there exists  $w \in W$  satisfying  $f = f_0 \circ w$ . We will say isomorphism instead of Weyl-isomorphism in the sequel. An isomorphism between two apartments  $\phi : A \rightarrow A'$  is a bijection such that ( $f \in \text{Isom}(\mathbb{A}, A)$  if, and only if,  $\phi \circ f \in \text{Isom}(\mathbb{A}, A')$ ). We extend all the notions that are

preserved by  $W$  to each apartment. Thus sectors, enclosures, faces and chimneys are well defined in any apartment of type  $\mathbb{A}$ .

**Definition 2.3.** *A measure of type  $\mathbb{A}$  is a set  $\mathcal{I}$  endowed with a covering  $\mathcal{A}$  of subsets called apartments such that:*

(MA1) *Any  $A \in \mathcal{A}$  admits a structure of an apartment of type  $\mathbb{A}$ .*

(MA2) *If  $F$  is a point, a germ of a preordered interval, a generic ray or a solid chimney in an apartment  $A$  and if  $A'$  is another apartment containing  $F$ , then  $A \cap A'$  contains the enclosure  $\text{cl}_A(F)$  of  $F$  and there exists an isomorphism from  $A$  onto  $A'$  fixing  $\text{cl}_A(F)$ .*

(MA3) *If  $\mathfrak{R}$  is the germ of a splayed chimney and if  $F$  is a face or a germ of a solid chimney, then there exists an apartment that contains  $\mathfrak{R}$  and  $F$ .*

(MA4) *If two apartments  $A, A'$  contain  $\mathfrak{R}$  and  $F$  as in (MA3), then there exists an isomorphism from  $A$  to  $A'$  fixing  $\text{cl}_A(\mathfrak{R} \cup F)$ .*

(MAO) *If  $x, y$  are two points contained in two apartments  $A$  and  $A'$ , and if  $x \leq_A y$  then the two segments  $[x, y]_A$  and  $[x, y]_{A'}$  are equal.*

In this definition, one says that an apartment contains a germ of a filter if it contains at least one element of this germ. One says that a map fixes a germ if it fixes at least one element of this germ.

Until the end of this article,  $\mathcal{I}$  will be a measure. We suppose that  $\mathcal{I}$  is thick of *finite thickness*: the number of chambers (=alcoves) containing a given panel has to be finite, greater or equal to 3.

We assume that there exists a group  $G$  of automorphisms of  $\mathcal{I}$  acting strongly transitively on  $\mathcal{I}$ , which means that all isomorphisms involved in the above axioms are induced by elements of  $G$ . We choose in  $\mathcal{I}$  a fundamental apartment, that we identify with  $\mathbb{A}$ . As  $G$  is strongly transitive, the apartments of  $\mathcal{I}$  are the sets  $g.\mathbb{A}$  for  $g \in G$ . The stabilizer  $N$  of  $\mathbb{A}$  induces a group  $\nu(N)$  of affine automorphisms of  $\mathbb{A}$  and we suppose that  $\nu(N) = W^v \ltimes Y$ .

An example of such a measure  $\mathcal{I}$  is the measure associated to a split Kac-Moody group  $G$  over a non-archimedean local field constructed in [GR08], and [Rou16], see Subsection 2.2.4. In this case,  $G$  acts strongly transitively on  $\mathcal{I}$ .

**Remark 2.4.**

1. Recently, the second author proposed a simplified axiomatic of measures, see Theorem 5.1 of [Héb17a]. To simplify the reasonings, the reader can assume that  $G$  is an affine Kac-Moody group. In this case, (MA2), (MA4) and (MAO) can be replaced by the following property (see Theorem 5.38 of [Héb17a]):

“If  $A$  and  $A'$  are two apartments of  $\mathcal{I}$ . Then  $A \cap A' = \text{cl}(A \cap A')$  and there exists an isomorphism from  $A$  to  $A'$  fixing  $A \cap A'$ .”

This assumption enables to get rid of the technical questions concerning the existence of an isomorphism fixing subsets of an intersection of apartments.

2. Let  $F$  be a local face of an apartment  $A$  and  $B$  be apartments containing  $F$ . Then  $F$  is a local face of  $B$  and there exists an isomorphism from  $A$  to  $B$  fixing  $F$ . Indeed, let  $x$  be the vertex of  $F$  and  $J$  be a germ of a preordered segment based at  $x$  and included in  $F$ . Then the enclosure of  $J$  contains  $F$  and thus (MA2) applied to  $J$  enables to conclude.
3. Let  $w \in W$ . Then if  $\mathcal{X}$  is a filter of  $\mathbb{A}$  fixed by  $w$ ,  $w$  fixes  $\text{cl}(\mathcal{X})$ . Thus, using the same reasoning as above, if  $F^l$  is a local face of  $\mathcal{I}$ , say  $F^l = F^l(x, F^v)$  and  $F = F(x, F^v)$ , the fixer of  $F^l$  in  $G$  equals the fixer of  $F$  in  $G$ .

**Definition 2.5.** (Pre-order on  $\mathcal{I}$ ) As the pre-order  $\leq$  on  $\mathbb{A}$  (induced by the Tits cone) is invariant under the action of  $W$ , we can equip each apartment  $A$  with a pre-order  $\leq_A$ . Let  $A$  be an apartment of  $\mathcal{I}$  and  $x, y \in A$  such that  $x \leq_A y$ . Then by Proposition 5.4 of [Rou11], if  $B$  is an apartment containing  $x$  and  $y$ ,  $x \leq_B y$ . This defines a relation  $\leq$  on  $\mathcal{I}$ . By Théorème 5.9 of [Rou11], this defines a  $G$ -invariant pre-order on  $\mathcal{I}$ .

## 2.2.4 Example: measure associated to a Kac-Moody group

We consider the group functor  $\mathbf{G}$  associated to the root generating system  $\mathcal{S}$  in [Tit87] and in Chapitre 8 of [Rém02]. This functor is a functor from the category of rings to the category of groups satisfying axioms (KMG1) to (KMG 9) of [Tit87]. When  $R$  is a field,  $\mathbf{G}(R)$  is uniquely determined by these axioms by Theorem 1' of [Tit87]. This functor contains a toric functor  $\mathbf{T}$ , from the category of rings to the category of commutative groups (denoted  $\mathcal{T}$  in [Rém02]) and two functors  $\mathbf{U}^+$  and  $\mathbf{U}^-$  from the category of rings to the category of groups.

Let  $\mathcal{K}$  be a non-archimedean local field,  $\mathcal{O}$  its ring of integers,  $q$  the residue cardinal and  $G = \mathbf{G}(\mathcal{K})$  (and  $U^+ = \mathbf{U}^+(\mathcal{K})$ , ...). For all  $\epsilon \in \{-, +\}$ , and all  $\alpha \in \Phi^\epsilon$ , we have an isomorphism  $x_\alpha$  from  $\mathcal{K}$  to a group  $U_\alpha$ . For all  $k \in \mathbb{Z}$ , one defines a subgroup  $U_{\alpha,k} := x_\alpha(\pi^k \mathcal{O})$ , where  $\pi$  is a uniformizer of  $\mathcal{O}$  (see 3.1 of [GR08]). Let  $\mathcal{I}$  be the measure associated to  $G$  constructed in [Rou17]. Then we have:

- the fixer of  $\mathbb{A}$  in  $G$  is  $H = \mathbf{T}(\mathcal{O})$  (by remark 3.2 of [GR08])
- the fixer of  $\{0\}$  in  $G$  is  $K_s = \mathbf{G}(\mathcal{O})$  (by example 3.14 of [GR08]). By (MA2) applied to  $\{0\}$  and the same reasoning as in 3 of Remark 2.4,  $K_s$  is also the fixer of  $F_0$  in  $G$ .
- for all  $\alpha \in \Phi$  and  $k \in \mathbb{Z}$ , the fixer of  $D(\alpha, k)$  in  $G$  is  $H.U_{\alpha,k}$  (by 4.2 7) of [GR08])
- for all  $\epsilon \in \{-, +\}$ ,  $U^\epsilon$  is the fixer of  $\epsilon\infty$  (by 4.2 4) of [GR08]).

Each panel is contained in  $1 + q$  chambers and thus  $\mathcal{I}$  is thick of finite thickness.

**Remark 2.6.** The group  $G$  is reductive if and only if  $W^v$  is finite. In this case,  $\mathcal{I}$  is the usual Bruhat-Tits building of  $G$ . Then one has  $\mathcal{T} = \mathbb{A}$  and thus  $Y^+ = Y$ .

## 3 A topological restriction on parahoric subgroups

In this section, we prove that there exists no topology of topological group on  $G$  for which  $K_s$  or  $K_I$  are compact and open.

Let  $F$  be a type 0 face of  $\mathbb{A}$ , i.e a local face whose vertex is in  $Y$ . Maybe considering  $h.F$  for some  $h \in G$ , one can suppose that  $F \subset \pm C_0^+$ . One supposes that  $F \subset C_0^+$  as the other case is similar. Let  $K_F$  be the fixer of  $F$  in  $G$ . In this subsection, we show that if  $W^v$  is infinite, there exists no topology of topological group on  $G$  such that  $K_F$  is open and compact. For this, we show that there exists  $g \in G$  such that  $K_F / (K_F \cap g.K_F.g^{-1})$  is infinite.

Let  $\alpha \in \Phi^+$  and  $i \in I$  such that  $\alpha = w.\alpha_i$ , for some  $w \in W^v$ . For  $l \in \mathbb{Z}$ , one sets  $M_l = \{t \in \mathbb{A} | \alpha(t) = l\}$  and  $D_l = \{t \in \mathbb{A} | \alpha(t) \leq l\}$  and one denotes by  $K_{\alpha,l}$  the fixer of  $D_l$  in  $G$ . For  $l \in \mathbb{Z}$ , one chooses a panel  $P_l$  in  $M_l$  and a chamber  $C_l$  dominating  $P_l$  and included in  $\text{conv}(M_l, M_{l+1})$ .

For  $i \in I$ , one denotes by  $1 + q_i$  the number of chambers containing  $P_0$  and by  $1 + q'_i$  the number of chambers containing  $P_1$ . By Proposition 2.9 of [Rou11] and Lemma 3.2 of [Héb16], this does not depend on the choices of  $P_0$  and  $P_1$  (this reasoning will be detailed in the proof of Lemma 3.3). As  $\alpha_i(\alpha_i^V) = 2$  and as there exists an element of  $G$  inducing a translation of



vector  $\alpha_i^\vee$  on  $\mathbb{A}$  (because the stabilizer  $N$  of  $\mathbb{A}$  induces the group  $W^v \rtimes Y$  on  $\mathbb{A}$ ),  $1 + q_i$  is the number of chambers containing  $P_{2l}$  and  $1 + q'_i$  is the number of chambers  $P_{2l+1}$  for all  $l \in \mathbb{Z}$ .

Let us explain the basic idea of the proof. Let  $g \in G$  such that  $g.0 \in C_f^v$  and  $F' = g.F$ . Then  $K_F/(K_F \cap K_{F'})$  is in bijection with  $K_{F'}.F'$ . Let  $\tilde{K}_\alpha = \bigcup_{l \in \mathbb{Z}} K_{\alpha,l}$ , then  $\mathbb{T}_\alpha := \tilde{K}_\alpha.\mathbb{A}$  is a semi-homogeneous extended tree with parameters  $q_i$  and  $q'_i$ . Let  $K_\alpha = K_{\alpha,1}$ . We deduce from the thickness of  $\mathcal{I}$  that if  $n_\alpha$  is the number of walls parallel to  $\alpha^{-1}(\{0\})$  between  $a$  and  $g.a$ ,  $|K_\alpha.g.a| \geq 2^{n_\alpha}$  and hence that  $|K_F.g.a| \geq 2^{n_\alpha}$ . As  $W^v$  is infinite,  $n_\alpha$  can be made arbitrarily large by changing  $\alpha$ .

One chooses a sector-germ  $\mathfrak{q}$  included in  $D_0$ . If  $x \in \mathcal{I}$ , there exists an apartment  $A_x$  containing  $x$  and  $\mathfrak{q}$  by (MA3). By (MA4), there exists an isomorphism  $\phi_x : A \rightarrow \mathbb{A}$  fixing  $\mathfrak{q}$ . By 2.6 of [Rou11],  $\phi_x(x)$  does not depend on the choice of  $A_x$  and  $\phi_x$  and we denote it  $\rho(x)$ . The map  $\rho_\mathfrak{q}$  is the retraction of  $\mathcal{I}$  onto  $\mathbb{A}$  centered at  $\mathfrak{q}$ . The restriction of  $\rho_\mathfrak{q}$  to  $\mathbb{T}_\alpha$  does not depend on the choice of  $\mathfrak{q} \subset D_0$ .

**Remark 3.1.** Let  $A$  be an apartment containing  $\mathfrak{q}$ . Let  $\phi = \rho_{\mathfrak{q}|A} : A \rightarrow \mathbb{A}$ . Then  $\phi$  is the unique isomorphism of apartments fixing  $A \cap \mathbb{A}$ . Indeed, by (MA4), there exists an isomorphism of apartments  $\psi : A \rightarrow \mathbb{A}$  fixing  $\mathfrak{q}$ . By definition  $\phi = \psi$  and thus  $\phi$  is an isomorphism of apartments. If  $x \in \mathbb{A}$ ,  $\rho_\mathfrak{q}(x) = x$  by definition of  $\rho_\mathfrak{q}$  and thus  $\phi$  fixes  $A \cap \mathbb{A}$ . Let  $\psi' : A \rightarrow \mathbb{A}$  be an isomorphism fixing  $A \cap \mathbb{A}$ . Then  $\psi' \circ \phi^{-1} : \mathbb{A} \rightarrow \mathbb{A}$  is an isomorphism of affine spaces fixing  $\mathfrak{q}$  and thus  $\psi' = \phi$ .

**Lemma 3.2.** *Let  $v \in \tilde{K}_\alpha$  and  $x \in \mathbb{A}$ . Suppose that  $v.x \in \mathbb{A}$ . Then  $v$  fixes  $D_{\lceil \alpha(x) \rceil}$ .*

*Proof.* Let  $A = v.\mathbb{A}$ . There exists  $l \in \mathbb{Z}$  such that  $A \cap \mathbb{A}$  contains  $D_l$ . By Lemma 3.2 of [Héb16],  $A \cap \mathbb{A}$  is a half-apartment: there exists  $k \in \mathbb{Z}$  such that  $A \cap \mathbb{A} = D_k$  (our definition of half-apartments is different from the one of [Héb16] and what we call a half-apartment is a true half-apartment for [Héb16]). Let  $\phi : \mathbb{A} \rightarrow A$  be the isomorphism induced by  $v$ . By Remark 3.1,  $\phi^{-1}$  fixes  $A \cap \mathbb{A} = D_k$ , thus  $v^{-1}$  fixes  $A \cap \mathbb{A}$  and hence  $v$  fixes  $A \cap \mathbb{A}$ . One has  $v.x \in A \cap \mathbb{A}$ . Consequently  $v.(v.x) = v.x$  and thus  $v.x = x$ . Therefore  $x \in D_k$ , thus  $D_{\lceil \alpha(x) \rceil} \subset D_k$  and hence  $v$  fixes  $D_{\lceil \alpha(x) \rceil}$ .  $\square$

**Lemma 3.3.** *Let  $x \in \mathbb{A}$  and  $M_x = \{t \in \mathbb{A} | \alpha(t) = \lceil \alpha(x) \rceil\}$ . Then the map  $f : \begin{array}{l} K_\alpha.x \rightarrow K_\alpha.M_x \\ u.x \mapsto u.M_x \end{array}$  is well defined and is a bijection.*

*Proof.* Let  $u, u' \in K_\alpha$  such  $u.x = u'.x$ . By Lemma 3.2,  $u'^{-1}.u$  fixes  $M_x$  and thus  $f$  is well-defined.

Let  $u, u' \in K_\alpha$  such that  $u.M_x = u'.M_x$ ,  $v = u'^{-1}.u$  and  $A = v.\mathbb{A}$ . Then  $v.M_x = M_x \subset A \cap \mathbb{A}$  and by Lemma 3.2,  $v$  fixes  $D_{\lceil \alpha(x) \rceil}$ . In particular  $v.x = x$  and  $f$  is injective. By definition  $f$  is surjective and the lemma follows.  $\square$

Let  $\alpha_{\mathcal{I}} = \alpha \circ \rho_\mathfrak{q}$ . If  $l \in \mathbb{N}$ , one denotes by  $\mathcal{C}_l$  the set of chambers  $C$  dominating an element of  $K_\alpha.P_l$  and satisfying  $\alpha_{\mathcal{I}}(C) > l$  (which means that there exists  $X \in C$  such that  $\alpha_{\mathcal{I}}(x) > l$  for all  $x \in X$ ). Let  $C_l$  be the chamber of  $\mathbb{A}$  dominating  $P_l$  and not included in  $D_l$ .

**Lemma 3.4.** *Let  $l \in \mathbb{N}$ . Then the map  $g : \begin{array}{l} K_\alpha.M_{l+1} \rightarrow \mathcal{C}_l \\ u.M_{l+1} \rightarrow u.C_l \end{array}$  is well defined and is a bijection.*

*Proof.* Let  $u, u' \in K_\alpha$  such that  $u.M_{l+1} = u'.M_{l+1}$ . Let  $v = u'^{-1}.u$ . Then  $v.M_{l+1} \subset \mathbb{A}$  and by Lemma 3.2,  $v.C_l = C_l$ . Moreover  $\alpha_{\mathcal{I}}(u.C_l) = \alpha(C_l) > l$  and thus  $g$  is well defined.

Let  $u, u' \in K_\alpha$  such that  $u.C_l = u'.C_l$ . Let  $v = u'^{-1}.u$  and  $X \in C_l$  such that  $v$  fixes  $X$ . Let  $x \in X$  such that  $\alpha(x) > l$ . Then by Lemma 3.2,  $v$  fixes  $M_{l+1}$  and thus  $g$  is injective. It

remains to show that  $\mathcal{C}_l = K_\alpha.C_l$ . Let  $C \in \mathcal{C}_l$ . Then  $C$  dominates  $u.P_l$  for some  $u \in K_\alpha$ . By Proposition 2.9 1 of [Rou11], there exists an apartment  $A$  containing  $u.D_l$  and  $C$ . Let  $\phi : A \rightarrow \mathbb{A}$  fixing  $A \cap \mathbb{A}$  (which exists by Remark 3.1) and  $v \in K_\alpha$  inducing  $\phi$ . Then  $\alpha_{\mathcal{I}}(C) = \alpha(v.C)$ , thus  $\alpha(v.C) > l$ . Moreover  $v.C \subset \mathbb{A}$  dominates  $P_l$  and thus  $v.C = C_l$ , which concludes the proof.  $\square$

By combining Lemma 3.3 and Lemma 3.4, we get the following corollary:

**Corollary 3.5.** *Let  $x \in \mathbb{A}$ . Then if  $l = \max(1, \lceil \alpha(x) \rceil)$ ,  $|K_\alpha.x| = q'_i q_i q'_i \dots (l - 1 \text{ factors})$*

We now suppose that  $W^v$  is infinite.

**Lemma 3.6.** *Let  $F$  be a type 0 face of  $\mathbb{A}$ . Then there exists  $g \in G$  such that if  $F' = g.F$ ,  $K_F/K_F \cap K_{F'}$  is infinite.*

*Proof.* Let  $g \in G$  such  $a := g.0 \in C_f^v$ . Let  $(\alpha_k) \in (\Phi^+)^{\mathbb{N}}$  be an injective sequence. For all  $k \in \mathbb{N}$ ,  $K_{\alpha_k} \subset K_F$  and thus  $|K_F.F'| \geq |K_{\alpha_k}.a|$ . By Corollary 3.5, it suffices to show that  $\alpha_k(a) \rightarrow +\infty$  (by thickness of  $\mathcal{I}$ ).

One has  $\alpha_k = \sum_{i \in I} \lambda_{i,k} \alpha_i$ , with  $\lambda_{i,k} \in \mathbb{N}$  for all  $(i, k) \in I \times \mathbb{N}$ . By injectivity of  $(\alpha_k)$ ,  $\sum_{i \in I} \lambda_{i,k} \rightarrow +\infty$ . Therefore,  $\alpha_k(a) \rightarrow +\infty$ , which proves the lemma.  $\square$

**Proposition 3.7.** *Let  $F$  be a type 0 face of  $\mathcal{I}$ . Then there exists no topology on  $G$ , inducing a structure of topological group on  $G$  such that  $K_F$  is open and compact.*

*Proof.* Suppose that such a topology exists. Let  $g \in G$  and  $F' = g.F$ . The group  $K_{F'} = g.K_F.g^{-1}$  is open and compact and thus  $K_F \cap K_{F'}$  is open and compact. Therefore  $K_F/K_F \cap K_{F'}$  is finite: a contradiction with Lemma 3.6.  $\square$

This proposition applies to  $K = K_s = K_{F_0}$  and to the Iwahori group  $K_I = K_{C_0^+}$ , which shows that reductive groups and (non-reductive) Kac-Moody groups are very different from this point of view.

## 4 Completed Iwahori-Hecke algebra

### 4.1 Iwahori-Hecke algebra

Let us recall briefly the construction of the Iwahori-Hecke algebra of [BPGR16]. We give Bernstein-Lusztig presentation. In [BPGR16], this algebra is first defined as an algebra of functions on pairs of type 0 chambers in a measure. This definition is recalled in Section 5. The definition we give is less general and imposes restrictions on the ring of scalars. It authorizes nevertheless  $\mathbb{C}$  and  $\mathbb{Z}[\frac{1}{\sqrt{q}}, \sqrt{q}]$  if  $G$  is a split Kac-Moody group over  $\mathcal{K}$ .

Let  $(\sigma_i)_{i \in I}, (\sigma'_i)_{i \in I}$  be indeterminates satisfying the following relations:

- if  $\alpha_i(Y) = \mathbb{Z}$ , then  $\sigma_i = \sigma'_i$
- if  $r_i, r_j$  ( $i, j \in I$ ) are conjugate (i.e if  $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$ ), then  $\sigma_i = \sigma_j = \sigma'_i = \sigma'_j$ .

Let  $\mathcal{R}_1 = \mathbb{Z}[\sigma_i, \sigma'_i | i \in I]$ . In order to define the Iwahori-Hecke algebra  $\mathcal{H}$  associated to  $\mathbb{A}$  and  $(\sigma_i)_{i \in I}, (\sigma'_i)_{i \in I}$ , we first introduce the Bernstein-Lusztig-Hecke algebra  ${}^{BL}\mathcal{H}$ . Let  ${}^{BL}\mathcal{H}$  be the free  $\mathcal{R}_1$ -module with basis  $(Z^\lambda H_w)_{\lambda \in Y, w \in W^v}$ . For short, we write  $H_i = H_{r_i}$ ,  $H_w = Z^0 H_w$  and  $Z^\lambda H_1 = Z^\lambda$ , for  $i \in I$ ,  $\lambda \in Y^+$  and  $w \in W^v$ . The Iwahori-Hecke algebra  ${}^{BL}\mathcal{H}$  is the module  ${}^{BL}\mathcal{H}$  equipped with the unique product  $*$  which makes it an associative algebra and satisfying the following relations (the Bernstein-Lusztig relations):

- (BL1)  $\forall \lambda \in Y, Z^\lambda * H_w = Z^\lambda H_w,$
- (BL2)  $\forall i \in I, \forall w \in W^v, H_i * H_w = \begin{cases} H_{r_i w} & \text{if } l(r_i w) = l(w) + 1 \\ (\sigma_i - \sigma_i^{-1})H_w + H_{r_i w} & \text{if } l(r_i w) = l(w) - 1 \end{cases}$
- (BL3)  $\forall (\lambda, \mu) \in Y^2, Z^\lambda * Z^\mu = Z^{\lambda+\mu},$
- (BL4)  $\forall \lambda \in Y, \forall i \in I, H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^\lambda - Z^{r_i(\lambda)}),$  where  $b(t, u; z) = \frac{(t-t^{-1})+(u-u^{-1})z}{1-z^2}.$

The existence and unicity of such a product is Theorem 6.2 of [BPGR16]. The Iwahori-Hecke algebra  $\mathcal{H}_{\mathcal{R}_1}$  associated to  $\mathbb{A}$  and  $(\sigma_i)_{i \in I}, (\sigma'_i)_{i \in I}$  over  $\mathcal{R}_1$  is the submodule spanned by  $(Z^\lambda H_w)_{\lambda \in Y^+, w \in W^v}$ , where  $Y^+ = Y \cap \mathcal{T}$  (where  $\mathcal{T}$  is the Tits cone). When  $G$  is reductive, we find the usual Iwahori-Hecke algebra of  $G$ .

**Extension of scalars** Let  $(\mathcal{R}, \phi)$  be a pair such that  $\mathcal{R}$  is a ring containing  $\mathbb{Z}$ ,  $\phi : \mathcal{R}_1 \rightarrow \mathcal{R}$  is a ring morphism and the  $\phi(\sigma_i)$  and  $\phi(\sigma'_i)$  are invertible in  $\mathcal{R}$  for all  $i \in I$ . The Iwahori-Hecke algebra associated to  $\mathbb{A}$  and  $(\phi(\sigma_i))_{i \in I}, (\phi(\sigma'_i))_{i \in I}$  over  $\mathcal{R}$  is  $\mathcal{H}_{\mathcal{R}} = \mathcal{R} \otimes_{\mathcal{R}_1} \mathcal{H}_{\mathcal{R}_1}.$

**Split Kac-Moody case** When  $G$  is a split Kac-Moody group over  $\mathcal{K}$ , one can choose  $\sigma_i = \sigma'_i = \sqrt{q}$ , for all  $i \in I$  and  $\mathcal{R} = \mathbb{Z}[\sqrt{q}^{\pm 1}].$

## 4.2 Looijenga's algebra and almost finite sets

We fix a ring  $\mathcal{R}$  as in the above paragraph. In this subsection, we introduce almost finite sets. We use them to define the Looijenga algebra  $\mathcal{R}[[Y]]$  and we will use them in Subsection 4.4 to define  $\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\mathcal{R}}.$

**Definition 4.1.** A set  $E \subset Y$  is said to be almost finite if there exists a finite set  $J \subset Y$  such that for all  $\lambda \in E$ , there exists  $\nu \in J$  such that  $\lambda \leq_{Q^\vee} \nu.$

**Remark 4.2.** We will also use almost finite sets included in  $Y^+$  and thus we could define an almost finite set of  $Y^+$  as follows: a set  $E \subset Y^+$  is almost finite if there exists a finite set  $J \subset Y^+$  such that for all  $\lambda \in E$ , there exists  $\nu \in J$  such that  $\lambda \leq_{Q^\vee} \nu.$  This is actually the same definition by the lemma below applied to  $F = Y^+.$

**Lemma 4.3.** *Let  $E \subset Y$  be an almost finite set and  $F \subset Y$ . Then there exists a finite set  $J \subset F$  such that  $F \cap E \subset \bigcup_{j \in J} j - Q_+^\vee.$*

*Proof.* One can suppose that  $E \subset y - Q_+^\vee$ , for some  $y \in Y$ . Let  $J$  be the set of elements of  $F \cap E$  which are maximal in  $F \cap E$  for  $\leq_{Q^\vee}$ . As  $E$  is almost finite, for all  $x \in E$ , there exists  $\nu \in J$  such that  $x \leq_{Q^\vee} \nu.$  It remains to prove that  $J$  is finite. Let  $J' = \{u \in Q^\vee \mid y - u \in J\}.$  One identifies  $Q^\vee$  and  $\mathbb{Z}^I$ . If  $x = (x_i)_{i \in I}$  and  $x' = (x'_i)_{i \in I}$  one says that  $x \prec x'$  if  $x_i \leq x'_i$  for all  $i \in I$  and  $x \neq x'.$  Then the elements of  $J'$  are pairwise non comparable. Therefore  $J'$  is finite by Lemma 2.2 of [Héb17b], which completes the proof.  $\square$

**Definition 4.4.** The Looijenga algebra  $\mathcal{R}[[Y]]$  of  $Y$  over  $\mathcal{R}$  (defined in [Loo80]) is the set of formal series  $\sum_{\lambda \in Y} a_\lambda e^\lambda$ , with  $(a_\lambda) \in \mathcal{R}^Y$  such that  $\text{supp}((a_\lambda)) \subset Y$  is almost finite and the  $e^\lambda$  are symbols satisfying  $e^\lambda e^\mu = e^{\lambda+\mu}$  for all  $\lambda, \mu \in Y.$

For all  $\lambda \in Y$ , one defines  $\pi_\lambda : \mathcal{R}[[Y]] \rightarrow \mathcal{R}$  by  $\pi_\lambda(\sum_{\mu \in Y} a_\mu e^\mu) = a_\lambda$ . One sets  $\mathcal{R}[[Y^+]] = \{a \in \mathcal{R}[[Y]] \mid \pi_\lambda(a) = 0 \ \forall \lambda \in Y \setminus Y^+\}$ . One also sets  $\mathcal{R}[[Y]]^{W^v} = \{a \in \mathcal{R}[[Y]] \mid \pi_\lambda(a) = \pi_{w(\lambda)}(a) \ \forall (\lambda, w) \in Y \times W^v\}$ . Then  $\mathcal{R}[[Y^+]]$  and  $\mathcal{R}[[Y]]^{W^v}$  are sub-algebras of  $\mathcal{R}[[Y]]$ .

One denotes by  $\text{AF}_{\mathcal{R}}(Y^{++})$  the set of elements of  $\mathcal{R}^{Y^{++}}$  having almost finite support. A family  $(a_j)_{j \in J} \in (\mathcal{R}[[Y]])^J$  is said to be summable if:

- for all  $\lambda \in Y$ ,  $\{j \in J \mid \pi_\lambda(a_j) \neq 0\}$  is finite
- the set  $\{\lambda \in Y \mid \exists j \in J \mid \pi_\lambda(a_j) \neq 0\}$  is almost finite.

In this case, one sets  $\sum_{j \in J} a_j = \sum_{\lambda \in Y} b_\lambda e^\lambda \in \mathcal{R}[[Y]]$ , where  $b_\lambda = \sum_{j \in J} \pi_\lambda(a_j)$  for all  $\lambda \in Y$ .

For all  $\lambda \in Y^{++}$ , one sets  $E(\lambda) = \sum_{\mu \in W^v \cdot \lambda} e^\mu \in \mathcal{R}[[Y]]$  (this is well defined by Lemma 2.1). Let  $\lambda \in \mathcal{T}$ . There exists a unique  $\mu \in \overline{C_f^v}$  such that  $W^v \cdot \lambda = W^v \cdot \mu$ . One defines  $\lambda^{++} = \mu$ . The following two results are proved (but not stated) in the proof of Theorem 5.4 of [GR14].

**Lemma 4.5.** *Let  $x \in Y$ . Then  $W^v \cdot x$  is majorized for  $\leq_{Q^v}$  if and only if  $x \in Y^+$ .*

*Proof.* If  $x \in Y^+$ , then  $W \cdot x$  is majorized by  $x^{++}$  by Lemma 2.1.

Let  $x \in Y$  such that  $W^v \cdot x$  is majorized. Let  $y \in W^v \cdot x$  be maximal for  $\leq_{Q^v}$  and  $i \in I$ . One has  $r_i(y) \leq_{Q^v} y$  and thus  $\alpha_i(y) \geq 0$ . Therefore  $y \in \overline{C_f^v}$ , which proves that  $x \in Y^+$ .  $\square$

**Proposition 4.6.** *The map  $E : \text{AF}_{\mathcal{R}}(Y^{++}) \rightarrow \mathcal{R}[[Y]]^{W^v}$  defined by*

$$E((x_\lambda)) = \sum_{\lambda \in Y^{++}} x_\lambda E(\lambda)$$

*is well defined and is a bijection. In particular,  $\mathcal{R}[[Y]]^{W^v} \subset \mathcal{R}[[Y^+]]$ .*

*Proof.* Let  $(x_\lambda) \in \text{AF}_{\mathcal{R}}(Y^{++})$  and  $J$  be a finite set such that for all  $\mu \in \text{supp}((x_\lambda))$  there exists  $j \in J$  such that  $\mu \leq_{Q^v} j$ . Let us prove that  $(x_\lambda E(\lambda))_{\lambda \in Y^+}$  is summable. Let  $\nu \in Y$  and  $F_\nu = \{\lambda \in Y^{++} \mid \pi_\nu(x_\lambda E(\lambda)) \neq 0\}$ . Let  $\lambda \in F_\nu$ . Then  $\nu \in W^v \cdot \lambda$  and by Lemma 2.1,  $\nu \leq_{Q^v} \lambda$ . Moreover,  $\lambda \leq_{Q^v} j$  for some  $j \in J$ , which proves that  $F_\nu$  is finite.

Let  $F = \{\nu \in Y \mid \exists \lambda \in Y^{++} \mid \pi_\nu(x_\lambda E(\lambda)) \neq 0\}$ . If  $\nu \in F$  then  $\nu \leq_{Q^v} j$  for some  $j \in J$  and thus  $F$  is almost finite: the family  $(x_\lambda E(\lambda))_{\lambda \in Y^{++}}$  is summable.

As for all  $\lambda \in Y^{++}$ ,  $E(\lambda) \in \mathcal{R}[[Y]]^{W^v}$ ,  $\sum_{\lambda \in Y^{++}} x_\lambda E(\lambda) \in \mathcal{R}[[Y]]^{W^v}$ . Therefore,  $E$  is well-defined.

Let  $(x_\lambda) \in \text{AF}_{\mathcal{R}}(Y^{++}) \setminus \{0\}$  and  $\mu \in Y^{++}$  such that  $x_\mu \neq 0$  and maximal for the  $Q^v$ -order for this property. Then  $\pi_\mu(E((x_\lambda))) = x_\mu \neq 0$ :  $E((x_\lambda)) \neq 0$ . Therefore  $E$  is injective.

Let  $u = \sum_{\lambda \in Y} u_\lambda e^\lambda \in \mathcal{R}[[Y]]^{W^v}$  and  $\lambda \in \text{supp } u$ . As  $\text{supp } u$  is almost finite,  $W^v \cdot \lambda$  is majorized and by Lemma 4.5,  $\lambda \in Y^+$ . Consequently  $\text{supp } u \subset Y^+$ . One has  $u = E((u_\lambda)_{\lambda \in (\text{supp } u)^{++}})$ , and the proof is complete.  $\square$

### 4.3 Examples of almost finite sets of $Y^+$

**Affine case** Suppose that  $\mathbb{A}$  is associated to an affine Kac-Moody matrix  $A$ . By 1.3 of [Rou11], if  $\delta$  is the smallest positive imaginary root of  $A$ ,  $\mathcal{T} = \delta^{-1}(\mathbb{R}_+^*) \sqcup \mathbb{A}_{in}$ . Moreover,  $\delta$  is  $W^v$ -invariant (see 5.10 of [Rou11]) and thus  $\delta(\alpha_i^\vee) = 0$  for all  $i \in I$ . Therefore an almost finite set of  $Y^+$  is a set  $E$  such that for some  $k \in \mathbb{N}$  and  $y_1, \dots, y_k \in Y^+$ ,  $E \subset \bigcup_{i=1}^k y_i - Q_+^\vee$ .

**Indefinite case** On the contrary to the finite and affine case, when  $\mathbb{A}$  is associated to an indefinite Kac-Moody matrix,  $y - Q_+^\vee \not\subseteq Y^+$  for all  $y \in Y$ . Indeed, by the Lemma of 2.9 of [GR14] and its proof, there exists a linear form  $\delta$  on  $\mathbb{A}$  such that  $\delta(\mathcal{T}) \geq 0$  and  $\delta(\alpha_i^\vee) < 0$  for all  $i \in I$ . If  $y \in Y$  and  $i \in I$ , then  $\delta(y - n\alpha_i^\vee) < 0$  for  $n \in \mathbb{N}$  large enough and thus  $y - Q_+^\vee \not\subseteq Y^+$ . However, the inclusion  $Q_-^\vee \supset Y^+$  can occur.

**Lemma 4.7.** *One has  $Q_-^\vee \supset Y^+$  if and only if  $Q_-^\vee \supset Y^{++}$ .*

*Proof.* One implication is clear because  $Y^{++} \subset Y^+$ . Suppose  $Q_-^\vee \supset Y^{++}$ . Let  $\lambda \in Y^+$ . By Lemma 2.1,  $\lambda \leq_{Q^\vee} \lambda^{++}$  and thus  $\lambda \in Q_-^\vee$ .  $\square$

Suppose that  $\mathbb{A}$  is essential and associated to a size 2 indefinite matrix  $A$ . One writes  $A = \begin{pmatrix} 2 & a_{1,2} \\ a_{2,1} & 2 \end{pmatrix}$ , with  $a_1, a_2 \in -\mathbb{N}^*$ . One has  $\mathbb{A} = \mathbb{R}\alpha_1^\vee \oplus \mathbb{R}\alpha_2^\vee$ . If  $(\lambda, \mu) \in \mathbb{Z}^2$  such that  $\lambda\mu < 0$  then  $(2\lambda + a_{1,2}\mu)(a_{2,1}\lambda + 2\mu) < 0$  and thus  $Y^{++} \subset Q_+^\vee \cup Q_-^\vee$ . By Theorem 4.3 of [Kac94],  $Q_+^\vee \cap Y^{++} = \{0\}$ , thus  $Q_-^\vee \supset Y^{++}$  and hence  $Q_-^\vee \supset Y^+$ . Therefore an almost finite set of  $Y^+$  is just a subset of  $Y^+$ .

However this property is not true in general when the size of  $A$  is greater or equal to 3. For example if  $A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -5 & 0 & 2 \end{pmatrix}$  and  $\mathbb{A}$  is the essential realization of  $A$ ,  $-2\alpha_1^\vee + \alpha_2^\vee - \alpha_3^\vee \in Y^{++} \setminus Q_-^\vee$ .

## 4.4 Completed Iwahori-Hecke algebra

In this subsection, we define the completed Iwahori-Hecke algebra  $\hat{\mathcal{H}}$ . We equip  $W^v$  with its Bruhat order  $\leq$ . One has  $1 \leq w$  for all  $w \in W^v$ . If  $u \in W^v$ , one sets  $[1, u] = \{w \in W^v | w \leq u\}$ .

Let  $\mathcal{B} = \prod_{w \in W^v, \lambda \in Y^+} \mathcal{R}$ . If  $f = (a_{\lambda,w}) \in \mathcal{B}$ , the set  $\{(\lambda, w) \in W^v \times Y^+ | a_{\lambda,w} \neq 0\}$  is called the support of  $f$  and is denoted by  $\text{supp} f$ , the set  $\{w \in W^v | \exists \lambda \in Y^+ | a_{\lambda,w} \neq 0\}$  is called the support of  $f$  along  $W^v$  and denoted  $\text{supp}_{W^v} f$ , and the set  $\{\lambda \in Y^+ | \exists w \in W^v | a_{\lambda,w} \neq 0\}$  is called the support of  $f$  along  $Y$  and denoted  $\text{supp}_Y f$ . A set  $Z \subset Y^+ \times W^v$  is said to be almost finite if  $\{w \in W^v | \exists \lambda \in Y^+ | (\lambda, w) \in Z\}$  is finite and for all  $w \in W^v$ ,  $\{\lambda \in Y^+ | (\lambda, w) \in Z\}$  is almost finite.

Let  $\hat{\mathcal{H}}$  be the set of  $a \in \mathcal{B}$  such that  $\text{supp} a$  is almost finite. If  $a = (a_{\lambda,w}) \in \hat{\mathcal{H}}$ , one writes  $a = \sum_{(\lambda,w) \in Y^+ \times W^v} a_{\lambda,w} Z^\lambda H_w$ . For  $(\lambda, w) \in Y^+ \times W^v$ , we define  $\pi_{\lambda,w} : \hat{\mathcal{H}} \rightarrow \mathcal{R}$  by  $\pi_{\lambda,w}(\sum a_{\lambda',w'} Z^{\lambda'} H_{w'}) = a_{\lambda,w}$ . In order to extend  $*$  to  $\hat{\mathcal{H}}$ , we first prove that if  $\sum a_{\lambda,w} Z^\lambda H_w, \sum b_{\lambda,w} Z^\lambda H_w \in \hat{\mathcal{H}}$  and  $(\mu, v) \in Y^+ \times W^v$ ,

$$\sum_{(\lambda,w), (\lambda',w') \in Y^+ \times W^v} \pi_{\mu,v}(a_{\lambda,w} b_{\lambda',w'} Z^\lambda H_w * Z^{\lambda'} H_{w'})$$

is well defined, i.e that only a finite number of  $\pi_{\mu,v}(a_{\lambda,w} b_{\lambda',w'} Z^\lambda H_w * Z^{\lambda'} H_{w'})$  are non-zero. The key ingredient to prove this is the fact that if  $w \in W^v$  and  $\lambda \in Y$ , the support of  $H_w * Z^\lambda$  along  $Y^+$  is in the convex hull of the  $u.\lambda$ , for  $u \leq w$  for the Bruhat order (this is Lemma 4.10).

For  $E \subset Y$  and  $i \in I$ , one sets  $R_i(E) = \text{conv}(\{E, r_i(E)\}) \subset E + Q^\vee$ . If  $E = \{\lambda\}$ , one writes  $R_i(\lambda)$  for short. Let  $w \in W^v$ , one sets  $R_w(\lambda) = \bigcup R_{i_1}(R_{i_2}(\dots(R_{i_k}(\lambda)\dots)))$  where the union is taken over all the reduced writings of  $w$ .

**Remark 4.8.** Let  $\lambda \in Y$  and  $w \in W^v$ ,  $R_w(\lambda)$  is finite. Indeed, if  $E$  is a finite set and  $i \in I$ ,  $R_i(E)$  is bounded and included in  $E + Q^\vee$ . Therefore  $R_i(E)$  is finite and by induction, if  $k \in \mathbb{N}^*$  and  $i_1, \dots, i_k \in I$ ,  $R_{i_1}(R_{i_2}(\dots(R_{i_k}(E))\dots))$  is finite. As there are at most  $|I|^{l(w)}$  reduced writings of  $w$ , we deduce that  $R_w(\lambda)$  is finite.

**Lemma 4.9.** *Let  $\nu \in Y$  and  $i \in I$ . Then*

$$H_i * Z^\nu \in \bigoplus_{\nu' \in R_i(\nu), t' \in \{1, r_i\}} \mathcal{R}.Z^{\nu'} H_{t'}.$$

*Proof.* Suppose  $\sigma_i = \sigma'_i$ . Then by (BL4), one has

$$H_i * Z^\nu = Z^{r_i(\nu)} * H_i + (\sigma_i - \sigma_i^{-1}) Z^\nu \frac{1 - Z^{-\alpha_i(\nu)\alpha_i^\vee}}{1 - Z^{-\alpha_i^\vee}}.$$

If  $\alpha_i(\nu) = 0$ ,  $H_i * Z^\nu = Z^\nu * H_i$ .

If  $\alpha_i(\nu) > 0$ ,  $H_i * Z^\nu = Z^{r_i(\nu)} * H_i + (\sigma_i - \sigma_i^{-1}) \sum_{h=0}^{\alpha_i(\nu)-1} Z^{\nu - h\alpha_i^\vee}$  and  $r_i(\nu), \nu - h\alpha_i^\vee \in R_i(\nu)$  for all  $h \in \llbracket 0, \alpha_i(\nu) - 1 \rrbracket$ .

If  $\alpha_i(\nu) < 0$ ,  $H_i * Z^\nu = Z^{r_i(\nu)} * H_i + (\sigma_i - \sigma_i^{-1}) \sum_{h=1}^{-\alpha_i(\nu)} Z^{\nu + h\alpha_i^\vee}$  and  $r_i(\nu), \nu + h\alpha_i^\vee \in R_i(\nu)$  for all  $h \in \llbracket 1, -\alpha_i(\nu) \rrbracket$ .

Consequently if  $\sigma_i = \sigma'_i$ ,  $H_i * Z^\nu \in \sum_{\nu' \in R_i(\nu), t' \in \{1, r_i\}} \mathcal{R}.Z^{\nu'} H_{t'}$ .

Suppose  $\sigma_i \neq \sigma'_i$ . Then  $\alpha_i(Y) = 2\mathbb{Z}$ . One has

$$H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + Z^\lambda ((\sigma_i - \sigma_i^{-1}) + (\sigma'_i - \sigma_i'^{-1}) Z^{-\alpha_i^\vee}) \frac{1 - Z^{-\alpha_i(\lambda)\alpha_i^\vee}}{1 - Z^{-2\alpha_i^\vee}}$$

The same computations as above complete the proof. □

**Lemma 4.10.** *For all  $u, v \in W^v$  and  $\mu \in Y$ ,*

$$H_u * Z^\mu H_v \in \bigoplus_{\nu \in R_u(\mu), t \in [1, u].v} \mathcal{R}.Z^\nu H_t.$$

*Proof.* We do it by induction on  $l(u)$ . Let  $v \in W^v$  and  $\mu \in Y$ . Let  $k \in \mathbb{N}^*$  and suppose that for all  $w \in W^v$  such that  $l(w) \leq k - 1$ ,  $H_w * Z^\mu H_v \in \bigoplus_{\nu \in R_w(\mu), t \in [1, w].v} \mathcal{R}.Z^\nu H_t$ . Let  $u \in W^v$  and suppose that  $l(u) = k$ . One writes  $u = r_i w$ , with  $i \in I$  and  $w \in W^v$  such that  $l(w) = k - 1$ . One has

$$H_u * Z^\mu H_v = H_i * H_w * Z^\mu H_v \in \sum_{\nu \in R_w(\mu), t \in [1, w].v} \mathcal{R}.H_i * Z^\nu H_t.$$

By Lemma 4.9,

$$\sum_{\nu \in R_w(\mu), t \in [1, w].v} \mathcal{R}.H_i * Z^\nu H_t \subset \sum_{\nu \in R_w(\mu), t \in [1, w].v} \sum_{\nu' \in R_i(\nu), t' \in \{1, r_i\}} \mathcal{R}.Z^{\nu'} * H_{t'} * H_t.$$

By (BL2), if  $t' \in \{1, r_i\}$  and  $t \in [1, w].v$ ,

$$H_{t'} * H_t \in \mathcal{R}.H_{r_i.t} \oplus \mathcal{R}.H_t \subset \sum_{u' \in [1, u].v} \mathcal{R}.H_{u'},$$

and the lemma follows. □

**Lemma 4.11.** *Let  $u \in W^v$  and  $\mu \in Y$ . Then for all  $\nu \in R_u(\mu)$ , there exists  $(\lambda_{u'})_{u' \leq u} \in [0, 1]^{\{u' \in W^v \mid u' \leq u\}}$  such that  $\sum_{u' \leq u} \lambda_{u'} = 1$  and  $\nu = \sum_{u' \leq u} \lambda_{u'} u' \cdot \mu$ .*

*Proof.* We do it by induction on  $l(u)$ . Let  $k \in \mathbb{N}$  and suppose this is true for all  $u$  having length  $k$ . Let  $\tilde{u} \in W^v$  such that  $l(\tilde{u}) = k + 1$ . Let  $\nu \in R_{\tilde{u}}(\mu)$ . Then  $\nu \in R_i(\nu')$  for some  $i \in I$  and  $\nu' \in R_u(\mu)$ , for some  $u \in W^v$  having length  $k$ . One writes  $\nu = s\nu' + (1-s)r_i.\nu'$ , with  $s \in [0, 1]$ . One writes  $\nu' = \sum_{u' \leq u} \lambda_{u'} u' \cdot \mu$ . One has  $\nu = s \sum_{u' \leq u} \lambda_{u'} u' \cdot \mu + (1-s) \sum_{u' \leq u} \lambda_{u'} r_i.u' \cdot \mu$ . As  $r_i.u' \leq \tilde{u}$  for all  $u' \leq u$ , one gets the lemma.  $\square$

**Lemma 4.12.** 1. *Let  $\lambda, \mu \in Y^+$ . Then  $(\lambda + \mu)^{++} \leq_{Q^v} \lambda^{++} + \mu^{++}$ .*

2. *Let  $\mu \in Y^+$  and  $v \in W^v$ . Then for all  $\nu \in R_v(\mu)$ ,  $\nu^{++} \leq_{Q^v} \mu^{++}$ .*

*Proof.* Let  $\lambda, \mu \in Y^+$ . Let  $w \in W^v$  such that  $(\lambda + \mu)^{++} = w.(\lambda + \mu)$ . By Lemma 2.1,  $w.\lambda \leq_{Q^v} \lambda^{++}$  and  $w.\mu \leq_{Q^v} \mu^{++}$  and thus we get 1.

The point 2 is a consequence of Remark 4.8, Lemma 4.11, Lemma 2.1 and point 1.  $\square$

If  $x = \sum_{i \in I} x_i \alpha_i^\vee \in Q^\vee$ , one sets  $h(x) = \sum_{i \in I} x_i$ . If  $\lambda \in Y^+$ , we denote by  $w_\lambda$  the element  $w$  of  $W^v$  of minimal length such that  $w^{-1}.\lambda \in \overline{C_f^v}$ . One has  $\lambda = w_\lambda.\lambda^{++}$  for all  $\lambda \in Y^+$ .

**Lemma 4.13.** *Let  $\lambda \in Y^{++}$  and  $(\mu_n) \in (W^v.\lambda)^\mathbb{N}$  such that  $l(w_{\mu_n}) \rightarrow +\infty$ . Then  $h(\mu_n - \lambda) \rightarrow -\infty$ .*

*Proof.* The fact that  $h(\mu_n - \lambda)$  is well defined is a consequence of Lemma 2.1. Suppose that  $h(\mu_n - \lambda)$  does not converge to  $-\infty$ . By Lemma 2.1,  $h(\mu_n - \lambda) \leq 0$  for all  $n \in \mathbb{N}$  and thus, maybe considering a subsequence of  $(\mu_n)$ , one can suppose that  $(\mu_n)$  is injective and that  $h(\mu_n - \lambda) \rightarrow k$ , for some  $k \in \mathbb{Z}$ . Let  $(h_i)_{i \in I}$  be the dual basis of the basis  $(\alpha_i^\vee)$  of  $Q^\vee$  and  $\underline{h} = (h_i)_{i \in I}$ . For all  $i \in I$  and  $n \in \mathbb{N}$ ,  $h_i(\mu_n - \lambda) \leq 0$  and  $(\underline{h}(\mu_n - \lambda))$  is injective. This is absurd and thus  $h(\mu_n - \lambda) \rightarrow -\infty$ .  $\square$

**Lemma 4.14.** *Let  $\lambda \in Y^{++}$ ,  $\nu \in Y^+$  and  $u \in W^v$ . Then  $F = \{\mu \in W^v.\lambda \mid \nu \in R_u(\mu)\}$  is finite.*

*Proof.* Let  $N \in \mathbb{N}$  such that for all  $\nu' \in W^v.\lambda$  satisfying  $l(w_{\nu'}) \geq N$ ,  $h(\nu' - \lambda) < h(\nu - \lambda)$  ( $N$  exists by Lemma 4.13).

Let  $\mu \in F$  and  $w = w_\mu$ . One writes  $\nu = \sum_{x \leq u} \lambda_x x \cdot \mu$ , with  $\lambda_x \in [0, 1]$  for all  $x \leq u$  and  $\sum_{x \leq u} \lambda_x = 1$ , which is possible by Lemma 4.11.

If  $u' \leq u$ , one sets  $v(u') = w_{u'}. \mu$ . Suppose that for all  $u' \leq u$ ,  $l(v(u')) \geq N$ . One has

$$\nu - \lambda = \sum_{u' \leq u} \lambda_{u'} (u' \cdot \mu - \lambda) = \sum_{u' \leq u} \lambda_{u'} (v(u') \cdot \mu - \lambda)$$

and thus

$$h(\nu - \lambda) = \sum_{u' \leq u} \lambda_{u'} h(v(u') \cdot \mu - \lambda) < \sum_{u' \leq u} \lambda_{u'} h(\nu - \lambda) = h(\nu - \lambda),$$

which is absurd. Therefore,  $l(v(u')) < N$ , for some  $u' \leq u$ .

One has  $u' \cdot \mu = v(u') \cdot \mu^{++}$ , thus  $u'^{-1}.v(u') \cdot \mu^{++} = \mu$  and hence  $l(u'^{-1}.v(u')) \geq l(w)$ , by definition of  $w$ . Therefore,  $l(v(u')) + l(u) \geq l(v(u')) + l(u') \geq l(w)$ . As a consequence,  $l(w) \leq N + l(u)$  and thus  $F$  is finite.  $\square$

**Definition 4.15.** A family  $(a_j)_{j \in J} \in \hat{\mathcal{H}}^J$  is said to be summable if:

- for all  $\lambda \in Y^+$ ,  $\{j \in J \mid \exists w \in W^v \mid \pi_{w,\lambda}(a_j) \neq 0\}$  is finite

- $\bigcup_{j \in J} \text{supp } a_j$  is almost finite.

When  $(a_j)_{j \in J} \in \hat{\mathcal{H}}^J$  is summable, one defines  $\sum_{j \in J} a_j \in \hat{\mathcal{H}}$  as follows:  $\sum_{j \in J} a_j = \sum_{\lambda, w} x_{\lambda, w} Z^\lambda H_w$  where  $x_{\lambda, w} = \sum_{j \in J} \pi_{\lambda, w}(a_j)$  for all  $(\lambda, w) \in Y^+ \times W^v$ .

**Lemma 4.16.** *Let  $(a_j)_{j \in J} \in (\mathcal{H})^J$ ,  $(b_k)_{k \in K} \in (\mathcal{H})^K$  be two summable families. Then  $(a_j * b_k)_{(j, k) \in J \times K}$  is summable. Moreover  $\sum_{(j, k)} a_j * b_k$  depends only on  $\sum_{j \in J} a_j$  and  $\sum_{k \in K} b_k$  and we denote it by  $a * b$ , if  $a = \sum_{j \in J} a_j$  and  $b = \sum_{k \in K} b_k$ .*

*Proof.* For  $j \in J$ ,  $k \in K$ , one writes

$$a_j = \sum_{(\lambda, u) \in Y^+ \times W^v} x_{j, \lambda, u} Z^\lambda H_u, \quad b_k = \sum_{(\mu, v) \in Y^+ \times W^v} y_{k, \mu, v} Z^\mu H_v.$$

If  $(u, \mu, v) \in W^v \times Y \times W^v$ , one denotes by  $(z_{\nu, t}^{u, \mu, v})_{\nu \in R_u(\mu), t \in [1, u] \cdot v} \in \mathcal{R}^{R_u(\mu) \times ([1, u] \cdot v)}$  the family such that

$$H_u * Z^\mu H_v = \sum_{\nu \in R_u(\mu), t \in [1, u] \cdot v} z_{\nu, t}^{u, \mu, v} Z^\nu H_t,$$

which exists by Lemma 4.10. One has

$$a_j * b_k = \sum_{(\lambda, u), (\mu, v) \in Y^+ \times W^v, \nu \in R_u(\mu), t \in [1, u] \cdot v} x_{j, \lambda, u} y_{k, \mu, v} z_{\nu, t}^{u, \mu, v} Z^{\lambda + \nu} H_t.$$

As a consequence, if  $S_j^a = \bigcup_{u \in \text{supp}_{W^v} a_j} [1, u]$  and  $S_k^b = \bigcup_{v \in \text{supp}_{W^v} b_k} [1, v]$ ,  $\text{supp}_{W^v}(a_j * b_k) \subset S_j^a \cdot S_k^b$ .

Thus

$$S_{W^v} := \bigcup_{(j', k') \in J \times K} \text{supp}_{W^v}(a_{j'} * b_{k'}) \subset \left( \bigcup_{j' \in J} S_{j'}^a \right) \cdot \left( \bigcup_{k' \in K} S_{k'}^b \right)$$

is finite.

Let  $(\rho, s) \in Y^+ \times W^v$ . One has

$$\pi_{\rho, s}(a_j * b_k) = \sum_{(\lambda, u) \in Y^+ \times W^v, (\mu, v) \in Y^+ \times W^v, \nu \in R_u(\mu), \lambda + \nu = \rho} x_{j, \lambda, u} y_{k, \mu, v} z_{\nu, s}^{u, \mu, v}$$

Let  $S = \bigcup_{j \in J} \text{supp } a_j \cup \bigcup_{k \in K} \text{supp } b_k$  and  $S_Y = \pi_Y(S)$ , where  $\pi_Y : Y \times W^v \rightarrow Y$  is the projection on the first coordinate. By hypothesis,  $S$  and  $S_Y$  are almost finite.

Let  $k \in \mathbb{N}$  and  $\kappa_1, \dots, \kappa_k \in Y^{++}$  such that for all  $\lambda \in S_Y$ ,  $\lambda^{++} \leq_{Q^v} \kappa_i$ , for some  $i \in \llbracket 1, k \rrbracket$ .

Let  $F(\rho) = \{(\lambda, \nu) \in S_Y \times Y^+ \mid \exists (\mu, u) \in S_Y \times S_{W^v} \mid \nu \in R_u(\mu), \text{ and } \lambda + \nu = \rho\}$ . Let  $(\lambda, \nu) \in F(\rho)$ ,  $(\mu, u) \in S_Y \times S_{W^v}$  such that  $\nu \in R_u(\mu)$ . By Lemma 4.12, one has  $\lambda \leq_{Q^v} \lambda^{++} \leq_{Q^v} \kappa_i$  and  $\nu \leq_{Q^v} \mu^{++} \leq_{Q^v} \kappa_j$  for some  $i, j \in \llbracket 1, k \rrbracket$ . Therefore,  $F(\rho)$  is finite.

Let  $F'(\rho) = \{\mu \in S_Y \mid \exists (u, \lambda, \nu) \in S_{W^v} \times F(\rho) \mid \nu \in R_u(\mu)\}$ . Let  $\mu \in F'(\rho)$  and  $(u, \lambda, \nu) \in S_{W^v} \times F(\rho)$  such that  $\nu \in R_u(\mu)$ . Then by Lemma 4.12,  $\nu^{++} \leq_{Q^v} \mu^{++} \leq_{Q^v} \kappa_i$ , for some  $i \in \llbracket 1, k \rrbracket$ . As a consequence,  $F'(\rho)^{++}$  is finite and by Lemma 4.14,  $F'(\rho)$  is finite.

If  $\lambda \in Y^+$ , one sets  $J(\lambda) = \{j \in J \mid \exists u \in W^v \mid x_{j, \lambda, u} \neq 0\}$  and  $K(\lambda) = \{k \in K \mid \exists u \in W^v \mid y_{k, \lambda, u} \neq 0\}$ . Let  $F_1(\rho) = \{\lambda \in Y^+ \mid \exists \nu \in Y^+ \mid (\lambda, \nu) \in F(\rho)\}$  and  $L(\rho) = \bigcup_{(\lambda, \mu) \in F_1(\rho) \times F'(\rho)} J(\lambda) \times K(\mu)$ . Then  $L(\rho)$  is finite and for all  $(j, k) \in J \times K$ ,  $\pi_{\rho, s}(a_j * b_k) \neq 0$  implies that  $(j, k) \in L(\rho)$ .

Let  $(\rho, s) \in \bigcup_{(j, k) \in J \times K} \text{supp}(a_j * b_k)$ . Then there exist  $(\lambda, \mu) \in S_Y^2$ ,  $u \in S_{W^v}$  and  $\nu \in R_u(\mu)$  such that  $\lambda + \nu = \rho$ . Thus  $\rho^{++} \leq_{Q^v} \lambda^{++} + \mu^{++} \leq_{Q^v} \kappa_i + \kappa_{i'}$  for some  $i, i' \in \llbracket 1, k \rrbracket$ . Consequently,  $\bigcup_{(j, k) \in J \times K} \text{supp}(a_j * b_k)$  is almost finite and  $(a_j * b_k)$  is summable.



Moreover,

$$\begin{aligned} \pi_{\rho,s} \left( \sum_{(j,k) \in J \times K} a_j * b_k \right) &= \sum_{(\lambda,u) \in Y^+ \times W^v, (\mu,v) \in Y^+ \times W^v, \nu \in R_u(\mu), \lambda + \nu = \rho} \left( \sum_{(j,k) \in J \times K} x_{j,\lambda,u} y_{k,\mu,v} z_{\nu,s}^{u,\mu,v} \right) \\ &= \sum_{(\lambda,u) \in Y^+ \times W^v, (\mu,v) \in Y^+ \times W^v, \nu \in R_u(\mu), \lambda + \nu = \rho} x_{\lambda,u} y_{\mu,v} z_{\nu,s}^{u,\mu,v}, \end{aligned}$$

where  $a = \sum_{(\lambda,u) \in Y^+ \times W^v} x_{\lambda,u} Z^\lambda H_u$  and  $b = \sum_{(\mu,v) \in Y^+ \times W^v} y_{\mu,v} Z^\mu H_v$ , which completes the proof.  $\square$

**Theorem 4.17.** *The convolution  $*$  equips  $\hat{\mathcal{H}}$  with a structure of associative algebra.*

*Proof.* By Lemma 4.16,  $(\hat{\mathcal{H}}, *)$  is an algebra. The associativity comes from Lemma 4.16 and from the associativity of  $\mathcal{H}$ .  $\square$

**Definition 4.18.** The algebra  $\hat{\mathcal{H}} = \hat{\mathcal{H}}_{\mathcal{R}}$  is the *completed Iwahori-Hecke algebra* of  $(\mathbb{A}, (\sigma_i)_{i \in I}, (\sigma'_i)_{i \in I})$  over  $\mathcal{R}$ .

**Example 4.19.** Let  $\mathcal{I}$  be a measure and suppose that  $\mathcal{I}$  is thick of finite thickness and that a group  $G$  acts strongly transitively on  $\mathcal{I}$ . Let  $i \in I$  and  $P_i$  (resp.  $P'_i$ ) be a panel of  $\{x \in \mathbb{A} \mid \alpha_i(x) = 0\}$  (resp.  $\{x \in \mathbb{A} \mid \alpha_i(x) = 1\}$ ). One denotes by  $1 + q_i$  (resp.  $1 + q'_i$ ) the number of chambers containing  $P_i$  (resp.  $P'_i$ ). One sets  $\sigma_i = \sqrt{q_i}$  and  $\sigma'_i = \sqrt{q'_i}$  for all  $i \in I$ . Then  $(\sigma_i)_{i \in I}, (\sigma'_i)_{i \in I}$  satisfy the conditions of the beginning of Section 4 and the completed Iwahori-Hecke algebra over  $\mathcal{R}$  associated to  $(\mathbb{A}, (\sigma_i)_{i \in I}, (\sigma'_i)_{i \in I})$  is the completed Iwahori-Hecke algebra of  $\mathcal{I}$  over  $\mathcal{R}$ .

## 4.5 Center of Iwahori-Hecke algebras

In this subsection, we determine the center  $\mathcal{Z}(\hat{\mathcal{H}})$  of  $\hat{\mathcal{H}}$ . For this we adapt the proof of Theorem 1.4 of [NR03].

### 4.5.1 Completed Bernstein-Lusztig bimodule

In order to determine  $\mathcal{Z}(\hat{\mathcal{H}})$ , we would like to compute  $Z^\mu * z * Z^{-\mu}$  if  $z \in \mathcal{Z}(\hat{\mathcal{H}})$  and  $\mu \in Y^+$ . However, left and right multiplication by  $Z^\mu$  is defined in  $\hat{\mathcal{H}}$  only when  $\mu \in Y^+$ . We need to extend this multiplication to arbitrary  $\mu \in Y$  in a compatible way with the multiplication in  $\hat{\mathcal{H}}$ . Obviously, multiplication by  $Z^\mu$  cannot stabilize  $\hat{\mathcal{H}}$  (because  $Z^\mu * 1 = Z^\mu \notin \hat{\mathcal{H}}$  if  $\mu \in Y \setminus Y^+$ ). Thus we define a ‘‘completion’’  ${}^{BL}\overline{\mathcal{H}}$  of  ${}^{BL}\mathcal{H}$  containing  $\hat{\mathcal{H}}$ . We do not equip  ${}^{BL}\overline{\mathcal{H}}$  with a structure of algebra but we equip it with a structure of  $Y$ -bimodule compatible with the convolution product on  $\hat{\mathcal{H}}$ .

If  $a = (a_{\lambda,w}) \in \mathcal{R}^{Y \times W^v}$ , one writes  $a = \sum_{(\lambda,w) \in Y \times W^v} a_{\lambda,w} Z^\lambda H_w$ . The *support of  $a$  along  $W^v$*  is  $\{w \in W^v \mid \exists \lambda \in Y \mid a_{\lambda,w} \neq 0\}$  and is denoted  $\text{supp}_{W^v}(a)$ .

Let  ${}^{BL}\overline{\mathcal{H}} = \{a \in \mathcal{R}^{W^v \times Y} \mid \text{supp}_{W^v}(a) \text{ is finite}\}$ . If  $(\rho, s) \in Y \times W^v$ , one defines  $\pi_{\rho,s} : {}^{BL}\overline{\mathcal{H}} \rightarrow \mathcal{R}$  by  $\pi_{\rho,s}(\sum_{(\lambda,w) \in Y \times W^v} a_{\lambda,w} Z^\lambda H_w) = a_{\rho,s}$  for all  $\sum_{(\lambda,w) \in Y \times W^v} a_{\lambda,w} Z^\lambda H_w \in {}^{BL}\overline{\mathcal{H}}$ .

One considers  $\hat{\mathcal{H}}$  as a subspace of  ${}^{BL}\overline{\mathcal{H}}$ .

**Definition 4.20.** A family  $(a_j)_{j \in J} \in ({}^{BL}\overline{\mathcal{H}})^J$  is said to be summable if:

- for all  $(s, \rho) \in W^v \times Y$ ,  $\{j \in J \mid \pi_{s,\rho}(a_j) \neq 0\}$  is finite
- $\bigcup_{j \in J} \text{supp}_{W^v}(a_j)$  is finite.

When  $(a_j)_{j \in J}$  is summable, one defines  $\sum_{j \in J} a_j \in {}^{BL}\overline{\mathcal{H}}$  by

$$\sum_{j \in J} a_j = \sum_{(\lambda, w) \in Y \times W^v} a_{\lambda, w} Z^\lambda H_w,$$

with  $a_{\lambda, w} = \sum_{j \in J} \pi_{\lambda, w}(a_j)$  for all  $(\lambda, w) \in Y \times W^v$ .

**Lemma 4.21.** *Let  $(a_j) \in ({}^{BL}\mathcal{H})^J$  be a summable family of  ${}^{BL}\overline{\mathcal{H}}$ ,  $\mu \in Y$  and  $a = \sum_{j \in J} a_j$ . Then  $(a_j * Z^\mu)$  and  $(Z^\mu * a_j)$  are summable and  $\sum_{j \in J} a_j * Z^\mu$ ,  $\sum_{j \in J} Z^\mu * a_j$  depends on  $a$  (and  $\mu$ ), but not on the choice of the family  $(a_j)_{j \in J}$ .*

*One sets  $a \overline{*} Z^\mu = \sum_{j \in J} a_j * Z^\mu$  and  $Z^\mu \overline{*} a = \sum_{j \in J} Z^\mu * a_j$ . Then this convolution equips  ${}^{BL}\overline{\mathcal{H}}$  with a structure of  $Y$ -bimodule.*

*Proof.* Let  $S = \bigcup_{j \in J} \text{supp}_{W^v}(a_j)$ . If  $(\lambda, w) \in Y \times W^v$ , one sets  $J(\lambda, w) = \{j \in J \mid \pi_{\lambda, w}(a_j) \neq 0\}$ .

Let  $(\rho, s) \in Y \times W^v$ . Let  $j \in J$ . One has  $\pi_{\rho, s}(Z^\mu * a_j) = \pi_{\rho - \mu, s}(a_j)$ , therefore  $\bigcup_{j \in J} \text{supp}_{W^v}(Z^\mu * a_j) = \bigcup_{j \in J} \text{supp}_{W^v}(a_j) = S$  is finite and  $\{j \in J \mid \pi_{\rho, s}(Z^\mu * a_j) \neq 0\} = J(\rho - \mu, s)$  is finite. Consequently  $(Z^\mu * a_j)$  is summable. Moreover  $\pi_{\rho, s}(\sum_{j \in J} Z^\mu * a_j) = \pi_{\rho - \mu, s}(a)$ , which depends only on  $a$ .

Let  $w \in W^v$ . By Lemma 4.10, there exists  $(z_{\nu, t}^w)_{(\nu, t) \in R_w(\mu) \times [1, w]} \in \mathcal{R}^{R_w(\mu) \times [1, w]}$  such that

$$H_w * Z^\mu = \sum_{\nu \in R_w(\mu), t \in [1, w]} z_{\nu, t}^w Z^\nu H_t.$$

Let  $j \in J$ . One writes  $a_j = \sum_{(\lambda, w) \in Y \times W^v} a_{j, \lambda, w} Z^\lambda H_w$ , with  $(a_{j, \lambda, w}) \in \mathcal{R}^{Y \times W^v}$ .

One has:

$$\begin{aligned} \pi_{\rho, s}(a_j * Z^\mu) &= \pi_{\rho, s} \left( \sum_{(\lambda, w) \in Y \times S} a_{j, \lambda, w} Z^\lambda H_w * Z^\mu \right) \\ &= \pi_{\rho, s} \left( \sum_{(\lambda, w) \in Y \times S} \left( \sum_{\nu \in R_w(\mu), t \in [1, w]} a_{j, \lambda, w} z_{\nu, t}^w Z^{\nu + \lambda} H_t \right) \right) \\ &= \sum_{(\lambda, w) \in Y \times S} \left( \sum_{\nu \in R_w(\mu), \nu + \lambda = \rho} a_{j, \lambda, w} z_{\nu, s}^w \right). \end{aligned}$$

Let  $F_{\rho, s} = \{j \in J \mid \pi_{\rho, s}(a_j * Z^\mu) \neq 0\}$ . Then  $F_{\rho, s} \subset \bigcup_{w \in S, \nu \in R_w(\mu)} J(\rho - \nu, w)$ , which is finite. Moreover  $\text{supp}_{W^v}(a_j * Z^\mu) \subset \bigcup_{w \in S} [1, w]$  and thus  $\bigcup_{j \in J} \text{supp}_{W^v}(a_j * Z^\mu)$  is finite:  $(a_j * Z^\mu)$  is summable. One has:

$$\begin{aligned} \pi_{\rho, s} \left( \sum_{j \in J} a_j * Z^\mu \right) &= \sum_{j \in J} \left( \sum_{(\lambda, w) \in Y \times S} \left( \sum_{\nu \in R_w(\mu), \nu + \lambda = \rho} a_{j, \lambda, w} z_{\nu, s}^w \right) \right) \\ &= \sum_{(\lambda, w) \in Y \times S} \left( \sum_{\nu \in R_w(\mu), \nu + \lambda = \rho} \left( \sum_{j \in J} a_{j, \lambda, w} z_{\nu, s}^w \right) \right) \\ &= \sum_{(\lambda, w) \in Y \times S} \sum_{\nu \in R_w(\mu), \nu + \lambda = \rho} a_{\lambda, w} z_{\nu, s}^w, \end{aligned}$$

if  $a = \sum_{(\lambda, w) \in Y \times W^v} a_{\lambda, w} Z^\lambda H_w$ .

Let  $b, \mu, \mu' \in Y$ . It remains to show that  $Z^\mu \overline{*} (Z^{\mu'} \overline{*} b) = (Z^{\mu + \mu'}) \overline{*} b$ ,  $(b \overline{*} Z^\mu) \overline{*} Z^{\mu'} = b \overline{*} (Z^{\mu + \mu'})$  and  $Z^\mu \overline{*} (b \overline{*} Z^{\mu'}) = (Z^\mu \overline{*} b) \overline{*} Z^{\mu'}$ . One writes  $b = \sum_{(w, \lambda) \in W^v \times Y} b_{w, \lambda} Z^\lambda H_w$  and

one applies the first part of the lemma with  $J = W^v \times Y$ , using the fact that if  $x \in {}^{BL}\mathcal{H}$ ,  $Z^\mu * (Z^{\mu'} * x) = (Z^{\mu+\mu'}) * x$ ,  $(x * Z^\mu) * Z^{\mu'} = x * (Z^{\mu+\mu'})$  and  $Z^\mu * (x * Z^{\mu'}) = (Z^\mu * x) * Z^{\mu'}$ , which is a consequence of the associativity of  $({}^{BL}\mathcal{H}, *)$ .  $\square$

**Corollary 4.22.** *Let  $a \in \hat{\mathcal{H}}$  and  $\mu \in Y^+$ . Then  $Z^\mu * a = Z^\mu \bar{*} a$  and  $a * Z^\mu = a \bar{*} Z^\mu$ .*

#### 4.5.2 Center of Iwahori-Hecke algebras

We now write  $*$  instead of  $\bar{*}$ . Let  $\mathcal{Z}(\hat{\mathcal{H}})$  be the center of  $\hat{\mathcal{H}}$ .

**Lemma 4.23.** *Let  $a \in \mathcal{Z}(\hat{\mathcal{H}})$  and  $\mu \in Y$ . Then  $a * Z^\mu = Z^\mu * a$ .*

*Proof.* One writes  $\mu = \mu_+ - \mu_-$ , with  $\mu_+, \mu_- \in Y^+$ .

One has  $Z^{\mu_-} * (Z^{-\mu_-} * a) = a$  and  $Z^{\mu_-} * (a * Z^{-\mu_-}) = a$ . Therefore  $Z^{-\mu_-} * a = a * Z^{-\mu_-}$ . Consequently,  $Z^\mu * a = Z^{\mu_+} * a * Z^{-\mu_-} = a * Z^\mu$ .  $\square$

Let  $w \in W^v$ . Let  ${}^{BL}\bar{\mathcal{H}}_{\neq w} = \{\sum_{(\lambda,v) \in Y \times W^v} a_{\lambda,v} Z^\lambda H_v \in {}^{BL}\bar{\mathcal{H}} \mid a_{\lambda,v} \neq 0 \Rightarrow v \not\leq w\}$ ,  $\hat{\mathcal{H}}_{\neq w} = \hat{\mathcal{H}} \cap {}^{BL}\bar{\mathcal{H}}_{\neq w}$ ,  ${}^{BL}\bar{\mathcal{H}}_{=w} = \{\sum_{(\lambda,v) \in Y \times W^v} a_{\lambda,v} Z^\lambda H_v \in {}^{BL}\bar{\mathcal{H}} \mid a_{\lambda,v} \neq 0 \Rightarrow w = v\}$  and  $\hat{\mathcal{H}}_{=w} = \hat{\mathcal{H}} \cap {}^{BL}\bar{\mathcal{H}}_{=w}$ .

**Lemma 4.24.** *Let  $w \in W^v$ . Then :*

1. *For all  $\lambda \in Y$ ,*

- ${}^{BL}\bar{\mathcal{H}}_{\neq w} * Z^\lambda \subset {}^{BL}\bar{\mathcal{H}}_{\neq w}$
- $Z^\lambda * {}^{BL}\bar{\mathcal{H}}_{\neq w} \subset {}^{BL}\bar{\mathcal{H}}_{\neq w}$
- $Z^\lambda * {}^{BL}\bar{\mathcal{H}}_{=w} \subset {}^{BL}\bar{\mathcal{H}}_{=w}$

2. *Let  $\lambda \in Y$ . Then there exists  $S \in {}^{BL}\bar{\mathcal{H}}_{\neq w}$  such that  $H_w * Z^\lambda = Z^{w(\lambda)} H_w + S$ .*

*Proof.* This is a consequence of Theorem 6.2 of [BPGR16] or of Lemma 4.10 and of Lemma 4.21.  $\square$

**Lemma 4.25.** *One has  $\mathcal{Z}(\mathcal{H}) = \mathcal{Z}(\hat{\mathcal{H}}) \cap \mathcal{H}$ .*

*Proof.* Let  $a \in \mathcal{Z}(\mathcal{H})$ . Then  $a * Z^\lambda H_w = Z^\lambda H_w * a$  for all  $(\lambda, w) \in Y \times W^v$ . By Lemma 4.16,  $a \in \mathcal{Z}(\hat{\mathcal{H}})$ . The other inclusion is clear.  $\square$

Let  $A_1, \dots, A_r$  be the indecomposable components of the matrix  $A$ . Let  $J^f$  be the set of  $j \in \llbracket 1, r \rrbracket$  such that  $A_j$  is of finite type (see Theorem 4.3 of [Kac94]) and  $J^\infty = \llbracket 1, r \rrbracket \setminus J^f$ . Let  $\mathbb{A}^f = \bigoplus_{j \in J^f} \mathbb{A}_j$ . If  $j \in \llbracket 1, r \rrbracket$ , one sets  $\mathbb{A}_{j,in} = \bigcap_{\phi \in \Phi_j} \ker \phi$ , where  $\Phi_j$  is the root system of  $\mathbb{A}_j$ . One sets  $\mathbb{A}_{in}^\infty = \bigoplus_{j \in J^\infty} \mathbb{A}_{j,in}$ ,  $Y^f = Y \cap \mathbb{A}^f$  and  $Y_{in}^\infty = Y \cap \mathbb{A}_{in}^\infty$ .

The following theorem is a generalization of a well-known theorem of Bernstein, whose first published version seems to be Theorem 8.1 of [Lus83]. A part of the proof is postponed at the end of Section 5 where we study the orbits of  $\mathbb{A}$  under the action of  $W^v$ .

**Theorem 4.26.** 1. *The center  $\mathcal{Z}(\hat{\mathcal{H}})$  of  $\hat{\mathcal{H}}$  is  $R[[Y]]^{W^v}$ .*

2. *The center  $\mathcal{Z}(\mathcal{H})$  of  $\mathcal{H}$  is  $R[Y^f \oplus Y_{in}^\infty]$ .*

*Proof.* We first prove 1. Let  $z \in R[[Y]]^{W^v} \subset \hat{\mathcal{H}}$ ,  $z = \sum_{\lambda \in Y^+} a_\lambda Z^\lambda$ . Let  $i \in I$ . One has  $z = x + y$ , with  $x = \sum_{\lambda \in Y^+ | \alpha_i(\lambda) = 0} a_\lambda Z^\lambda$  and  $y = \sum_{\lambda \in Y^+ | \alpha_i(\lambda) > 0} a_\lambda (Z^\lambda + Z^{r_i(\lambda)})$ . As  $H_i * x = x * H_i$  and  $H_i * y = y * H_i$ , we get that  $z \in \mathcal{Z}(\hat{\mathcal{H}})$  and thus  $R[[Y]]^{W^v} \subset \mathcal{Z}(\hat{\mathcal{H}})$ .

Let  $z \in \mathcal{Z}(\hat{\mathcal{H}})$ . One writes  $z = \sum_{\lambda \in Y, w \in W^v} c_{\lambda, w} Z^\lambda H_w \in {}^{BL}\overline{\mathcal{H}}$ . Suppose that there exists  $w \in W^v \setminus \{1\}$  such that for some  $\lambda \in Y$ ,  $\pi_{\lambda, w}(z) \neq 0$ . Let  $m \in W^v$  maximal (for the Bruhat order) for this property. One writes  $z = x + y$  with  $x \in \hat{\mathcal{H}}_{=m}$  and  $y \in \hat{\mathcal{H}}_{\neq m}$ . One writes  $x = \sum_{\lambda \in Y} c_{\lambda, m} Z^\lambda H_m$ . By Lemma 4.23 and Lemma 4.24, if  $\mu \in Y$ ,

$$z = Z^\mu * z * Z^{-\mu} = \sum_{\lambda \in Y} c_{\lambda, m} Z^{\lambda + \mu - m(\mu)} H_m + y',$$

with  $y' \in {}^{BL}\overline{\mathcal{H}}_{\neq m}$ .

By projecting on  ${}^{BL}\overline{\mathcal{H}}_{=m}$ , we get that  $x = \sum_{\lambda \in Y} c_{\lambda, m} Z^{\lambda + \mu - m(\mu)} H_m \in \hat{\mathcal{H}}_{=m}$ . Let  $J \subset Y$  finite such that for all  $(w, \lambda) \in W^v \times Y$ ,  $c_{\lambda, w} \neq 0$  implies that there exists  $\nu \in J$  such that  $\lambda \leq \nu$ . Let  $\gamma \in Y$  such that  $c_{\gamma, m} \neq 0$ . For all  $\mu \in Y$ , one has  $\pi_{\gamma + \mu - m(\mu), m}(z) \neq 0$  therefore  $\gamma + \mu - m(\mu) \leq_{Q^v} \nu$  for some  $\nu \in J$  for all  $\mu \in Y$ . Let  $\mu \in Y \cap C_f^v$ . Let  $\nu \in J$  such that for some  $\sigma : \mathbb{N} \rightarrow \mathbb{N}$  such that  $\sigma(n) \rightarrow +\infty$ ,  $\gamma + \sigma(n)(\mu - m(\mu)) \leq_{Q^v} \nu$  for all  $n \in \mathbb{N}$ . In particular  $\gamma + \sigma(1)(\mu - m(\mu)) - \nu \in Q^v$ . By Lemma 2.1,  $\mu - m(\mu) \in Q_+^v \setminus \{0\}$  and thus for  $n$  large enough  $\gamma + \sigma(n)(\mu - m(\mu)) = \gamma + \sigma(1)(\mu - m(\mu)) + (\sigma(n) - \sigma(1))(\mu - m(\mu)) >_{Q^v} \nu$ , which is absurd. Therefore  $\mathcal{Z}(\hat{\mathcal{H}}) \subset R[[Y]]$ .

Let  $z \in \mathcal{Z}(\hat{\mathcal{H}})$ . One writes  $z = \sum_{\lambda \in Y} c_\lambda Z^\lambda$ . Let  $w \in W^v$ . By Lemma 4.24, one has  $H_w z = \sum_{\lambda \in Y} Z^{w(\lambda)} H_w + y$ , with  $y \in {}^{BL}\overline{\mathcal{H}}_{\neq w}$ . But  $H_w * z = z * H_w = \sum_{\lambda \in Y} c_\lambda Z^\lambda H_w$ . By projecting on  $\hat{\mathcal{H}}_{=w}$ , we get that  $\sum_{\lambda \in Y} c_\lambda Z^{w(\lambda)} H_w = \sum_{\lambda \in Y} c_\lambda Z^\lambda H_w$ . Therefore,  $z \in R[[Y]]^{W^v}$ .

To prove 2, Lemma 4.25 shows that  $\mathcal{Z}(\mathcal{H}) = \mathcal{H} \cap R[[Y]]^{W^v}$ . We then use Corollary 5.21 to conclude.  $\square$

**Module over the center** When  $W^v$  is finite, it is well known that  $\mathcal{H}$  is a finitely generated  $\mathcal{Z}(\mathcal{H})$ -module. Suppose that  $W^v$  is infinite and let  $\mathcal{Z} = \mathcal{Z}(\hat{\mathcal{H}})$ . Then  $\hat{\mathcal{H}}$  is not finitely generated as  $\mathcal{Z}$ -module. Indeed, let  $J$  be a finite set and  $(h_j) \in \hat{\mathcal{H}}^J$ . Then for all  $(a_j) \in \mathcal{Z}^J$ ,  $\text{supp}_{W^v}(\sum_{j \in J} a_j h_j) \subset \bigcup_{j \in J} \text{supp}_{W^v}(h_j) \subsetneq W^v$  and thus  $(h_j)$  does not span  $\hat{\mathcal{H}}$ .

## 4.6 Case of a reductive group

In this subsection, we study the case where  $G$  is reductive.

In [GR14], an almost finite set is a set  $E$  such that  $E \subset (\bigcup_{i=1}^k y_i - Q_+^v) \cap Y^{++}$  for some  $y_1, \dots, y_k \in Y$ . If  $G$  is reductive, then such a set is finite. Indeed the Kac-Moody matrix  $A$  of 2.1.1 is a Cartan matrix: it satisfies condition (FIN) of Theorem 4.3 of [Kac94]. In particular,  $Y^{++} \subset Q_+^v \oplus \mathbb{A}_{in}$ , where  $\mathbb{A}_{in} = \bigcap_{i \in I} \ker \alpha_i$ , which proves our claim.

However, the algebra  $\hat{\mathcal{H}}$  that we define is different from  $\mathcal{H}$  even in the reductive case. If  $G$  is reductive,  $\mathcal{T} = \mathbb{A}$  and thus  $Y^+ = Y$ . For instance,  $\sum_{\mu \in Q_+^v} Z^{-\mu} \in \hat{\mathcal{H}} \setminus \mathcal{H}$ .

**Proposition 4.27.** *Let  $R$  be a ring. Then  $R[[Y]]^{W^v} = R[Y]^{W^v}$  if and only if  $W^v$  is finite.*

*Proof.* Suppose that  $W^v$  is infinite. Let  $y \in Y \cap C_f^v$ . Then  $\sum_{w \in W^v} e^{w \cdot y} \in R[[Y]]^{W^v} \setminus R[Y]^{W^v}$ .

Suppose that  $W^v$  is finite. Let  $w_0$  be the longest element of  $W^v$ . By the paragraph after Theorem of Section 1.8 of [Hum92],  $w_0 \cdot Q_+^v = Q_-^v$ . Let  $E \subset Y$  be an almost finite set invariant under the action of  $W^v$ . One has  $E \subset \bigcup_{j \in J} w_0 \cdot y_j - Q_+^v$  for some finite set  $J$ . Therefore  $E = w_0 \cdot E \subset \bigcup_{j \in J} w_0 \cdot y_j + Q_+^v$ . Consequently, for all  $x \in E$ , there exists  $j, j' \in J$  such that  $w_0 \cdot y_{j'} \leq_{Q^v} x \leq_{Q^v} y_j$  and hence  $E$  is finite, which completes the proof.  $\square$

By Theorem 8.1 of [Lus83] and Theorem 4.26, when  $W^v$  is finite, one has:

$$\mathcal{Z}(\hat{\mathcal{H}}) = R[Y]^{W^v} = \mathcal{Z}(\mathcal{H}).$$

## 5 Hecke algebra associated to a parahoric subgroup

In this section, we associate Hecke algebras to subgroups more general than  $K = K_I$  (the Iwahori subgroup). This generalizes constructions of [BKP16] and [BPGR16].

### 5.1 Motivation from the reductive case

This subsection uses I 3.3 of [Vig96]. Assume that  $G$  is reductive and let  $K$  be an open compact subgroup of  $G$ . Let  $\mathbb{Z}_c(G/K)$  be the space of functions from  $G$  to  $\mathbb{Z}$  which are  $K$ -invariant under right multiplication and have compact support. One defines an action of  $G$  on this set as follows:  $g.f(x) = f(g.x)$  for all  $g \in G$ ,  $f \in \mathbb{Z}_c(G/K)$  and  $x \in G/K$ . The *Hecke algebra of  $G$  relative to  $K$*  is the algebra  $H(G, K) = \text{End}_G \mathbb{Z}_c(G/K)$  of  $G$ -equivariant endomorphisms of  $\mathbb{Z}_c(G/K)$ . Let  $\mathbb{Z}_c(G//K)$  be the ring of functions from  $G$  to  $\mathbb{Z}$ , with compact support, which are invariant under the action of  $K$  on the left and on the right. We have an isomorphism of  $\mathbb{Z}$ -modules  $\Upsilon : H(G, K) \rightarrow \mathbb{Z}_c(G//K)$  defined by  $\Upsilon(\phi) = \phi(\mathbf{1}_K)$  for all  $\phi \in H(G, K)$ . Therefore,  $H(G, K)$  is a free  $\mathbb{Z}$ -algebra, with canonical basis  $(e_g)_{g \in K \backslash G/K}$ , where  $e_g = \mathbf{1}_{KgK}$  for all  $g \in G$ . If  $\mathcal{R}$  is a commutative ring, one defines  $H_{\mathcal{R}}(G, K) = H(G, K) \otimes_{\mathbb{Z}} \mathcal{R}$ : this is the *Hecke algebra over  $\mathcal{R}$  of  $G$  relative to  $K$* .

If  $g, g' \in K \backslash G/K$ ,  $e_g e_{g'} = \sum_{g'' \in K \backslash G/K} m(g, g'; g'') e_{g''}$ , where

$$m(g, g'; g'') = |(KgK \cap g''Kg'^{-1}K)/K|$$

for all  $g'' \in K \backslash G/K$  ( $m(g, g'; g'') \neq 0$  implies  $Kg''K \subset KgKg'K$  for all  $g, g', g'' \in K \backslash G/K$ ).

We no more suppose  $G$  to be reductive. We want to define Hecke algebras relative to some subgroups of  $G$ . As there is (up to now ?) no topology on  $G$  similar to the topology of reductive groups, we cannot define “open compact” in  $G$ . However we can still define special parahoric subgroups, which are fixers of type 0 faces (whose vertices are in  $G.0$ ) in the measure  $\mathcal{I}$ . Let  $K = K_F$  be the fixer in  $G$  of some type 0 face  $F$  such that  $F_0 \subset \overline{F} \subset \overline{C_0^+}$ , where  $F_0 = F^l(0, \mathbb{A}_{in})$  and  $C_0^+ = F(0, C_f^v)$ . Using the method of Bardy-Panse, Gaussent and Rousseau of [GR14] and [BPGR16], we view the  $(KgK \cap g''Kg'^{-1}K)/K$  as intersections of “spheres” in  $\mathcal{I}$ . We prove that when  $F$  is spherical, these intersections are finite. We then define the Hecke algebra  ${}^F\mathcal{H}$  of  $G$  relative to  $K$  as follows:  ${}^F\mathcal{H}$  is the free module over  $\mathbb{Z}$  with basis  $e_g = \mathbf{1}_{KgK}$ ,  $g \in G^+$ , where  $G^+ = \{g \in G | g.0 \geq 0\}$ , with convolution product  $e_g * e_{g'} = \sum_{g'' \in K \backslash G^+/K} m(g, g'; g'') e_{g''}$ , where  $m(g, g'; g'') = |(KgK \cap g''Kg'^{-1}K)/K|$  for all  $g'' \in K \backslash G^+/K$ . To prove that this formula indeed defines an algebra, we need to prove finiteness results. We prove these results by using the fact that they are true when  $F$  is a type 0 chamber, which was proved by Bardy-Panse, Gaussent and Rousseau to define the Iwahori-Hecke algebra, and the fact that the number of type 0 chambers dominating  $F$  is finite. The reason why one uses  $G^+$  instead of  $G$  is linked to the fact that two points are not always in a same apartment. This was already done in [BK11], [GR14], [BKP16] and [BPGR16].

We also prove that when  $F$  is a type 0 face such that  $F_0 \subset \overline{F} \subset \overline{C_0^+}$ ,  $F \neq F_0$  is non-spherical, there exists  $g \in G$  such that  $(KgK \cap g''Kg'^{-1}K)/K$  is infinite and thus this method fails to associate a Hecke algebra to  $F$ .

## 5.2 Distance and spheres associated to a type 0 face

In this subsection, we define an “ $F$ -distance” (or a  $W_F$ -distance, where  $W_F$  is the fixer of  $F$  in  $W^v$ ) for each type 0 face  $F$  between  $F_0$  and  $C_0^+$ , generalizing the  $W^v$ -distance of [GR14] and the  $W$ -distance of [BPGR16].

If  $A$  (resp.  $A'$ ) is an apartment of  $\mathcal{I}$ ,  $E_1, \dots, E_k$  (resp.  $E'_1, \dots, E'_k$ ) are subsets or filters of  $A$  (resp.  $A'$ ), the notation  $\phi : (A, E_1, \dots, E_k) \mapsto (A', E'_1, \dots, E'_k)$  means that  $\phi$  is an isomorphism from  $A$  to  $A'$  induced by an element of  $G$  and that  $\phi(E_i) = E'_i$  for all  $i \in \llbracket 1, k \rrbracket$ . When we do not want to precise  $A$  and  $A'$ , we write  $\phi : (E_1, \dots, E_k) \rightarrow (E'_1, \dots, E'_k)$ .

Let  $F$  be type 0 face of  $\mathcal{I}$  such that  $F \subset \overline{C_0^+}$ . Let  $W_F$  (resp.  $K_F$ ) be the fixer of  $F$  in  $W^v$  (resp. in  $G$ ). Let  $\Delta_F = G.F$ . We have a bijection  $\Upsilon : G/K_F \rightarrow \Delta_F$  mapping each  $g.K_F$  to  $g.F$ .

If  $F_1, F_2 \in \Delta_F$ , one writes  $F_1 \leq F_2$  if  $a_1 \leq a_2$ , where  $a_1$  and  $a_2$  are the vertices of  $F_1$  and  $F_2$ . One denotes by  $\Delta_F \times_{\leq} \Delta_F$  the set  $\{(F_1, F_2) \in \Delta_F^2 | F_1 \leq F_2\}$ .

**Definition/Proposition 5.1.** If  $F' \in \Delta_F$  such that  $F' \subset \mathbb{A}$ , one sets  $[F'] = W_F.F'$ .

Let  $(F_1, F_2) \in \Delta_F \times_{\leq} \Delta_F$ . Then there exists an apartment  $A$  containing  $F_1$  and  $F_2$  and an isomorphism  $\phi : (A, F_1) \mapsto (\mathbb{A}, F)$ . One sets  $d^F(F_1, F_2) = [\phi(F_2)]$ . This does not depend on the choices we made.

*Proof.* Proposition 5.1 of [Rou11] yields the existence of  $A$ .

By definition,  $F_1 = g.F$  for some  $g \in G$ . Let  $A' = g.\mathbb{A}$ . By (MA2), there exists  $\psi : (A, F_1) \mapsto (A', F_1)$  and if  $\psi' = g|_{\mathbb{A}}^{A'}$ , then  $\phi := \psi'^{-1} \circ \psi : (A, F_1) \mapsto (\mathbb{A}, F)$ :  $\phi$  has the desired property.

Suppose  $A_1$  is an apartment containing  $F_1, F_2$  and  $\phi_1 : (A_1, F_1) \mapsto (\mathbb{A}, F)$ . By Theorem 5.18 of [Héb17a], there exists  $f : (A, F_1, F_2) \mapsto (A_1, F_1, F_2)$ . Then one has the following commutative diagram:

$$\begin{array}{ccc} (\mathbb{A}, F_1, F_2) & \xrightarrow{f} & (A_1, F_1, F_2) \\ \downarrow \phi & & \downarrow \phi_1 \\ (\mathbb{A}, F, \phi(F_2)) & \longrightarrow & (\mathbb{A}, F, \phi_1(F_2)), \end{array}$$

and the lower horizontal arrow is in  $W_F$ , which completes the proof.  $\square$

**Remark 5.2.** 1. Suppose that  $F = F_0$ . Using the natural bijection  $\Delta_{F_0} \simeq \mathcal{I}_0$ , where  $\mathcal{I}_0 = G.0$  and  $Y^{++} \simeq Y^+/W^v$ , we get that  $d^{F_0}$  is the “vectorial distance”  $d^v$  of [GR14].

Suppose that  $F = C_0^+$ . Then  $W_{C_0^+} = \{1\}$ . One has  $[C] = \{C\}$  for all  $C \in \Delta_{C_0^+}$ . Therefore the distance  $d^{C_0^+}$  is the distance  $d^W$  of [BPGR16], by identifying each element  $w$  of  $W$  to the type 0 chamber  $w.C_0^+$ .

2. We did not assume  $F$  to be spherical, which is the most important case for us. When  $F$  is spherical, one can use Proposition 1.10c) of [BPGR16], instead of Theorem 5.18 of [Héb17a] in the proof of Definition/Proposition 5.1. When  $F$  is not spherical, we will only use the  $F$ -distance for pairs of type 0 faces based at the same vertex. Therefore we could use Proposition 5.2 of [Rou11] instead of Theorem 5.18 of [Héb17a].

Let  $\Delta_{\geq F}^{\mathbb{A}} = \{E \in \Delta_F | E \subset \mathbb{A} \text{ and } E \geq F\}$ . Let  $[\Delta_F] = \{[F'] | F' \in \Delta_{\geq F}^{\mathbb{A}}\}$ . If  $E \in \Delta_F$  and  $[R] \in [\Delta_F]$ , one sets  $\mathcal{S}^F(E, [R]) = \{E' \in \Delta_F | E' \geq E \text{ and } d^F(E, E') = [R]\}$  and  $\mathcal{S}_{op}^F(E, [R]) = \{E' \in \Delta_F | E' \leq E \text{ and } d^F(E', E) = [R]\}$ . If  $E \in \Delta_{\geq F}^{\mathbb{A}}$ , one chooses  $g_E \in N$  such that  $E = g_E.F$ . Such a  $g_E$  exists: let  $g \in G$  such that  $E = g.F$  and  $A = g.\mathbb{A}$ . By (MA2) and

2.2.1) of [Rou11], there exists  $\phi : (A, g.F) \mapsto (\mathbb{A}, g.F)$ . Let  $\psi = g|_{\mathbb{A}}$ . Then  $\phi \circ \psi \in N$  and  $\phi \circ \psi(F) = \phi(E) = E$ .

**Lemma 5.3.** *Let  $[R] \in [\Delta_F]$  and  $\Upsilon : G/K_F \xrightarrow{\sim} \Delta_F$ . Then  $\Upsilon^{-1}(\mathcal{S}^F(F, [R])) = K_F g_R K_F / K_F$  and  $\Upsilon^{-1}(\mathcal{S}_{op}^F(F, [R])) = K_F g_R^{-1} K_F / K_F$ .*

*Proof.* Let  $E \in \mathcal{S}^F(F, [R])$ . Then there exists  $g \in K_F$  such that  $g.E = R = g_R.F$ . Thus  $\Upsilon^{-1}(E) \in K_F g_R K_F / K_F$ . Let  $x \in K_F g_R K_F$ ,  $x = k_1 g_R k_2$ , with  $k_1, k_2 \in K_F$ . Then  $\Upsilon(x) = k_1 g_R.F = k_1.R$ . As  $d^F$  is  $G$ -invariant,  $d^F(k_1.F, k_1.R) = d^F(F, R) = d^F(F, k_1.R)$ , and thus  $x \in \Upsilon^{-1}(\mathcal{S}^F(F, [R]))$ . The proof of the second statement is similar.  $\square$

### 5.3 Hecke algebra associated to a spherical type 0 face

In this subsection we define the Hecke algebra associated to a spherical type 0 face  $F$  between  $F_0$  and  $C_0^+$  (or to  $K_F$ ).

Let  $C, C'$  be two positive type 0 chambers based at some  $x \in \mathcal{I}_0$ . One identifies the type 0 chambers of  $\mathbb{A}$  whose vertex are in  $Y^+$  and  $W$ . Then  $d^W(C, C') (=d^{C_0^+}(C, C'))$  is in  $W^v$ . One sets  $d(C, C') = l(d^W(C, C'))$ .

**Lemma 5.4.** *Let  $C$  be a positive type 0 chamber of  $\mathcal{I}$ ,  $n \in \mathbb{N}$  and  $x$  be the vertex of  $C$ . Let  $B_n(C)$  be the set of positive type 0 chambers  $C'$  of  $\mathcal{I}$  based at  $x$  and such that  $d(C, C') \leq n$ . Then  $B_n(C)$  is finite.*

*Proof.* We do it by induction on  $n$ . The set  $B_1(F')$  is finite for all  $F' \in G.C_0^+$  because  $\mathcal{I}$  is of finite thickness. Let  $n \in \mathbb{N}$ . Suppose that  $B_k(F')$  is finite for all  $k \leq n$  and  $F' \in G.C_0^+$ . Let  $C' \in B_{n+1}(C)$ . Let  $A$  be an apartment containing  $C'$  and  $C$ , which exists by Proposition 5.1 of [Rou11] and  $\phi : A \rightarrow \mathbb{A}$  be an isomorphism of apartments such that  $\phi(C) = C_0^+$ . One has  $\phi(C') = w.C_0^+$ , with  $w \in W^v$  and  $l(w) = n + 1$ . Let  $\tilde{w} \in W^v$  such that  $l(\tilde{w}) = n$  and  $d(\tilde{w}.C_0^+, \phi(C')) = 1$ . Let  $\tilde{C} = \phi^{-1}(\tilde{w}.C_0^+)$ . Then  $d(C, \tilde{C}) = 1$  and thus  $B_{n+1}(C) \subset \bigcup_{C'' \in B_n(C)} B_1(C'')$ , which is finite.  $\square$

**Type of a type 0 face** Let  $\mathcal{F}_{\mathbb{A}}^v$  be the set of positive vectorial faces of  $\mathbb{A}$  and  $\mathcal{F}_{\mathbb{A}}^0$  be the set of positive type 0 faces of  $\mathbb{A}$  based at 0.

**Lemma 5.5.** *The map  $f : \mathcal{F}_{\mathbb{A}}^v \rightarrow \mathcal{F}_{\mathbb{A}}^0$  mapping each  $F^v \in \mathcal{F}_{\mathbb{A}}^v$  on  $F^l(0, F^v)$  is a bijection.*

*Proof.* By definition of local faces,  $f$  is surjective. Let  $F_1^v, F_2^v \in \mathcal{F}_{\mathbb{A}}^v$  such that  $F_1^v \neq F_2^v$ . As 0 is special,  $F_1^v \in f(F_1^v)$  and  $F_2^v \in f(F_2^v)$ . As  $F_1^v \cap F_2^v = \emptyset$ ,  $f(F_1^v) \neq f(F_2^v)$  (for otherwise, one would have  $\emptyset \in f(F_1^v)$ ) and thus  $f$  is a bijection.  $\square$

Let  $F$  be a positive type 0 face of  $\mathcal{I}$ . One has  $F = g_1.F_1$  for some type 0 face  $F_1 \leq C_0^+$ . Let  $J \subset I$  such that  $F_1 = F^l(0, F^v(J))$  (see Subsection 2.1.2 for the definition of  $F^v(J)$ ). The type of  $F$ , denoted  $\text{type}(F)$  is  $J$  and it is well-defined. Indeed, suppose  $F = g_2.F^l(0, F^v(J_2))$ , for some  $g_2 \in G$  and  $J_2 \subset I$ . Let  $g = g_2^{-1}.g_1$ . One has  $F^l(0, F^v(J)) = g.F^l(0, F^v(J_2))$ . By (MA2) and 2.2.1) of [Rou11], one can suppose that  $g \in N$ , thus  $g|_{\mathbb{A}} \in W^v$  and by Lemma 5.5,  $F^v(J) = g.F^v(J_2)$ . By Section 1.3 of [Rou11]  $J = J_2$  and the type is well defined.

The type is invariant under the action of  $G$  and if  $C$  is a type 0 chamber, there exists exactly one sub-face of  $C$  of type  $J$  for each  $J \subset I$ .

We now fix a spherical type 0 face  $F$  between  $F_0$  and  $C_0^+$ .

**Lemma 5.6.** *Let  $F' \in \Delta_F$  and  $\mathcal{C}_{F'}$  be the set of type 0 chambers of  $\mathcal{I}$  containing  $F'$ . Then  $\mathcal{C}_{F'}$  is finite.*

*Proof.* We fix  $C \in \mathcal{C}_{F'}$ . Let  $x$  be the vertex of  $C$  and  $C' \in \mathcal{C}_{F'}$ . Let  $A$  be an apartment containing  $C$  and  $C'$  (such an apartment exists by Proposition 5.1 of [Rou11]). We identify  $A$  and  $\mathbb{A}$  and we fix the origin of  $\mathbb{A}$  in  $x$ . There exists  $w \in W^v$  such that  $C' = w.C$ . Let  $J$  be the type of  $F'$ . Then  $w.F'$  is the face of  $C'$  of type  $J$  and thus  $w.F' = F'$ . Therefore  $w \in W_{F'}$  where  $W_{F'}$  is the fixer of  $F'$  in  $W^v$ . Therefore  $d(C, C') \leq \max\{l(x) \mid x \in W_{F'}\}$  ( $W_{F'}$  is finite by definition of spherical). Lemma 5.4 concludes the proof.  $\square$

**Lemma 5.7.** *Let  $(E_1, E_2), (E'_1, E'_2) \in \Delta_F \times_{\leq} \Delta_F$ . Then  $d^F(E_1, E_2) = d^F(E'_1, E'_2)$  if and only if there exists an isomorphism  $\phi : (E_1, E_2) \mapsto (E'_1, E'_2)$ .*

*Proof.* Suppose that  $d^F(E_1, E_2) = d^F(E'_1, E'_2) = [R]$ . Let  $\psi : (E_1, E_2) \mapsto (F, R)$  and  $\psi' : (E'_1, E'_2) \mapsto (F, R)$ . Then  $\phi = \psi'^{-1} \circ \psi : (E_1, E_2) \mapsto (E'_1, E'_2)$ .

Suppose that there exists an isomorphism  $\phi : (E_1, E_2) \mapsto (E'_1, E'_2)$ . Let  $R \in d^F(E_1, E_2)$  and  $\psi : (E_1, E_2) \mapsto (F, R)$ . Then  $\phi^{-1} \circ \psi : (E'_1, E'_2) \mapsto (F, R)$  and thus  $d^F(E'_1, E'_2) = [R] = d^F(E_1, E_2)$ , which proves the lemma.  $\square$

**Lemma 5.8.** *Let  $(F_1, F_2) \in \Delta_F \times_{\leq} \Delta_F$  and  $r := d^F(F_1, F_2)$ . Let  $R \in \Delta_{\mathbb{A}}^{\mathbb{A}}_{>F}$  such that  $r = W_F.R = [R]$ . Then if  $C_1, C_2$  are two type 0 chambers dominating  $F_1$  and  $F_2$ ,  $\bar{d}^W(C_1, C_2) \in \mathcal{C}_{\mathbb{A}}(r)$  where  $\mathcal{C}_{\mathbb{A}}(r)$  is the set of chambers of  $\mathbb{A}$  containing an element of  $r = W_F.R$ . Moreover  $\mathcal{C}_{\mathbb{A}}(r)$  is finite.*

*Proof.* Let  $A$  be an apartment containing  $C_1$  and  $C_2$  and  $\phi : (A, C_1) \mapsto (\mathbb{A}, C_0^+)$ . Then  $\phi(F_1)$  is the face of  $C_0^+$  of type  $\text{type}(F)$ :  $\phi(F_1) = F$ . Therefore  $\phi(F_2) \in W_F.R = r$  and thus  $d^W(C_1, C_2) \in \mathcal{C}_{\mathbb{A}}(r)$ .

Using the type, we get that if  $w \in W_F$ , the set of type 0 chambers of  $\mathbb{A}$  containing  $w.R$  is in bijection with the fixer of  $R$  in  $W$ , which is conjugated to  $W_F$  and the lemma follows.  $\square$

**Lemma 5.9.** *Let  $(F_1, F_2) \in \Delta_F \times_{\leq} \Delta_F$  and  $r_1, r_2 \in [\Delta_F]$ . Then the set  $\mathcal{S}^F(F_1, r_1) \cap \mathcal{S}_{op}^F(F_2, r_2)$  is finite. Its cardinal depends only on  $r_1, r_2$  and  $r := d^F(F_1, F_2)$  and we denote it by  $a_{r_1, r_2}^r$ .*

*Proof.* Let  $\mathcal{S}$  be the set of type 0 chambers containing an element of  $\mathcal{S}^F(F_1, r_1) \cap \mathcal{S}_{op}^F(F_2, r_2)$ . Let  $C_1$  (resp.  $C_2$ ) be a type 0 chamber containing  $F_1$  (resp.  $F_2$ ).

By Lemma 5.8, if  $C \in \mathcal{S}$ , one has  $d^W(C_1, C) \in \mathcal{C}_{\mathbb{A}}(r_1)$  and  $d^W(C, C_2) \in \mathcal{C}_{\mathbb{A}}(r_2)$ . Consequently,

$$\mathcal{S} \subset \bigcup_{w_1 \in \mathcal{C}_{\mathbb{A}}(r_1), w_2 \in \mathcal{C}_{\mathbb{A}}(r_2)} \{C \in \mathcal{C}_0^+ \mid C_1 \leq C \leq C_2, d^W(C_1, C) = w_1 \text{ and } d^W(C, C_2) = w_2\}.$$

By Lemma 5.8 and Proposition 2.3 of [BPG16],  $\mathcal{S}$  is finite. Hence

$$\mathcal{S}^F(F_1, r_1) \cap \mathcal{S}_{op}^F(F_2, r_2)$$

is finite.

It remains to prove the invariance of the cardinal. Let  $(F'_1, F'_2) \in \Delta_F \times_{\leq} \Delta_F$  such that  $d^F(F'_1, F'_2) = r$  and  $\phi : (F_1, F_2) \mapsto (F'_1, F'_2)$ , which exists by Lemma 5.7. Then  $\mathcal{S}^F(F'_1, r_1) \cap \mathcal{S}_{op}^F(F'_2, r_2) = \phi(\mathcal{S}^F(F_1, r_1) \cap \mathcal{S}_{op}^F(F_2, r_2))$ , which proves the lemma.  $\square$

**Lemma 5.10.** *Let  $r_1, r_2 \in [\Delta_F]$  and*

$$P_{r_1, r_2} := \{d^F(F_1, F_2) \mid (F_1, F', F_2) \in \Delta_F \times_{\leq} \Delta_F \times_{\leq} \Delta_F, d^F(F_1, F') = r_1 \text{ and } d^F(F', F_2) = r_2\}.$$

*Then  $P_{r_1, r_2}$  is finite.*



*Proof.* Let  $\mathcal{E}$  be the set of triples  $(C_1, C', C_2)$  of type 0 chambers such that for some faces  $F_1, F'$  and  $F_2$  (with  $F_1 \subset C_1, \dots$ ) of these chambers,  $d^F(F_1, F') = r_1$  and  $d^F(F', F_2) = r_2$ .

Let  $(C_1, C', C_2) \in \mathcal{E}$ . By Lemma 5.8,  $d^W(C_1, C') \in \mathcal{C}_\mathbb{A}(r_1)$  and  $d^W(C', C_2) \in \mathcal{C}_\mathbb{A}(r_2)$ . Therefore,

$$P := \{d^W(C_1, C_2) | (C_1, C', C_2) \in \mathcal{E}\} \subset \bigcup_{\mathbf{w}_1 \in \mathcal{C}_\mathbb{A}(r_1), \mathbf{w}_2 \in \mathcal{C}_\mathbb{A}(r_2)} P_{\mathbf{w}_1, \mathbf{w}_2},$$

where the  $P_{\mathbf{w}_1, \mathbf{w}_2}$  are as in Proposition 2.2 of [BPGR16] (or in the statement of this lemma). Thus  $P$  is finite.

Let  $(F_1, F', F_2) \in \Delta_F \times_{\leq} \Delta_F \times_{\leq} \Delta_F$  such that  $d^F(F_1, F') = r_1$  and  $d^F(F', F_2) = r_2$ . Then  $F_1$  and  $F_2$  are some faces of  $C_1$  and  $C_2$ , for some  $(C_1, C', C_2) \in \mathcal{E}$ . The distance  $d^F(F_1, F_2)$  is  $W_F.F''$  for some face  $F''$  of  $d^W(C_1, C_2)$ , which proves the lemma.  $\square$

Let  $\mathcal{R}$  be a unitary and commutative ring (we do not make the additional assumptions of Section 4 on  $\mathcal{R}$ ) and let  ${}^F\mathcal{H} = {}^F\mathcal{H}_{\mathcal{R}}^{\mathcal{I}}$  be the set of functions from  $G \setminus \Delta_F \times_{\leq} \Delta_F$  to  $\mathcal{R}$ . Let  $r \in [\Delta_F]$ . One defines  $T_r : \Delta_F \times_{\leq} \Delta_F \rightarrow \mathcal{R}$  by  $T_r(F_1, F_2) = \delta_{d^F(F_1, F_2), r}$  for all  $(F_1, F_2) \in \Delta_F \times_{\leq} \Delta_F$ . Then  ${}^F\mathcal{H}$  is a free  $\mathcal{R}$ -module with basis  $T_r$ , for  $r \in [\Delta_F]$ .

**Theorem 5.11.** *We equip  ${}^F\mathcal{H}$  with a product  $*$  :  ${}^F\mathcal{H} \times {}^F\mathcal{H} \rightarrow {}^F\mathcal{H}$  defined as follows: if  $\phi_1, \phi_2 \in {}^F\mathcal{H}_{\mathcal{R}}^{\mathcal{I}}$ ,*

$$\phi_1 * \phi_2(F_1, F_2) = \sum_{F' \in \Delta_F | F_1 \leq F' \leq F_2} \phi_1(F_1, F') \phi_2(F', F_2)$$

*for all  $(F_1, F_2) \in \Delta_F \times_{\leq} \Delta_F$ . This product is well defined and equips  ${}^F\mathcal{H}$  with a structure of associative algebra with identity element  $T_{\{F\}}$ . Moreover, if  $r_1, r_2 \in [\Delta_F]$ ,*

$$T_{r_1} * T_{r_2} = \sum_{r \in P_{r_1, r_2}} a_{r_1, r_2}^r T_r.$$

*Proof.* The fact that  $*$  is well defined and the expression of  $T_{r_1} * T_{r_2}$  are consequences of Lemma 5.9 and of Lemma 5.10. The associativity is clear from the definition. The fact that  $T_{\{F\}}$  is the identity element comes from the fact that  $\mathcal{S}^F(F_1, [F]) = \{F_1\}$  for all  $F_1 \in \Delta_F$ .  $\square$

**Definition 5.12.** The algebra  ${}^F\mathcal{H} = {}^F\mathcal{H}_{\mathcal{R}}^{\mathcal{I}}$  is the Hecke algebra of  $\mathcal{I}$  associated to  $F$  over  $\mathcal{R}$ .

**Remark 5.13.** Let  $g \in G^+$ , then  $\{F' \in K_F g K_F . F | F' \subset \mathbb{A}\}$  is of the form  $[R_g]$  for some  $R_g \in \Delta_{\geq F}^{\mathbb{A}}$ .

One has a bijection  $f : G \setminus \Delta_F \times_{\leq} \Delta_F \xrightarrow{\sim} K_F \setminus G^+ / K_F$ . This map is defined as follows: let  $(F_1, F_2) \in G \setminus \Delta_F \times_{\leq} \Delta_F$ . One can suppose that  $F_1 = F$ . One has  $F_2 = g.F$  for some  $g \in G$  and one sets  $f(g) = K_F g K_F$ . Then it is easy to see that  $f$  is well defined and is a bijection. One identifies  ${}^F\mathcal{H}$  and the set of functions from  $K_F \setminus G^+ / K_F$  to  $\mathcal{R}$ .

Through this identification,  $e_g = \mathbb{1}_{K_F g K_F}$  corresponds to  $T_{[R_g]}$  for all  $g \in G^+$ . If  $g, g' \in K_F \setminus G^+ / K_F$ , one has  $e_g * e_{g'} = \sum_{g'' \in K_F \setminus G^+ / K_F} m(g, g'; g'') e_{g''}$ , where  $m(g, g'; g'') = a_{[R_g], [R_{g'}]}^{[R_{g''}]}$  for all  $g'' \in K_F \setminus G^+ / K_F$ .

By Lemma 5.3 and Lemma 5.9,  $m(g, g'; g'') = |(KgK \cap g''Kg'^{-1}K)/K|$  for all  $g'' \in K_F \setminus G^+ / K_F$ .

## 5.4 Case of a non-spherical type 0 face

In [GR14], Gaussent and Rousseau associated an algebra (the spherical Hecke algebra) to the type 0 face  $F_0$ . By Remark 5.2, their distance  $d^v$  correspond to  $d^{F_0}$ . It seems natural to try to associate a Hecke algebra to each type 0 face  $F$  between  $F_0$  and  $C_0^+$ . Let  $F$  be a non-spherical type 0 face such that  $F_0 \subsetneq F \subsetneq C_0^+$  (as a consequence  $A$  is an indefinite Kac-Moody matrix of size at least 3 because if  $A$  is of finite type, all type 0 faces are spherical and if  $A$  is of affine type,  $F_0$  is the only non-spherical type 0 face of  $C_0^+$ ). As we will see in this section, the definition of the product as above leads to infinite coefficients. To prove this, we use the fact that the restriction map which associates  $w|_{Q^v}$  to each  $w \in W^v$  is injective, which is proved in [Kac94]. As this is shown for less general realizations  $\mathbb{A}$  of the Kac-Moody matrix  $A$  than we use, we need to extend this result to our framework, which is the aim of the next subsection.

### 5.4.1 Realization of a Kac-Moody matrix

In this subsection,  $\mathbb{A}$  is no more the standard apartment of  $\mathcal{I}$ .

Let  $A = (a_{i,j})_{i,j \in [1,n]}$  be a Kac-Moody matrix. A *realization* of  $A$  (see chapter 1 of [Kac94]) is a triple  $(\mathbb{A}, \Pi, \Pi^\vee)$  where  $\mathbb{A}$  is a vectorial space over  $\mathbb{R}$  (in [Kac94], Kac uses  $\mathbb{C}$  instead of  $\mathbb{R}$ ),  $\Pi = \{\alpha_1, \dots, \alpha_n\} \subset \mathbb{A}^*$  and  $\Pi^\vee = \{\alpha_1^\vee, \dots, \alpha_n^\vee\} \subset \mathbb{A}$  are sets with cardinality  $n$  satisfying the following conditions:

(F): both sets  $\Pi$  and  $\Pi^\vee$  are linearly independent

(C):  $\alpha_j(\alpha_i^\vee) = a_{i,j}$  for all  $i, j \in [1, n]$

(D):  $n - \text{rk}(A) = \dim \mathbb{A} - n$ ;

A *generalized free realization* of  $A$  is a triple  $(\mathbb{A}, \Pi, \Pi^\vee)$  as above satisfying (F) and (C). Two realizations  $(\mathbb{A}_1, \Pi_1, \Pi_1^\vee)$ ,  $(\mathbb{A}_2, \Pi_2, \Pi_2^\vee)$  are called isomorphic if there exists a vector space isomorphism  $\phi : \mathbb{A}_1 \rightarrow \mathbb{A}_2$  such that  $\phi(\Pi_1) = \Pi_2$  and  $\phi^*(\Pi_1^\vee) = \Pi_2^\vee$ . Proposition 1.1 of [Kac94] asserts in particular that up to isomorphism,  $A$  admits a unique realization  $(\mathbb{A}_0, \Pi_0, \Pi_0^\vee)$ .

If  $\mathbb{A}$  is a generalized free realization of  $A$ , the space  $\bigcap_{i=1}^n \ker \alpha_i$  is the *inessential part* of  $\mathbb{A}$  and is denoted  $\mathbb{A}_{in}$ . The following lemma is easy to prove.

**Lemma 5.14.** *Let  $\mathbb{A}$  be a generalized free realization of  $A$ . Then there exists  $\mathbb{A}' \subset \mathbb{A}$  and  $B \subset \mathbb{A}_{in}$  such that  $\mathbb{A}'$  is isomorphic to  $\mathbb{A}_0$  (as a realization of  $A$ ),  $Q_{\mathbb{A}'}^\vee \subset \mathbb{A}'$  and  $\mathbb{A} = \mathbb{A}' \oplus B$ .*

**Lemma 5.15.** *Let  $\mathbb{A}$  be a generalized free realization of  $A$ . Then the map  $W_{\mathbb{A}}^v \rightarrow \text{Aut}_{\mathbb{Z}}(Q_{\mathbb{A}}^\vee)$   
 $w \mapsto w|_{Q^v}$*

*is injective.*

*Proof.* One writes  $\mathbb{A} = \mathbb{A}' \oplus B$ , with  $\mathbb{A}'$  and  $B$  as in Lemma 5.14. For all  $x \in \mathbb{A}$  and  $w \in W_{\mathbb{A}}^v$ ,  $w(x) - x \in Q_{\mathbb{R}, \mathbb{A}}^\vee$ , where  $Q_{\mathbb{R}, \mathbb{A}}^\vee = \bigoplus_{i=1}^n \mathbb{R}\alpha_i^\vee$ . Therefore,  $\mathbb{A}'$  is stable by  $W_{\mathbb{A}}^v$ . Moreover, for all  $x \in \mathbb{A}_{in}$ ,  $w(x) = x$ . Hence the restriction map  $W_{\mathbb{A}}^v \rightarrow W_{\mathbb{A}'}^v$  is an isomorphism. As a consequence, one can suppose that  $\mathbb{A} = \mathbb{A}_0$ . But by the assertion (3.12.1) of proof of Proposition 3.12 of [Kac94] (applied to  $\Delta^\vee$  instead of  $\Delta$ ), if  $w \in W_{\mathbb{A}_0}^v$  satisfies  $w|_{\Delta^\vee} = 1$  then  $w = 1$  (where  $\Delta^\vee$  is a set included in  $Q^\vee$ ). This proves the lemma.  $\square$

### 5.4.2 Infinite intersection of spheres

In this subsection  $\mathbb{A}$  denotes the standard apartment of  $\mathcal{I}$ .

We suppose that there exists a non-spherical type 0 face  $F$  of  $\mathbb{A}$  satisfying  $F_0 \subsetneq F \subsetneq C_0^+$ .

**Lemma 5.16.** *Let  $F$  be a non-spherical type 0 face of  $\mathbb{A}$  of type 0. Then its fixer  $W_F$  in  $W$  is infinite.*

*Proof.* One can suppose that the vertex of  $F$  is 0. Then  $W_F \subset W^v$ . Let  $F^v$  be the vectorial face such that  $F = F^l(0, F^v)$ . Let us prove that  $W_F$  is the fixer  $W_{F^v}$  of  $F^v$ . Let  $w \in W_F$ . Let  $X \in F$  such that  $w$  fixes  $X$ . Then  $w$  fixes  $\mathbb{R}_+^* \cdot X \supset F^v$  and thus  $w \in W_{F^v}$ . Let  $w \in W_{F^v}$ . As 0 is special,  $F^v \in F$ . Therefore  $W_{F^v} \subset W_F$  and thus  $W_F = W_{F^v}$ . As  $F^v$  is non spherical,  $W_{F^v}$  is infinite and the lemma follows.  $\square$

**Remark 5.17.** By 1.3 of [Rou11] the vectorial faces based at 0 form a partition of the Tits cone. Therefore, if  $F^v$  is a vectorial face and if for some  $u \in F^v$  and  $w \in W^v$ ,  $w.u \in F^v$ , then  $w.F^v = F^v$ . Hence if  $W' \subset W^v$ ,  $W'.F^v$  is infinite if and only if  $W'.u$  is infinite for some  $u \in F^v$  if and only if  $W'.u$  is infinite for all  $u \in F^v$ .

For the next proposition, we use the graph of the matrix  $A$ , whose vertices are the  $i \in I$  and whose arrows are the  $\{i, j\}$  such that  $a_{i,j} \neq 0$ .

**Lemma 5.18.** *Suppose that the matrix  $A$  is indecomposable. Let  $F$  be a non-spherical type 0 face  $F$  of  $\mathbb{A}$  satisfying  $F_0 \subsetneq F \subsetneq C_0^+$ . Then there exists  $w \in W^v$  such that  $W_F.w.F$  is infinite.*

*Proof.* One writes  $F = F^l(0, F^v)$ , with  $F^v = \{u \in \mathbb{A} \mid \alpha_i(x) > 0 \ \forall i \in J \text{ and } \alpha_i(x) = 0 \ \forall i \in I \setminus J\}$  for some  $J \subset I$ . By Lemma 5.15, there exists  $k \in I$  such that  $W_F.\alpha_k^\vee$  is infinite.

Let  $i \in J$  ( $J \neq \emptyset$  because  $F_0 \subsetneq F$ ). By 4.7 of [Kac94], the graph of  $A$  is connected. Therefore, there exists a sequence  $j_1 = i, \dots, j_l = k$  such that  $a_{j_1, j_2} a_{j_2, j_3} \dots a_{j_{l-1}, j_l} \neq 0$ .

Let  $u \in F^v$ . Let us show that there exists  $w \in W^v$  such that  $\alpha_k(w.u) \neq 0$ . If  $x \in \mathbb{A}$  and  $m \in \llbracket 1, l \rrbracket$ , one says that  $x$  satisfies  $P_m$  if for all  $m' \in \llbracket m+1, l \rrbracket$ ,  $\alpha_{j_{m'}}(x) = 0$  and  $\alpha_{j_m}(x) \neq 0$ . Let  $x \in \mathbb{A}$ ,  $m \in \llbracket 1, l-1 \rrbracket$  and suppose that  $x$  satisfies  $P_m$ . Let  $x' = r_{j_m}(x) = x - \alpha_{j_m}(x)\alpha_{j_m}^\vee$ . Then  $\alpha_{j_{m+1}}(x') = -\alpha_{j_m}(x)a_{j_m, j_{m+1}} \neq 0$  and thus  $x'$  satisfies  $P_{m'}$  for some  $m' \in \llbracket m+1, l \rrbracket$ . As  $i = j_1 \in J$ ,  $\alpha_{j_1}(u) \neq 0$  and hence  $u$  satisfies  $P_m$  for some  $m \in \llbracket 1, l \rrbracket$ . Therefore, there exists  $w \in W^v$  such that  $w(u)$  satisfies  $P_l$ :  $\alpha_k(w(u)) \neq 0$ .

If  $W_F.w(u)$  is finite,  $W_F.r_k(w(u)) = W_F.(u - \alpha_k(w(u))\alpha_k^\vee)$  is infinite and thus at least one of the sets  $W_F.w(u)$  and  $W_F.r_k(w(u))$  is infinite. This proves the lemma by Remark 5.17.  $\square$

Let  $A_1, \dots, A_r$  be the indecomposable components of the Kac-Moody matrix  $A$ . One writes  $\mathbb{A} = \mathbb{A}_1 \oplus \dots \oplus \mathbb{A}_r$ , where  $\mathbb{A}_i$  is a realization of  $A_i$  for all  $i \in \llbracket 1, r \rrbracket$ . One has  $W^v = W_1^v \times \dots \times W_r^v$ , where for all  $i \in \llbracket 1, r \rrbracket$ ,  $W_i^v$  is the vectorial Weyl group of  $\mathbb{A}_i$ .

**Proposition 5.19.** *Let  $F$  be a type 0 face of  $\mathbb{A}$ . One writes  $F = \bigoplus_{i=1}^r F_i$ . Then there exists  $w \in W^v$  such that  $W_F.w.F$  is infinite if and only if there exists  $i \in \llbracket 1, r \rrbracket$  such that  $F_i$  is non-spherical and different from  $F_{i,0}$ , where  $F_{i,0}$  is the minimal type 0 face based at 0 of  $\mathbb{A}_i$  ( $F_{i,0} = F^l(0, \mathbb{A}_{i,in})$ ).*

*Proof.* With obvious notation, one has  $W_F = W_{F_1} \times \dots \times W_{F_r}$ . Suppose that there exists  $w \in W^v$  such that  $W_F.w.F$  is infinite. One writes  $w = (w_1, \dots, w_r)$ . Then  $W_F.w.F = W_{F_1}.w_1.F_1 \oplus \dots \oplus W_{F_r}.w_r.F_r$  and thus for some  $i \in \llbracket 1, r \rrbracket$ ,  $W_{F_i}.w_i.F_i$  is infinite. If  $F_i$  is spherical,  $W_{F_i}$  is finite and if  $F_i = F_{i,0}$ ,  $W_{F_i}.w_i.F_i = F_{i,0}$  and thus  $F_i$  is non-spherical and different from  $F_{i,0}$ .

The reciprocal is a consequence of Lemma 5.18.  $\square$

The following proposition is a counterexample to Lemma 5.9 when we do not consider spherical type 0 faces.

**Proposition 5.20.** *Let  $F$  be a face based at 0 such that for some  $w \in W^v$ ,  $W_F.w.F$  is infinite. Then*

$$\mathcal{S}^F(F, [w.F]) \cap \mathcal{S}_{op}^F(F, [w^{-1}.F])$$

*is infinite.*

*Proof.* Let us prove that  $W_F.w.F \subset \mathcal{S}^F(F, [w.F]) \cap \mathcal{S}_{op}^F(F, [w^{-1}.F])$ . Let  $E \in W_F.w.F$ . Then  $F \leq E \leq F$ . By definition,  $d^F(F, E) = [E] = [w.F]$ . One writes  $E = w_F.w.F$ , with  $w_F \in W_F$ . The map  $(w_F.w)^{-1} : \mathbb{A} \rightarrow \mathbb{A}$  sends  $E$  on  $F$  and  $F$  on  $w^{-1}.w_F^{-1}.F = w^{-1}.F$  and thus  $d^F(E, F) = [w^{-1}.F]$ . Therefore  $W_F.w.F \subset \mathcal{S}^F(F, [w.F]) \cap \mathcal{S}_{op}^F(F, [w^{-1}.F])$  and the proposition follows.  $\square$

For the next corollary, we use the same notation as in the paragraph above Theorem 4.26.

**Corollary 5.21.** *Let  $\lambda \in Y^+$ . Then  $W^v.\lambda$  is finite if and only if  $\lambda \in Y^f \oplus Y_{in}^\infty$ .*

*Proof.* Let  $\lambda \in Y^+$ . One writes  $\lambda = \sum_{j \in \llbracket 1, r \rrbracket} \lambda_j$ , with  $\lambda_j \in \mathbb{A}_j$  for all  $j \in \llbracket 1, r \rrbracket$ .

Suppose that  $\lambda \in Y^f \oplus Y_{in}^\infty$ . Then  $W^v.\lambda = \bigoplus_{j \in J^f} W_j^v.\lambda_j \oplus \bigoplus_{j \in J^\infty} \lambda_j$ . As  $W_j^v$  is finite for all  $j \in J^f$ , one gets one implication.

Suppose  $\lambda \notin Y^f \oplus Y_{in}^\infty$ . Let  $j \in J^\infty$  such that  $\lambda_j \notin \mathbb{A}_{j,in}$ . Let  $F_j^v$  be the vectorial face of  $\mathbb{A}_j$  containing  $\lambda_j$ . By Remark 5.17, the map  $\begin{array}{c} W_j^v.F_j^v \rightarrow W_j^v.\lambda_j \\ w.F_j^v \rightarrow w.\lambda_j \end{array}$  is well defined and is a bijection. If  $F_j^v$  is spherical,  $W_j^v.F_j^v$  is infinite because the stabilizer of  $F_j^v$  is finite. Suppose  $F_j^v$  non-spherical. Then by Lemma 5.18, there exists  $w_j \in W_j^v$  such that  $W_{F_j}.w_j.F_j^v$  is infinite, where  $W_{F_j}$  is the fixer of  $F_j^v$  in  $W_j^v$ . Therefore  $W_j^v.F_j^v$  is infinite and the lemma follows.  $\square$

## References

- [BK11] Alexander Braverman and David Kazhdan. The spherical Hecke algebra for affine Kac-Moody groups I. *Annals of mathematics*, pages 1603–1642, 2011.
- [BKP16] Alexander Braverman, David Kazhdan, and Manish M Patnaik. Iwahori–Hecke algebras for p-adic loop groups. *Inventiones mathematicae*, 204(2):347–442, 2016.
- [BPGR16] Nicole Bardy-Panse, Stéphane Gaussent, and Guy Rousseau. Iwahori-Hecke algebras for Kac-Moody groups over local fields. *Pacific J. Math.*, 285(1):1–61, 2016.
- [BT72] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local. *Publications Mathématiques de l’IHÉS*, 41(1):5–251, 1972.
- [BT84] François Bruhat and Jacques Tits. Groupes réductifs sur un corps local. *Publications Mathématiques de l’IHÉS*, 60(1):5–184, 1984.
- [GR08] Stéphane Gaussent and Guy Rousseau. Kac-Moody groups, hovels and Littelmann paths. In *Annales de l’institut Fourier*, volume 58, pages 2605–2657, 2008.
- [GR14] Stéphane Gaussent and Guy Rousseau. Spherical Hecke algebras for Kac-Moody groups over local fields. *Annals of Mathematics*, 180(3):1051–1087, 2014.
- [Héb16] Auguste Hébert. Distances on a measure (affine ordered hovel). *arXiv preprint arXiv:1611.06105*, 2016.
- [Héb17a] Auguste Hébert. Convexity in a measure. *arXiv preprint arXiv:1710.09272*, 2017.
- [Héb17b] Auguste Hébert. Gindikin–Karpelevich finiteness for Kac–Moody groups over local fields. *International Mathematics Research Notices*, 2017(22):7028–7049, 2017.

- [Hum92] James E Humphreys. *Reflection groups and Coxeter groups*, volume 29. Cambridge university press, 1992.
- [Kac94] Victor G Kac. *Infinite-dimensional Lie algebras*, volume 44. Cambridge university press, 1994.
- [Loo80] Eduard Looijenga. Invariant theory for generalized root systems. *Inventiones mathematicae*, 61(1):1–32, 1980.
- [Lus83] George Lusztig. Singularities, character formulas, and a q-analog of weight multiplicities. *Astérisque*, 101(102):208–229, 1983.
- [NR03] Kendra Nelsen and Arun Ram. Kostka-Foulkes polynomials and Macdonald spherical functions. In *Surveys in combinatorics, 2003 (Bangor)*, volume 307 of *London Math. Soc. Lecture Note Ser.*, pages 325–370. Cambridge Univ. Press, Cambridge, 2003.
- [Rém02] Bertrand Rémy. Groupes de Kac-Moody déployés et presque déployés. *Astérisque*, (277):viii+348, 2002.
- [Rou11] Guy Rousseau. Mesures affines. *Pure and Applied Mathematics Quarterly*, 7(3):859–921, 2011.
- [Rou16] Guy Rousseau. Groupes de Kac-Moody déployés sur un corps local II. Mesures ordonnées. *Bull. Soc. Math. France*, 144(4):613–692, 2016.
- [Rou17] Guy Rousseau. Almost split Kac–Moody groups over ultrametric fields. *Groups Geometry, and Dynamics*, 11:891–975, 2017.
- [Tit87] Jacques Tits. Uniqueness and presentation of Kac-Moody groups over fields. *J. Algebra*, 105(2):542–573, 1987.
- [Vig96] Marie-France Vignéras. *Représentations modulaires des groupes réductifs p-adiques. Représentations cuspidales de  $GL(n)$* , volume 137. Springer Science & Business Media, 1996.