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Nevanlinna classes associated to a closed set on $\partial\mathbb{D}$.

Eric Amar

Abstract

We introduce Nevanlinna classes of holomorphic functions associated to a closed set on the boundary of the unit disc in the complex plane and we get Blaschke type theorems relative to these classes by use of several complex variables methods. This gives alternative proofs of some results of Favorov & Golinskii, useful, in particular, for the study of eigenvalues of non self adjoint Schrödinger operators.

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1 Introduction.

We shall study classes of holomorphic functions whose zeros may appear as eigenvalues of a Schrödinger operator with a non self adjoint potential. For instance Frank and Sabin [7] use the work of Boritchev, Golinskii and Kupin [3] to get interesting estimates this way.

The aim of this work is to study Blaschke type conditions relative to Nevanlinna classes associated to a closed set on the torus. In order to do this we shall use the "way of thinking of several complex variables".

The methods used in several complex variables already proved their usefulness in the one variable case. For instance:

- the corona theorem of Carleson [5] is easier to prove and to understand thanks to the proof of T. Wolff based on L. Hörmander [8] ;
- the characterization of interpolating sequences by Carleson for H^∞ and by Shapiro & Shields for H^p are also easier to prove by these methods (see [1], last section, where they allow me to get the bounded linear extension property for the case H^p ; the H^∞ case being done by Pehr Beurling [2]).

So it is not too surprising that in the case of zero sets, they can also be useful.

In this work we shall define Nevanlinna classes of holomorphic functions in the unit disc \mathbb{D} of \mathbb{C} associated to a closed set E in the torus \mathbb{T} and we show that the zero set of functions in these Nevanlinna classes must satisfy a Blaschke type condition.

In fact, the only thing we use with respect to $u = \log |f(z)|$ is the fact that u is a subharmonic function such that $u(0) = 0$. So we can replace $\log |f(z)|$ by any subharmonic function u in the unit disc and the "zeros formula" $\Delta \log |f| = \sum_{a \in Z(f)} \delta_a$ by the Riesz measure associated to u , $d\mu := \Delta u$,

which is a positive measure.

As an application we get an alternative proof of results by Favorov & Golinskii [6]. See also Boritchev, Golinskii and Kupin [4].

Let $E = \bar{E} \subset \mathbb{T}$ be a closed set and $p \geq 0$, $q > 0$ real numbers ; set $\forall z \in \mathbb{D}$, $d(z, E)$ the euclidean distance from z to E and $\varphi(z) := d(z, E)^q$. Then we define a Nevanlinna class of functions associated to E , p , q this way. For $p > 0$:

Definition 1.1 Let $E = \bar{E} \subset \mathbb{T}$. We say that an holomorphic function f in \mathbb{D} is in the generalised Nevanlinna class $\mathcal{N}_{\varphi,p}(\mathbb{D})$ for $p > 0$ if $\exists \delta > 0$, $\delta < 1$ such that

$$\|f\|_{\mathcal{N}_{\varphi,p}} := \sup_{1-\delta < s < 1} \int_{\mathbb{D}} (1 - |z|)^{p-1} \varphi(sz) \log^+ |f(sz)| < \infty.$$

And, for $p = 0$,

Definition 1.2 Let $E = \bar{E} \subset \mathbb{T}$. We say that an holomorphic function f is in the generalised Nevanlinna class $\mathcal{N}_{d(\cdot,E)^q,0}$ if $\exists \delta > 0$, $\delta < 1$ such that

$$\begin{aligned} \|f\|_{\mathcal{N}_{d(\cdot,E)^q,0}} := & \sup_{1-\delta < s < 1} \left\{ \int_{\mathbb{T}} d(se^{i\theta}, E)^q \log^+ |f(se^{i\theta})| + \right. \\ & \left. + \int_{\mathbb{D}} d(sz, E)^{q-1} \log^+ |f(sz)| + \int_{\mathbb{D}} (1 - |sz|^2)^{q-1} \log^+ |f(sz)| \right\} < \infty. \end{aligned}$$

And we prove the Blaschke type condition, for $p \geq 0$,

Theorem 1.3 Let $E = \bar{E} \subset \mathbb{T}$. Suppose $q > 0$ and $f \in \mathcal{N}_{\varphi,p}(\mathbb{D})$ with $|f(0)| = 1$, then

$$\sum_{a \in Z(f)} (1 - |a|^2)^{1+p} \varphi(a) \leq c(\varphi) \|f\|_{\mathcal{N}_{\varphi,p}}.$$

As an application we get also the following results, which are special cases of results of Favorov & Golinskii [6]. See also Boritchev, Golinskii and Kupin [4].

Theorem 1.4 Suppose that $f \in \mathcal{H}(\mathbb{D})$, $|f(0)| = 1$ and

$$\forall z \in \mathbb{D}, \log^+ |f(z)| \leq \frac{K}{(1 - |z|^2)^p} \frac{1}{d(z, E)^q},$$

then we have, with any $\epsilon > 0$, $p > 0$,

$$\sum_{a \in Z(f)} (1 - |a|^2)^{1+p} d(a, E)^{(q-\alpha(E)+\epsilon)_+} \leq c(p, q, \epsilon)K.$$

And in the case $p = 0$,

Theorem 1.5 Suppose that $f \in \mathcal{H}(\mathbb{D})$, $|f(0)| = 1$ and

$$\forall z \in \mathbb{D}, \log^+ |f(z)| \leq K \frac{1}{d(z, E)^q},$$

then

$$\sum_{a \in Z(f)} (1 - |a|^2) d(a, E)^{(q-\alpha(E)+\epsilon)_+} \leq c(q, \epsilon)K.$$

1.1 Notations.

Let $E = \bar{E} \subset \mathbb{T}$ be a closed set ; we have $\mathbb{T} \setminus E = \bigcup_{j \in \mathbb{N}} (\alpha_j, \beta_j)$ where the $F_j := (\alpha_j, \beta_j)$ are the contiguous intervals to E . Set $2\delta_j$ the length of the arc F_j .

Let $\Gamma_j := \{z = re^{i\psi} \in \mathbb{D} :: \psi \in (\alpha_j, \beta_j)\}$ the conical set based on F_j and $\Gamma_E := \{z = re^{i\psi} \in \mathbb{D} :: \psi \in E\}$.

Let

$$\chi \in \mathcal{C}^\infty(\mathbb{R}), \quad t \leq 2 \Rightarrow \chi(t) = 0, \quad t \geq 3 \Rightarrow \chi(t) = 1.$$

Now we define:

$$\forall z \in \Gamma_j, \quad \psi_j(z) := \frac{|z - \alpha_j|^2 |z - \beta_j|^2}{\delta_j^2}, \quad \eta_j(z) := \chi\left(\frac{|z - \alpha_j|^2}{(1 - |z|^2)^2}\right) \chi\left(\frac{|z - \beta_j|^2}{(1 - |z|^2)^2}\right),$$

and

$$\forall z \in \Gamma_j, \quad \varphi_j(z) := \eta_j(z) \psi_j(z)^q + (1 - |z|^2)^{2q}, \quad \forall z \in \Gamma_E, \quad \varphi_E(z) := (1 - |z|^2)^{2q}.$$

Lemma 1.6 We have

$$\forall z \in \Gamma_j, \quad \varphi_j(z) \geq \frac{1}{3^q} d(z, \{\alpha_j, \beta_j\})^{2q}$$

and

$$\forall z \in \Gamma_j, \quad \varphi_j(z) \leq (4^q + 2^q) d(z, \{\alpha_j, \beta_j\})^{2q}.$$

Proof.

We have

$$\forall z \in \Gamma_j, \quad d(z, \{\alpha_j, \beta_j\}) = \min(|z - \alpha_j|, |z - \beta_j|).$$

Suppose that $d(z, \{\alpha_j, \beta_j\}) = |z - \alpha_j|$ then $|z - \beta_j| \geq |z - \alpha_j|$ hence $|z - \beta_j| \geq \delta_j$. So

$$\psi_j(z) := \frac{|z - \alpha_j|^2 |z - \beta_j|^2}{\delta_j^2} \geq |z - \alpha_j|^2 = d(z, \{\alpha_j, \beta_j\})^2.$$

Now

• if $\eta_j(z) = 1$, then

$$\forall z \in \Gamma_j, \quad d(z, \{\alpha_j, \beta_j\})^{2q} \leq \psi_j(z)^q \leq \eta_j(z) \psi_j(z)^q + (1 - |z|^2)^{2q} = \varphi_j(z).$$

• Suppose $d(z, \{\alpha_j, \beta_j\}) = |z - \alpha_j|$ then if $\eta_j(z) < 1$ then either $|z - \alpha|^2 \leq 3(1 - |z|^2)^2$ or $|z - \beta|^2 \leq 3(1 - |z|^2)^2$. Suppose that $|z - \alpha|^2 \leq 3(1 - |z|^2)^2$ we have

$$d(z, \{\alpha_j, \beta_j\})^2 = |z - \alpha_j|^2 \leq 3(1 - |z|^2)^2 \Rightarrow (1 - |z|^2)^2 \geq \frac{1}{3}d(z, \{\alpha_j, \beta_j\})^2$$

hence

$$\varphi_j(z) = \eta_j(z)\psi_j(z)^q + (1 - |z|^2)^{2q} \geq (1 - |z|^2)^{2q} \geq \frac{1}{3^q}d(z, \{\alpha_j, \beta_j\})^{2q}.$$

If $|z - \beta|^2 \leq 3(1 - |z|^2)^2$ still with $d(z, \{\alpha_j, \beta_j\}) = |z - \alpha_j|$ then

$$|z - \alpha_j| \leq |z - \beta_j| \leq 3(1 - |z|^2)^2 \Rightarrow (1 - |z|^2)^2 \geq \frac{1}{3}d(z, \{\alpha_j, \beta_j\})^2$$

and again $\varphi_j(z) \geq \frac{1}{3^q}d(z, \{\alpha_j, \beta_j\})^{2q}$.

Hence in any cases we have $\varphi_j(z) \geq 3^{-q}d(z, \{\alpha_j, \beta_j\})^{2q}$.

For the other way, still with $d(z, \{\alpha_j, \beta_j\}) = |z - \alpha_j|$, we have, if $\eta_j(z) > 0$, that $|z - \alpha|^2 \geq 2(1 - |z|^2)^2$ hence, with $z = \rho e^{i\theta}$,

$$|z - \beta|^2 = (1 - \rho)^2 + |e^{i\theta} - \beta|^2 \geq |z - \alpha|^2 \geq 2(1 - \rho^2)^2$$

hence

$$|e^{i\theta} - \beta|^2 \geq (1 - \rho^2)^2 \Rightarrow (1 - \rho)^2 \leq |e^{i\theta} - \beta|^2.$$

So

$$|z - \beta|^2 = (1 - \rho)^2 + |e^{i\theta} - \beta|^2 \leq |e^{i\theta} - \beta|^2 + |e^{i\theta} - \beta|^2 = 2|e^{i\theta} - \beta|^2 \leq 2\delta^2. \quad (1.1)$$

Putting it in ψ we get

$$\psi_j(z) := \frac{|z - \alpha_j|^2 |z - \beta_j|^2}{\delta_j^2} \leq 2|z - \alpha_j|^2 = 2d(z, \{\alpha_j, \beta_j\})^2 \quad (1.2)$$

hence

$$\eta_j(z)\psi_j(z) \leq 2d(z, \{\alpha_j, \beta_j\})^2.$$

Because $(1 - |z|^2)^2 \leq 4d(z, \{\alpha_j, \beta_j\})^2$ we get

$$\varphi_j(z) = \eta_j(z)\psi_j(z)^q + (1 - |z|^2)^{2q} \leq (4^q + 2^q)d(z, \{\alpha_j, \beta_j\})^{2q}. \quad \blacksquare$$

Lemma 1.7 *There is a function $\varphi \in \mathcal{C}^\infty(\mathbb{D})$ such that φ coincides with φ_j and φ_E in their domains of definition.*

Proof.

Clearly $\eta_j(z)\psi_j(z)^q$ is in $\mathcal{C}^\infty(\Gamma_j)$ so the question is between Γ_j and Γ_E . But for any $s < 1$ and $z \in \Gamma_j \cap D(0, s)$ we have that, for any multi index $\alpha \in \mathbb{N}^2$, $\partial^\alpha[\eta_j(z)\psi_j(z)^q] \rightarrow 0$ when $z \rightarrow z_0 \in$

$\partial\Gamma_j \cap D(0, s)$ because $\chi\left(\frac{|z - \alpha_j|^2}{(1 - |z|^2)^2}\right)$ goes to 0 with all its derivatives when $\frac{|z - \alpha_j|^2}{(1 - |z|^2)^2} \rightarrow 0$. The

same for $\chi\left(\frac{|z - \beta_j|^2}{(1 - |z|^2)^2}\right)$. So $\eta_j(z)\psi_j(z)^q$ extends \mathcal{C}^∞ by 0 to $\Gamma_E \cap D(0, s)$. And $\varphi_E(z) := (1 - |z|^2)^{2q}$

is already global and $\mathcal{C}^\infty(\mathbb{D})$. (Not in $\mathcal{C}^\infty(\bar{\mathbb{D}})$!)

So φ_j being the sum of these functions extends \mathcal{C}^∞ to the open disc. \blacksquare

Now we set, for $0 \leq s < 1$ and $q > 0$, $g_s(z) := (1 - |z|^2)^{p+1} \varphi(sz) \in \mathcal{C}^\infty(\bar{\mathbb{D}})$ so we can apply the Green formula to it. Recall that $f_s(z) := f(sz)$.

In fact in the case of $\log |f_s|$, even if this function is not \mathcal{C}^2 , this is quite well known but for sake of completeness we give a proof as lemma 7.9. Now, because everything works exactly the same way if we replace $\log |f_s|$ by $v(sz)$ where v is a subharmonic function in the unit disc \mathbb{D} , we give also a proof of the Green formula in that case in lemma 7.10, in the appendix. Throughout this work we let $\log |f|$ instead of a general subharmonic function v because it is the most interesting case.

With the "zero" formula: $\Delta \log |f_s| = \sum_{a \in Z(f_s)} \delta_a$ we get

$$\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log |f(sz)| \Delta g_s(z) + \int_{\mathbb{T}} (g_s \partial_n \log |f(sz)| - \log |f(sz)| \partial_n g_s).$$

Because $g_s = 0$ on \mathbb{T} , we get:

$$\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log |f(sz)| \Delta g_s(z) - \int_{\mathbb{T}} \log |f(se^{i\theta})| \partial_n g_s(e^{i\theta}).$$

If, moreover $p > 0$, $\partial_n g_s = 0$ on \mathbb{T} , hence $\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log |f(sz)| \Delta g_s(z)$. So we proved

Lemma 1.8 *Let $p > 0$ and $f \in \mathcal{H}(\mathbb{D})$ we have*

$$\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log |f(sz)| \Delta g_s(z).$$

We have to compute

$$\Delta g_s(z) \log |f(sz)| = \Delta g_s(z) \log^+ |f(sz)| - \Delta g_s(z) \log^- |f(sz)|.$$

We have $\Delta g_s = 4\bar{\partial}\partial g_s$ hence

$$\begin{aligned} \Delta g_s(z) &= \Delta[(1 - |z|^2)^{p+1} \varphi(sz)] = \varphi(sz) \Delta[(1 - |z|^2)^{p+1}] + (1 - |z|^2)^{p+1} \Delta[\varphi(sz)] + \\ &\quad + 8\Re[\partial((1 - |z|^2)^{p+1}) \bar{\partial}(\varphi(sz))]. \end{aligned}$$

Recall that, with $\varphi_{A,j}(z) := \eta_j(z) \psi_j(z)^q$ and $\varphi_{C,j} := (1 - |z|^2)^{2q}$,

$$\forall z \in \Gamma_j, \varphi_j(z) := \varphi_{A,j}(z) + \varphi_{C,j}(z);$$

we start with the last term.

2 Estimates on $\varphi_{C,j}(z) := (1 - |z|^2)^{2q}$.

In this case

$$g_{C,s}(z) := (1 - |z|^2)^{p+1} \varphi_C(sz) = (1 - |z|^2)^{p+1} (1 - |sz|^2)^{2q}.$$

So we have to compute $\Delta[(1 - |z|^2)^{p+1} \varphi_C(z)] = A_1 + A_2 + A_3$ with:

$$\begin{aligned} A_1 &:= (1 - |sz|^2)^{2q} \Delta((1 - |z|^2)^{p+1}) = \\ &= -4(p+1)(1 - |z|^2)^p (1 - |sz|^2)^{2q} + 4p(p+1)(1 - |z|^2)^{p-1} |z|^2 (1 - |sz|^2)^{2q}; \\ A_2 &:= (1 - |z|^2)^{p+1} \Delta((1 - |sz|^2)^{2q}) = \\ &= -8sq(1 - |z|^2)^{p+1} (1 - |sz|^2)^{2q-1} + 8q(2q-1)(1 - |z|^2)^{p+1} |z|^2 (1 - |sz|^2)^{2q-2}; \\ A_3 &:= 8\Re[\partial((1 - |z|^2)^{p+1}) \bar{\partial}((1 - |sz|^2)^{2q})] = \\ &= 16sq(p+1)(1 - |z|^2)^p (1 - |sz|^2)^{2q-1} |z|^2. \end{aligned}$$

We shall consider the terms $\Delta g_{C,s}(z) \log^+ |f(sz)|$. We shall use

Lemma 2.1 For $p > 0$ we have:

$$\forall z \in \mathbb{D}, \Delta g_{C,s}(z) \leq c(p, q)(1 - |z|^2)^{p-1} |z|^2 (1 - |sz|^2)^{2q}.$$

And for $p = 0$ we have:

$$\forall z \in \mathbb{D}, \Delta g_{C,s}(z) \leq c(q) |z|^2 (1 - |sz|^2)^{2q-1},$$

with $c(q) := 8q(2q - 1) + 16q(p + 1)$ (hence $c(0) = 0$).

Proof.

We have

$$A_1 \leq 4p(p + 1)(1 - |z|^2)^{p-1} |z|^2 (1 - |sz|^2)^{2q},$$

because $-4(p + 1)(1 - |z|^2)^p (1 - |sz|^2)^{2q}$ is negative.

$$A_2 \leq 8q(2q - 1)(1 - |z|^2)^{p+1} |z|^2 (1 - |sz|^2)^{2q-2},$$

because $-8sq(1 - |z|^2)^{p+1} (1 - |sz|^2)^{2q-1}$ is negative. So adding, we get

$$\begin{aligned} \Delta g_{C,s}(z) &\leq 4p(p + 1)(1 - |z|^2)^{p-1} |z|^2 (1 - |sz|^2)^{2q} + \\ &\quad + 8q(2q - 1)(1 - |z|^2)^{p+1} |z|^2 (1 - |sz|^2)^{2q-2} + \\ &\quad + 16sq(p + 1)(1 - |z|^2)^p (1 - |sz|^2)^{2q-1} |z|^2. \end{aligned}$$

If $p > 0$ we use $(1 - |z|^2) \leq (1 - |sz|^2)$ to get

$$(1 - |z|^2)^{p+1} |z|^2 (1 - |sz|^2)^{2q-2} \leq (1 - |z|^2)^{p-1} |z|^2 (1 - |sz|^2)^{2q},$$

and

$$(1 - |z|^2)^p (1 - |sz|^2)^{2q-1} |z|^2 \leq (1 - |z|^2)^{p-1} |z|^2 (1 - |sz|^2)^{2q}.$$

If $p = 0$ we keep

$$(1 - |z|^2) |z|^2 (1 - |sz|^2)^{2q-2} \leq |z|^2 (1 - |sz|^2)^{2q-1},$$

So, setting, for $p > 0$,

$$c(p, q) := 4p(p + 1) + 8q(2q - 1) + 16q(p + 1),$$

and

$$c(q) := 8q(2q - 1) + 16q(p + 1),$$

which ends the proof of the lemma. ■

Proposition 2.2 We have, for $p > 0$,

$$\int_{\mathbb{D}} \Delta g_{C,s}(z) \log^+ |f(sz)| \leq c(p, q) \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \varphi_C(sz) \log^+ |f(sz)|.$$

And for $p = 0$,

$$\int_{\mathbb{D}} \Delta g_{C,s}(z) \log^+ |f(sz)| \leq c(q) \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)|$$

with $c(0) = 0$.

Proof.

Integrating the estimates of lemma 2.1 we get the proposition. ■

In order to consider the terms containing $\log^- |f(sz)|$ we shall need:

Lemma 2.3 We have, for $p \geq 0$ and any $s \geq 1/2$,

$$\begin{aligned} \forall z \in \mathbb{D}, -\Delta g_{C,s}(z) &\leq -4p(p + 1)(1 - |z|^2)^{p-1} |z|^2 (1 - |sz|^2)^{2q} + \\ &\quad + 4[(p + 1) + 2sq](1 - |z|^2)^p (1 - |sz|^2)^{2q}. \end{aligned}$$

Proof.

With $-\Delta g_{C,s}(z)$ we get:

$$-A_1 \leq 4(p + 1)(1 - |z|^2)^p (1 - |sz|^2)^{2q} -$$

$$\begin{aligned}
& -4p(p+1)(1-|z|^2)^{p-1}|z|^2(1-|sz|^2)^{2q}; \\
-A_2 &= 8sq(1-|z|^2)^{p+1}(1-|sz|^2)^{2q-1} - \\
& -8q(2q-1)(1-|z|^2)^{p+1}|z|^2(1-|sz|^2)^{2q-2};
\end{aligned}$$

and,

$$-A_3 = -16sq(p+1)(1-|z|^2)^p(1-|sz|^2)^{2q-1}|z|^2.$$

We have two cases:

- $2q-1 \geq 0$ then $-A_2 \leq 8sq(1-|z|^2)^{p+1}(1-|sz|^2)^{2q-1}$.
- $2q-1 < 0$ then:

$$\begin{aligned}
& 8q(1-2q)(1-|z|^2)^p|z|^2(1-|sz|^2)^{2q-1} - \\
& -16sq(p+1)(1-|z|^2)^p(1-|sz|^2)^{2q-1}|z|^2 = \\
& = -[16sq(p+1) - 8q(1-2q)](1-|z|^2)^p(1-|sz|^2)^{2q-1}|z|^2.
\end{aligned}$$

But

$$[16sq(p+1) - 8q(1-2q)] = 16sq - 8q + 16sqp + 16q^2 \geq 8q(2s-1) \geq 0$$

provided that $s \geq 1/2$.

So in any cases, with $s \geq 1/2$, we get for $p > 0$,

$$\begin{aligned}
\forall z \in \mathbb{D}, -\Delta g_{C,s}(z) &\leq -4p(p+1)(1-|z|^2)^{p-1}|z|^2(1-|sz|^2)^{2q} + \\
& + 4(p+1)(1-|z|^2)^p(1-|sz|^2)^{2q} + \\
& + 8sq(1-|z|^2)^{p+1}(1-|sz|^2)^{2q-1} \leq \\
& \leq -4p(p+1)(1-|z|^2)^{p-1}|z|^2(1-|sz|^2)^{2q} + \\
& + 4[(p+1) + 2sq](1-|z|^2)^p(1-|sz|^2)^{2q}.
\end{aligned}$$

And for $p = 0$,

$$\forall z \in \mathbb{D}, -\Delta g_{C,s}(z) \leq 4[1 + 2sq](1-|sz|^2)^{2q}.$$

So we proved the lemma. ■

Proposition 2.4 *We have with $|f(0)| = 1$ and $p \geq 0$,*

$$-\int_{\mathbb{D}} \Delta g_{C,s}(z) \log^- |f(sz)| \leq 4[(p+1) + 2sq] \int_{\mathbb{D}} (1-|z|^2)^p(1-|sz|^2)^{2q} \log^+ |f(sz)|.$$

Proof.

Passing in polar coordinates we get

$$\int_{\mathbb{D}} (1-|z|^2)^p(1-|sz|^2)^{2q} \log^- |f(sz)| = \int_0^1 (1-\rho^2)^p(1-s^2\rho^2)^{2q} \left\{ \int_{\mathbb{T}} \log^- |f(s\rho e^{i\theta})| \right\} \rho d\rho;$$

by the subharmonicity of $\log |f(sz)|$ and the fact $|f(0)| = 1$, we get

$$\int_{\mathbb{T}} \log^- |f(s\rho e^{i\theta})| \leq \int_{\mathbb{T}} \log^+ |f(s\rho e^{i\theta})|$$

so

$$\int_{\mathbb{D}} (1-|z|^2)^p(1-|sz|^2)^{2q} \log^- |f(sz)| \leq \int_{\mathbb{D}} (1-|z|^2)^p(1-|sz|^2)^{2q} \log^+ |f(sz)|.$$

Now using lemma 2.3,

$$\begin{aligned}
-\int_{\mathbb{D}} \Delta g_{C,s}(z) \log^- |f(sz)| &\leq -4p(p+1) \int_{\mathbb{D}} (1-|z|^2)^{p-1}|z|^2(1-|sz|^2)^{2q} \log^- |f(sz)| + \\
& + 4[(p+1) + 2sq] \int_{\mathbb{D}} (1-|z|^2)^p(1-|sz|^2)^{2q} \log^- |f(sz)| \leq \\
& \leq 4[(p+1) + 2sq] \int_{\mathbb{D}} (1-|z|^2)^p(1-|sz|^2)^{2q} \log^- |f(sz)| \leq \\
& \leq 4[(p+1) + 2sq] \int_{\mathbb{D}} (1-|z|^2)^p(1-|sz|^2)^{2q} \log^+ |f(sz)|.
\end{aligned}$$

This ends the proof of the proposition. ■

3 Estimates on $\varphi_{A,j}(z) := \psi_j(sz)^q \eta_j(sz)$.

We set

$$g_{A,s}(z) := (1 - |z|^2)^{p+1} \sum_{j \in \mathbb{N}} \mathbb{1}_{\Gamma_j}(z) \varphi_{A,j}(sz),$$

and we have seen that $g_{A,s}(z) \in \mathcal{C}^\infty(\mathbb{D})$

We shall compute

$$\Delta g_{A,s}(z) \log |f(sz)| = \Delta g_{A,s}(z) \log^+ |f(sz)| - \Delta g_{A,s}(z) \log^- |f(sz)|.$$

We have $\Delta g_{A,s} = 4\partial\bar{\partial}g_{A,s}$ hence here:

$$\begin{aligned} \forall z \in \Gamma_j, \Delta[(1 - |z|^2)^{p+1} \varphi_{A,j}(sz)] &= \varphi_{A,j}(sz) \Delta[(1 - |z|^2)^{p+1}] + \\ &\quad + (1 - |z|^2)^{p+1} \Delta[\varphi_{A,j}(sz)] + \\ &\quad + 8\Re[\partial((1 - |z|^2)^{p+1}) \bar{\partial}(\varphi_{A,j}(sz))] =: \\ &=: A_1 + A_2 + A_3. \end{aligned}$$

Lemma 3.1 *We have:*

$$\begin{aligned} A_1 &:= \varphi_{A,j}(sz) \Delta[(1 - |z|^2)^{p+1}] = 4p(p+1)(1 - |z|^2)^{p-1} |z|^2 \varphi_{A,j}(sz) - \\ &\quad - 4(p+1)(1 - |z|^2)^p \varphi_{A,j}(sz) =: \\ &=: A'_1 - A''_1 \end{aligned}$$

with, for $z \in \Gamma_j$,

$$\begin{aligned} A'_1 &:= 4p(p+1)(1 - |z|^2)^{p-1} |z|^2 \varphi_{A,j}(sz) \leq 2^{2q} \times 4p(p+1)(1 - |z|^2)^{p-1} |z|^2 d(sz, E)^{2q}. \\ A''_1 &:= 4(p+1)(1 - |z|^2)^p \varphi_{A,j}(sz) \leq 2^{2q} \times 4(p+1)(1 - |z|^2)^p d(sz, E)^{2q}. \end{aligned}$$

Proof.

A simple computation of $\Delta[(1 - |z|^2)^{p+1}] = 4\partial\bar{\partial}[(1 - |z|^2)^{p+1}]$ with lemma 1.6 gives the result. ■

We have, just using $\Delta = 4\partial\bar{\partial}$,

$$\begin{aligned} A_2 &:= (1 - |z|^2)^{p+1} \Delta[\varphi_{A,j}(sz)] = (1 - |z|^2)^{p+1} \Delta[\eta_j(sz) \psi_j(sz)^q] = \\ &= (1 - |z|^2)^{p+1} \eta_j(sz) \Delta[\psi_j(sz)^q] + \\ &\quad + (1 - |z|^2)^{p+1} \psi_j(sz)^q \Delta[\eta_j(sz)] + \\ &\quad + (1 - |z|^2)^{p+1} 8\Re[\partial(\eta_j(sz)) \bar{\partial}(\psi_j(sz)^q)] =: A_{2,1} + A_{2,2} + A_{2,3}. \end{aligned}$$

Lemma 3.2 *We have*

$$\begin{aligned} \forall z \in \Gamma_j, A_{2,1}(s, z) &:= (1 - |z|^2)^{p+1} \eta_j(sz) \Delta[\psi_j(sz)^q] = \\ &= 4q^2 (1 - |z|^2)^{p+1} \eta_j(sz) \frac{|sz - \alpha_j|^{2q-2} |sz - \beta_j|^{2q-2}}{\delta_j^{2q}} \{ |sz - \beta_j|^2 + |sz - \alpha_j|^2 + 2\Re[(sz - \alpha_j)(\bar{z} - \bar{\beta}_j)] \}. \end{aligned}$$

Hence $\forall z \in \Gamma_j, A_{2,1}(s, z) \geq 0$ and, for $s \geq 1/2$,

$$\forall z \in \Gamma_j, A_{2,1}(s, z) \leq 4^{2q+2} q^2 (1 - |z|^2)^p cd(sz, E)^{2q-1}.$$

Proof.

We just apply lemma 7.1 with $\Delta = 4\partial\bar{\partial}$, to get the first assertion. Then we apply remark 7.2 to get $\forall z \in \Gamma_j, A_{2,1}(z) \geq 0$. Now for the third assertion we notice that $\eta_j(sz) \leq 1$ then, using (1.1) in lemma 1.6 with $0 < \eta_j(z)$, we get $|sz - \alpha_j| \leq \sqrt{2}\delta_j$ and $|sz - \beta_j| \leq \sqrt{2}\delta_j$, hence

$$\forall z \in \Gamma_j, |A_{2,1}(s, z)| \leq 8 \times 4^{2q} q^2 (1 - |z|^2)^{p+1} cd(sz, E)^{2q} \{ |sz - \beta_j|^{-2} + |sz - \alpha_j|^{-2} \}$$

using lemma 1.6. But $(1 - |z|^2) \leq 2|z - \gamma|$ for any $\gamma \in \mathbb{T}$ so we get

$$\forall z \in \Gamma_j, |A_{2,1}(s, z)| \leq 4^{2q+2} q^2 (1 - |z|^2)^p cd(sz, E)^{2q-1},$$

which ends the proof of the lemma. ■

We set $\chi_\alpha(z) := \chi\left(\frac{|z - \alpha|^2}{(1 - |z|^2)^2}\right)$, $\chi_\beta(z) := \chi\left(\frac{|z - \beta|^2}{(1 - |z|^2)^2}\right)$ and we set $|\chi'| := \max(|\chi'_\alpha|, |\chi'_\beta|)$ and $|\chi''| := \max(|\chi''_\alpha|, |\chi''_\beta|)$.

Lemma 3.3 *We have*

$$\begin{aligned} \forall z \in \Gamma_j, A_{2,2}(s, z) &:= (1 - |z|^2)^{p+1} \psi_j(sz)^q \Delta[\eta_j(sz)] \Rightarrow \\ &\Rightarrow |A_{2,2}(s, z)| \lesssim (|\chi'| + |\chi''|)(1 - |z|^2)^p \psi_j(sz)^{q-1/2} \lesssim \\ &\lesssim (|\chi'| + |\chi''|)(1 - |z|^2)^p d(sz, E)^{2q-1}. \end{aligned}$$

And, for $p > 0$,

$$|A_{2,2}(s, z)| \lesssim (|\chi'| + |\chi''|)(1 - |z|^2)^{p-1} d(sz, E)^{2q}.$$

Proof.

We have

$$\partial \bar{\partial}[\eta_j(sz)] = \chi_\alpha(z) \partial \bar{\partial}[\chi_\beta(z)] + \chi_\beta(z) \partial \bar{\partial}[\chi_\alpha(z)] + 2\Re[\partial \chi_\alpha(z) \bar{\partial}[\chi_\beta(z)]].$$

and by lemma 7.4:

$$\begin{aligned} |\bar{\partial}[\chi_\alpha(z)]| &\leq 3|\chi'|(\lambda + 1)(1 - |z|^2)^{-1}. \\ |\partial \bar{\partial}[\chi_\beta]| &\lesssim (|\chi'| + |\chi''|)(1 - |z|^2)^{-2}. \end{aligned}$$

So

$$|\Delta \eta_j| \lesssim (|\chi'| + |\chi''|)(1 - |z|^2)^{-2}$$

hence

$$\begin{aligned} |A_{2,2}(s, z)| &= (1 - |z|^2)^{p+1} \psi_j(sz)^q |\Delta[\eta_j(sz)]| \lesssim \\ &\lesssim (1 - |z|^2)^{p+1} \psi_j(sz)^q (|\chi'| + |\chi''|)(1 - |sz|^2)^{-2}. \end{aligned}$$

Because $(1 - |z|^2) \leq (1 - |sz|^2)$ we get

$$|A_{2,2}(s, z)| \lesssim (|\chi'| + |\chi''|)(1 - |z|^2)^{p-1} \psi_j(sz)^q.$$

Now by lemma 7.3 we get, if $\Delta \eta_j \neq 0$,

$$\forall z \in \Gamma_j, 2(1 - |z|^2)^2 \leq \psi_j(z) \leq 3(1 - |z|^2)^2$$

and

$$|A_{2,2}(s, z)| \lesssim (|\chi'| + |\chi''|)(1 - |z|^2)^p \psi_j(sz)^{q-1/2}.$$

Because $(1 - |z|^2) \leq d(z, E)$ we get, for $p > 0$, $(1 - |z|^2)^p \psi_j(sz)^{q-1/2} \leq (1 - |z|^2)^{p-1} d(sz, E)^{2q}$.

It remains to use lemma 1.6 to get the result. ■

Lemma 3.4 *We have*

$$\begin{aligned} \forall z \in \Gamma_j, A_{2,3} &:= (1 - |z|^2)^{p+1} 8\Re[\partial(\eta_j(sz)) \bar{\partial}(\psi_j(sz)^q)] \Rightarrow \\ &\Rightarrow |A_{2,3}| \lesssim |\chi'| (1 - |z|^2)^p \psi_j(sz)^{q-1/2} \lesssim |\chi'| (1 - |z|^2)^p d(sz, E)^{2q-1}. \end{aligned}$$

Proof.

We use exactly the same estimates as above for $\partial \eta_j$ and $\bar{\partial} \psi_j$. ■

Lemma 3.5 *We have*

$$\begin{aligned} A_3 &:= 8\Re[\partial((1 - |z|^2)^{p+1}) \bar{\partial}(\varphi_{A,j}(sz))] \leq \\ &\lesssim |\chi'| (1 - |z|^2)^p \psi_j^{q-1/2} + 16q(p+1)(1 - |z|^2)^p \psi_j^{q-1/2} \lesssim \end{aligned}$$

$$\lesssim |\chi'| (1 - |z|^2)^p d(sz, E)^{2q-1} + 16q(p+1)(1 - |z|^2)^p d(sz, E)^{2q-1}.$$

and

$$\begin{aligned} -A_3 &\lesssim |\chi'| (1 - |z|^2)^p \psi_j^{q-1/2} + 8(p+1)(1 - |z|^2)^{p-1/2} q \psi_j^q \lesssim \\ &\lesssim |\chi'| (1 - |z|^2)^p d(sz, E)^{2q-1} + 8q(p+1)(1 - |z|^2)^{p-1/2} d(sz, E)^{2q}. \end{aligned}$$

Proof.

We have

$$\bar{\partial}(\varphi_{A,j}(sz)) = \psi_j^q \bar{\partial} \eta_j + \eta_j \bar{\partial}(\psi_j^q)$$

For the term $\psi_j \bar{\partial} \eta_j$ we proceed exactly as in lemma 3.3 to get

$$|\psi_j^q \bar{\partial} \eta_j| \lesssim |\chi'| (1 - |z|^2)^p \psi_j^{q-1/2}.$$

So it remains

$$\begin{aligned} B &:= 8\eta_j \Re[\partial((1 - |z|^2)^{p+1}) \bar{\partial}(\psi_j^q)(sz)] = \\ &= -8(p+1)(1 - |z|^2)^p \eta_j(sz) \Re[\bar{z} \bar{\partial}(\psi_j^q)(sz)]. \end{aligned}$$

For this term we have by lemma 7.1

$$\bar{\partial}(\psi_j^q)(z) = q \frac{(z - \alpha_j) |z - \alpha_j|^{2q-2} |z - \beta_j|^{2q}}{\delta_j^{2q}} + q \frac{(z - \beta_j) |z - \alpha_j|^{2q} |z - \beta_j|^{2q-2}}{\delta_j^{2q}}$$

hence $B = B_1 + B_2$ with

$$B_1 := -8(p+1)(1 - |z|^2)^p \eta_j(sz) q \frac{|sz - \alpha_j|^{2q-2} |sz - \beta_j|^{2q}}{\delta_j^{2q}} \Re[\bar{z}(sz - \alpha_j)]$$

and

$$B_2 := -8(p+1)(1 - |z|^2)^p \eta_j(sz) q \frac{|sz - \alpha_j|^{2q} |sz - \beta_j|^{2q-2}}{\delta_j^{2q}} \Re[\bar{z}(sz - \beta_j)].$$

Now we shall apply lemma 7.5 to get that $\Re[\bar{z}(sz - \alpha_j)] \leq 0$ iff $D(\frac{\alpha_j}{2}, \frac{1}{2})$ so

$$B_1 \geq 0 \iff z \in \Gamma_j \cap D(\frac{\alpha_j}{2}, \frac{1}{2}).$$

The same way

$$B_2 \geq 0 \iff z \in \Gamma_j \cap D(\frac{\beta_j}{2}, \frac{1}{2}).$$

If $z \notin D(\frac{\alpha_j}{2}, \frac{1}{2})$, then we have that $(1 - |z|^2) \leq 2|z - \alpha_j|^2$ so we get

$$\begin{aligned} \forall z \in \Gamma_j \cap D(\frac{\alpha_j}{2}, \frac{1}{2})^c, \quad -B_1 &\leq 8q(p+1)(1 - |z|^2)^p \psi_j^q |(sz - \alpha_j)|^{-1} \leq \\ &\leq 8q(p+1)(1 - |z|^2)^{p-1/2} \psi_j^q. \end{aligned}$$

The same way:

$$\begin{aligned} \forall z \in \Gamma_j \cap D(\frac{\beta_j}{2}, \frac{1}{2})^c, \quad -B_2 &\leq 8q(p+1)(1 - |z|^2)^p \psi_j^q |(sz - \beta_j)|^{-1} \leq \\ &\leq 8q(p+1)(1 - |z|^2)^{p-1/2} \psi_j^q. \end{aligned}$$

Hence we get

$$\forall z \in \Gamma_j, \quad -B \leq 16q(p+1)(1 - |z|^2)^{p-1/2} \psi_j^q.$$

Now we have $B_1 \geq 0 \iff z \in \Gamma_j \cap D(\frac{\alpha_j}{2}, \frac{1}{2})$, so

$$\begin{aligned} B_1 &\leq 8(p+1)(1 - |z|^2)^p \eta_j(sz) q \frac{|sz - \alpha_j|^{2q-2} |sz - \beta_j|^{2q}}{\delta_j^{2q}} |sz - \alpha_j| \leq \\ &\leq 8(p+1)(1 - |z|^2)^p q |(sz - \alpha_j)|^{-1} \psi_j^q. \end{aligned}$$

And the same way

$$\begin{aligned} B_2 &\leq 8(p+1)(1-|z|^2)^p \eta_j(sz) q \frac{|sz - \beta_j|^{2q-2} |sz - \alpha_j|^{2q}}{\delta_j^{2q}} |sz - \beta_j| \leq \\ &\leq 8(p+1)(1-|z|^2)^p q |(sz - \beta_j)|^{-1} \psi_j^q. \end{aligned}$$

Hence

$$B \leq 8(p+1)(1-|z|^2)^p q (|(sz - \alpha_j)|^{-1} + |(sz - \beta_j)|^{-1}) \psi_j^q.$$

So we get

$$\begin{aligned} \forall z \in \Gamma_j, A_3 &:= 8\Re[\partial((1-|z|^2)^{p+1})\bar{\partial}(\varphi_{A,j}(sz))] \lesssim \\ &\lesssim |\chi'| (1-|z|^2)^p \psi_j^{q-1/2} + 16q(p+1)(1-|z|^2)^{p-1/2} \psi_j^q. \end{aligned}$$

And

$$-A_3 \lesssim |\chi'| (1-|z|^2)^p \psi_j^{q-1/2} + 8q(p+1)(1-|z|^2)^{p-1/2} \psi_j^q.$$

It remains to use lemma 1.6 to get the result. ■

We shall estimate $\Delta g_{A,s}(z) \log^+ |f(sz)|$.

Proposition 3.6 *We have*

$$\Delta g_{A,s}(z) \lesssim 4p(p+1)(1-|z|^2)^{p-1} d(sz, E)^{2q} + (1-|z|^2)^p d(sz, E)^{2q-1}.$$

Proof.

By use of $\Delta g_{A,s}(z) = A_1 + A_2 + A_3$ and by the previous lemmas, we get for $z \in \Gamma_j$,

$$\begin{aligned} A_1 &= A'_1 - A''_1 \leq A'_1 = 4p(p+1)(1-|z|^2)^{p-1} |z|^2 \eta_j(sz) \psi_j^q(sz) \leq \\ &\leq 4p(p+1)(1-|z|^2)^{p-1} d(sz, E)^{2q}. \end{aligned}$$

Then

$$A_2 = A_{2,1} + A_{2,2} + A_{2,3},$$

and, for $s \geq 1/2$,

$$\begin{aligned} 0 \leq A_{2,1}(s, z) &\leq 4^{2q+2} q^2 (1-|z|^2)^p c(\lambda) d(sz, E)^{2q-1}. \\ |A_{2,2}(s, z)| &\lesssim (1-|z|^2)^p d(sz, E)^{2q-1}. \\ |A_{2,3}| &\lesssim |\chi'| (1-|z|^2)^p d(sz, E)^{2q-1}. \end{aligned}$$

Hence, for $\forall z \in \Gamma_j$,

$$A_2 \lesssim (1-|z|^2)^p d(sz, E)^{2q-1}.$$

Finally

$$A_3 \lesssim (1-|z|^2)^p d(sz, E)^{2q-1} + 16q(p+1)(1-|z|^2)^{p-1/2} d(sz, E)^{2q} \lesssim (1-|z|^2)^p d(sz, E)^{2q-1}.$$

So we get

$$\Delta g_{A,s}(z) \lesssim 4p(p+1)(1-|z|^2)^{p-1} d(sz, E)^{2q} + (1-|z|^2)^p d(sz, E)^{2q-1},$$

which proves the proposition. ■

Now we shall estimate $-\Delta g_{A,s}(z) \log^- |f(sz)|$. We set:

$$\begin{aligned} P_{\mathbb{D},A,-}(s) &:= \int_{\mathbb{D}} (1-|z|^2)^{p-1} |z|^2 \varphi_A(sz) \log^- |f(sz)|, \\ P_{\mathbb{D},A,+}(s) &:= \int_{\mathbb{D}} (1-|z|^2)^{p-1} \varphi_A(sz) \log^+ |f(sz)|, \\ P_-(\delta, u, s) &:= \int_{\mathbb{D} \setminus D(0,u)} (1-|z|^2)^{p-1+\delta} \varphi_A(sz) \log^- |f(sz)|. \end{aligned}$$

and

$$P_+(\delta, u, s) := \int_{D(0,u)} (1 - |z|^2)^{p-1+\delta} \varphi_A(sz) \log^+ |f(sz)|.$$

Proposition 3.7 *We have, for $p > 0$,*

$$-\int_{\mathbb{D}} \Delta g_{A,s}(z) \log^- |f(sz)| \leq 2^{2q} P_{\mathbb{D},A,+}(s) + 2 \times 4^q (1 - u^2)^{-2q} P_+(\frac{1}{2}, u, s).$$

Proof.

By use of $\Delta g_{A,s}(z) = A_1 + A_2 + A_3$ and by the previous lemmas, we get for $\forall z \in \Gamma_j$,

$$-A_1 =: -A_1' + A_1'' = -4p(p+1)(1 - |z|^2)^{p-1} |z|^2 \varphi_{A,j}(sz) + 4(p+1)(1 - |z|^2)^p d(sz, E)^{2q}.$$

Now

$$-A_2 = -A_{2,1} - A_{2,2} - A_{2,3} \leq -A_{2,2} - A_{2,3},$$

because $A_{2,1} \leq 0$.

We have

$$|A_{2,2}| \lesssim (|\chi'| + |\chi''|)(1 - |z|^2)^p d(sz, E)^{2q-1}$$

and

$$|A_{2,3}| \lesssim |\chi'| (1 - |z|^2)^p d(sz, E)^{2q-1}$$

so

$$-A_2 \lesssim (|\chi'| + |\chi''|)(1 - |z|^2)^p d(sz, E)^{2q-1}.$$

Now

$$-A_3 \lesssim |\chi'| (1 - |z|^2)^p d(sz, E)^{2q-1} + 8q(p+1)(1 - |z|^2)^{p-1/2} d(sz, E)^{2q}.$$

So grouping the terms we get

$$-\Delta g_{A,s}(z) \lesssim (|\chi'| + |\chi''|)(1 - |z|^2)^p d(sz, E)^{2q-1} + 8q(p+1)(1 - |z|^2)^{p-1/2} d(sz, E)^{2q} - 4p(p+1)(1 - |z|^2)^{p-1} |z|^2 \varphi_{A,j}(sz) + 4(p+1)(1 - |z|^2)^p \varphi_{A,j}(sz).$$

and

$$-\Delta g_{A,s}(z) \log^- |f(sz)| \lesssim (|\chi'| + |\chi''|)(1 - |z|^2)^p d(sz, E)^{2q-1} \log^- |f(sz)| + 8q(p+1)(1 - |z|^2)^{p-1/2} d(sz, E)^{2q} \log^- |f(sz)| - 4p(p+1)(1 - |z|^2)^{p-1} |z|^2 \varphi_{A,j}(sz) \log^- |f(sz)| + 4(p+1)(1 - |z|^2)^p \varphi_{A,j}(sz) \log^- |f(sz)|.$$

For the first term, because on $(|\chi'| + |\chi''|) \neq 0$ we have $d(sz, E) \leq 3(1 - |sz|^2)$, we get

$$B_1 := (|\chi'| + |\chi''|)(1 - |z|^2)^p d(sz, E)^{2q-1} \log^- |f(sz)| \lesssim (1 - |z|^2)^p (1 - |sz|^2)^{2q-1} \log^- |f(sz)|$$

hence, passing in polar coordinates,

$$\int_{\mathbb{D}} B_1 \lesssim \int_0^1 (1 - \rho^2)^p (1 - s^2 \rho^2)^{2q-1} \left\{ \int_{\mathbb{T}} \log^- |f(s\rho e^{i\theta})| \right\} \rho d\rho.$$

By the subharmonicity of $\log |f(sz)|$ and $|f(0)| = 1$, we get

$$\int_{\mathbb{T}} \log^- |f(s\rho e^{i\theta})| \leq \int_{\mathbb{T}} \log^+ |f(s\rho e^{i\theta})|,$$

hence

$$\int_{\mathbb{D}} B_1 \lesssim \int_{\mathbb{D}} (1 - |z|^2)^p (1 - |sz|^2)^{2q-1} \log^+ |f(sz)|.$$

For $B_2 := 8q(p+1)(1 - |z|^2)^{p-1/2} d(sz, E)^{2q} \log^- |f(sz)|$, we use the substitution lemma 7.7 with $\delta = 1/2$, to get:

$$\int_{\mathbb{D}} B_2 \leq 4^q (1 - u^2)^{-2q} P_+(\frac{1}{2}, u) + (1 - u^2)^{1/4} u^{-2} P_-(\frac{1}{4}, u, s).$$

For $p > 0$, because $\varphi_A(z) \lesssim d(z, E)^{2q}$, we get:

$$\begin{aligned} \int_{\mathbb{D}} B_2 &\lesssim 4^q(1-u^2)^{-2q} \int_{D(0,u)} (1-|z|^2)^{p-1/2} d(sz, E)^{2q} \log^+ |f(sz)| + \\ &\quad + (1-u^2)^{1/4} u^{-2} \int_{\mathbb{D}} (1-|z|^2)^{p-3/4} |z|^2 \varphi_A(sz) \log^- |f(sz)|. \end{aligned}$$

The same for the last term with $\delta = 1$ and we get that

$$B_4 := 4(p+1)(1-|z|^2)^p d(sz, E)^{2q} \log^- |f(sz)|$$

verifies

$$\int_{\mathbb{D}} B_4 \lesssim 4^q(1-u^2)^{-2q} P_+(1, u, s) + (1-u^2)^{1/2} u^{-2} P_-\left(\frac{1}{2}, u, s\right).$$

Now it remains the "good" term

$$B_3 := -4p(p+1)(1-|z|^2)^{p-1} |z|^2 \varphi_{A,j}(sz) \log^- |f(sz)|$$

and, if $p > 0$, we choose $1-u^2$ small enough to get that

$$\begin{aligned} (1-u^2)^{1/4} u^{-2} \int_{\mathbb{D}} (1-|z|^2)^{p-3/4} |z|^2 \varphi_A(sz) \log^- |f(sz)| + \\ + (1-u^2)^{1/2} u^{-2} \int_{\mathbb{D}} (1-|z|^2)^{p-1/2} |z|^2 \varphi_A(sz) \log^- |f(sz)| - \\ - 4p(p+1) \int_{\mathbb{D}} (1-|z|^2)^{p-1} |z|^2 \varphi_A(sz) \log^- |f(sz)| \leq 0. \end{aligned}$$

So it remains:

$$\begin{aligned} - \int_{\mathbb{D}} \Delta g_{A,s}(z) \log^- |f(sz)| &\leq \int_{\mathbb{D}} (1-|z|^2)^p (1-|sz|^2)^{2q-1} \log^+ |f(sz)| + \\ &\quad + 2^q (1-u^2)^{-2q} P_+\left(\frac{1}{2}, u, s\right) + \\ &\quad + 2^q (1-u^2)^{-2q} P_+(1, u, s) \leq \\ &\leq \int_{\mathbb{D}} (1-|z|^2)^p (1-|sz|^2)^{2q-1} \log^+ |f(sz)| + \\ &\quad + 2 \times 4^q (1-u^2)^{-2q} P_+\left(\frac{1}{2}, u, s\right). \end{aligned}$$

Now $(1-|z|^2) \leq (1-|sz|^2)$, and $(1-|sz|^2)^{2q} \leq 2^{2q} \varphi_A(sz)$ so we get

$$\int_{\mathbb{D}} (1-|z|^2)^p (1-|sz|^2)^{2q-1} \log^+ |f(sz)| \leq 2^{2q} \int_{\mathbb{D}} (1-|z|^2)^{p-1} \varphi_A(sz) \log^+ |f(sz)|,$$

so putting it, we get

$$\begin{aligned} - \int_{\mathbb{D}} \Delta g_{A,s}(z) \log^- |f(sz)| &\leq 2^{2q} \int_{\mathbb{D}} (1-|z|^2)^{p-1} \varphi_A(sz) \log^+ |f(sz)| + \\ &\quad + 2 \times 4^q (1-u^2)^{-2q} P_+\left(\frac{1}{2}, u\right), \end{aligned}$$

which ends the proof. ■

Proposition 3.8 *We have, for $p = 0$,*

$$\begin{aligned} - \int_{\mathbb{D}} \Delta g_{A,s}(z) \log^- |f(sz)| &\lesssim \int_{\mathbb{D}} (1-|sz|^2)^{2q-1} \log^+ |f(sz)| + \\ &\quad + 2 \times 4^q (1-u^2)^{-2q} \int_{D(0,u)} (1-|z|^2)^{-1/2} d(sz, E)^{2q} \log^+ |f(sz)| + \\ &\quad + 2(1-u^2)^{1/4} u^{-2} \int_{\mathbb{D} \setminus D(0,u)} (1-|z|^2)^{-3/4} |z|^2 \varphi_A(sz) \log^- |f(sz)|. \end{aligned}$$

Proof.

For $p = 0$, there is no "good" term and we have

$$\int_{\mathbb{D}} B_1 \lesssim \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)|.$$

And

$$\begin{aligned} \int_{\mathbb{D}} B_2 \lesssim & 4^q (1 - u^2)^{-2q} \int_{D(0,u)} (1 - |z|^2)^{-1/2} d(sz, E)^{2q} \log^+ |fsz| + \\ & + (1 - u^2)^{1/4} u^{-2} \int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{-3/4} |z|^2 \varphi_A(sz) \log^- |fsz|. \end{aligned}$$

The same for the last term with $\delta = 1$ and we get

$$B_4 := 4d(sz, E)^{2q} \log^- |f(sz)|$$

verifies

$$\int_{\mathbb{D}} B_4 \lesssim 4^q (1 - u^2)^{-2q} P_+(1, u) + (1 - u^2)^{1/2} u^{-2} P_-\left(\frac{1}{2}, u, s\right).$$

So adding:

$$\begin{aligned} - \int_{\mathbb{D}} \Delta g_{A,s}(z) \log^- |f(sz)| \lesssim & \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)| + \\ & + 4^q (1 - u^2)^{-2q} \int_{D(0,u)} (1 - |z|^2)^{-1/2} d(sz, E)^{2q} \log^+ |fsz| + \\ & + 4^q (1 - u^2)^{-2q} \int_{D(0,u)} d(sz, E)^{2q} \log^+ |fsz| + \\ & + (1 - u^2)^{1/4} u^{-2} \int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{-3/4} |z|^2 \varphi_A(sz) \log^- |fsz| + \\ & + (1 - u^2)^{1/2} u^{-2} \int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{-1/2} |z|^2 \varphi_A(sz) \log^- |fsz|. \end{aligned}$$

And

$$\begin{aligned} - \int_{\mathbb{D}} \Delta g_{A,s}(z) \log^- |f(sz)| \lesssim & \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)| + \\ & + 2 \times 4^q (1 - u^2)^{-2q} \int_{D(0,u)} (1 - |z|^2)^{-1/2} d(sz, E)^{2q} \log^+ |fsz| + \\ & + 2(1 - u^2)^{1/4} u^{-2} \int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{-3/4} |z|^2 \varphi_A(sz) \log^- |fsz|, \end{aligned}$$

which ends the proof. ■

Proposition 3.9 *We have, for $p > 0$,*

$$\int_{\mathbb{D}} \Delta g_{A,s}(z) \log |f(sz)| \lesssim [2^{2q} + 4p(p+1) + 2] P_{\mathbb{D},A,+}(s) + 2 \times 4^q (1 - u^2)^{-2q} P_+\left(\frac{1}{2}, u, s\right),$$

Proof.

From

$$\Delta g_{A,s}(z) \log |f(sz)| = \Delta g_{A,s}(z) \log^+ |f(sz)| - \Delta g_{A,s}(z) \log^- |f(sz)|$$

using proposition 3.6 we have, using $(1 - |z|^2) \leq 2d(sz, E)$,

$$\begin{aligned} \Delta g_{A,s}(z) \log^+ |f(sz)| \lesssim & 4p(p+1)(1 - |z|^2)^{p-1} d(sz, E)^{2q} \log^+ |fsz| + \\ & + (1 - |z|^2)^p d(sz, E)^{2q-1} \log^+ |fsz| \leq \\ \leq & [4p(p+1) + 2](1 - |z|^2)^{p-1} d(sz, E)^{2q} \log^+ |fsz|. \end{aligned}$$

And using proposition 3.7 we have

$$- \int_{\mathbb{D}} \Delta g_{A,s}(z) \log^- |f(sz)| \leq 2^{2q} P_{\mathbb{D},A,+}(s) + 2 \times 4^q (1-u^2)^{-2q} P_+(\frac{1}{2}, u, s).$$

Hence

$$\int_{\mathbb{D}} \Delta g_{A,s}(z) \log |f(sz)| \lesssim 2^{2q} P_{\mathbb{D},A,+}(s) + 2 \times 4^q (1-u^2)^{-2q} P_+(\frac{1}{2}, u, s) + [4p(p+1) + 2] \int_{\mathbb{D}} (1-|z|^2)^{p-1} d(sz, E)^{2q} \log^+ |f(sz)|,$$

so

$$\int_{\mathbb{D}} \Delta g_{A,s}(z) \log |f(sz)| \lesssim [2^{2q} + 4p(p+1) + 2] P_{\mathbb{D},A,+}(s) + 2 \times 4^q (1-u^2)^{-2q} P_+(\frac{1}{2}, u, s),$$

which ends the proof. ■

4 The case $p > 0$.

Recall that

$$\forall z \in \Gamma_j, \varphi_j(z) := \eta_j(z) \psi_j(z)^q + (1-|z|^2)^{2q}, \quad \forall z \in \Gamma_E, \varphi_E(z) := (1-|z|^2)^{2q},$$

and by lemma 1.7 we have that there is a function $\varphi \in \mathcal{C}^\infty(\mathbb{D})$ such that φ coincides with φ_j and φ_E in their domains of definition. Moreover we have for $0 \leq s < 1$ and $q > 0$, $g_s(z) := (1-|z|^2)^{p+1} \varphi(sz) \in \mathcal{C}^\infty(\bar{\mathbb{D}})$ so we can apply the Green formula to it. Recall that $f_s(z) := f(sz)$.

With the "zero" formula: $\Delta \log |f_s| = \sum_{a \in Z(f_s)} \delta_a$ we get

$$\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log |f(sz)| \Delta g_s(z) + \int_{\mathbb{T}} (g_s \partial_n \log |f(sz)| - \log |f(sz)| \partial_n g_s).$$

So, because $g_s = 0$ on \mathbb{T} ,

$$\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log |f(sz)| \Delta g_s(z) - \int_{\mathbb{T}} \log |f(se^{i\theta})| \partial_n g_s(e^{i\theta}).$$

If, moreover $p > 0$, $\partial_n g_s = 0$ on \mathbb{T} , hence $\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log |f(sz)| \Delta g_s(z)$.

So we have to compute

$$\Delta g_s(z) \log |f(sz)| = \Delta g_s(z) \log^+ |f(sz)| - \Delta g_s(z) \log^- |f(sz)|.$$

We have $\Delta g_s = 4\bar{\partial}\partial g_s$ hence

$$\Delta g_s(z) = \Delta[(1-|z|^2)^{p+1} \varphi(sz)] = \varphi(sz) \Delta[(1-|z|^2)^{p+1}] + (1-|z|^2)^{p+1} \Delta[\varphi(sz)] + 8\Re[\partial((1-|z|^2)^{p+1}) \bar{\partial}(\varphi(sz))].$$

Recall that, with $\varphi_{A,j}(z) := \eta_j(z) \psi_j(z)^q$ and $\varphi_{C,j}(z) := (1-|z|^2)^{2q}$, we have

$$\forall z \in \Gamma_j, \varphi_j(z) := \varphi_{A,j}(z) + \varphi_{C,j}(z),$$

and

$$g_{C,s}(z) := (1-|z|^2)^{p+1} \varphi_C(sz) = (1-|z|^2)^{p+1} (1-|sz|^2)^{2q},$$

and

$$g_{A,s}(z) := (1-|z|^2)^{p+1} \sum_{j \in \mathbb{N}} \mathbb{1}_{\Gamma_j}(z) \varphi_{A,j}(sz).$$

Now we are in position to apply the previous results. By proposition 2.2 we get:

$$\int_{\mathbb{D}} \Delta g_{C,s}(z) \log^+ |f(sz)| \leq c(p, q) \int_{\mathbb{D}} (1-|z|^2)^{p-1} \varphi_C(sz) \log^+ |f(sz)|.$$

And by proposition 2.4 we get:

$$-\int_{\mathbb{D}} \Delta g_{C,s}(z) \log^- |f(sz)| \leq 4[(p+1) + 2sq] \int_{\mathbb{D}} (1-|z|^2)^p (1-|sz|^2)^{2q} \log^+ |f(sz)|.$$

So adding:

$$\begin{aligned} \int_{\mathbb{D}} \Delta g_{C,s}(z) \log |f(sz)| &\leq c(p, q) \int_{\mathbb{D}} (1-|z|^2)^{p-1} \varphi_C(sz) \log^+ |f(sz)| + \\ &\quad + 4[(p+1) + 2sq] \int_{\mathbb{D}} (1-|z|^2)^p (1-|sz|^2)^{2q} \log^+ |f(sz)|. \end{aligned}$$

Now by proposition 3.9 we get:

$$\begin{aligned} \int_{\mathbb{D}} \Delta g_{A,s}(z) \log |f(sz)| &\lesssim [2^{2q} + 4p(p+1) + 2] \int_{\mathbb{D}} (1-|z|^2)^{p-1} \varphi_A(sz) \log^+ |f(sz)| + \\ &\quad + 2 \times 4^q (1-u^2)^{-2q} \int_{D(0,u)} (1-|z|^2)^{p-1/2} \varphi_A(sz) \log^+ |f(sz)|. \end{aligned}$$

Adding, because $\varphi = \varphi_A + \varphi_C$, we get

Theorem 4.1 *We have:*

$$\int_{\mathbb{D}} \Delta g_s(z) \log |f(sz)| \lesssim \int_{\mathbb{D}} (1-|z|^2)^{p-1} \varphi(sz) \log^+ |f(sz)|.$$

Proof.

This is clear because, in the second term:

$$(1-|z|^2)^{p-1/2} \varphi(sz) \log^+ |f(sz)| \leq (1-|z|^2)^{p-1} \varphi(sz) \log^+ |f(sz)|. \quad \blacksquare$$

So we are lead to

Definition 4.2 *Let $E = \bar{E} \subset \mathbb{T}$. We say that an holomorphic function f is in the generalised Nevanlinna class $\mathcal{N}_{\varphi,p}(\mathbb{D})$ for $p > 0$ if $\exists \delta > 0$, $\delta < 1$ such that*

$$\|f\|_{\mathcal{N}_{\varphi,p}} := \sup_{1-\delta < s < 1} \int_{\mathbb{D}} (1-|z|)^{p-1} \varphi(sz) \log^+ |f(sz)| < \infty.$$

And we proved the Blaschke type condition:

Theorem 4.3 *Let $E = \bar{E} \subset \mathbb{T}$. Suppose $q > 0$ and $f \in \mathcal{N}_{\varphi,p}(\mathbb{D})$ with $|f(0)| = 1$, then*

$$\sum_{a \in Z(f)} (1-|a|^2)^{1+p} \varphi(a) \leq c(\varphi) \|f\|_{\mathcal{N}_{\varphi,p}}.$$

Corollary 4.4 *Let $E = \bar{E} \subset \mathbb{T}$. Suppose $q \in \mathbb{R}$ and $f \in \mathcal{N}_{d(\cdot, E)^q, p}(\mathbb{D})$ with $|f(0)| = 1$, then*

$$\sum_{a \in Z(f)} (1-|a|^2)^{1+p} d(a, E)^q \leq c(\varphi) \|f\|_{\mathcal{N}_{d(\cdot, E)^q, p}}.$$

Proof.

By use of lemma 1.8, we have

$$\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log |f(sz)| \Delta g_s(z),$$

hence, by theorem 4.1,

$$\sum_{a \in Z(f_s)} g_s(a) \lesssim \int_{\mathbb{D}} (1-|z|^2)^{p-1} \varphi(sz) \log^+ |f(sz)|,$$

the constant in \lesssim being independent of $s < 1$. It remains to apply lemma 7.11 to get that, for any $1 > \delta > 0$ we have:

$$\sum_{a \in Z(f)} (1 - |a|^2)^{p+1} \varphi(a) \leq \sup_{1-\delta < s < 1} \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \varphi(sz) \log^+ |f(sz)|,$$

which ends the proof of theorem 4.3. ■

To prove corollary 4.4, we use lemma 1.6 and lemma 1.7 which give that $\varphi(z)$ is equivalent to $d(z, E)^{2q}$. ■

5 The case $p = 0$.

This time we have $g_s(z) := (1 - |z|^2)\varphi(sz) \in \mathcal{C}^\infty(\bar{\mathbb{D}})$ hence

$$\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log |f(sz)| \Delta g_s(z) - \int_{\mathbb{T}} \log |f(se^{i\theta})| \partial_n g_s(e^{i\theta}),$$

with

$$\partial_n g_s(z) = -2\varphi(sz) + (1 - |z|^2)\partial_n \varphi(sz),$$

so

$$\forall e^{i\theta} \in \mathbb{T}, \partial_n g_s(e^{i\theta}) = -2\varphi(se^{i\theta})$$

hence

$$\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log |f(sz)| \Delta g_s(z) + 2 \int_{\mathbb{T}} \varphi(e^{i\theta}) \log |f(se^{i\theta})|. \quad (5.3)$$

Now we are in position to apply the previous results. By proposition 2.2, we get:

$$\int_{\mathbb{D}} \Delta g_{C,s}(z) \log^+ |f(sz)| \leq c(q) \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)|.$$

By proposition 2.4, with $p = 0$, we get:

$$- \int_{\mathbb{D}} \Delta g_{C,s}(z) \log^- |f(sz)| \leq 4[1 + 2sq] \int_{\mathbb{D}} (1 - |sz|^2)^{2q} \log^+ |f(sz)|.$$

So, adding:

$$\int_{\mathbb{D}} \Delta g_{C,s}(z) \log |f(sz)| \leq c(q) \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)|.$$

By proposition 3.6 with $p = 0$, we get:

$$\Delta g_{A,s}(z) \lesssim d(sz, E)^{2q-1},$$

hence

$$\int_{\mathbb{D}} \Delta g_{A,s}(z) \log^+ |f(sz)| \lesssim \int_{\mathbb{D}} d(sz, E)^{2q-1} \log^+ |f(sz)|.$$

By proposition 3.8 we get:

$$\begin{aligned} - \int_{\mathbb{D}} \Delta g_{A,s}(z) \log^- |f(sz)| &\lesssim \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)| + \\ &+ 2 \times 4^q (1 - u^2)^{-2q} \int_{D(0,u)} (1 - |z|^2)^{-1/2} d(sz, E)^{2q} \log^+ |f(sz)| + \\ &+ 2(1 - u^2)^{1/4} u^{-2} \int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{-3/4} |z|^2 \varphi_A(sz) \log^- |f(sz)|. \end{aligned}$$

So adding we get

$$\begin{aligned}
\int_{\mathbb{D}} \Delta g_{A,s}(z) \log |f(sz)| &\lesssim \int_{\mathbb{D}} d(sz, E)^{2q-1} \log^+ |f(sz)| + \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)| + \\
&+ 2 \times 4^q (1 - u^2)^{-2q} \int_{D(0,u)} (1 - |z|^2)^{-1/2} d(sz, E)^{2q} \log^+ |f(sz)| + \\
&+ 2(1 - u^2)^{1/4} u^{-2} \int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{-3/4} |z|^2 \varphi_A(sz) \log^- |f(sz)|.
\end{aligned}$$

Combining these results, we proved:

Proposition 5.1 *We have:*

$$\begin{aligned}
\int_{\mathbb{D}} \Delta g_s(z) \log |f(sz)| &\lesssim \int_{\mathbb{D}} d(sz, E)^{2q-1} \log^+ |f(sz)| + \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)| + \\
&+ 2 \times 4^q (1 - u^2)^{-2q} \int_{D(0,u)} (1 - |z|^2)^{-1/2} d(sz, E)^{2q} \log^+ |f(sz)| + \\
&+ 4(1 - u^2)^{1/2} \sup_{su < t < s} \int_{\mathbb{T}} \varphi_A(te^{i\theta}) \log^- |f(te^{i\theta})| d\theta.
\end{aligned}$$

Proof.

It remains to deal with the term in $\log^- |f(sz)|$. We have, passing in polar coordinates,

$$\begin{aligned}
\int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{-3/4} |z|^2 \varphi_A(sz) \log^- |f(sz)| &= \\
&\int_u^1 (1 - \rho^2)^{-3/4} \left\{ \int_{\mathbb{T}} \varphi_A(s\rho e^{i\theta}) \log^- |f(s\rho e^{i\theta})| \rho d\rho \right\} \leq \\
&\leq \sup_{su < t < s} \int_{\mathbb{T}} \varphi_A(te^{i\theta}) \log^- |f(te^{i\theta})| d\theta \int_u^1 (1 - \rho^2)^{-3/4} \rho d\rho \leq \\
&\leq 2(1 - u^2)^{1/4} \sup_{su < t < s} \int_{\mathbb{T}} \varphi_A(te^{i\theta}) \log^- |f(te^{i\theta})| d\theta.
\end{aligned}$$

Hence we get

$$\begin{aligned}
2(1 - u^2)^{1/4} u^{-2} \int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{-3/4} |z|^2 \varphi_A(sz) \log^- |f(sz)| &\leq \\
&\leq 4(1 - u^2)^{1/2} \sup_{su < t < s} \int_{\mathbb{T}} \varphi_A(te^{i\theta}) \log^- |f(te^{i\theta})| d\theta
\end{aligned}$$

which ends the proof. ■

Now we shall use the relation (5.3) which says:

$$\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log |f(sz)| \Delta g_s(z) + 2 \int_{\mathbb{T}} \varphi(e^{i\theta}) \log |f(se^{i\theta})|.$$

So we have, using proposition 5.1,

$$\begin{aligned}
\sum_{a \in Z(f_s)} g_s(a) &\lesssim \int_{\mathbb{D}} d(sz, E)^{2q-1} \log^+ |f(sz)| + \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)| + \\
&+ 2 \times 4^q (1 - u^2)^{-2q} \int_{D(0,u)} (1 - |z|^2)^{-1/2} d(sz, E)^{2q} \log^+ |f(sz)| + \\
&+ 4(1 - u^2)^{1/2} \sup_{su < t < s} \int_{\mathbb{T}} \varphi_A(te^{i\theta}) \log^- |f(te^{i\theta})| d\theta + \\
&+ 2 \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^+ |f(se^{i\theta})| - 2 \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^- |f(se^{i\theta})|.
\end{aligned}$$

So the "good" term is now $-2 \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^- |f(se^{i\theta})|$.

We shall set

$$P_{\mathbb{T},+}(t_0) := \sup_{0 \leq s \leq t_0} \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^+ |f(se^{i\theta})|.$$

and

$$P_{\mathbb{T},-}(t_0) := \sup_{0 \leq s \leq t_0} \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^- |f(se^{i\theta})|.$$

Because $\gamma(s) := \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^- |f(se^{i\theta})|$ is continuous for $s \in [0, t_0]$ by lemma 7.8 in the appendix, the sup is achieved for a $s_0 \in [0, t_0]$ and we have

$$P_{\mathbb{T},-}(t_0) = \int_{\mathbb{T}} \varphi(s_0 e^{i\theta}) \log^- |f(s_0 e^{i\theta})|.$$

Fix $t_0 < 1$ and set:

$$\begin{aligned} P_{\mathbb{D},+}(s) &:= \int_{\mathbb{D}} d(sz, E)^{2q-1} \log^+ |f(sz)| + \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)| + \\ &\quad + 2 \times 4^q (1 - u^2)^{-2q} \int_{D(0,u)} (1 - |z|^2)^{-1/2} d(sz, E)^{2q} \log^+ |f(sz)| + \\ &\quad + 2 \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^+ |f(se^{i\theta})|. \end{aligned}$$

Then we get, with the $s_0 \leq t_0$ associated to t_0 ,

$$\begin{aligned} \sum_{a \in Z(f_s)} g_{t_0}(a) + \sum_{a \in Z(f_s)} g_{s_0}(a) &\lesssim P_{\mathbb{D},+}(t_0) + P_{\mathbb{D},+}(s_0) + \\ &\quad + 4(1 - u^2)^{1/2} \sup_{t_0 u < t < t_0} \int_{\mathbb{T}} \varphi_A(te^{i\theta}) \log^- |f(te^{i\theta})| d\theta + \\ &\quad + 4(1 - u^2)^{1/2} \sup_{s_0 u < t < s_0} \int_{\mathbb{T}} \varphi_A(te^{i\theta}) \log^- |f(te^{i\theta})| d\theta - \\ &\quad - 2 \int_{\mathbb{T}} \varphi(t_0 e^{i\theta}) \log^- |f(t_0 e^{i\theta})| - 2 \int_{\mathbb{T}} \varphi(s_0 e^{i\theta}) \log^- |f(s_0 e^{i\theta})|. \end{aligned}$$

But, because $P_{\mathbb{T},-}(t_0) = \int_{\mathbb{T}} \varphi(s_0 e^{i\theta}) \log^- |f(s_0 e^{i\theta})|$ and $s_0 \leq t_0$, we get:

$$\sup_{t_0 u < t < t_0} \int_{\mathbb{T}} \varphi_A(te^{i\theta}) \log^- |f(te^{i\theta})| d\theta \leq P_{\mathbb{T},-}(t_0)$$

and

$$\sup_{s_0 u < t < s_0} \int_{\mathbb{T}} \varphi_A(te^{i\theta}) \log^- |f(te^{i\theta})| d\theta \leq P_{\mathbb{T},-}(t_0),$$

so

$$\begin{aligned} 8(1 - u^2)^{1/2} P_{\mathbb{T},-}(t_0) - 2 \int_{\mathbb{T}} \varphi(s_0 e^{i\theta}) \log^- |f(s_0 e^{i\theta})| &\leq \\ &\leq (8(1 - u^2)^{1/2} - 2) \int_{\mathbb{T}} \varphi(s_0 e^{i\theta}) \log^- |f(s_0 e^{i\theta})|. \end{aligned}$$

So choosing $u < 1$ such that $8(1 - u^2)^{1/2} - 2 \leq 0$, i.e. $u \geq \sqrt{\frac{15}{16}}$ which is independent of t_0 , we get

$$\forall t_0 < 1, \quad \sum_{a \in Z(f_s)} g_{t_0}(a) \leq \sum_{a \in Z(f_s)} g_{t_0}(a) + \sum_{a \in Z(f_s)} g_{s_0}(a) \lesssim P_{\mathbb{D},+}(t_0) + P_{\mathbb{D},+}(s_0). \quad (5.4)$$

In fact we have, for $u = \sqrt{\frac{15}{16}}$,

$$\begin{aligned} 2 \times 4^q (1 - u^2)^{-2q} \int_{D(0,u)} (1 - |z|^2)^{-1/2} d(sz, E)^{2q} \log^+ |f sz| &\leq \\ &\leq 2 \times 4^q \left(\frac{16}{15}\right)^q \left[\int_{\mathbb{D}} d(sz, E)^{2q-1} \log^+ |f(sz)| + \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)| \right]. \end{aligned}$$

So

$$P_{\mathbb{D},+}(s) \leq c(q) \left[\int_{\mathbb{D}} d(sz, E)^{2q-1} \log^+ |f(sz)| + \int_{\mathbb{D}} (1 - |sz|^2)^{2q-1} \log^+ |f(sz)| \right] + P_{\mathbb{T},+}(s).$$

So we are lead to the definition, replacing $2q$ by q :

Definition 5.2 Let $E = \bar{E} \subset \mathbb{T}$. We say that an holomorphic function f is in the generalised Nevanlinna class $\mathcal{N}_{d(\cdot, E)^q, 0}$ if $\exists \delta > 0$, $\delta < 1$ such that

$$\begin{aligned} \|f\|_{\mathcal{N}_{d(\cdot, E)^q, 0}} &:= \sup_{1-\delta < s < 1} \left\{ \int_{\mathbb{T}} d(se^{i\theta}, E)^q \log^+ |f(se^{i\theta})| + \right. \\ &\quad \left. + \int_{\mathbb{D}} d(sz, E)^{q-1} \log^+ |f(sz)| + \int_{\mathbb{D}} (1 - |sz|^2)^{q-1} \log^+ |f(sz)| \right\} < \infty. \end{aligned}$$

And we proved the Blaschke type condition:

Theorem 5.3 Let $E = \bar{E} \subset \mathbb{T}$. Suppose $q > 0$ and $f \in \mathcal{N}_{\varphi, 0}(\mathbb{D})$ with $|f(0)| = 1$, then

$$\sum_{a \in Z(f)} (1 - |a|^2) \varphi(a) \leq c(\varphi) \|f\|_{\mathcal{N}_{\varphi, 0}}.$$

Corollary 5.4 Let $E = \bar{E} \subset \mathbb{T}$. Suppose $q > 0$ and $f \in \mathcal{N}_{d(\cdot, E)^q, 0}(\mathbb{D})$ with $|f(0)| = 1$, then

$$\sum_{a \in Z(f)} (1 - |a|^2) d(a, E)^q \leq c(E, q) \|f\|_{\mathcal{N}_{d(\cdot, E)^q, 0}}.$$

Proof.

For the theorem we apply inequality (5.4)

$$\forall t < 1, \quad \sum_{a \in Z(f_t)} g_t(a) \lesssim P_{\mathbb{D},+}(t) + P_{\mathbb{D},+}(s),$$

and for the corollary we recall that $\varphi(z) \simeq d(z, E)^q$. ■

6 Application : L^∞ bounds.

We shall examine two cases.

- Case $p > 0$.

Let $E = \bar{E} \subset \mathbb{T}$; its Ahern-Clark type $\alpha(E)$ is defined the following way:

$$\alpha(E) := \sup \{ \alpha \in \mathbb{R} :: |\{t \in \mathbb{T} :: d(t, E) < x\}| = \mathcal{O}(x^\alpha), x \rightarrow +0 \},$$

where $|A|$ denotes the Lebesgue measure of the set A .

Our hypothesis is

$$\log |f(z)| \leq \frac{K}{(1 - |z|)^p} \frac{1}{d(z, E)^q}, \quad z \in \mathbb{D}, \quad p, q \geq 0.$$

We want to apply corollary 4.4 so we have, with $\varphi(z) := d(z, E)^{q-\alpha(E)+\epsilon}$:

$$\sum_{a \in Z(f)} (1 - |a|^2)^{1+p} \varphi(a) \leq c(\varphi) \|f\|_{\mathcal{N}_{\varphi,p}},$$

and we shall compute $\|f\|_{\mathcal{N}_{\varphi,p}}$, i.e.

$$\|f\|_{\mathcal{N}_{\varphi,p}} := \sup_{1-\delta \leq s < 1} \int_{\mathbb{D}} (1 - |z|^2)^{p-1} d(sz, E)^{q-\alpha(E)+\epsilon} \log^+ |f(sz)|.$$

The hypothesis gives

$$\forall z \in \mathbb{D}, \log^+ |f(z)| \leq \frac{K}{(1 - |z|^2)^p} \frac{1}{d(z, E)^q},$$

so we have

$$\int_{\mathbb{D}} (1 - |z|^2)^{p-1} d(sz, E)^{q-\alpha(E)+\epsilon} \log^+ |f(sz)| \leq K \int_{\mathbb{D}} (1 - |z|^2)^{\epsilon-1} d(sz, E)^{-\alpha(E)}.$$

We set $\Gamma_n := E_n \times (1 - 2^{-n}, 1)$ and $\gamma_n := \Gamma_n \setminus \Gamma_{n+1}$. Then we get

$$\begin{aligned} \int_{\mathbb{D}} (1 - |z|^2)^{\epsilon-1} d(sz, E)^{-\alpha(E)} &= \sum_{n \in \mathbb{N}} \int_{\gamma_n} (1 - |z|^2)^{\epsilon-1} d(sz, E)^{-\alpha(E)} \leq \\ &\sum_{n \in \mathbb{N}} 2^{-(\epsilon-1)n} 2^{n\alpha(E)} \int_{\gamma_n} dm(z) \leq \sum_{n \in \mathbb{N}} 2^{-(\epsilon-1)n} 2^{n\alpha(E)} |E_n| 2^{-n} = \sum_{n \in \mathbb{N}} 2^{-\epsilon n} =: c(\epsilon) < \infty \end{aligned}$$

because $\epsilon > 0$.

So corollary 4.4 gives

$$\sum_{a \in Z(f)} (1 - |a|^2)^{1+p} d(a, E)^{q-\alpha(E)+\epsilon} \leq c(p, q, \epsilon) \|f\|_{\mathcal{N}_{\varphi,p}},$$

hence we get

$$\sum_{a \in Z(f)} (1 - |a|^2)^{1+p} d(a, E)^{q-\alpha(E)+\epsilon} \leq c(p, q, \epsilon) \|f\|_{\mathcal{N}_{\varphi,p}} \leq K c(\epsilon) c(p, q, \epsilon).$$

So we proved:

Theorem 6.1 *Suppose that $f \in \mathcal{H}(\mathbb{D})$, $|f(0)| = 1$ and*

$$\forall z \in \mathbb{D}, \log^+ |f(z)| \leq \frac{K}{(1 - |z|^2)^p} \frac{1}{d(z, E)^q},$$

then we have, with any $\epsilon > 0$,

$$\sum_{a \in Z(f)} (1 - |a|^2)^{1+p} d(a, E)^{(q-\alpha(E)+\epsilon)_+} \leq c(p, q, \epsilon) K.$$

• **Case $p = 0$.**

For this case we want to apply corollary 5.4. So let

$$\forall z \in \mathbb{D}, \log^+ |f(z)| \leq K \frac{1}{d(z, E)^q}.$$

We have

Theorem 6.2 *Suppose that $f \in \mathcal{H}(\mathbb{D})$, $|f(0)| = 1$ and*

$$\forall z \in \mathbb{D}, \log^+ |f(z)| \leq K \frac{1}{d(z, E)^q},$$

then

$$\sum_{a \in Z(f)} (1 - |a|^2) d(a, E)^{(q-\alpha(E)+\epsilon)_+} \leq c(q, \epsilon) K.$$

Proof.

We have to verify

$$\sup_{1-\delta \leq s < 1} \int_{\mathbb{T}} d(se^{i\theta}, E)^{q-\alpha(E)+\epsilon} \log^+ |f(se^{i\theta})| d\theta < \infty$$

and

$$\sup_{1-\delta \leq s < 1} \int_{\mathbb{D}} d(sz, E)^{q-\alpha(E)-1+\epsilon} \log^+ |f(sz)| < \infty.$$

For the first one, we have

$$\int_{\mathbb{T}} d(se^{i\theta}, E)^{q-\alpha(E)+\epsilon} \log^+ |f(se^{i\theta})| \leq K \int_{\mathbb{T}} d(se^{i\theta}, E)^{\epsilon-\alpha(E)}.$$

Set $E_n := \{x \in \mathbb{T} :: d(x, E) \geq 2^{-n}\}$ and $F_n := E_n \setminus E_{n+1}$. we have

$$\begin{aligned} \int_{\mathbb{T}} d(se^{i\theta}, E)^{\epsilon-\alpha(E)} d\theta &= \sum_{n \in \mathbb{N}} \int_{F_n} d(se^{i\theta}, E)^{\epsilon-\alpha(E)} d\theta \leq \sum_{n \in \mathbb{N}} \int_{F_n} 2^{-n(\epsilon-\alpha(E))} d\theta \leq \\ &\sum_{n \in \mathbb{N}} 2^{-n(\epsilon-\alpha(E))} \int_{F_n} d\theta \leq \sum_{n \in \mathbb{N}} 2^{-n\epsilon} < \infty \end{aligned}$$

by the very definition of $\alpha(E)$ and because $\epsilon > 0$.

For the second one we set $\Gamma_n := E_n \times (1 - 2^{-n}, 1)$ and $\gamma_n := \Gamma_n \setminus \Gamma_{n+1}$.

We get

$$\int_{\mathbb{D}} d(sz, E)^{q-\alpha(E)-1+\epsilon} \log^+ |f(sz)| = \int_{\mathbb{D}} d(sz, E)^{q-\alpha(E)-1+\epsilon} \log^+ |f(sz)| \leq \int_{\mathbb{D}} d(sz, E)^{\epsilon-\alpha(E)-1}.$$

But

$$\begin{aligned} \sum_{n \in \mathbb{N}} \int_{\gamma_n} d(sz, E)^{\epsilon-\alpha(E)-1} &\leq \sum_{n \in \mathbb{N}} \int_{\gamma_n} 2^{-n(\epsilon-\alpha(E)-1)} \leq \sum_{n \in \mathbb{N}} 2^{-n(\epsilon-\alpha(E)-1)} \int_{\gamma_n} dm(z) \leq \\ &\sum_{n \in \mathbb{N}} 2^{-n(\epsilon-\alpha(E)-1)} |E_n| \times (2^{-n}) \leq \sum_{n \in \mathbb{N}} 2^{-n\epsilon} < \infty, \end{aligned}$$

because $\epsilon > 0$. We end the proof as in the case $p > 0$. ■

These results give alternative proofs of some of the results by Favorov & Golinskii [6].

7 Appendix.

When there is no ambiguities, we shall forget the index j .

Lemma 7.1 *We have*

$$\bar{\partial}(\psi_j)^q(z) = q \frac{(z - \alpha_j) |z - \alpha_j|^{2q-2} |z - \beta_j|^{2q}}{\delta_j^{2q}} + q \frac{(z - \beta_j) |z - \alpha_j|^{2q} |z - \beta_j|^{2q-2}}{\delta_j^{2q}}.$$

And

$$\partial \bar{\partial}(\psi_j)^q = q^2 \frac{|z - \alpha_j|^{2q-2} |z - \beta_j|^{2q-2}}{\delta_j^{2q}} \{ |z - \alpha_j|^2 + |z - \beta_j|^2 + 2\Re[(z - \alpha_j)(\bar{z} - \bar{\beta}_j)] \}.$$

Proof.

We have $\forall z \in \Gamma$, $\psi(z)^q = \frac{|z - \alpha|^{2q} |z - \beta|^{2q}}{\delta^{2q}}$ so

$$\bar{\partial}(\psi)^q(z) = q \frac{(z - \alpha) |z - \alpha|^{2q-2} |z - \beta|^{2q}}{\delta^{2q}} + q \frac{(z - \beta) |z - \alpha|^{2q} |z - \beta|^{2q-2}}{\delta^{2q}}.$$

And

$$\begin{aligned}
\partial\bar{\partial}(\psi)^q(z) &= q^2 \frac{|z - \alpha|^{2q-2} |z - \beta|^{2q}}{\delta^{2q}} + q^2 \frac{|z - \alpha|^{2q} |z - \beta|^{2q-2}}{\delta^{2q}} + \\
&\quad + 2q^2 \frac{|z - \alpha|^{2q-2} |z - \beta|^{2q-2}}{\delta^{2q}} \Re[(z - \alpha)(\bar{z} - \bar{\beta})] = \\
&= q^2 \frac{|z - \alpha|^{2q-2} |z - \beta|^{2q-2}}{\delta^{2q}} \{|z - \beta|^2 + |z - \alpha|^2 + 2\Re[(z - \alpha)(\bar{z} - \bar{\beta})]\}. \quad \blacksquare
\end{aligned}$$

Remark 7.2 We notice that: $\partial\bar{\partial}(\psi_j)^q(z) \geq 0$ because

$$|z - \beta|^2 + |z - \alpha|^2 + 2\Re[(z - \alpha)(\bar{z} - \bar{\beta})] \geq |z - \beta|^2 + |z - \alpha|^2 - 2|z - \alpha||z - \beta| \geq 0.$$

Lemma 7.3 If $\eta'_j \neq 0$ or if $\eta''_j \neq 0$, we have:

$$\forall z \in \Gamma_j, \quad 2(1 - |z|^2)^2 \leq \psi_j(z) \leq 3(1 - |z|^2)^2.$$

Proof.

If $\eta'_j \neq 0$ we have

- $\chi'_j \neq 0$ which implies $2 \leq \frac{\psi_j(z)}{(1 - |z|^2)^2} \leq 3$ hence

$$2(1 - |z|^2)^2 \leq \psi_j(z) \leq 3(1 - |z|^2)^2.$$

The same for the second derivatives, which ends the proof of the lemma. \blacksquare

Lemma 7.4 We have

$$\left| \bar{\partial} \left[\chi \left(\frac{|z - \alpha|^2}{(1 - |z|^2)^2} \right) \right] \right| \leq 9 |\chi'(\alpha)| (1 - |z|^2)^{-1}.$$

and

$$|\partial\bar{\partial}\chi| \lesssim (|\chi'| + |\chi''|)(1 - |z|^2)^{-2}.$$

Proof.

We have

$$\bar{\partial} \left[\chi \left(\frac{|z - \alpha_j|^2}{(1 - |z|^2)^2} \right) \right] = \chi'(\alpha_j) \bar{\partial} \left[\frac{|z - \alpha_j|^2}{(1 - |z|^2)^2} \right] = \chi'(\alpha_j) \left[\frac{(z - \alpha_j)}{(1 - |z|^2)^2} + 2z \frac{|z - \alpha_j|^2}{(1 - |z|^2)^3} \right].$$

But if $\chi'(\alpha_j) \neq 0$ then $2 \leq \frac{|z - \alpha_j|^2}{(1 - |z|^2)^2} \leq 3$ hence

$$\left| \bar{\partial} \left[\chi \left(\frac{|z - \alpha_j|^2}{(1 - |z|^2)^2} \right) \right] \right| \leq |\chi'(\alpha_j)| [\sqrt{3} + 6] (1 - |z|^2)^{-1} \leq 9 |\chi'(\alpha_j)| (1 - |z|^2)^{-1}.$$

Now

$$\begin{aligned}
\partial\bar{\partial} \left[\chi \left(\frac{|z - \alpha_j|^2}{(1 - |z|^2)^2} \right) \right] &= \partial \left\{ \chi'(\alpha_j) \left[\frac{(z - \alpha_j)}{(1 - |z|^2)^2} + 2z \frac{|z - \alpha_j|^2}{(1 - |z|^2)^3} \right] \right\} = \\
&= \partial \{ \chi'(\alpha_j) \} \left[\frac{(z - \alpha_j)}{(1 - |z|^2)^2} + 2z \frac{|z - \alpha_j|^2}{(1 - |z|^2)^3} \right] + \chi'(\alpha_j) \partial \left\{ \frac{(z - \alpha_j)}{(1 - |z|^2)^2} + 2z \frac{|z - \alpha_j|^2}{(1 - |z|^2)^3} \right\}.
\end{aligned}$$

And

$$\partial \chi'(\alpha_j) = \chi''(\alpha_j) \left[\frac{(\bar{z} - \bar{\alpha}_j)}{(1 - |z|^2)^2} + 2\bar{z} \frac{|z - \alpha_j|^2}{(1 - |z|^2)^3} \right].$$

so

$$|\partial \chi'(\alpha_j)| \leq 9 |\chi''(\alpha_j)| (1 - |z|^2)^{-1}.$$

And a straightforward computation gives

$$\left| \partial \left\{ \frac{(z - \alpha_j)}{(1 - |z|^2)^2} + 2z \frac{|z - \alpha_j|^2}{(1 - |z|^2)^3} \right\} \right| \lesssim (1 - |z|^2)^{-2}.$$

So the lemma is proved. ■

Lemma 7.5 *Let $\eta \in \mathbb{T}$, then we have $\Re(\bar{z}(z - \eta)) \leq 0$ iff $z \in \mathbb{D} \cap D(\frac{\eta}{2}, \frac{1}{2})$.*

Proof.

We set $z = \eta t$, then we have

$$\bar{z}(z - \eta) = \bar{\eta} \bar{t}(\eta t - \eta) = \bar{t}(t - 1).$$

Hence

$$\Re(\bar{z}(z - \eta)) = \Re(\bar{t}(t - 1)) = \Re(r^2 - r e^{i\theta}) = r^2 - r \cos \theta.$$

Hence with $t = x + iy = r e^{i\theta}$, $x = r \cos \theta$, $y = r \sin \theta$, we get

$$\Re(\bar{t}(t - 1)) \leq 0 \iff x^2 + y^2 - x \leq 0$$

which means $(x, y) \in D(\frac{1}{2}, \frac{1}{2})$ hence $z \in \mathbb{D} \cap D(\frac{\eta}{2}, \frac{1}{2})$. ■

Lemma 7.6 *(Substitution 1) We have, for $\delta > 0$ and $u \in]0, 1[$, and $|f(0)| = 1$,*

$$\begin{aligned} \int_{\mathbb{D}} (1 - |z|^2)^{p-1+\delta} (1 - |sz|^2)^{2q} \log^- |f(sz)| &\leq \\ &\leq (1 - u^2)^\delta u^{-2} \int_{\mathbb{D}} (1 - |z|^2)^{p-1} |z|^2 (1 - |sz|^2)^{2q} \log^- |f(sz)| + \\ &\quad + \int_{\mathbb{D}} (1 - |z|^2)^{p-1} (1 - |sz|^2)^{2q} \log^+ |f(sz)|. \end{aligned}$$

Proof.

We have

$$\begin{aligned} \int_{\mathbb{D}} (1 - |z|^2)^{p-1+\delta} (1 - |sz|^2)^{2q} \log^- |f(sz)| &= \\ &= \int_{D(0,u)} (1 - |z|^2)^{p-1+\delta} (1 - |sz|^2)^{2q} \log^- |f(sz)| + \\ &\quad + \int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{p-1+\delta} (1 - |sz|^2)^{2q} \log^- |f(sz)|. \end{aligned}$$

For the first term, passing in polar coordinates, we get

$$I_1 = \int_0^u (1 - \rho^2)^{p-1+\delta} (1 - s^2 \rho^2)^{2q} \left\{ \int_{\mathbb{T}} \log^- |f(s\rho e^{i\theta})| \right\} \rho d\rho. \quad (7.5)$$

The subharmonicity of $\log |f(sz)|$ gives

$$0 = \log |f(0)| \leq \int_{\mathbb{T}} \log |f(s\rho e^{i\theta})| = \int_{\mathbb{T}} \log^+ |f(s\rho e^{i\theta})| - \int_{\mathbb{T}} \log^- |f(s\rho e^{i\theta})|,$$

hence

$$\int_{\mathbb{T}} \log^- |f(s\rho e^{i\theta})| \leq \int_{\mathbb{T}} \log^+ |f(s\rho e^{i\theta})|.$$

Putting it in (7.5) we get

$$I_1 \leq \int_0^u (1 - \rho^2)^{p-1+\delta} (1 - s^2 \rho^2)^{2q} \left\{ \int_{\mathbb{T}} \log^+ |f(s\rho e^{i\theta})| \right\} \rho d\rho \leq$$

$$\leq \int_{D(0,u)} (1 - |z|^2)^{p-1+\delta} (1 - |sz|^2)^{2q} \log^+ |f(sz)|. \quad (7.6)$$

For the second term, we have

$$\begin{aligned} I_2 &:= \int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{p-1+\delta} (1 - |sz|^2)^{2q} \log^- |f(sz)| \leq \\ &\leq (1 - u^2)^\delta u^{-2} \int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{p-1} |z|^2 (1 - |sz|^2)^{2q} \log^- |f(sz)|. \end{aligned}$$

This ends the proof. ■

Lemma 7.7 (*Substitution 2*) *We have, for $\delta > 0$ and any u , $0 < u < 1$,*

$$\begin{aligned} \int_{\mathbb{D}} (1 - |z|^2)^{p-1+\delta} \varphi_A(sz) \log^- |fsz| &\leq \\ &\leq 4^q (1 - u^2)^{-2q} \int_{D(0,u)} (1 - |z|^2)^{p-1+\delta} \varphi_A(sz) \log^+ |f(sz)| + \\ &\quad + (1 - u^2)^{\delta/2} u^{-2} \int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{p+\delta/2-1} |z|^2 \varphi_A(sz) \log^- |f(sz)|. \end{aligned}$$

Proof.

We have:

$$\begin{aligned} \int_{\mathbb{D}} (1 - |z|^2)^{p-1+\delta} \varphi_A(sz) \log^- |f(sz)| &= \\ &= \int_{D(0,u)} (1 - |z|^2)^{p-1+\delta} \varphi_A(sz) \log^- |f(sz)| + \\ &\quad + \int_{\mathbb{D} \setminus D(0,u)} (1 - |z|^2)^{p-1+\delta} \varphi_A(sz) \log^- |f(sz)|. \end{aligned}$$

For the first term we get

$$I_1 := \int_{D(0,u)} (1 - |z|^2)^{p-1+\delta} \varphi_A(sz) \log^- |f(sz)|.$$

Because $\forall \alpha \in \mathbb{T}$, $|sz - \alpha| \leq 2$ we get $\varphi_{A,j}(sz) = \eta_j(z) \frac{|z - \alpha_j|^{2q} |z - \beta_j|^{2q}}{\delta_j^{2q}}$, hence, in order to have

$\eta_j(z) \neq 0$, we have $|z - \alpha|^2 \geq 2(1 - |z|^2)^2$ and $|z - \beta|^2 \geq \lambda(1 - |z|^2)^2$. But, with (1.2),

$$\psi_j(z) := \frac{|z - \alpha_j|^2 |z - \beta_j|^2}{\delta_j^2} \leq 2|z - \alpha_j|^2 \leq 4$$

hence

$$\varphi_{A,j}(sz) = \eta_j(z) \psi_j(z)^q \leq 4^q.$$

So we get

$$I_1 \leq 4^q \int_{D(0,u)} (1 - |z|^2)^{p-1+\delta} \log^- |f(sz)|.$$

and we can apply inequality (7.6) to get

$$I_1 \leq 4^q \int_{D(0,u)} (1 - |z|^2)^{p-1+\delta} \log^+ |f(sz)|;$$

but

$$\forall z \in D(0, u), \quad 1 \leq (1 - u^2)^{-2q} \varphi_A(sz)$$

so

$$\begin{aligned} I_1 &\leq 4^q(1-u^2)^{-2q} \int_{D(0,u)} (1-|z|^2)^{p-1+\delta} \varphi_A(sz) \log^+ |f(sz)| \leq \\ &\leq 4^q(1-u^2)^{-2q} P_+(\delta, u). \end{aligned}$$

For the second one

$$\begin{aligned} I_2 &:= \int_{\mathbb{D} \setminus D(0,u)} (1-|z|^2)^{p-1+\delta} \varphi_A(sz) \log^- |f(sz)| \leq \\ &\leq (1-u^2)^{\delta/2} u^{-2} \int_{\mathbb{D} \setminus D(0,u)} (1-|z|^2)^{p+\delta/2-1} |z|^2 \varphi_A(sz) \log^- |f(sz)|. \end{aligned}$$

Adding we get

$$\begin{aligned} \int_{\mathbb{D}} (1-|z|^2)^{p-1+\delta} \varphi_A(sz) \log^- |f(sz)| &\leq 4^q(1-u^2)^{-2q} \int_{D(0,u)} (1-|z|^2)^{p-1+\delta} \log^+ |f(sz)| + \\ &+ (1-u^2)^{\delta/2} u^{-2} \int_{\mathbb{D} \setminus D(0,u)} (1-|z|^2)^{p+\delta/2-1} |z|^2 \varphi_A(sz) \log^- |f(sz)|. \end{aligned}$$

Which ends the proof of the lemma. ■

Lemma 7.8 *Let φ be a continuous function in the unit disc \mathbb{D} . We have that:*

$$s \leq t \in]0, 1[\rightarrow \gamma(s) := \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^- |f(se^{i\theta})| d\theta$$

is a continuous function of $s \in [0, t]$.

Proof.

Because $s \leq t < 1$, the holomorphic function in the unit disc $f(se^{i\theta})$ has only a finite number of zeroes say $N(t)$. As usual we can factor out the zeros of f to get

$$f(z) = \prod_{j=1}^N (z - a_j) g(z)$$

where $g(z)$ has no zeros in the disc $\bar{D}(0, t)$. Hence we get

$$\log |f(z)| = \sum_{j=1}^N \log |z - a_j| + \log |g(z)|.$$

Let $a_j = r_j e^{i\alpha_j}$, $r_j > 0$ because $|f(0)| = 1$, then it suffices to show that

$$\gamma(s) := \int_{\mathbb{T}} \varphi(se^{i\theta}) \log^- |se^{i\theta} - re^{i\alpha}| d\theta$$

is continuous in s near $s = r$, because $\int_{\mathbb{T}} \varphi(se^{i\theta}) \log^- |g(se^{i\theta})| d\theta$ is clearly continuous.

To see that $\gamma(s)$ is continuous at $s = r$, it suffices to show

$$\gamma(s_n) \rightarrow \gamma(r) \text{ when } s_n \rightarrow r.$$

But

$$\forall \theta \neq 0, \varphi(se^{i\theta}) \log |se^{i\theta} - r| \rightarrow \varphi(re^{i\theta}) \log |re^{i\theta} - r|$$

and $\log \frac{1}{|se^{i\theta} - r|} \leq c_\epsilon |se^{i\theta} - r|^{-\epsilon}$ with $\epsilon > 0$. So choosing $\epsilon < 1$, we get that $\log \frac{1}{|se^{i\theta} - r|} \in L^1(\mathbb{T})$

uniformly in s . Because $\varphi(se^{i\theta})$ is continuous uniformly in $s \in [0, t]$ we get also $\varphi(se^{i\theta}) \log \frac{1}{|se^{i\theta} - r|} \in$

$L^1(\mathbb{T})$ uniformly in s . So we can apply the dominated convergence theorem of Lebesgue to get the result. ■

Lemma 7.9 *Suppose that $g_s(z) \in \mathcal{C}^\infty(\bar{\mathbb{D}})$ and $f \in \mathcal{H}(\mathbb{D})$ then, with $s < 1$, $f_s(z) := f(sz)$, we have:*

$$\sum_{a \in Z(f_s)} g_s(a) = \int_{\mathbb{D}} \log |f_s(z)| \Delta g_s(z) + \int_{\mathbb{T}} (g_s \partial_n \log |f_s(z)| - \log |f_s(z)| \partial_n g_s).$$

Proof.

To apply the Green formula we need $\mathcal{C}^2(\bar{\mathbb{D}})$ functions, so we shall use an approximation of $\log |f_s(z)|$. First because $s < 1$, we have that f_s has a finite number of zeroes in \mathbb{D} and we take an $\epsilon > 0$ small enough to have the discs $\forall a \in Z(f_s)$, $D(a, \epsilon)$ disjoint. Then we consider

$$u_\epsilon(z) := \log |f_s(z)| \left(1 - \sum_{a \in Z(f_s)} \chi_a(z, \epsilon)\right),$$

with $\chi_a(z, \epsilon) := 0$ for $z \notin D(a, \epsilon)$, $\chi_a(z, \epsilon) = 1$ for $z \in D(a, \epsilon/2)$, $0 \leq \chi_a(z, \epsilon) \leq 1$ and $\chi_a(z, \epsilon) \in \mathcal{C}^\infty(\bar{\mathbb{D}})$.

Then, because $Z(f_s)$ is finite, we have that u_ϵ is in $\mathcal{C}^\infty(\bar{\mathbb{D}})$ and we can apply the Green formula to g_s and u_ϵ . we have

$$\int_{\mathbb{D}} (g_s(z) \Delta u_\epsilon(z) - u_\epsilon(z) \Delta g_s(z)) = \int_{\mathbb{T}} (g_s(e^{i\theta}) \partial_n u_\epsilon(e^{i\theta}) - u_\epsilon(e^{i\theta}) \partial_n g_s(e^{i\theta})).$$

Clearly $\Delta u_\epsilon = 0$ outside $\bigcup_{a \in Z(f_s)} D(a, \epsilon)$ and in $D(a, \epsilon)$ we get, because $g_s(z)$ is continuous in $\bar{\mathbb{D}}$,

$$\int_{D(a, \epsilon)} g_s(z) \Delta u_\epsilon(z) \xrightarrow{\epsilon \rightarrow 0} g_s(a).$$

We have also

$$\int_{\mathbb{D}} u_\epsilon(z) \Delta g_s(z) \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{D}} \log |f_s(z)| \Delta g_s(z),$$

$$\int_{\mathbb{T}} u_\epsilon(e^{i\theta}) \partial_n g_s(e^{i\theta}) \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{T}} \log |f_s(z)| \partial_n g_s,$$

and

$$\int_{\mathbb{T}} g_s(e^{i\theta}) \partial_n u_\epsilon(e^{i\theta}) \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{T}} (g_s \partial_n \log |f_s(z)|),$$

which prove the lemma. ■

Lemma 7.10 *Suppose that $g_s(z) \in \mathcal{C}^\infty(\bar{\mathbb{D}})$ and u is a subharmonic function in the disc \mathbb{D} ; then, with $\forall s < 1$, $u_s(z) := u(sz)$, we have:*

$$\int_{\mathbb{D}} g_s(z) d\mu(z) = \int_{\mathbb{D}} u_s(z) \Delta g_s(z) + \int_{\mathbb{T}} (g_s \partial_n u_s - u_s \partial_n g_s),$$

where $\mu_s := \Delta u_s$ is the positive Riesz measure associated to u_s .

Proof.

First recall that $\mu := \Delta u$, the Riesz measure associated to the subharmonic non trivial function u in the disc \mathbb{D} , is finite on the compact sets of \mathbb{D} because $u \in L^1_{loc}(\mathbb{D})$ implies that $u \in \mathcal{D}'(\mathbb{D})$ hence $\Delta u \in \mathcal{D}'(\mathbb{D})$; so take a function $\varphi \in \mathcal{D}(\mathbb{D})$ which is 1 on the compact $K \Subset \mathbb{D}$ and $\varphi \geq 0$. Then, because Δu is a positive measure, we get

$$\langle \Delta u, \varphi \rangle = \int_{\mathbb{D}} \varphi(z) d\mu(z) \geq \int_K d\mu(z)$$

hence $\mu(K) \leq \langle \Delta u, \varphi \rangle < \infty$.

The idea is to start with the measure $\mu := \Delta u$ and, because $s < 1$, we can cut it by a smooth function $\gamma_s(z) \in \mathcal{C}_c^\infty(\mathbb{D})$, such that $\gamma(z) = 1$ in $D(0, s)$. Then we regularise $\gamma\mu$ by convolution with :

$$\chi_\epsilon(\rho e^{i\theta}) := a_\epsilon(\rho) b_\epsilon(\theta),$$

with

$$a_\epsilon(\rho) := \frac{1}{\epsilon} a\left(\frac{|\rho|}{\epsilon}\right), \quad 0 \leq a(t) \leq 1, \quad a \in \mathcal{C}_c^\infty([0, 1[), \quad t \leq 1/2 \Rightarrow a(t) = 1.$$

And

$$b_\epsilon(\rho) := \frac{1}{\epsilon} a\left(\frac{|\theta|}{\epsilon}\right), \quad 0 \leq b(t) \leq 1, \quad a \in \mathcal{C}_c^\infty([0, 2\pi[), \quad t \leq 1/2 \Rightarrow b(t) = 1.$$

So we set the potential:

$$U(z) := \int_{\mathbb{D}} \log |z - \zeta| \gamma d\mu(\zeta) = \log |\cdot| * (\gamma\mu)$$

and we have $\Delta U(z) = \gamma(z)\mu(z)$ in distributions sense, and we regularise

$$U_\epsilon := \chi_\epsilon * U \Rightarrow \Delta U_\epsilon = \chi_\epsilon * \Delta U.$$

Now we have that $\Delta(u - U) = \mu - \gamma\mu = 0$ in $D(0, s)$ so $H := u - U$ is harmonic in $D(0, s)$ hence smooth.

On the other hand we have, because U_ϵ is \mathcal{C}^∞ , that the Green formula is applicable so

$$\int_{\mathbb{D}} (g_s(z) \Delta U_\epsilon(sz) - U_\epsilon(sz) \Delta g_s(z)) = \int_{\mathbb{T}} (g_s(e^{i\theta}) \partial_n U_\epsilon(se^{i\theta}) - U_\epsilon(se^{i\theta}) \partial_n g_s(e^{i\theta})).$$

And from $u = U + H$, we get $u = H + \lim_{\epsilon \rightarrow 0} U_\epsilon$ so it remains to see what happen to each term.

For the first one

$$\int_{\mathbb{D}} g_s(z) \Delta U_\epsilon(sz) = \int_{\mathbb{D}} g_s(z) (\chi_\epsilon * \Delta U)(sz) = \int_{\mathbb{D}} (g_s * \chi_\epsilon)(\zeta) \Delta U(s\zeta).$$

But $(g_s * \chi_\epsilon)(\zeta) \rightarrow g_s(\zeta)$ uniformly in $\bar{\mathbb{D}}$, because g_s is smooth on $\bar{\mathbb{D}}$, and $\Delta U(sz) = \gamma(sz)\mu(sz) = \mu(sz)$ is a bounded measure in $\bar{\mathbb{D}}$ so we get

$$\int_{\mathbb{D}} g_s(z) \Delta U_\epsilon(sz) \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{D}} g_s(z) \Delta U_\epsilon(sz) = \int_{\mathbb{D}} g_s(z) d\mu(sz).$$

For the second one:

$$\int_{\mathbb{D}} U_\epsilon(sz) \Delta g_s(z) = \int_{\mathbb{D}} U(s\zeta) (\Delta g_s * \chi_\epsilon) \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{D}} U(sz) \Delta g_s(z),$$

as above because $(\Delta g_s * \chi_\epsilon)(\zeta) \rightarrow \Delta g_s(\zeta)$ uniformly in $\bar{\mathbb{D}}$, because Δg_s is smooth on $\bar{\mathbb{D}}$.

For the third term

$$\int_{\mathbb{T}} g_s(e^{i\theta}) \partial_n U_\epsilon(se^{i\theta}) = \int_{\mathbb{T}} g_s(e^{i\theta}) (\chi_\epsilon * \partial_n U)(se^{i\theta})$$

and here we use the special form of $\chi_\epsilon(\rho e^{i\theta}) := a_\epsilon(\rho) b_\epsilon(\theta)$ to get

$$(\chi_\epsilon * \partial_n U)(sz) = \int_{\mathbb{D}} \chi_\epsilon(\zeta - z) \partial_n U(\zeta) = \int_0^1 a_\epsilon(\rho - s) \left\{ \int_{\mathbb{T}} b_\epsilon(\varphi - \theta) \partial_n U(\rho e^{i\varphi}) d\varphi \right\} \rho d\rho,$$

so by Fubini we get

$$\begin{aligned} \int_{\mathbb{T}} g_s(e^{i\theta}) (\chi_\epsilon * \partial_n U)(se^{i\theta}) d\theta &= \\ &= \int_{\mathbb{T}} \left\{ \int_{\mathbb{T}} g_s(e^{i\theta}) b_\epsilon(\varphi - \theta) d\theta \left[\int_0^1 a_\epsilon(\rho - s) \partial_n U(\rho e^{i\varphi}) \right] \rho d\rho \right\} d\varphi. \end{aligned}$$

But

$$\int_{\mathbb{T}} g_s(e^{i\theta}) b_\epsilon(\varphi - \theta) d\theta = (g_s * b_\epsilon)(\varphi)$$

and

$$\int_0^1 a_\epsilon(\rho - s) \partial_n U(\rho e^{i\varphi}) \rho d\rho \xrightarrow{\epsilon \rightarrow 0} \partial_n U(se^{i\varphi})$$

as a measure on \mathbb{T} , so, because $(g_s * b_\epsilon)(\varphi) \xrightarrow{\epsilon \rightarrow 0} g_s(e^{i\varphi})$ uniformly on \mathbb{T} because $g_s(e^{i\theta}) \in \mathcal{C}^\infty(\mathbb{T})$, we get

$$\int_{\mathbb{T}} g_s(e^{i\theta}) (\chi_\epsilon * \partial_n U)(se^{i\theta}) d\theta \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{T}} g_s(e^{i\theta}) \partial_n U(se^{i\theta}) d\theta.$$

For the last term, we get the same way:

$$\int_{\mathbb{T}} U_\epsilon(se^{i\theta}) \partial_n g_s(e^{i\theta}) \xrightarrow{\epsilon \rightarrow 0} \int_{\mathbb{T}} \partial_n g_s(e^{i\theta}) U(se^{i\theta}) d\theta.$$

So we get

$$\int_{\mathbb{D}} (g_s(z) \Delta U(sz) - U(sz) \Delta g_s(z)) = \int_{\mathbb{T}} (g_s(e^{i\theta}) \partial_n U(se^{i\theta}) - U(se^{i\theta}) \partial_n g_s(e^{i\theta})).$$

Now we replace U by $U = u + H$ with H harmonic in $\bar{D}(0, s)$ to get

$$\begin{aligned} \int_{\mathbb{D}} (g_s(z) \Delta u(sz) - [u + H](sz) \Delta g_s(z)) &= \\ &= \int_{\mathbb{T}} (g_s(e^{i\theta}) \partial_n [u + H](se^{i\theta}) - [u + H](se^{i\theta}) \partial_n g_s(e^{i\theta})), \end{aligned}$$

but because H is \mathcal{C}^∞ we get, applying the Green formula to it

$$\int_{\mathbb{D}} -H(sz) \Delta g_s(z) = \int_{\mathbb{T}} (g_s(e^{i\theta}) \partial_n H(se^{i\theta}) - H(se^{i\theta}) \partial_n g_s(e^{i\theta})),$$

so it remains

$$\int_{\mathbb{D}} (g_s(z) \Delta u(sz) - u(sz) \Delta g_s(z)) = \int_{\mathbb{T}} (g_s(e^{i\theta}) \partial_n u(se^{i\theta}) - u(se^{i\theta}) \partial_n g_s(e^{i\theta})),$$

which proves the lemma. ■

Lemma 7.11 *Let $\varphi(z)$ be a positive function in \mathbb{D} and $f \in \mathcal{H}(\mathbb{D})$; set $f_s(z) := f(sz)$ and suppose that:*

$$\forall s < 1, \quad \sum_{a \in Z(f_s)} (1 - |a|^2)^{p+1} \varphi(sa) \leq \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \varphi(sz) \log^+ |f(sz)|,$$

then, for any $1 > \delta > 0$ we have

$$\sum_{a \in Z(f)} (1 - |a|^2)^{p+1} \varphi(a) \leq \sup_{1-\delta < s < 1} \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \varphi(sz) \log^+ |f(sz)|.$$

We have also:

let $\varphi(z)$, $\psi(z)$ be positive continuous functions in \mathbb{D} and $f \in \mathcal{H}(\mathbb{D})$ such that:

$$\forall s < 1, \quad \sum_{a \in Z(f) \cap D(0, s)} (1 - |a|^2) \varphi(sa) \leq \int_{\mathbb{D}} \varphi(sz) \log^+ |f(sz)| + \int_{\mathbb{T}} \psi(se^{i\theta}) \log^+ |f(se^{i\theta})|$$

then, for any $1 > \delta > 0$ we have

$$\sum_{a \in Z(f)} (1 - |a|^2) \varphi(a) \leq \sup_{1-\delta < s < 1} \int_{\mathbb{D}} \varphi(sz) \log^+ |f(sz)| + \sup_{1-\delta < s < 1} \int_{\mathbb{T}} \psi(se^{i\theta}) \log^+ |f(sz)|.$$

Proof.

We have $a \in Z(f_s) \iff f(sa) = 0$, i.e. $b := sa \in Z(f) \cap D(0, s)$. Hence the hypothesis is

$$\forall s < 1, \quad \sum_{a \in Z(f) \cap D(0, s)} (1 - \left| \frac{a}{s} \right|^2)^{p+1} \varphi(a) \leq \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \varphi(sz) \log^+ |f(sz)|.$$

We fix $1 - \delta < r < 1$, $r < s < 1$, then, because $Z(f) \cap D(0, r) \subset Z(f) \cap D(0, s)$ and $\varphi \geq 0$, we have

$$\begin{aligned} \sum_{a \in Z(f) \cap D(0,r)} \left(1 - \left|\frac{a}{s}\right|^2\right)^{p+1} \varphi(a) &\leq \sum_{a \in Z(f) \cap D(0,s)} \left(1 - \left|\frac{a}{s}\right|^2\right)^{p+1} \varphi(a) \leq \\ &\leq \sup_{1-\delta < s < 1} \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \varphi(z) \log^+ |f(z)|. \end{aligned}$$

In $D(0, r)$ we have a finite fixed number of zeroes of f , and, because $(1 - |\frac{a}{s}|^2)^{p+1}$ is continuous in $s \leq 1$ for $a \in \mathbb{D}$, we have

$$\forall a \in Z(f) \cap D(0, r), \lim_{s \rightarrow 1} \left(1 - \left|\frac{a}{s}\right|^2\right)^{p+1} = (1 - |a|^2)^{p+1}.$$

Hence

$$\sum_{a \in Z(f) \cap D(0,r)} (1 - |a|^2)^{p+1} \varphi(a) \leq \sup_{1-\delta < s < 1} \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \varphi(sz) \log^+ |f(sz)|.$$

Because the right hand side is independent of $r < 1$ and φ is positive in \mathbb{D} so the sequence

$$S(r) := \sum_{a \in Z(f) \cap D(0,r)} (1 - |a|^2)^{p+1} \varphi(a)$$

is increasing with r , we get

$$\sum_{a \in Z(f)} (1 - |a|^2)^{p+1} \varphi(a) \leq \sup_{1-\delta < s < 1} \int_{\mathbb{D}} (1 - |z|^2)^{p-1} \varphi(sz) \log^+ |f(sz)|.$$

This proves the first part. The proof of the second one is just identical. ■

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