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The discontinuous Galerkin gradient discretisation

Robert Eymard^{} and Cindy Guichard[†]*

Abstract

The Symmetric Interior Penalty Galerkin (SIPG) method, based on Discontinuous Galerkin approximations, is shown to be included in the Gradient Discretisation Method (GDM) framework. Therefore, it can take benefit from the general properties of the GDM, since we prove that it meets the main mathematical gradient discretisation properties on any kind of polytopal mesh. For this proof, we adapt discrete functional analysis properties to our precise geometrical hypotheses. We illustrate this inheritance property on the case of the p -Laplace problem. A short numerical study shows the effect of the numerical parameter included in the scheme.

Keywords: Gradient Discretisation method, Discontinuous Galerkin method, Symmetric Interior Penalty Galerkin scheme, discrete functional analysis, polytopal meshes

1 Introduction

Discontinuous Galerkin (DG) methods are being more and more studied. They present the advantage to be suited to elliptic and parabolic problems, while opening the possibility to closely approximate weakly regular functions on general meshes. Although the convergence of DG methods has been proved on a variety of problems (see [3] and references therein), note that the stabilisation of DG schemes for elliptic or parabolic problems has to be specified with respect to the problem, and that there are numerous possible choices [1].

On the other hand, convergence and error estimate results for a wide variety of numerical methods applied to some elliptic, parabolic, coupled, linear and nonlinear problems are proved on the generic “gradient scheme” issued from the Gradient Discretisation Method framework (see [4] and references therein). This framework is shown to include conforming Galerkin methods with or without mass lumping, nonconforming \mathbb{P}^1 finite elements, mixed finite elements and a variety of schemes issued from extensions of the finite volume method. Convergence and error estimate results are then proved in [4] for the following problems:

1. elliptic problem with constant or unknown dependent diffusion,
2. steady or transient p -Laplace problem and more generally Leray-Lions problem,
3. parabolic problem with constant or unknown dependent diffusion,
4. degenerate parabolic (Richards or Stefan-type) problems,

provided that the Gradient Discretisation is coercive, GD-consistent, limit-conforming, compact and in some cases with piecewise constant reconstruction (the precise mathematical meaning of these properties is recalled in Section 2 of this paper). Our aim is to show that, from the DG setting, we can build a Gradient Discretisation, that we call the Discontinuous Galerkin Gradient Discretisation (DGGD), which satisfies all these properties on general polytopal meshes in any space dimension. This immediately extends the range of problems which can be handled by Discontinuous Galerkin methods to all the above problems, as we show in this paper by considering the case of the p -Laplace problem for which we prove a convergence and error estimate result (derived from the Gradient Discretisation Method framework).

It is then interesting to remark that the gradient scheme resulting from the DGGD is identical to the Symmetric Interior Penalty Galerkin scheme in the case of linear diffusion problems, as we detail in section 6.

This paper is organised as follows. Section 2 includes the main definitions for the Gradient Discretisation Method in the case of homogeneous Dirichlet boundary conditions. In Section 3, we give a gradient discretisation version of Discontinuous Galerkin schemes suited to polytopal meshes in any space dimension. We then prove in Section 4 that this gradient discretisation satisfies the core properties which are sufficient for convergence and error estimates results. We then take an example of application of the preceding results in Section 5, where the gradient scheme issued from the DGGD method is shown to satisfy a convergence and error estimate in the case of the p -Laplace problem. We then handle in Section 6 the case of linear elliptic problems, and we show that the scheme issued from the DGGD is identical to the Symmetric Interior Penalty Galerkin method. Then a numerical example shows the role the numerical parameter used in the design of the scheme plays in its accuracy. A short conclusion is then proposed, before the adaptation of the Sobolev inequalities proved in [2, 3] to our setting and definition of the DG norm in Appendix A. We then get that the constant involved in these inequalities do not depend on the regularity factor of the mesh (we only use that the polytopal grid blocks are strictly star-shaped with respect to some point).

2 Main definitions of Gradient Discretisations for homogeneous Dirichlet boundary conditions

In this paper, we make the following assumptions: $p \in]1, +\infty[$ is given and

$$\Omega \text{ is an open bounded polytopal connected subset of } \mathbb{R}^d \text{ (} d \in \mathbb{N}^* \text{),} \quad (1)$$

where polytopal means that it is the union of a finite number of simplices.

The following definitions, first introduced in [6], are given in [4] for a larger variety of boundary conditions.

Definition 2.1 (GD, homogeneous Dirichlet BCs):

A gradient discretisation \mathcal{D} for homogeneous Dirichlet conditions is defined by $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$, where:

1. the set of discrete unknowns $X_{\mathcal{D},0}$ is a finite dimensional real vector space,
2. the function reconstruction $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \rightarrow L^p(\Omega)$ is a linear mapping that reconstructs, from an element of $X_{\mathcal{D},0}$, a function over Ω ,
3. the gradient reconstruction $\nabla_{\mathcal{D}} : X_{\mathcal{D},0} \rightarrow L^p(\Omega)^d$ is a linear mapping which reconstructs, from an element of $X_{\mathcal{D},0}$, a “gradient” (vector-valued function) over Ω . This gradient reconstruction must be chosen such that $\|\nabla_{\mathcal{D}} \cdot\|_{L^p(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$.

Definition 2.2 (Coercivity): If \mathcal{D} is a gradient discretisation in the sense of Definition 2.1, define $C_{\mathcal{D}}$ as the norm of the linear mapping $\Pi_{\mathcal{D}}$:

$$C_{\mathcal{D}} = \max_{v \in X_{\mathcal{D},0} \setminus \{0\}} \frac{\|\Pi_{\mathcal{D}} v\|_{L^p(\Omega)}}{\|\nabla_{\mathcal{D}} v\|_{L^p(\Omega)^d}}. \quad (2)$$

A sequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$ of gradient discretisations in the sense of Definition 2.1 is coercive if there exists $C_P \in \mathbb{R}_+$ such that $C_{\mathcal{D}_m} \leq C_P$ for all $m \in \mathbb{N}$.

The consistency properties indicate how accurately a regular function and its gradient are approximated by some discrete function and gradient which are reconstructed from the space $X_{\mathcal{D},0}$.

Definition 2.3 (GD-consistency): If \mathcal{D} is a gradient discretisation in the sense of Definition 2.1, define $S_{\mathcal{D}} : W_0^{1,p}(\Omega) \rightarrow [0, +\infty)$ by

$$\forall \varphi \in W_0^{1,p}(\Omega), S_{\mathcal{D}}(\varphi) = \min_{v \in X_{\mathcal{D},0}} \left(\|\Pi_{\mathcal{D}} v - \varphi\|_{L^p(\Omega)} + \|\nabla_{\mathcal{D}} v - \nabla \varphi\|_{L^p(\Omega)^d} \right). \quad (3)$$

A sequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$ of gradient discretisations in the sense of Definition 2.1 is GD-consistent, or consistent for short, if

$$\forall \varphi \in W_0^{1,p}(\Omega), \lim_{m \rightarrow \infty} S_{\mathcal{D}_m}(\varphi) = 0. \quad (4)$$

The concept of limit-conformity which we now introduce states that the discrete gradient and divergence operator satisfy this property asymptotically. Since we shall be dealing with non linear problems, we introduce, for any $q \in (1, +\infty)$, the space $W_{\text{div}}^q(\Omega)$ of functions in $(L^q(\Omega))^d$ with divergence in $L^q(\Omega)$:

$$W_{\text{div}}^q(\Omega) = \{\varphi \in L^q(\Omega)^d : \text{div}\varphi \in L^q(\Omega)\}. \quad (5)$$

Definition 2.4 (Limit-conformity): If \mathcal{D} is a gradient discretisation in the sense of Definition 2.1, let $p' = \frac{p}{p-1}$ and define $W_{\mathcal{D}}$: $W_{\text{div}}^{p'}(\Omega) \rightarrow [0, +\infty)$ by

$$\forall \varphi \in W_{\text{div}}^{p'}(\Omega), \quad W_{\mathcal{D}}(\varphi) = \sup_{v \in X_{\mathcal{D},0} \setminus \{0\}} \frac{\left| \int_{\Omega} (\nabla_{\mathcal{D}} v(\mathbf{x}) \cdot \varphi(\mathbf{x}) + \Pi_{\mathcal{D}} v(\mathbf{x}) \text{div}\varphi(\mathbf{x})) \, d\mathbf{x} \right|}{\|\nabla_{\mathcal{D}} v\|_{L^p(\Omega)^d}}. \quad (6)$$

A sequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$ of gradient discretisations is limit-conforming if

$$\forall \varphi \in W_{\text{div}}^{p'}(\Omega), \quad \lim_{m \rightarrow \infty} W_{\mathcal{D}_m}(\varphi) = 0. \quad (7)$$

Dealing with generic non-linearity often requires additional compactness properties on the scheme.

Definition 2.5 (Compactness): A sequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$ of gradient discretisations in the sense of Definition 2.1 is compact if, for any sequence $u_m \in X_{\mathcal{D}_m,0}$ such that $(\|u_m\|_{\mathcal{D}_m})_{m \in \mathbb{N}}$ is bounded, the sequence $(\Pi_{\mathcal{D}_m} u_m)_{m \in \mathbb{N}}$ is relatively compact in $L^p(\Omega)$.

Definition 2.6 (Piecewise constant reconstruction): Let $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ be a gradient discretisation in the sense of Definition 2.1. The operator $\Pi_{\mathcal{D}} : X_{\mathcal{D},0} \rightarrow L^p(\Omega)$ is a piecewise constant reconstruction if there exists a basis $(e_i)_{i \in B}$ of $X_{\mathcal{D},0}$ and a family of disjoint subsets $(\Omega_i)_{i \in B}$ of Ω such that $\Pi_{\mathcal{D}} u = \sum_{i \in B} u_i \mathbf{1}_{\Omega_i}$ for all $u = \sum_{i \in B} u_i e_i \in X_{\mathcal{D},0}$, where $\mathbf{1}_{\Omega_i}$ is the characteristic function of Ω_i .

In other words, $\Pi_{\mathcal{D}} u$ is the piecewise constant function equal to u_i on Ω_i , for all $i \in B$.

It is shown in [4] that all the considered examples of Gradient Discretisations (as listed in the introduction of this paper) meet four of the core properties (coercivity, GD-consistency, limit-conformity, compactness), and that mass-lumped versions satisfy the piece-wise constant reconstruction property. They therefore satisfy convergence and error estimates for the range of problems passed into review in the introduction of this paper. The next sections aim to build a Gradient Discretisation with the discontinuous Galerkin setting, and then to prove that it satisfies the core properties as well.

3 Discontinuous Galerkin Gradient Discretisation (DGGD)

Let us provide a definition for a polytopal mesh of Ω , which is a slightly simplified version of that given in [4].

Definition 3.1 (Polytopal mesh): A polytopal mesh of Ω is given by $\mathfrak{T} = (\mathcal{M}, \mathcal{F}, \mathcal{P})$, where:

1. \mathcal{M} is a finite family of non empty connected polytopal open disjoint subsets of Ω (the ‘‘cells’’) such that $\bar{\Omega} = \cup_{K \in \mathcal{M}} \bar{K}$. For any $K \in \mathcal{M}$, let $\partial K = \bar{K} \setminus K$ be the boundary of K , $|K| > 0$ is the measure of K and h_K denotes the diameter of K , that is the maximum distance between two points of \bar{K} .
2. $\mathcal{F} = \mathcal{F}_{\text{int}} \cup \mathcal{F}_{\text{ext}}$ is a finite family of disjoint subsets of $\bar{\Omega}$ (the ‘‘faces’’ of the mesh – ‘‘edges’’ in 2D), such that, for all $\sigma \in \mathcal{F}_{\text{int}}$, σ is a non empty open subset of a hyperplane of \mathbb{R}^d included in Ω and, for all $\sigma \in \mathcal{F}_{\text{ext}}$, σ is a non empty open subset of $\partial\Omega$; furthermore, the $(d-1)$ -dimensional measure $|\sigma|$ of any $\sigma \in \mathcal{F}$ is strictly positive. We assume that, for all $K \in \mathcal{M}$, there exists a subset \mathcal{F}_K of \mathcal{F} such that $\partial K = \cup_{\sigma \in \mathcal{F}_K} \bar{\sigma}$. We then denote by $\mathcal{M}_{\sigma} = \{K \in \mathcal{M}, \sigma \in \mathcal{F}_K\}$. We then assume that, for all $\sigma \in \mathcal{F}$, either \mathcal{M}_{σ} has exactly one element and then $\sigma \in \mathcal{F}_{\text{ext}}$ or \mathcal{M}_{σ} has exactly two elements and then $\sigma \in \mathcal{F}_{\text{int}}$. For all $K \in \mathcal{M}$ and for any $\sigma \in \mathcal{F}_K$, we denote by $\mathbf{n}_{K,\sigma}$ the (constant) unit vector normal to σ outward to K .

3. \mathcal{P} is a family of points of Ω indexed by \mathcal{M} , denoted by $\mathcal{P} = (\mathbf{x}_K)_{K \in \mathcal{M}}$, such that for all $K \in \mathcal{M}$, $\mathbf{x}_K \in K$. We then denote by $d_{K,\sigma}$ the signed orthogonal distance between \mathbf{x}_K and $\sigma \in \mathcal{F}_K$ (see Figure 1), that is:

$$d_{K,\sigma} = (\mathbf{x} - \mathbf{x}_K) \cdot \mathbf{n}_{K,\sigma}, \text{ for all } \mathbf{x} \in \sigma. \quad (8)$$

We then assume that each cell $K \in \mathcal{M}$ is strictly star-shaped with respect to \mathbf{x}_K , that is $d_{K,\sigma} > 0$ for all $\sigma \in \mathcal{F}_K$. This implies that for all $\mathbf{x} \in K$, the line segment $[\mathbf{x}_K, \mathbf{x}]$ is included in K .

For all $K \in \mathcal{M}$ and $\sigma \in \mathcal{F}_K$, we denote by $D_{K,\sigma}$ the cone with vertex \mathbf{x}_K and basis σ , that is

$$D_{K,\sigma} = \{\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K), s \in]0, 1[, \mathbf{y} \in \sigma\}. \quad (9)$$

The size of the polytopal mesh is defined by:

$$h_{\mathcal{M}} = \sup\{h_K, K \in \mathcal{M}\}. \quad (10)$$

Finally, for a given polytopal mesh \mathfrak{T} we define a number that measures the regularity properties of the mesh:

$$\eta_{\mathfrak{T}} = \max \left(\left\{ \frac{h_K}{h_L} + \frac{h_L}{h_K}, \sigma \in \mathcal{F}_{\text{int}}, \mathcal{M}_{\sigma} = \{K, L\}\right\} \cup \left\{ \frac{h_K}{d_{K,\sigma}}, K \in \mathcal{M}, \sigma \in \mathcal{F}_K \right\} \right). \quad (11)$$

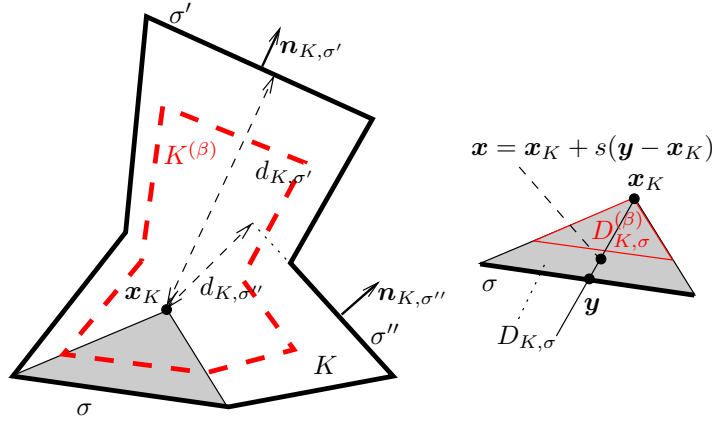


Fig. 1: A cell K of a polytopal mesh and notation on $D_{K,\sigma}$

Let us now define, using the discontinuous Galerkin framework, a gradient discretisation in the sense of Definition 2.1.

Definition 3.2: [Discontinuous Galerkin Gradient Discretisation

(DGGD)] Let $\mathfrak{T} = (\mathcal{M}, \mathcal{F}, \mathcal{P}, \mathcal{V})$ be a polytopal mesh of Ω in the sense of Definition 3.1. Define the Discontinuous Galerkin Gradient Discretisation $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ by the following.

1. For a given value $k \in \mathbb{N}^*$, we consider the space W_h , defined by

$$W_h = \{u \in L^p(\Omega) : u|_K \in \mathbb{P}_k(K), \forall K \in \mathcal{M}\}. \quad (12)$$

Recall that the dimension of $\mathbb{P}_k(\mathbb{R}^d)$ is $\frac{(k+d)!}{k!d!}$, and therefore the dimension of W_h is equal to $\frac{(k+d)!}{k!d!} \#\mathcal{M}$. Let $(\chi_i)_{i \in I}$ be a family of piecewise polynomial basis functions of degree k on each cell, spanning W_h . We set

$$X_{\mathcal{D},0} = \{v = (v_i)_{i \in I} : v_i \in \mathbb{R} \text{ for all } i \in I\}. \quad (13)$$

2. The operator $\Pi_{\mathcal{D}}$ is the reconstruction in $L^p(\Omega)$ of the elements of $X_{\mathcal{D},0}$:

$$\forall v \in X_{\mathcal{D},0}, \Pi_{\mathcal{D}}v = \sum_{i \in \mathcal{I}} v_i \chi_i. \quad (14)$$

In this paper, we denote, for all $K \in \mathcal{M}$ and $v \in X_{\mathcal{D},0}$, by $\Pi_{\overline{K}}v \in \mathbb{P}_k(\overline{K})$ the piecewise polynomial defined by $\Pi_{\mathcal{D}}v$ on K , extended on \overline{K} , and we denote by $\nabla_{\overline{K}}v = \nabla \Pi_{\overline{K}}v$.

3. For $v \in X_{\mathcal{D},0}$, for $K \in \mathcal{M}$ and for any $\sigma \in \mathcal{F}_K$, we set, for a.e. $\mathbf{x} \in D_{K,\sigma}$,

$$\begin{aligned} \nabla_{\mathcal{D}}v(\mathbf{x}) &= \nabla_{\overline{K}}v(\mathbf{x}) + \psi(s) \frac{[v]_{K,\sigma}(\mathbf{y})}{d_{K,\sigma}} \mathbf{n}_{K,\sigma} \\ \text{letting } \mathbf{x} &= \mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K) \text{ with } s \in]0, 1[, \mathbf{y} \in \sigma, \end{aligned} \quad (15)$$

where (see Figure 1):

- for all $K \in \mathcal{M}$ and all $\sigma \in \mathcal{F}_K$, we denote by

$$\begin{aligned} \forall \mathbf{y} \in \sigma, \quad \text{if } \mathcal{M}_\sigma &= \{K, L\}, [v]_{K,\sigma}(\mathbf{y}) = \frac{1}{2}(\Pi_{\overline{L}}v(\mathbf{y}) - \Pi_{\overline{K}}v(\mathbf{y})) \\ \text{else if } \mathcal{M}_\sigma &= \{K\}, [v]_{K,\sigma}(\mathbf{y}) = 0 - \Pi_{\overline{K}}v(\mathbf{y}), \end{aligned} \quad (16)$$

- Let $\beta \in]0, 1[$ be given. Let $\psi :]0, 1[\rightarrow \mathbb{R}$ be the function such that $\psi(s) = 0$ on $]0, \beta[$, $\psi|_{[\beta, 1]} \in \mathbb{P}_{k-1}([\beta, 1])$ and

$$\int_{\beta}^1 \psi(s) s^{d-1} ds = 1, \quad (17a)$$

$$\forall i = 1, \dots, k-1, \int_{\beta}^1 (1-s)^i \psi(s) s^{d-1} ds = 0. \quad (17b)$$

In the case $k = 1$, the function $\psi|_{[\beta, 1]} \in \mathbb{P}_0([\beta, 1])$ has the constant value $\frac{d}{1-\beta^d}$.

For the general case $k \in \mathbb{N}^*$, considering the basis $(1, (1-s), \dots, (1-s)^{k-1})$ of $\mathbb{P}_{k-1}([\beta, 1])$, and writing the function ψ as $\psi(s) = \sum_{i=1}^k \alpha_i (1-s)^{i-1}$, we see that the matrix A of the linear system issued from (17), with unknowns $(\alpha_i)_{i=1, \dots, k}$, is such that

$$A_{i,j} = \int_{\beta}^1 (1-s)^{i+j-2} s^{d-1} ds.$$

Considering a vector U such that $U^t = (u_i)_{i=1, \dots, k}$, we note that

$$U^t A U = \int_{\beta}^1 \left(\sum_{i=1}^k u_i (1-s)^{i-1} \right)^2 s^{d-1} ds,$$

which implies that A is symmetric definite positive, hence leading to the existence and uniqueness of ψ such that (17) holds.

We split $D_{K,\sigma}$ into $D_{K,\sigma}^{(\beta)}$ and $D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}$ with

$$\begin{aligned} D_{K,\sigma}^{(\beta)} &:= \{\mathbf{x} \in D_{K,\sigma}, \mathbf{x} = \mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K), s \in]0, \beta], \mathbf{y} \in \sigma\} \\ \text{and } K^{(\beta)} &= \bigcup_{\sigma \in \mathcal{F}_K} D_{K,\sigma}^{(\beta)}, \end{aligned} \quad (18)$$

(note that we have $|D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}| = \frac{1-\beta^d}{d} d_{K,\sigma} |\sigma|$). Throughout this paper, we denote by s, \mathbf{y} the functions $s : D_{K,\sigma} \rightarrow]0, 1]$ and $\mathbf{y} : D_{K,\sigma} \rightarrow \sigma$ such that $\mathbf{x} = \mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)$, which means

$$s(\mathbf{x}) = \frac{(\mathbf{x} - \mathbf{x}_K) \cdot \mathbf{n}_{K,\sigma}}{d_{K,\sigma}},$$

and

$$\mathbf{y}(\mathbf{x}) = \mathbf{x}_K + \frac{\mathbf{x} - \mathbf{x}_K}{s(\mathbf{x})}.$$

Remark 3.3: The above definition for the discrete gradient $\nabla_{\mathcal{D}}v$ can be seen as a regularisation of the gradient of v in the distribution sense, by replacing the Dirac mass on the faces of the mesh by a function with integral equal to 1.

Remark 3.4: It is possible to consider $\beta_{K,\sigma}$ instead of a constant β , without changing the mathematical analysis done in this paper.

Remark 3.5 (Piecewise constant reconstruction): One can for example replace $\Pi_{\mathcal{D}}$ by $\widehat{\Pi}_{\mathcal{D}}$ such that, for all $K \in \mathcal{M}$, and a.e. $\mathbf{x} \in K$, $\widehat{\Pi}_{\mathcal{D}}v(\mathbf{x}) = \frac{1}{|K|} \int_K \overline{\Pi}_K v(\mathbf{x}) d\mathbf{x}$, which provides a piecewise constant reconstruction in the sense of Definition 2.6, choosing a basis $(\chi_i)_{i \in I}$ such that, for each $K \in \mathcal{M}$, there exists $i \in I$ with $v_i = \frac{1}{|K|} \int_K \overline{\Pi}_K v(\mathbf{x}) d\mathbf{x}$ for all $v \in X_{\mathcal{D},0}$.

Remark 3.6 (Definition of the jump at the faces of the mesh): Note that, in Definition 3.2, the jump at the faces is divided by 2 at the interior faces. This allows to keep the same definition for $\nabla_{\mathcal{D}}$ on all $D_{K,\sigma}$ in the two cases, $\sigma \in \mathcal{F}_{\text{int}}$ and $\sigma \in \mathcal{F}_{\text{ext}}$.

4 Mathematical properties of the DGGD method

We can now state and prove that $\|\nabla_{\mathcal{D}} \cdot\|_{L^p(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$.

Lemma 4.1: Let \mathcal{D} be a DGGD in the sense of Definition 3.2. Then $\|\nabla_{\mathcal{D}} \cdot\|_{L^p(\Omega)^d}$ is a norm on $X_{\mathcal{D},0}$.

Proof. It suffices to check that, if $v \in X_{\mathcal{D},0}$ is such that $\|\nabla_{\mathcal{D}}v\|_{L^p(\Omega)^d} = 0$, then $v = 0$. Indeed, thanks to Lemma A.3, we get that $\|v\|_{\text{DG},p} = 0$. We can apply Lemma A.7, since in the case $d = 1$ or $d = 2$, it applies without restriction to $q = p$, and in the case $d \geq 3$, there holds $p < p^* = pd/(d-p)$. Hence we deduce that $\|\Pi_{\mathcal{D}}v\|_{L^p(\Omega)} = 0$, which concludes the proof. ■

We now prove the core properties of the DGGD, as described in Section 2, gathered in the following theorem.

Theorem 4.2 (Properties of DGGDs): Let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be a sequence of DGGDs in the sense of Definition 3.2, defined from underlying polytopal meshes $(\mathfrak{T}_m)_{m \in \mathbb{N}}$. Assume that $(\eta_{\mathfrak{T}_m})_{m \in \mathbb{N}}$ is bounded (see (11)), and that $h_{\mathcal{M}_m} \rightarrow 0$ as $m \rightarrow \infty$.

Then the sequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$ is coercive, GD-consistent, limit-conforming and compact in the sense of Definitions 2.2, 2.3, 2.4 and 2.5.

Proof. The limit-conformity, coercivity, GD-consistency and compactness are obtained by applying Lemmas 4.7, 4.3, 4.5, 4.4. ■

Let us now prove each of the lemmas involved in the proof of the above theorem.

Lemma 4.3 (Coercivity): Let \mathcal{D} be a DGGD in the sense of Definition 3.2. Let $C_{\mathcal{D}} \geq 0$ be defined by (2). Then there exists C_P depending only on $|\Omega|$, β , p , k and d such that $C_P \geq C_{\mathcal{D}}$, which means that any sequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$ is coercive in the sense of Definition 2.2.

Proof. We again apply Lemma A.7, since we noticed in the proof of Lemma 4.1 that it applies to the case $q = p$. ■

Lemma 4.4 (Compactness): Let $(\mathcal{D}_m)_{m \in \mathbb{N}}$ be a sequence of DGGDs in the sense of Definition 3.2. Then, for all $(v_m)_{m \in \mathbb{N}}$ such that, for all $m \in \mathbb{N}$, $v_m \in X_{\mathcal{D}_m,0}$ and such that the sequence $(\|\nabla_{\mathcal{D}_m}v_m\|_{L^p(\Omega)})_{m \in \mathbb{N}}$ is bounded, the sequence $(\Pi_{\mathcal{D}_m}v_m)_{m \in \mathbb{N}}$ is relatively compact in $L^p(\Omega)$, which means that any sequence $(\mathcal{D}_m)_{m \in \mathbb{N}}$ is compact in the sense of Definition 2.5.

Proof. We first extend $\Pi_{\mathcal{D}_m} v_m$ by 0 in $\mathbb{R}^d \setminus \Omega$. We then have, for all $m \in \mathbb{N}$, applying Lemma A.6,

$$\begin{aligned} \forall \xi \in \mathbb{R}^d, \|\Pi_{\mathcal{D}_m} v_m(\cdot + \xi) - \Pi_{\mathcal{D}_m} v_m\|_{L^1(\mathbb{R}^d)} &\leq \left(\sum_{i=1}^d |\xi_i| \right) \|\Pi_{\mathcal{D}_m} v_m\|_{\text{BV}} \\ &\leq \left(\sum_{i=1}^d |\xi_i| \right) C d ((d+1)|\Omega|)^{\frac{p-1}{p}} \|\Pi_{\mathcal{D}_m} v_m\|_{\text{DG},p}. \end{aligned}$$

This implies that the sequence $(\Pi_{\mathcal{D}_m} v_m)_{m \in \mathbb{N}}$ is relatively compact in $L^1(\mathbb{R}^d)$. Thanks to Lemma A.7, the sequence $(\Pi_{\mathcal{D}_m} v_m)_{m \in \mathbb{N}}$ is bounded in $L^q(\mathbb{R}^d)$ for some $q > p$. We thus deduce that the sequence $(\Pi_{\mathcal{D}_m} v_m)_{m \in \mathbb{N}}$ is relatively compact in $L^p(\mathbb{R}^d)$, which concludes the proof of the lemma. \blacksquare

Lemma 4.5 (GD-consistency): Let \mathcal{D} be a DGGD in the sense of Definition 3.2. Take $\varrho \geq \eta_{\bar{\varepsilon}}$ (see (11)). Then there exists $C_1 > 0$, depending only on $|\Omega|, \beta, p, k, d$ and ϱ , such that

$$\forall \varphi \in C^{k+1}(\bar{\Omega}) \cap W_0^{1,p}(\Omega), S_{\mathcal{D}}(\varphi) \leq C_1 h_{\mathcal{M}}^k \|\varphi\|_{k+1,\infty}, \quad (19)$$

where $S_{\mathcal{D}}$ is defined by (3).

As a consequence, any sequence $(\mathcal{D})_{m \in \mathbb{N}}$ of DGGDs such that $h_{\bar{\varepsilon}_m}$ tends to 0 as $m \rightarrow \infty$ while $\theta_{\bar{\varepsilon}_m} + \eta_{\bar{\varepsilon}_m}$ remains bounded is GD-consistent in the sense of Definition 2.3.

Proof.

In this proof, we denote by C_i various values depending only on $|\Omega|, \beta, p, k, d$ and ϱ . Let $\varphi \in C^{k+1}(\bar{\Omega}) \cap W_0^{1,p}(\Omega)$. For any $\mathbf{x} \in \bar{\Omega}, \mathbf{y} \in \mathbb{R}^d$ and $m = 1 \dots, k$, we denote by

$$D^m \varphi(\mathbf{x}) \cdot \mathbf{y}^m = \sum_{1 \leq i_1, \dots, i_m \leq d} y_{i_1} \dots y_{i_m} \partial_{i_1 \dots i_m}^m \varphi(\mathbf{x}).$$

We consider $v \in X_{\mathcal{D},0}$ defined by

$$\forall K \in \mathcal{M}, \forall \mathbf{x} \in K, \Pi_{\bar{K}} v(\mathbf{x}) = \varphi(\mathbf{x}_K) + \sum_{m=1}^k \frac{1}{m!} D^m \varphi(\mathbf{x}_K) \cdot (\mathbf{x} - \mathbf{x}_K)^m.$$

Thanks to results on Taylor expansions, for any $\mathbf{x} \in \bar{K}$, there holds

$$|\varphi(\mathbf{x}) - \Pi_{\bar{K}} v(\mathbf{x})| \leq h_K^{k+1} \|\varphi\|_{k+1,\infty}.$$

which provides $\|\varphi - \Pi_{\mathcal{D}} v\|_{L^p(\Omega)} \leq h_K^{k+1} \|\varphi\|_{k+1,\infty} (|\Omega|)^{1/p}$. Let $\sigma \in \mathcal{F}$ with $\mathcal{M}_\sigma = \{K, L\}$, and let $\mathbf{y} \in \sigma$. Since $\mathbf{y} \in \bar{K}$, we get

$$|\varphi(\mathbf{y}) - \Pi_{\bar{K}} v(\mathbf{y})| \leq h_K^{k+1} \|\varphi\|_{k+1,\infty}.$$

This leads to the existence of $C_2 \geq 1$ depending only on ϱ such that

$$|[v]_{K,\sigma}(\mathbf{y})| \leq (h_K^{k+1} + h_L^{k+1}) \|\varphi\|_{k+1,\infty} \leq C_2 h_K^{k+1} \|\varphi\|_{k+1,\infty}.$$

In the case $\mathcal{M}_\sigma = \{K\}$, we get

$$|[v]_{K,\sigma}(\mathbf{y})| \leq h_K^{k+1} \|\varphi\|_{k+1,\infty} \leq C_2 h_K^{k+1} \|\varphi\|_{k+1,\infty}.$$

Moreover, we have

$$|\nabla \varphi(\mathbf{x}) - \nabla_{\bar{K}} v(\mathbf{x})| \leq h_K^k \|\varphi\|_{k+1,\infty}.$$

This provides, considering $\mathbf{x} \in D_{K,\sigma}$, the existence of C_3 depending only on ϱ, β, d and k such that

$$|\nabla \varphi(\mathbf{x}) - \nabla_{\mathcal{D}} v(\mathbf{x})| \leq h_K^k \|\varphi\|_{k+1,\infty} + \frac{C_2 h_K^{k+1} \|\varphi\|_{k+1,\infty}}{d_{K,\sigma}} |\psi(s(\mathbf{x}))| \leq C_3 h_K^k \|\varphi\|_{k+1,\infty}.$$

Gathering the above inequalities shows that

$$\|\nabla\varphi - \nabla_{\mathcal{D}}v\|_{L^p(\Omega)^d} \leq C_3 \|\varphi\|_{k+1,\infty} h_{\mathcal{M}}^k (|\Omega|)^{1/p},$$

which proves (19).

We then show that any sequence $(\mathcal{D})_{m \in \mathbb{N}}$ of DGGDs such that $h_{\overline{\mathcal{S}}_m}$ tends to 0 as $m \rightarrow \infty$ while $\theta_{\overline{\mathcal{S}}_m}, \eta_{\overline{\mathcal{S}}_m}$ remain bounded is GD-consistent in the sense of Definition 2.3, by applying [6, Lemma 2.4] since $C^{k+1}(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ is dense in $W_0^{1,p}(\Omega)$. \blacksquare

We now provide an improved estimate on $S_{\mathcal{D}}$.

Lemma 4.6 (Estimate on $S_{\mathcal{D}}$ in the $W^{2,p}$ case): Let \mathcal{D} be a DGGD in the sense of Definition 3.2. Assume $p > d/2$ and take $\varrho \geq \eta_{\overline{\mathcal{S}}}$ (see (11)). Then there exists $C_4 > 0$, depending only on $|\Omega|, \beta, p, k, d$ and ϱ , such that

$$\forall \varphi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega), \quad S_{\mathcal{D}}(\varphi) \leq C_4 h_{\mathcal{M}} \|\varphi\|_{W^{2,p}(\Omega)},$$

where $S_{\mathcal{D}}$ is defined by (3).

Proof. In this proof, we again denote by C_i various increasing functions with respect to $\theta_{\overline{\mathcal{S}}}$ and $\eta_{\overline{\mathcal{S}}}$, and also depending on $|\Omega|, \beta, p, k$ and d . For a given $\varphi \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$ and for any $K \in \mathcal{M}$, we denote by $\Pi_{\overline{K}}v \in \mathbb{P}_k(\mathbb{R}^d)$ (since $k \geq 1$) the function $A_{\varphi} : K \rightarrow \mathbb{R}$ provided by Lemma A.10 letting $V = K$, such that (56) and (57) hold. From (56), raising to the power p and summing on $K \in \mathcal{M}$, we get the existence of C_5 such that

$$\|\varphi - \Pi_{\mathcal{D}}v\|_{L^p(\Omega)} \leq C_5 h_{\mathcal{M}}^2 \| |D^2\varphi| \|_{L^p(\Omega)}.$$

Owing to Hölder's inequality, we have, for $K \in \mathcal{M}$ and $\sigma \in \mathcal{F}_K$, thanks to $\mathbf{x} = \mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)$,

$$\begin{aligned} & \int_{D_{K,\sigma}} |\nabla\varphi(\mathbf{x}) - \nabla_{\mathcal{D}}v(\mathbf{x})|^p d\mathbf{x} \leq \\ & 2^{p-1} \left(\int_{D_{K,\sigma}} |\nabla\varphi(\mathbf{x}) - \nabla_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x} + \int_{D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}} \left| \frac{[v]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \psi(s(\mathbf{x})) \right|^p d\mathbf{x} \right), \end{aligned}$$

which gives, by summing on $\sigma \in \mathcal{F}_K$,

$$\begin{aligned} & \int_K |\nabla\varphi(\mathbf{x}) - \nabla_{\mathcal{D}}v(\mathbf{x})|^p d\mathbf{x} \leq 2^{p-1} \int_K |\nabla\varphi(\mathbf{x}) - \nabla_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x} \\ & + \sum_{\sigma \in \mathcal{F}_K} \left(\frac{2}{d_{K,\sigma}} \right)^{p-1} \int_{\sigma} |[v]_{K,\sigma}(\mathbf{y})|^p d\gamma(\mathbf{y}) \int_{\beta}^1 |\psi(s)|^p s^{d-1} ds. \end{aligned} \quad (20)$$

Let $\sigma \in \mathcal{F}$ with $\mathcal{M}_{\sigma} = \{K, L\}$, and let $\mathbf{y} \in \sigma$. Since $\mathbf{y} \in \overline{K}$, we get from (56)

$$|\varphi(\mathbf{y}) - \Pi_{\overline{K}}v(\mathbf{y})| \leq C_{15} h_K^{2-\frac{d}{p}} \| |D^2\varphi| \|_{L^p(K)},$$

and therefore

$$|[v]_{K,\sigma}(\mathbf{y})| \leq C_{15} (h_K^{2-\frac{d}{p}} \| |D^2\varphi| \|_{L^p(K)} + h_L^{2-\frac{d}{p}} \| |D^2\varphi| \|_{L^p(L)}).$$

In the case $\mathcal{M}_{\sigma} = \{K\}$, we get

$$|[v]_{K,\sigma}(\mathbf{y})| \leq C_{15} h_K^{2-\frac{d}{p}} \| |D^2\varphi| \|_{L^p(K)}.$$

This leads to the existence of C_6 such that

$$\begin{aligned} & \sum_{\sigma \in \mathcal{F}_K} \left(\frac{2}{d_{K,\sigma}} \right)^{p-1} \int_{\sigma} |[v]_{K,\sigma}(\mathbf{y})|^p d\gamma(\mathbf{y}) \int_{\beta}^1 |\psi(s)|^p s^{d-1} ds \leq \\ & C_6 h_{\mathcal{M}}^p \left(\| |D^2\varphi| \|_{L^p(K)}^p + \sum_{\sigma \in \mathcal{F}_K, \mathcal{M}_{\sigma} = \{K,L\}} \| |D^2\varphi| \|_{L^p(L)}^p \right) \end{aligned}$$

Remarking that $\theta_{\mathfrak{I}}$ includes a bound on the number of neighbors of any $K \in \mathcal{M}$, we obtain that

$$\begin{aligned} \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} \left(\frac{2}{d_{K,\sigma}} \right)^{p-1} \int_{\sigma} |[v]_{K,\sigma}(\mathbf{y})|^p d\gamma(\mathbf{y}) \int_{\beta} |\psi(s)|^p s^{d-1} ds \leq \\ C_6 h_{\mathcal{M}}^p (\theta_{\mathfrak{I}} + 1) \| |D^2 \varphi| \|_{L^p(\Omega)}^p. \end{aligned}$$

Reporting the above inequality in (20) and using (57), we get the existence of C_7 such that

$$\int_{\Omega} |\nabla \varphi(\mathbf{x}) - \nabla_{\mathcal{D}} v(\mathbf{x})|^p d\mathbf{x} \leq C_7 h_{\mathcal{M}}^p \| |D^2 \varphi| \|_{L^p(\Omega)}^p \quad (21)$$

which concludes the proof of the lemma. \blacksquare

Lemma 4.7 (Limit conformity): Let \mathcal{D} be a DGGD in the sense of Definition 3.2. Then, denoting by $p' = \frac{p}{p-1}$, there exists C_8 , depending only on $|\Omega|$, β , p , k and d , such that

$$\forall \varphi \in W^{1,p'}(\Omega)^d, |W_{\mathcal{D}}(\varphi)| \leq C_8 \| \varphi \|_{W^{1,p'}(\Omega)^d} h_{\mathcal{M}}, \quad (22)$$

where $W_{\mathcal{D}}$ is defined by (6).

As a consequence, any sequence $(\mathcal{D})_{m \in \mathbb{N}}$ of DGGDs in the sense of Definition 3.2 such that $h_{\mathfrak{I}_m}$ tends to 0 as $m \rightarrow \infty$ is limit-conforming in the sense of Definition 2.4.

Proof. Let $\varphi \in C^\infty(\mathbb{R}^d)^d$ be given. We compute, for $v \in X_{\mathcal{D},0}$,

$$\begin{aligned} \int_{\Omega} (\nabla_{\mathcal{D}} v(\mathbf{x}) \cdot \varphi(\mathbf{x}) + \Pi_{\mathcal{D}} v(\mathbf{x}) \operatorname{div} \varphi(\mathbf{x})) d\mathbf{x} &= \sum_{K \in \mathcal{M}} \int_K (\nabla_{\overline{K}} v(\mathbf{x}) \cdot \varphi(\mathbf{x}) + \Pi_{\overline{K}} v(\mathbf{x}) \operatorname{div} \varphi(\mathbf{x})) d\mathbf{x} \\ &\quad + \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} \int_{D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}} \frac{[v]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \psi(s(\mathbf{x})) \mathbf{n}_{K,\sigma} \cdot \varphi(\mathbf{x}) d\mathbf{x} \\ &= \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} \left(\int_{\sigma} \Pi_{\overline{K}} v(\mathbf{y}) \mathbf{n}_{K,\sigma} \cdot \varphi(\mathbf{y}) d\gamma(\mathbf{y}) + \int_{D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}} \frac{[v]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \psi(s(\mathbf{x})) \mathbf{n}_{K,\sigma} \cdot \varphi(\mathbf{x}) d\mathbf{x} \right). \end{aligned}$$

Using Definition (16) of $[v]_{K,\sigma}(\mathbf{y})$ which includes a factor $\frac{1}{2}$ on interior faces and $\mathbf{n}_{K,\sigma} + \mathbf{n}_{L,\sigma} = 0$ if $\mathcal{M}_\sigma = \{K, L\}$, we get $\Pi_{\overline{K}} v(\mathbf{y}) \mathbf{n}_{K,\sigma} + \Pi_{\overline{L}} v(\mathbf{y}) \mathbf{n}_{L,\sigma} = -[v]_{K,\sigma}(\mathbf{y}) \mathbf{n}_{K,\sigma} - [v]_{L,\sigma}(\mathbf{y}) \mathbf{n}_{L,\sigma}$, as well as $\Pi_{\overline{K}} v(\mathbf{y}) \mathbf{n}_{K,\sigma} = -[v]_{K,\sigma}(\mathbf{y}) \mathbf{n}_{K,\sigma}$ in the case of exterior faces with $\mathcal{M}_\sigma = \{K\}$. Therefore we can write

$$\begin{aligned} \int_{\Omega} (\nabla_{\mathcal{D}} v(\mathbf{x}) \cdot \varphi(\mathbf{x}) + \Pi_{\mathcal{D}} v(\mathbf{x}) \operatorname{div} \varphi(\mathbf{x})) d\mathbf{x} \\ = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} \left(- \int_{\sigma} [v]_{K,\sigma}(\mathbf{y}) \mathbf{n}_{K,\sigma} \cdot \varphi(\mathbf{y}) d\gamma(\mathbf{y}) + \int_{D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}} \frac{[v]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \psi(s(\mathbf{x})) \mathbf{n}_{K,\sigma} \cdot \varphi(\mathbf{x}) d\mathbf{x} \right). \end{aligned}$$

Owing to the change of variable $\mathbf{x} = \mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)$, we obtain

$$\begin{aligned} \int_{\Omega} (\nabla_{\mathcal{D}} v(\mathbf{x}) \cdot \varphi(\mathbf{x}) + \Pi_{\mathcal{D}} v(\mathbf{x}) \operatorname{div} \varphi(\mathbf{x})) d\mathbf{x} \\ = \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} \int_{\sigma} \frac{[v]_{K,\sigma}(\mathbf{y})}{d_{K,\sigma}} \int_{\beta} (\varphi(\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)) - \varphi(\mathbf{y})) \cdot \mathbf{n}_{K,\sigma} d_{K,\sigma} \psi(s) s^{d-1} ds d\gamma(\mathbf{y}). \end{aligned} \quad (23)$$

This provides, thanks to Young's inequality, the existence of C_9 depending only on β , d and k , such that

$$\left| \int_{\Omega} (\nabla_{\mathcal{D}} v(\mathbf{x}) \cdot \varphi(\mathbf{x}) + \Pi_{\mathcal{D}} v(\mathbf{x}) \operatorname{div} \varphi(\mathbf{x})) d\mathbf{x} \right| \leq C_9 \left(\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} A_{K,\sigma} \right)^{1/p} \left(\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} B_{K,\sigma} \right)^{1/p'},$$

with

$$\begin{aligned}
A_{K,\sigma} &= \frac{1}{d_{K,\sigma}^{p-1}} \int_{\sigma} ([v]_{K,\sigma}(\mathbf{y}))^p d\gamma(\mathbf{y}), \\
B_{K,\sigma} &= \int_{\sigma} \int_{\beta}^{1} |(\varphi(\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)) - \varphi(\mathbf{y}))|^{p'} d_{K,\sigma} s^{d-1} ds d\gamma(\mathbf{y}).
\end{aligned} \tag{24}$$

Let us denote by φ_i , for $i = 1, \dots, d$ the components of φ . We have $B_{K,\sigma} \leq d^{1/(p-1)} \sum_{i=1}^d B_{K,\sigma,i}$ with

$$B_{K,\sigma,i} = \int_{\sigma} \int_{\beta}^{1} |\varphi_i(\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)) - \varphi_i(\mathbf{y})|^{p'} d_{K,\sigma} s^{d-1} ds d\gamma(\mathbf{y}).$$

We have

$$B_{K,\sigma,i} = \int_{\sigma} \int_{\beta}^{1} \left| \int_s^1 \nabla \varphi_i(\mathbf{x}_K + t(\mathbf{y} - \mathbf{x}_K)) \cdot (\mathbf{y} - \mathbf{x}_K) dt \right|^{p'} d_{K,\sigma} s^{d-1} ds d\gamma(\mathbf{y}),$$

which leads to

$$B_{K,\sigma,i} \leq \int_{\sigma} \int_{\beta}^{1} \left| \int_s^1 \nabla \varphi_i(\mathbf{x}_K + t(\mathbf{y} - \mathbf{x}_K)) \cdot (\mathbf{y} - \mathbf{x}_K) dt \right|^{p'} d_{K,\sigma} s^{d-1} ds d\gamma(\mathbf{y}).$$

Since we have, for any $s \in [\beta, 1]$,

$$\left| \int_s^1 \nabla \varphi_i(\mathbf{x}_K + t(\mathbf{y} - \mathbf{x}_K)) \cdot (\mathbf{y} - \mathbf{x}_K) dt \right|^{p'} \leq h_K^{p'} \int_{\beta}^1 |\nabla \varphi_i(\mathbf{x}_K + t(\mathbf{y} - \mathbf{x}_K))|^{p'} dt,$$

we get

$$B_{K,\sigma,i} \leq h_K^{p'} \int_{\sigma} \int_{\beta}^{1} |\nabla \varphi_i(\mathbf{x}_K + t(\mathbf{y} - \mathbf{x}_K))|^{p'} dt d_{K,\sigma} d\gamma(\mathbf{y}),$$

which provides

$$B_{K,\sigma,i} \leq \frac{h_K^{p'}}{\beta^{d-1}} \int_{\sigma} \int_{\beta}^{1} |\nabla \varphi_i(\mathbf{x}_K + t(\mathbf{y} - \mathbf{x}_K))|^{p'} t^{d-1} dt d_{K,\sigma} d\gamma(\mathbf{y}),$$

and the change of variable $\mathbf{x} = \mathbf{x}_K + t(\mathbf{y} - \mathbf{x}_K)$ gives

$$B_{K,\sigma,i} \leq \frac{h_K^{p'}}{\beta^{d-1}} \int_{D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}} |\nabla \varphi_i(\mathbf{x})|^{p'} d\mathbf{x}.$$

Since Definition (42) of the discontinuous Galerkin norm implies

$$\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} A_{K,\sigma} \leq \|v\|_{\text{DG},p}^p, \tag{25}$$

this concludes the proof of (22) for $\varphi \in C^\infty(\mathbb{R}^d)^d$. We then conclude (22) by density of $C^\infty(\mathbb{R}^d)^d$ in $W^{1,p'}(\Omega)^d$ since Ω is locally star-shaped.

Then, considering a sequence $(\mathcal{D})_{m \in \mathbb{N}}$ of DGGDs in the sense of Definition 3.2 such that $h_{\bar{x}_m}$ tends to 0 as $m \rightarrow \infty$, we get that it is limit-conforming in the sense of Definition 2.4 by density and applying [6, Lemma 2.4] (this is possible thanks to Lemma 4.3 which states that $(\mathcal{D})_{m \in \mathbb{N}}$ is coercive). ■

Lemma 4.8 (Higher order estimate for $W_{\mathcal{D}}(\varphi)$): Let \mathcal{D} be a DGGD in the sense of Definition 3.2. Then there exists C_{10} , depending only on $|\Omega|$, β , p , k and d , such that

$$\forall \varphi \in C^k(\bar{\Omega})^d, |W_{\mathcal{D}}(\varphi)| \leq C_{10} h_{\mathcal{D}}^k \|\varphi\|_{k,\infty}, \tag{26}$$

where $W_{\mathcal{D}}$ is defined by (6).

Proof. Let $\varphi \in C^k(\overline{\Omega})^d$. For any $K \in \mathcal{M}$, $\sigma \in \mathcal{F}_K$, $\mathbf{x} \in D_{K,\sigma}$ with $\mathbf{x} = \mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)$ with $s \in]0, 1]$ and $\mathbf{y} \in \sigma$, let us denote by $\varphi(s) = (\varphi(\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)) - \varphi(\mathbf{y})) \cdot \mathbf{n}_{K,\sigma}$. Thanks to a Taylor expansion, we get that there exists $\theta_s \in [s, 1]$ such that

$$\varphi(s) = \varphi(1) + \sum_{m=1}^{k-1} \frac{1}{m!} \varphi^{(m)}(1)(s-1)^m + \frac{1}{k!} \varphi^{(k)}(\theta_s)(s-1)^k.$$

We then remark that $\varphi(1) = 0$. Owing to (17), we have $\int_{\beta}^1 (s-1)^m \psi(s) s^{d-1} ds = 0$ for $m = 1, \dots, k-1$. Using

$$|\varphi^{(k)}(\theta_s)(s-1)^k| \leq h_K^k \|\varphi\|_{k,\infty},$$

we get from (23) that

$$\begin{aligned} & \left| \int_{\Omega} (\nabla_{\mathcal{D}} v(\mathbf{x}) \cdot \varphi(\mathbf{x}) + \Pi_{\mathcal{D}} v(\mathbf{x}) \operatorname{div} \varphi(\mathbf{x})) d\mathbf{x} \right| \\ & \leq \sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} \int_{\sigma} \frac{|[v]_{K,\sigma}(\mathbf{y})|}{d_{K,\sigma}} \\ & \quad h_K^k \|\varphi\|_{k,\infty} d_{K,\sigma} \int_{\beta}^1 |\psi(s)| s^{d-1} ds d\gamma(\mathbf{y}). \end{aligned}$$

As in the proof of Lemma 4.7, we get, thanks to Young's inequality, the existence of C_{11} depending only on β , d and k , such that

$$\begin{aligned} & \left| \int_{\Omega} (\nabla_{\mathcal{D}} v(\mathbf{x}) \cdot \varphi(\mathbf{x}) + \Pi_{\mathcal{D}} v(\mathbf{x}) \operatorname{div} \varphi(\mathbf{x})) d\mathbf{x} \right| \\ & \leq h_{\mathcal{M}}^k \|\varphi\|_{k,\infty} C_{11} \left(\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} A_{K,\sigma} \right)^{1/p} \left(\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} d_{K,\sigma} |\sigma| \right)^{1/p'}, \end{aligned}$$

with $A_{K,\sigma}$ defined by (24) as in the proof of Lemma 4.7. The conclusion follows from $\sum_{K \in \mathcal{M}} \sum_{\sigma \in \mathcal{F}_K} d_{K,\sigma} |\sigma| = d|\Omega|$ and from (25), which holds as well. \blacksquare

Note that the application of Lemmas 4.6 and 4.7 to the error estimate [6, Lemma 2.2, eqns. (6) and (7)] provides an error in $h_{\mathcal{M}}$ in the case of the linear elliptic problem (37) under Hypotheses (38) in one, two or three space dimensions, when the exact solution belongs to $H^2(\Omega)$.

5 Convergence and error estimate, p -Laplace problem

This section shows, on the example of the p -Laplace equation, the interest of plugging the DG method in the GDM framework. We consider the following problem.

$$-\operatorname{div}(|\nabla \bar{u}|^{p-2} \nabla \bar{u}) = f + \operatorname{div}(\mathbf{F}) \text{ in } \Omega, \quad (27a)$$

with boundary conditions

$$\bar{u} = 0 \text{ on } \partial\Omega, \quad (27b)$$

under the following assumption, in addition to $p \in]1, +\infty[$ and (1):

$$f \in L^{p'}(\Omega), \quad \mathbf{F} \in L^{p'}(\Omega)^d. \quad (28)$$

Under these hypotheses, the weak solution of (27) is the unique function \bar{u} satisfying:

$$\begin{aligned} & \bar{u} \in W_0^{1,p}(\Omega) \text{ and, for all } v \in W_0^{1,p}(\Omega), \\ & \int_{\Omega} |\nabla \bar{u}|^{p-2} \nabla \bar{u}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) v(\mathbf{x}) d\mathbf{x} - \int_{\Omega} \mathbf{F}(\mathbf{x}) \cdot \nabla v(\mathbf{x}) d\mathbf{x}. \end{aligned} \quad (29)$$

Definition 5.1 (Gradient scheme for the p -Laplace problem): Let $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ be a Gradient Discretisation in the sense of Definition 2.1. The corresponding gradient scheme for Problem (29) is defined by

$$\text{Find } u \in X_{\mathcal{D},0} \text{ such that, for any } v \in X_{\mathcal{D},0}, \quad (30)$$

$$\int_{\Omega} |\nabla_{\mathcal{D}} u(\mathbf{x})|^{p-2} \nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) \Pi_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} - \int_{\Omega} \mathbf{F}(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x}.$$

The following lemma, which is a consequence of the underlying minimisation problems and is proved in [4], establishes the existence and uniqueness of the solutions to (29) and (30), as well as estimates on these solutions.

Lemma 5.2: Under Hypotheses (28), there exists one and only one solution to each of the problems (29) and (30). These solutions moreover satisfy

$$\|\nabla \bar{u}\|_{L^p(\Omega)^d} \leq (C_{P,p} \|f\|_{L^{p'}(\Omega)} + \|\mathbf{F}\|_{L^{p'}(\Omega)^d})^{\frac{1}{p-1}} \quad (31)$$

and

$$\|\nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)^d} \leq (C_{\mathcal{D}} \|f\|_{L^{p'}(\Omega)} + \|\mathbf{F}\|_{L^{p'}(\Omega)^d})^{\frac{1}{p-1}}, \quad (32)$$

where $C_{P,p}$ is the continuous Poincaré's constant in $W_0^{1,p}(\Omega)$, and $C_{\mathcal{D}}$ is defined by (2).

Theorem 5.3 (Control of the approximation error): Under Hypotheses (28), let $\bar{u} \in W_0^{1,p}(\Omega)$ be the solution of Problem (29), let \mathcal{D} be a GD in the sense of Definition 2.1, and let $u_{\mathcal{D}} \in X_{\mathcal{D},0}$ be the solution to the gradient scheme (30). Then there exists $C_{12} > 0$, depending only on p such that:

1. If $p \in (1, 2]$,

$$\begin{aligned} \|\nabla \bar{u} - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)^d} &\leq S_{\mathcal{D}}(\bar{u}) + C_{12} [W_{\mathcal{D}}(|\nabla \bar{u}|^{p-2} \nabla \bar{u} + \mathbf{F}) + S_{\mathcal{D}}(\bar{u})^{p-1}] \\ &\quad \times \left[S_{\mathcal{D}}(\bar{u})^p + [(C_{\mathcal{D}} + C_{P,p}) \|f\|_{L^{p'}(\Omega)} + \|\mathbf{F}\|_{L^{p'}(\Omega)^d}]^{\frac{p}{p-1}} \right]^{\frac{2-p}{2}}. \end{aligned} \quad (33)$$

2. If $p \in (2, +\infty)$,

$$\begin{aligned} \|\nabla \bar{u} - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)^d} &\leq S_{\mathcal{D}}(\bar{u}) + C_{12} \left[W_{\mathcal{D}}(|\nabla \bar{u}|^{p-2} \nabla \bar{u} + \mathbf{F}) \right. \\ &\quad \left. + S_{\mathcal{D}}(\bar{u}) [(C_{P,p} \|f\|_{L^{p'}(\Omega)} + \|\mathbf{F}\|_{L^{p'}(\Omega)^d})^{\frac{1}{p-1}} + S_{\mathcal{D}}(\bar{u})]^{p-2} \right]^{\frac{1}{p-1}}. \end{aligned} \quad (34)$$

As a consequence of (33)–(34), we have the following error estimate:

$$\|\bar{u} - \Pi_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)} \leq S_{\mathcal{D}}(\bar{u}) + C_{\mathcal{D}} (S_{\mathcal{D}}(\bar{u}) + \|\nabla \bar{u} - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)^d}). \quad (35)$$

The application of the previous theorem and of Lemmas proved in Section 4 allows to state and prove the following corollaries.

Corollary 5.4 (Convergence of the discontinuous Galerkin method for the p -Laplace problem):

Under Hypotheses (28), let $(\mathcal{D})_{m \in \mathbb{N}}$ be a sequence of DGGDs such that $h_{\mathcal{D}_m}$ tends to 0 as $m \rightarrow \infty$ while $\eta_{\mathcal{D}_m}$ remains bounded. Let \bar{u} and $u_{\mathcal{D}_m}$ be the respective solution to Problems (29) and (30) for all $m \in \mathbb{N}$. Then $\Pi_{\mathcal{D}_m} u_{\mathcal{D}_m}$ (resp. $\nabla_{\mathcal{D}_m} u_{\mathcal{D}_m}$) converge to \bar{u} in $L^p(\Omega)$ (resp. to $\nabla \bar{u}$ in $L^p(\Omega)^d$) as $m \rightarrow \infty$.

Proof. Owing to Lemma 4.3, $C_{\mathcal{D}_m}$ remains bounded, whereas Lemmas 4.5 and 4.7 prove that $S_{\mathcal{D}_m}(\bar{u})$ and $W_{\mathcal{D}_m}(|\nabla \bar{u}|^{p-2} \nabla \bar{u} + \mathbf{F})$ tend to 0 as $m \rightarrow \infty$. Hence (33) or (34) on one hand, and (35) on the other hand, show the conclusion of the corollary. \blacksquare

Corollary 5.5 (Error estimate of the discontinuous Galerkin method for the p -Laplace problem):

Under Hypotheses (28), let us assume that $p \in]1, 2]$, $\Omega =]0, 1[$, $f \in L^\infty(\Omega)$ and $\mathbf{F}(x) = 0$ for all $x \in \Omega$. Let \mathcal{D} be a DGGD in the sense of Definition 3.2 where the polytopal mesh of Ω is uniform. Let \bar{u} and $u_{\mathcal{D}}$ be the respective solution to Problems (29) and (30). Then there holds

$$\|\bar{u} - \Pi_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)} + \|\nabla \bar{u} - \nabla_{\mathcal{D}} u_{\mathcal{D}}\|_{L^p(\Omega)^d} \leq C h_{\mathcal{M}}^{p-1}, \quad (36)$$

where C only depends on f , β , p and k .

Proof. In this particular case, the continuous solution \bar{u} satisfies that there exists $C \in \mathbb{R}$ with

$$\forall x \in [0, 1], \quad |\bar{u}'(x)|^{p-2} \bar{u}'(x) = C - \int_0^x f(s) ds,$$

and therefore

$$\forall x \in [0, 1], \quad \bar{u}'(x) = \left| C - \int_0^x f(s) ds \right|^{\frac{1}{p-1}} \text{sign}\left(C - \int_0^x f(s) ds\right).$$

Then, since $\frac{1}{p-1} > 1$, we get that $\bar{u} \in W_0^{1,p}(\Omega) \cap W^{2,\infty}(\Omega)$ and $|\bar{u}'|^{p-2} \bar{u}' \in W^{1,\infty}(\mathbb{R})$ (extending f by 0 outside Ω). Therefore we apply Lemmas 4.3, 4.6 and 4.7, which, in addition to (33), complete the proof of (36). \blacksquare

Note that this error estimate provides, in the case $p = 2$, the order 1 for the error estimate of the derivative of the solution, which is confirmed by the numerical results below. But it does not provide the order 2 observed on the error estimate of the solution.

6 The case of a linear elliptic problem

6.1 Link with the SIPG scheme

Let us prove that the gradient scheme issued from the DGGD defined in Section 3 is identical to the Symmetric Interior Penalty Galerkin (SIPG) method [5] for the following elliptic problem:

$$\bar{u} \in H_0^1(\Omega), \quad \forall \bar{v} \in H_0^1(\Omega), \quad \int_{\Omega} \Lambda(\mathbf{x}) \nabla \bar{u}(\mathbf{x}) \cdot \nabla \bar{v}(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) \bar{v}(\mathbf{x}) d\mathbf{x}, \quad (37)$$

where the assumptions are:

- Λ is a measurable function from Ω to the set of $d \times d$ symmetric matrices and there exists $\underline{\lambda}, \bar{\lambda} > 0$ such that, for a.e. $\mathbf{x} \in \Omega$, $\Lambda(\mathbf{x})$ has eigenvalues in $[\underline{\lambda}, \bar{\lambda}]$, (38a)
- $f \in L^2(\Omega)$. (38b)

Using the DGGD $\mathcal{D} = (X_{\mathcal{D},0}, \Pi_{\mathcal{D}}, \nabla_{\mathcal{D}})$ in the sense of Definition 3.2, the gradient scheme for the discretisation of (37) is given by:

$$u \in X_{\mathcal{D},0}, \quad \forall v \in X_{\mathcal{D},0}, \quad \int_{\Omega} \Lambda(\mathbf{x}) \nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} = \int_{\Omega} f(\mathbf{x}) \Pi_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x}. \quad (39)$$

Let us assume that Λ restricted to $K \in \mathcal{M}$ is constant in K . Then the left hand side of (39) can be computed in this particular case, changing the variable $\mathbf{x} \in D_{K,\sigma}$ in $(\mathbf{y}, s) \in \sigma \times]0, 1[$ using $\mathbf{x} = \mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)$ (we then have $d\mathbf{x} = d_{K,\sigma} s^{d-1} d\gamma(\mathbf{y}) ds$). Indeed, we can first write

$$\begin{aligned} & \int_{D_{K,\sigma}} \Lambda(\mathbf{x}) \psi(s(\mathbf{x})) \frac{[u]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \mathbf{n}_{K,\sigma} \cdot \nabla_{\bar{K}} v(\mathbf{x}) d\mathbf{x} \\ &= \int_{\sigma} \Lambda_K \frac{[u]_{K,\sigma}(\mathbf{y})}{d_{K,\sigma}} \mathbf{n}_{K,\sigma} \cdot \int_{\beta}^1 \nabla_{\bar{K}} v(\mathbf{x}(\mathbf{y}, s)) \psi(s) s^{d-1} ds d\gamma(\mathbf{y}) d_{K,\sigma}. \end{aligned}$$

Let us now remark that the function $\varphi(s) = \nabla_{\bar{K}} v(\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)) \cdot \mathbf{n}_{K,\sigma}$ is such that

$$\varphi(s) = \nabla_{\bar{K}} v(\mathbf{y}) \cdot \mathbf{n}_{K,\sigma} + \sum_{m=1}^{k-1} p_m(\mathbf{y}) (1-s)^m,$$

where, for $m = 1, \dots, k-1$, $p_m(\mathbf{y})$ is a polynomial with degree less or equal to $k-1$ with respect to the coordinates of \mathbf{y} . Therefore, (17) implies that $\int_{\beta}^1 \nabla_{\overline{K}} v(\mathbf{x}(\mathbf{y}, s)) \cdot \mathbf{n}_{K,\sigma} \psi(s) s^{d-1} ds = \nabla_{\overline{K}} v(\mathbf{y}) \cdot \mathbf{n}_{K,\sigma}$. Hence we get

$$\begin{aligned} & \int_{D_{K,\sigma}} \Lambda(\mathbf{x}) \psi(s(\mathbf{x})) \frac{[u]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \mathbf{n}_{K,\sigma} \cdot \nabla_{\overline{K}} v(\mathbf{x}) d\mathbf{x} \\ &= \int_{\sigma} \Lambda_K [u]_{K,\sigma}(\mathbf{y}) \mathbf{n}_{K,\sigma} \cdot \nabla_{\overline{K}} v(\mathbf{y}) d\gamma(\mathbf{y}), \end{aligned}$$

We also have

$$\begin{aligned} & \int_{D_{K,\sigma}} \Lambda(\mathbf{x}) \frac{[u]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \mathbf{n}_{K,\sigma} \cdot \frac{[v]_{K,\sigma}(\mathbf{x})}{d_{K,\sigma}} \mathbf{n}_{K,\sigma} \psi(s(\mathbf{x}))^2 d\mathbf{x} \\ &= \frac{\Lambda_K \mathbf{n}_{K,\sigma} \cdot \mathbf{n}_{K,\sigma}}{d_{K,\sigma}} \int_{\beta}^1 \psi(s)^2 s^{d-1} ds \int_{\sigma} [u]_{K,\sigma}(\mathbf{y}) [v]_{K,\sigma}(\mathbf{y}) d\gamma(\mathbf{y}). \end{aligned}$$

Then the bilinear form involved in Gradient Scheme (39) satisfies

$$\begin{aligned} & \int_{\Omega} \Lambda(\mathbf{x}) \nabla_{\mathcal{D}} u(\mathbf{x}) \cdot \nabla_{\mathcal{D}} v(\mathbf{x}) d\mathbf{x} \\ &= \sum_{K \in \mathcal{M}} \left(\int_K \Lambda_K \nabla_{\overline{K}} u(\mathbf{x}) \cdot \nabla_{\overline{K}} v(\mathbf{x}) d\mathbf{x} \right. \\ & \quad + \sum_{\sigma \in \mathcal{F}_K} \left(\int_{\sigma} \Lambda_K ([u]_{K,\sigma}(\mathbf{y}) \nabla_{\overline{K}} v(\mathbf{y}) + [v]_{K,\sigma}(\mathbf{y}) \nabla_{\overline{K}} u(\mathbf{y})) \cdot \mathbf{n}_{K,\sigma} d\gamma(\mathbf{y}) \right. \\ & \quad \left. \left. + \frac{\Lambda_K \mathbf{n}_{K,\sigma} \cdot \mathbf{n}_{K,\sigma}}{d_{K,\sigma}} \int_{\beta}^1 \psi(s)^2 s^{d-1} ds \int_{\sigma} [u]_{K,\sigma}(\mathbf{y}) [v]_{K,\sigma}(\mathbf{y}) d\gamma(\mathbf{y}) \right) \right). \end{aligned}$$

We then recover the SIPG scheme as presented in [5] or [3], the penalty coefficient τ_{σ} (term $\frac{\sigma_{\varepsilon}}{|\varepsilon|^{\beta_0}}$ of [5, eqn. (11)], term $\frac{\eta}{h_F}$ of [3, eqn. (4.12)]) being equal, in the preceding relation, to

$$\tau_{\sigma} = \frac{1}{4} \int_{\beta}^1 \psi(s)^2 s^{d-1} ds \left(\frac{\Lambda_K \mathbf{n}_{K,\sigma} \cdot \mathbf{n}_{K,\sigma}}{d_{K,\sigma}} + \frac{\Lambda_L \mathbf{n}_{L,\sigma} \cdot \mathbf{n}_{L,\sigma}}{d_{L,\sigma}} \right) \text{ if } \mathcal{M}_{\sigma} = \{K, L\},$$

and

$$\tau_{\sigma} = \int_{\beta}^1 \psi(s)^2 s^{d-1} ds \frac{\Lambda_K \mathbf{n}_{K,\sigma} \cdot \mathbf{n}_{K,\sigma}}{d_{K,\sigma}} \text{ if } \mathcal{M}_{\sigma} = \{K\}.$$

Note that, since the Cauchy-Schwarz inequality implies

$$\left(\int_{\beta}^1 \psi(s) s^{d-1} ds \right)^2 \leq \int_{\beta}^1 \psi(s)^2 s^{d-1} ds \int_{\beta}^1 s^{d-1} ds,$$

we get that

$$\int_{\beta}^1 \psi(s)^2 s^{d-1} ds \geq \frac{d}{1 - \beta^d} \geq d,$$

which provides a minimum value for τ_{σ} letting $\beta \rightarrow 0$ (such minimum values are given, for example, by [3, Lemma 4.12]). In our setting, it does not depend on the regularity of the mesh nor on the maximum cardinal of \mathcal{F}_K (in the DGGD scheme, we don't handle separately the case $d = 1$ and the cases $d > 1$).

6.2 Numerical results

The aim of this section is to assess the influence of the parameter $\beta \in]0, 1[$ on the accuracy of the gradient scheme (39) issued from the DGGD for the discretisation of (37). We consider the 1D case $\Omega =]0, 1[$, and the polytopal mesh \mathfrak{T} defined, for $N \in \mathbb{N}^*$ and $h = \frac{1}{N}$, by $\mathcal{M} = \{(i-1)h, ih[, i = 1, \dots, N\}$, $\mathcal{F} = \{ih\}$, $i = 0, \dots, N\}$, $\mathcal{P} = \{(i - \frac{1}{2})h, i = 1, \dots, N\}$. We consider one of the test cases studied in [5], that is Problem (37) with $\Lambda = \text{Id}$ and $\bar{u}(x) = \cos(8\pi x) - 1$ (hence $f(x) = (8\pi)^2 \cos(8\pi x)$). Considering

first degree polynomials, the set $X_{\mathcal{D},0}$ is a vector space with dimension $2N$. In the following tables (where “order” is the convergence order with respect to the size of the mesh), the columns “FE” correspond to the conforming \mathbb{P}^1 Finite Element solution, and we check that the results provided by “[5]” with $\sigma_n = 4.5$, which corresponds to $\beta = 1 - 1/\sigma_n$ for the interior faces, and $\beta = 1 - 2/\sigma_n$ for the exterior faces, are close to ours:

$N \setminus \beta$	0	0.5	0.9	0.99	FE	[5]
10	0.496	0.241	0.347	0.394	0.399	0.247
order	1.438	1.529	1.734	1.843	1.855	
20	0.183	0.083	0.104	0.110	0.110	0.083
order	1.092	1.706	1.909	1.959	1.964	
40	0.086	0.026	0.028	0.028	0.028	0.024
order	1.013	1.894	1.973	1.989	1.991	
80	0.043	0.007	0.007	0.007	0.007	
order	0.999	1.967	1.992	1.997	1.998	
160	0.021	0.002	0.002	0.002	0.002	

 L^2 error of the solution

$N \setminus \beta$	0	0.5	0.9	0.99	FE	[5]
10	13.233	11.533	11.360	11.349	11.348	11.777
order	0.172	0.781	0.862	0.863	0.863	
20	11.743	6.714	6.251	6.240	6.240	6.421
order	0.010	1.004	0.966	0.965	0.965	
40	11.666	3.348	3.199	3.197	3.197	3.253
order	-0.008	1.034	0.992	0.991	0.991	
80	11.728	1.635	1.609	1.608	1.608	
order	-0.007	1.014	0.998	0.998	0.998	
160	11.781	0.810	0.805	0.805	0.805	

 L^2 error of the broken gradient

Although we did not prove that the linear systems are invertible when $\beta = 0$, we note that in practice a solution is obtained but that the broken gradient does not seem to converge. In this very regular case, the L^2 error is the lowest for $\beta = 0.5$ but the convergence seems slightly better for β closer to 1, and it tends to the results of the finite element method as $\beta \rightarrow 1$.

7 Conclusion

The version of the DG method included in the GDM framework has the advantages to be samely defined for $d = 1$ and $d > 1$, to hold on any polytopal mesh provided that the grid block are strictly star-shaped, to involve Discrete Functional Analysis results which do not depend on the regularity of the mesh, and to apply on any problem on which the GDM is shown to converge (like the example of the p -Laplace problem taken in this paper). Since it is identical to the SIPG method, it enlarges the application domain of this method to all problems for which the GDM is proved to be applicable.

A Discrete functional analysis

Since our geometric hypotheses are different from those done in [2, 3], we now provide the Sobolev inequalities suited to Definition 3.1 of polytopal meshes. The techniques of proof are identical to that of [3, Theorem 5.3]. In this section, we denote, for $\xi \in \mathbb{R}^d$ by $|\xi| = (\sum_{i=1}^d \xi_i^2)^{1/2}$ the Euclidean norm of ξ .

The constant $C_{q,n} > 0$, introduced by the following definition, is used in the course of this mathematical study for $(q, n) = (p, k - 1)$, and $(q, n) = (\frac{pd}{d-p}, k)$ when $p \in]1, d[$.

Definition A.1 (Comparison of norms on \mathbb{R}^{n+1}): Let $d \in \mathbb{N}^*$, $n \in \mathbb{N}$ and $q > 0$ be given. We denote by $C_{q,n} > 0$ the greatest constant, depending only on n , q and d , such that

$$\forall (a_0, \dots, a_n) \in \mathbb{R}^{n+1}, \int_0^1 \left| \sum_{i=0}^n a_i s^i \right|^q s^{d-1} ds \geq C_{q,n} \sum_{i=0}^n |a_i|^q. \quad (40)$$

The following lemma plays an essential role in the study of $\|\nabla_{\mathcal{D}} \cdot\|_{L^p(\Omega)^d}$.

Lemma A.2: Let $n \in \mathbb{N}$ and $\beta \in]0, 1[$ be given. Let \mathfrak{T} be a polytopal mesh in the sense of Definition 3.1. Then there holds

$$\forall v \in \mathbb{P}_n(\mathbb{R}^d), \forall K \in \mathcal{M}, \forall \sigma \in \mathcal{F}_K, \\ \int_{D_{K,\sigma}} |v(\mathbf{x})|^p d\mathbf{x} \leq \frac{(n+1)^{p-1}}{\beta^{d+pn} C_{p,n}} \int_{D_{K,\sigma}^{(\beta)}} |v(\mathbf{x})|^p d\mathbf{x},$$

where $C_{p,n}$ is defined in Definition A.1 with $q = p$, and where $D_{K,\sigma}^{(\beta)}$ is defined by (18).

Proof. For $K \in \mathcal{M}$ and $\sigma \in \mathcal{F}_K$, we compute $\int_{D_{K,\sigma}^{(\beta)}} |v(\mathbf{x})|^p d\mathbf{x}$ with making the change of variable $\mathbf{x} = \mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)$, where $\mathbf{y} \in \sigma$ and $s \in]0, \beta[$. We then have $d\mathbf{x} = d_{K,\sigma} s^{d-1} d\gamma(\mathbf{y}) ds$, which leads to

$$\int_{D_{K,\sigma}^{(\beta)}} |v(\mathbf{x})|^p d\mathbf{x} = \int_{\sigma} \int_0^{\beta} |v(\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K))|^p d_{K,\sigma} s^{d-1} ds d\gamma(\mathbf{y}).$$

For a given $\mathbf{y} \in \sigma$, $s \mapsto v(\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K))$ is a polynomial with respect to s with degree lower or equal to n , that we write under the form $v(\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)) = \sum_{i=0}^n a_i(\mathbf{y}) s^i$. We then use the notation introduced in Definition A.1, which provides

$$\begin{aligned} \int_0^{\beta} \left| \sum_{i=0}^n a_i(\mathbf{y}) s^i \right|^p s^{d-1} ds &= \beta^d \int_0^1 \left| \sum_{i=0}^n a_i(\mathbf{y}) \beta^i t^i \right|^p t^{d-1} dt \\ &\geq \beta^d C_{p,n} \sum_{i=0}^n |a_i(\mathbf{y}) \beta^i|^p \geq \beta^{d+pn} C_{p,n} \sum_{i=0}^n |a_i(\mathbf{y})|^p. \end{aligned}$$

This leads to

$$\int_{D_{K,\sigma}^{(\beta)}} |v(\mathbf{x})|^p d\mathbf{x} \geq \beta^{d+pn} C_{p,n} \int_{\sigma} d_{K,\sigma} \left(\sum_{i=0}^n |a_i(\mathbf{y})|^p \right) d\gamma(\mathbf{y}).$$

On the other hand, we have

$$\begin{aligned} \int_{D_{K,\sigma}} |v(\mathbf{x})|^p d\mathbf{x} &= \int_{\sigma} \int_0^1 |v(\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K))|^p d_{K,\sigma} s^{d-1} ds d\gamma(\mathbf{y}) \\ &= \int_{\sigma} \int_0^1 \left| \sum_{i=0}^n a_i(\mathbf{y}) s^i \right|^p d_{K,\sigma} s^{d-1} ds d\gamma(\mathbf{y}) \\ &\leq (n+1)^{p-1} \int_{\sigma} \left(\sum_{i=0}^n |a_i(\mathbf{y})|^p \right) d_{K,\sigma} d\gamma(\mathbf{y}). \end{aligned}$$

Hence, gathering the above relations, the lemma is proved. \blacksquare

We can now state and prove the following lemma, which provides a connection between the discrete gradient defined by (15) to a norm suited for the study of discontinuous Galerkin methods in the framework of elliptic problems.

Lemma A.3: Let \mathcal{D} be a DGGD in the sense of Definition 3.2. Then there exists $A > 0$, depending only on β , p , k and d , such that

$$\forall v \in X_{\mathcal{D},0}, \quad \frac{1}{A} \|v\|_{\text{DG},p} \leq \|\nabla_{\mathcal{D}} v\|_{L^p(\Omega)^d} \leq A \|v\|_{\text{DG},p}, \quad (41)$$

where

$$\|v\|_{\text{DG},p}^p = \sum_{K \in \mathcal{M}} \left(\int_K |\nabla_{\overline{K}} v(\mathbf{x})|^p d\mathbf{x} + \sum_{\sigma \in \mathcal{F}_K} \frac{1}{d_{K,\sigma}^{p-1}} \int_{\sigma} |[v]_{K,\sigma}(\mathbf{y})|^p d\gamma(\mathbf{y}) \right). \quad (42)$$

Remark A.4 (DG norm): Note that Definition (42) for the DG norm is slightly different from [2, eqn. (5)] or [3, eqn. (5.1)], with the use of $d_{K,\sigma}$ instead that of $\text{diam}(\sigma)$, and with notation (16) for the jump at the faces of the mesh. This allows the application of discrete functional analysis results without regularity hypotheses on the polytopal mesh.

Proof. Let $K \in \mathcal{M}$ and $\sigma \in \mathcal{F}_K$. Using $\mathbf{x} = \mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)$, and using, for some $c > 0$ to be chosen later, $|a + b|^p \leq (1 + c^p)^{p-1} (|a|^p + |\frac{b}{c}|^p)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $a = \nabla_{\overline{K}} v(\mathbf{x}) + \frac{[v]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \psi(s(\mathbf{x})) \mathbf{n}_{K,\sigma}$ and

$b = -\nabla_{\overline{K}}v(\mathbf{x})$, we have

$$\begin{aligned} \int_K |\nabla_{\mathcal{D}}v(\mathbf{x})|^p d\mathbf{x} &= \int_{K^{(\beta)}} |\nabla_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x} + \sum_{\sigma \in \mathcal{F}_K} \int_{D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}} \left| \nabla_{\overline{K}}v(\mathbf{x}) + \frac{[v]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \psi(s(\mathbf{x})) \mathbf{n}_{K,\sigma} \right|^p d\mathbf{x} \\ &\geq \int_{K^{(\beta)}} |\nabla_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x} - \frac{1}{c^p} \int_{K \setminus K^{(\beta)}} |\nabla_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x} \\ &\quad + \frac{1}{(1+c^{p'})^{p-1}} \sum_{\sigma \in \mathcal{F}_K} \int_{D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}} \left| \frac{[v]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \psi(s(\mathbf{x})) \mathbf{n}_{K,\sigma} \right|^p d\mathbf{x}. \end{aligned}$$

Writing

$$\frac{1}{d} \sum_{i=1}^d |\partial_i \Pi_{\overline{K}}v|^p \leq |\nabla_{\overline{K}}v|^p \leq d^{p/2} \sum_{i=1}^d |\partial_i \Pi_{\overline{K}}v|^p \quad (43)$$

and applying Lemma A.2 to the polynomial $\partial_i \Pi_{\overline{K}}v \in \mathbb{P}_{k-1}(\mathbb{R}^d)$ for $i = 1, \dots, d$, we can write

$$\int_{K^{(\beta)}} |\nabla_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x} \geq \frac{\beta^{d+p(k-1)} C_{p,k-1}}{dk^{p-1}} \sum_{i=1}^d \int_K |\partial_i \Pi_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x} \geq C'_{\beta,q} \int_K |\nabla_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x},$$

denoting by $C'_{\beta,k} = \frac{\beta^{d+p(k-1)} C_{p,k-1}}{d^{1+p/2} k^{p-1}}$. We then define c by $\frac{1}{c^p} = \frac{1}{2} C'_{\beta,k}$. Using $|\mathbf{n}_{K,\sigma}| = 1$, we get

$$\int_K |\nabla_{\mathcal{D}}v(\mathbf{x})|^p d\mathbf{x} \geq \frac{1}{2} C'_{\beta,k} \int_K |\nabla_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x} + \frac{1}{(1+c^{p'})^{p-1}} \sum_{\sigma \in \mathcal{F}_K} \int_{D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}} \left| \frac{[v]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \psi(s(\mathbf{x})) \right|^p d\mathbf{x}.$$

We again make the change of variable $\mathbf{x} = \mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)$. It yields

$$\begin{aligned} \int_{D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}} \left| \frac{[v]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \psi(s(\mathbf{x})) \right|^p d\mathbf{x} &= d_{K,\sigma} \int_{\sigma} \frac{|[v]_{K,\sigma}(\mathbf{y})|^p}{d_{K,\sigma}^p} d\gamma(\mathbf{y}) \int_{\beta}^1 |\psi(s)|^p s^{d-1} ds \\ &= \frac{1}{d_{K,\sigma}^{p-1}} \int_{\sigma} |[v]_{K,\sigma}(\mathbf{y})|^p d\gamma(\mathbf{y}) \int_{\beta}^1 |\psi(s)|^p s^{d-1} ds. \end{aligned}$$

This completes the proof of the left inequality in (41). On the other hand, we get

$$\begin{aligned} \int_{D_{K,\sigma}} |\nabla_{\mathcal{D}}v(\mathbf{x})|^p d\mathbf{x} &\leq 2^{p-1} \int_{D_{K,\sigma}} |\nabla_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x} + 2^{p-1} \int_{D_{K,\sigma} \setminus D_{K,\sigma}^{(\beta)}} \left| \frac{[v]_{K,\sigma}(\mathbf{y}(\mathbf{x}))}{d_{K,\sigma}} \psi(s(\mathbf{x})) \right|^p d\mathbf{x} \\ &= 2^{p-1} \int_{D_{K,\sigma}} |\nabla_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x} + \left(\frac{2}{d_{K,\sigma}} \right)^{p-1} \int_{\sigma} |[v]_{K,\sigma}(\mathbf{y})|^p d\gamma(\mathbf{y}) \int_{\beta}^1 |\psi(s)|^p s^{d-1} ds, \end{aligned}$$

which completes the proof of the right inequality in (41). \blacksquare

Since our geometric hypotheses are different from those done in [2, 3], we now provide the Sobolev inequalities suited to Definition 3.1 of polytopal meshes. The techniques of proof are identical to that of [3, Theorem 5.3].

Lemma A.5 (Comparison of DG norms): Let \mathcal{D} be a DGGD in the sense of Definition 3.2. Then, for $1 \leq p \leq q < +\infty$, there holds

$$\forall v \in X_{\mathcal{D},0}, \quad \|v\|_{\text{DG},p} \leq ((d+1)|\Omega|)^{\frac{q-p}{pq}} \|v\|_{\text{DG},q}. \quad (44)$$

Proof. We have

$$\|v\|_{\text{DG},p}^p = \sum_{K \in \mathcal{M}} \left(\int_K |\nabla_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x} + \sum_{\sigma \in \mathcal{F}_K} \frac{1}{d_{K,\sigma}^{p-1}} \int_{\sigma} |[v]_{K,\sigma}(\mathbf{y})|^p d\gamma(\mathbf{y}) \right).$$

Thanks to Young's inequality, we have

$$\begin{aligned} \|v\|_{\text{DG},p}^p &\leq \left(\sum_{K \in \mathcal{M}} \left(\int_K |\nabla_{\overline{K}} v(\mathbf{x})|^q d\mathbf{x} + \sum_{\sigma \in \mathcal{F}_K} \frac{1}{d_{K,\sigma}^{q-1}} \int_{\sigma} |[v]_{K,\sigma}(\mathbf{y})|^q d\gamma(\mathbf{y}) \right) \right)^{p/q} \\ &\quad \times \left(\sum_{K \in \mathcal{M}} \left(|K| + \sum_{\sigma \in \mathcal{F}_K} d_{K,\sigma} |\sigma| \right) \right)^{(q-p)/q}, \end{aligned}$$

which leads, since $\sum_{\sigma \in \mathcal{F}_K} d_{K,\sigma} |\sigma| = d|K|$, to the result. \blacksquare

Lemma A.6 (Comparison of DG norm and BV norm): For $\bar{v} \in L^1(\mathbb{R}^d)$, let us define

$$\|\bar{v}\|_{\text{BV}} = \sum_{i=1}^d \sup \left\{ \int_{\mathbb{R}^d} \bar{v}(\mathbf{x}) \partial_i \varphi(\mathbf{x}) d\mathbf{x}, \varphi \in C_c^1(\mathbb{R}^d), \|\varphi\|_{\infty} \leq 1 \right\} \in (\mathbb{R}^+ \cup \{+\infty\}).$$

Then, extending $\Pi_{\mathcal{D}} v$ for all $v \in X_{\mathcal{D},0}$ by 0 outside Ω , we have

$$\forall p \in [1, +\infty[, \forall v \in X_{\mathcal{D},0}, \|\Pi_{\mathcal{D}} v\|_{\text{BV}} \leq d((d+1)|\Omega|)^{\frac{p-1}{p}} \|v\|_{\text{DG},p} \quad (45)$$

Proof. We have, for all $v \in X_{\mathcal{D},0}$, thanks again to (16),

$$\|\Pi_{\mathcal{D}} v\|_{\text{BV}} \leq d \sum_{K \in \mathcal{M}} \left(\int_K |\nabla_{\overline{K}} v(\mathbf{x})| d\mathbf{x} + \sum_{\sigma \in \mathcal{F}_K} \int_{\sigma} |[v]_{K,\sigma}(\mathbf{y})| d\gamma(\mathbf{y}) \right).$$

Thanks to Young's inequality, we have

$$\begin{aligned} \|\Pi_{\mathcal{D}} v\|_{\text{BV}} &\leq d \left(\sum_{K \in \mathcal{M}} \left(\int_K |\nabla_{\overline{K}} v(\mathbf{x})|^p d\mathbf{x} + \sum_{\sigma \in \mathcal{F}_K} \frac{1}{d_{K,\sigma}^{p-1}} \int_{\sigma} |[v]_{K,\sigma}(\mathbf{y})|^p d\gamma(\mathbf{y}) \right) \right)^{1/p} \\ &\quad \times \left(\sum_{K \in \mathcal{M}} \left(|K| + \sum_{\sigma \in \mathcal{F}_K} d_{K,\sigma} |\sigma| \right) \right)^{(p-1)/p}. \end{aligned}$$

The results follows. \blacksquare

We now state the discrete Sobolev inequalities result.

Lemma A.7 (Discrete Sobolev inequalities): Let \mathcal{D} be a DGGD in the sense of Definition 3.2. Then, for all $q \in [1, \frac{pd}{d-p}]$ if $1 \leq p < d$ and $q \in [1, +\infty[$ otherwise, there exists C , depending only on $|\Omega|$, k , p , q and d , such that

$$\forall v \in X_{\mathcal{D},0}, \|\Pi_{\mathcal{D}} v\|_{L^q(\Omega)} \leq C \|v\|_{\text{DG},p}. \quad (46)$$

Proof. Let us first assume that $d = 1$. Then we have

$$\forall \bar{v} \in L^q(\mathbb{R}), \|\bar{v}\|_{L^\infty(\mathbb{R})} \leq \|\bar{v}\|_{\text{BV}},$$

which provides (46) thanks to Lemma A.6.

We now assume that $d \geq 2$.

As in [2], we follow Nirenberg's technique. First remark that, for all $q \in [1, d/(d-1)]$,

$$\forall \bar{v} \in L^q(\mathbb{R}^d), \|\bar{v}\|_{L^q(\mathbb{R}^d)} \leq \frac{1}{2d} \|\bar{v}\|_{\text{BV}}. \quad (47)$$

This inequality implies, thanks to Lemma A.6,

$$\forall v \in X_{\mathcal{D},0}, \quad \|\Pi_{\mathcal{D}}v\|_{L^q(\mathbb{R}^d)} \leq \frac{1}{2d} \|\Pi_{\mathcal{D}}v\|_{\text{BV}} \leq \frac{1}{2} \|v\|_{\text{DG},1}. \quad (48)$$

We now handle the case $1 < p < d$. We define $\alpha = \frac{p(d-1)}{d-p} > 1$ and $p^* = \frac{pd}{d-p}$. For $v \in X_{\mathcal{D},0}$, we apply (47) to $|\Pi_{\mathcal{D}}v|^\alpha$ and $q = d/(d-1)$. It yields

$$\left(\int_{\Omega} |\Pi_{\mathcal{D}}v(\mathbf{x})|^{p^*} d\mathbf{x} \right)^{\frac{d-1}{d}} \leq \frac{1}{2d} \sum_{i=1}^d \left(\sum_{K \in \mathcal{M}} \int_K |\partial_i |\Pi_{\overline{K}}v(\mathbf{x})|^\alpha| d\mathbf{x} + \sum_{\sigma \in \mathcal{F}} \int_{\sigma} |[v]^\alpha|_{\sigma}(\mathbf{y})| d\gamma(\mathbf{y}) \right),$$

denoting by $|[w]_{\sigma}(\mathbf{y})|$ the absolute value of the jump of w at $\mathbf{y} \in \sigma$. We observe that, for a.e. $\mathbf{x} \in K$, we have $|\partial_i |\Pi_{\overline{K}}v(\mathbf{x})|^\alpha| = \alpha |\Pi_{\overline{K}}v(\mathbf{x})|^{\alpha-1} |\partial_i \Pi_{\overline{K}}v(\mathbf{x})| \leq \alpha |\Pi_{\overline{K}}v(\mathbf{x})|^{\alpha-1} |\nabla_{\overline{K}}v(\mathbf{x})|$. On the other hand, we have, for $\mathcal{M}_{\sigma} = \{K, L\}$, $|[v]^\alpha|_{\sigma}(\mathbf{y})| \leq 2\alpha (|\Pi_{\overline{K}}v(\mathbf{y})|^{\alpha-1} + |\Pi_{\overline{L}}v(\mathbf{y})|^{\alpha-1}) \frac{|\Pi_{\overline{K}}v(\mathbf{y}) - \Pi_{\overline{L}}v(\mathbf{y})|}{2}$, and, for $\mathcal{M}_{\sigma} = \{K\}$, $|[v]^\alpha|_{\sigma}(\mathbf{y})| \leq \alpha |\Pi_{\overline{K}}v(\mathbf{y})|^{\alpha-1} |\Pi_{\overline{K}}v(\mathbf{y}) - 0|$. We then have, using notation (16),

$$\left(\int_{\Omega} |\Pi_{\mathcal{D}}v(\mathbf{x})|^{p^*} d\mathbf{x} \right)^{\frac{d-1}{d}} \leq \alpha \sum_{K \in \mathcal{M}} \left(\int_K |\Pi_{\overline{K}}v(\mathbf{x})|^{\alpha-1} |\nabla_{\overline{K}}v(\mathbf{x})| d\mathbf{x} + \sum_{\sigma \in \mathcal{F}_K} \int_{\sigma} |\Pi_{\overline{K}}v(\mathbf{y})|^{\alpha-1} |[v]_{K,\sigma}(\mathbf{y})| d\gamma(\mathbf{y}) \right).$$

Hence we get, from Young's inequality, since $(\alpha-1)p' = p^*$ with $p' = p/(p-1)$,

$$\left(\int_{\Omega} |\Pi_{\mathcal{D}}v(\mathbf{x})|^{p^*} d\mathbf{x} \right)^{\frac{d-1}{d}} \leq \alpha (T_1)^{1/p} (T_2)^{(p-1)/p},$$

with

$$T_1 = \sum_{K \in \mathcal{M}} \left(\int_K |\nabla_{\overline{K}}v(\mathbf{x})|^p d\mathbf{x} + \sum_{\sigma \in \mathcal{F}_K} \int_{\sigma} \frac{|[v]_{K,\sigma}(\mathbf{y})|^p}{d_{K,\sigma}^{p-1}} d\gamma(\mathbf{y}) \right) = \|v\|_{\text{DG},p}^p,$$

and

$$T_2 = \sum_{K \in \mathcal{M}} \left(\int_K |\Pi_{\overline{K}}v(\mathbf{x})|^{p^*} d\mathbf{x} + \sum_{\sigma \in \mathcal{F}_K} d_{K,\sigma} \int_{\sigma} |\Pi_{\overline{K}}v(\mathbf{y})|^{p^*} d\gamma(\mathbf{y}) \right).$$

Let us now observe that

$$\int_K |\Pi_{\overline{K}}v(\mathbf{x})|^{p^*} d\mathbf{x} = \sum_{\sigma \in \mathcal{F}_K} \int_{D_{K,\sigma}} |\Pi_{\overline{K}}v(\mathbf{x})|^{p^*} d\mathbf{x}.$$

We write, as in the proof of Lemma A.2,

$$\int_{D_{K,\sigma}} |\Pi_{\overline{K}}v(\mathbf{x})|^{p^*} d\mathbf{x} = d_{K,\sigma} \int_{\sigma} \int_0^1 |\Pi_{\overline{K}}v(\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K))|^{p^*} s^{d-1} ds d\gamma(\mathbf{y}).$$

For a given $\mathbf{y} \in \sigma$, $s \mapsto \Pi_{\overline{K}}v(\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K))$ is a polynomial with respect to s with degree lower or equal to k , that we write under the form $\Pi_{\overline{K}}v(\mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)) = \sum_{i=0}^k a_i(\mathbf{y}) s^i$. We then use the constant introduced in Definition A.1, which provides

$$\int_0^1 \left| \sum_{i=0}^k a_i(\mathbf{y}) s^i \right|^{p^*} s^{d-1} ds \geq C_{p^*,k} \sum_{i=0}^k |a_i(\mathbf{y})|^{p^*},$$

and therefore

$$\int_{D_{K,\sigma}} |\Pi_{\overline{K}}v(\mathbf{x})|^{p^*} d\mathbf{x} \geq C_{p^*,k} d_{K,\sigma} \int_{\sigma} \left(\sum_{i=0}^k |a_i(\mathbf{y})|^{p^*} \right) d\gamma(\mathbf{y}).$$

Remarking that $\mathbf{y} = \mathbf{x}_K + s(\mathbf{y} - \mathbf{x}_K)$ for $s = 1$, we have

$$\int_{\sigma} |\Pi_{\overline{K}}v(\mathbf{y})|^{p^*} d\gamma(\mathbf{y}) = \int_{\sigma} \left| \sum_{i=0}^k a_i(\mathbf{y}) \right|^{p^*} d\gamma(\mathbf{y}) \leq (k+1)^{p^*-1} \int_{\sigma} \left(\sum_{i=0}^k |a_i(\mathbf{y})|^{p^*} \right) d\gamma(\mathbf{y}).$$

This gives

$$d_{K,\sigma} \int_{\sigma} |\Pi_{\overline{K}} v(\mathbf{y})|^{p^*} d\gamma(\mathbf{y}) \leq \frac{(k+1)^{p^*-1}}{C_{p^*,k}} \int_{D_{K,\sigma}} |\Pi_{\overline{K}} v(\mathbf{x})|^{p^*} d\mathbf{x},$$

and concludes the proof that

$$T_2 \leq \left(1 + \frac{(k+1)^{p^*-1}}{C_{p^*,k}}\right) \int_{\Omega} |\Pi_{\mathcal{D}} v(\mathbf{x})|^{p^*} d\mathbf{x}.$$

Hence we conclude (46) for $1 < p < d$.

Let us finally consider the case $d \leq p$. We select any real value $q_1 > p$, and we set $p_1 = dq_1/(d+q_1)$. Then we have $1 < p_1 < d \leq p$ and $p_1^* = q_1$. We apply the result proved for $1 < p < d$ above, which yields (46), replacing p by p_1 and p_1^* by q_1 . We then apply Lemma A.5, which allows to bound $\|v\|_{DG,p_1}$ by $\|v\|_{DG,p}$. ■

Lemma A.8: Let $V \subset \mathbb{R}^d$ be an open bounded set that is star-shaped with respect to all points in a ball $B \subset V$. Let $p \in [1, +\infty)$. There exists C_{13} depending only on d and p such that, for any $\varphi \in W^{1,p}(V)$,

$$\left\| \varphi - \frac{1}{|B|} \int_B \varphi(\mathbf{x}) d\mathbf{x} \right\|_{L^p(V)} \leq C_{13} \frac{\text{diam}(V)^{\frac{d}{p}+1}}{\text{diam}(B)^{\frac{d}{p}}} \|\nabla \varphi\|_{L^p(V)}. \quad (49)$$

Proof. Since $C^\infty(V) \cap W^{1,p}(V)$ is dense in $W^{1,p}(V)$, we only need to prove the result for $\varphi \in C^\infty(V) \cap W^{1,p}(V)$, and the conclusion follows by density. To simplify the notations we let $h_V = \text{diam}(V)$. For all $(\mathbf{x}, \mathbf{y}) \in V \times B$, since V is star-shaped with respect to \mathbf{y} the segment $[\mathbf{x}, \mathbf{y}]$ belongs to V and we can write

$$\varphi(\mathbf{x}) - \varphi(\mathbf{y}) = \int_0^1 \nabla \varphi(t\mathbf{x} + (1-t)\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) dt.$$

Taking the average value with respect to $\mathbf{y} \in B$ and writing $|\mathbf{x} - \mathbf{y}| \leq h_V$ gives

$$\begin{aligned} \left| \varphi(\mathbf{x}) - \frac{1}{|B|} \int_B \varphi(\mathbf{y}) d\mathbf{y} \right| &= \left| \frac{1}{|B|} \int_B \int_0^1 \nabla \varphi(t\mathbf{x} + (1-t)\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) dt d\mathbf{y} \right| \\ &\leq \frac{h_V}{|B|} \int_B \int_0^1 |\nabla \varphi(t\mathbf{x} + (1-t)\mathbf{y})| dt d\mathbf{y}. \end{aligned}$$

Taking the power p , using the Jensen inequality with the convex function, and integrating with respect to $\mathbf{x} \in V$, we get

$$\int_V \left| \varphi(\mathbf{x}) - \frac{1}{|B|} \int_B \varphi(\mathbf{y}) d\mathbf{y} \right|^p d\mathbf{x} \leq \frac{h_V^p}{|B|} \int_V \int_B \int_0^1 |\nabla \varphi(t\mathbf{x} + (1-t)\mathbf{y})|^p dt d\mathbf{y} d\mathbf{x}. \quad (50)$$

We then apply the change of variable $\mathbf{x} \in V \rightarrow \mathbf{z} = t\mathbf{x} + (1-t)\mathbf{y}$, which has values in V since V is star-shaped with respect to all points in B . This gives

$$\int_V \int_B \int_0^1 |\nabla \varphi(t\mathbf{x} + (1-t)\mathbf{y})|^p dt d\mathbf{y} d\mathbf{x} \leq \int_V |\nabla \varphi(\mathbf{z})|^p \int_B \int_{I(\mathbf{z}, \mathbf{y})} t^{-d} dt d\mathbf{y} d\mathbf{z}, \quad (51)$$

where $I(\mathbf{z}, \mathbf{y}) = \{t \in (0, 1) : \exists \mathbf{x} \in V, t\mathbf{x} + (1-t)\mathbf{y} = \mathbf{z}\}$. For $t \in I(\mathbf{z}, \mathbf{y})$ we have $t(\mathbf{x} - \mathbf{y}) = \mathbf{z} - \mathbf{y}$ for some $\mathbf{x} \in V$ and therefore $h_V t \geq |\mathbf{z} - \mathbf{y}|$. Hence $I(\mathbf{z}, \mathbf{y}) \subset [\frac{|\mathbf{z} - \mathbf{y}|}{h_V}, 1]$ and we deduce that (for $d > 1$)

$$\int_{I(\mathbf{z}, \mathbf{y})} t^{-d} dt \leq \int_{\frac{|\mathbf{z} - \mathbf{y}|}{h_V}}^1 t^{-d} dt \leq \frac{1}{d-1} \frac{h_V^{d-1}}{|\mathbf{z} - \mathbf{y}|^{d-1}}. \quad (52)$$

Thus, denoting by ω_d the area of the unit sphere in \mathbb{R}^d , since $B \subset V \subset B(\mathbf{z}, h_V)$ for all $\mathbf{z} \in V$,

$$\int_B \int_{I(\mathbf{z}, \mathbf{y})} t^{-d} dt d\mathbf{y} \leq \frac{h_V^{d-1}}{d-1} \int_B \frac{1}{|\mathbf{z} - \mathbf{y}|^{d-1}} d\mathbf{y}$$

$$\begin{aligned}
&\leq \frac{h_V^{d-1}}{d-1} \int_{B(\mathbf{z}, h_V)} |\mathbf{z} - \mathbf{y}|^{1-d} d\mathbf{y} \\
&\leq \frac{h_V^{d-1}}{d-1} \omega_d \int_0^{h_V} \rho^{1-d} \rho^{d-1} d\rho \leq \frac{h_V^d}{d-1} \omega_d.
\end{aligned} \tag{53}$$

The proof is complete by plugging this estimate into (51), by using the resulting inequality in (50), and by recalling that

$$|B| = |B(0, 1)| \left(\frac{\text{diam}(B)}{2} \right)^d.$$

Note that in the case $d = 1$, (52) is modified and involves $\ln(\frac{h}{|\mathbf{z}-\mathbf{y}|})$ but the final estimate (53) is still in $\mathcal{O}(h^d)$. \blacksquare

The following lemma is a simple technical result used in Lemma A.10 below.

Lemma A.9: Let $h > 0$, $d \in \mathbb{N}^*$, $\mathbf{x} \in \mathbb{R}^d$ and let us define the function $F_{\mathbf{x}, h} : B(\mathbf{x}, h) \rightarrow \mathbb{R}$ by

$$\forall \mathbf{z} \in B(\mathbf{x}, h), F_{\mathbf{x}, h}(\mathbf{z}) = \int_{\frac{|\mathbf{x}-\mathbf{z}|}{h}}^1 s^{1-d} ds. \tag{54}$$

Let $q \in [1, +\infty]$ if $d = 1$, $q \in [1, +\infty)$ if $d = 2$, and $q \in [1, \frac{d}{d-2})$ if $d \geq 3$. Then, there exists $C_{14} > 0$ depending only on d and q such that

$$\|F_{\mathbf{x}, h}\|_{L^q(B(\mathbf{x}, h))} \leq C_{14} h^{d/q}. \tag{55}$$

Proof.

CASE $d = 1$.

We have $|F_{\mathbf{x}, h}(\mathbf{z})| \leq 1$ and therefore (55) is satisfied with $C_{14} = 2^{1/q}$.

CASE $d = 2$.

We have $F_{\mathbf{x}, h}(\mathbf{z}) = \ln\left(\frac{h}{|\mathbf{x}-\mathbf{z}|}\right)$ and therefore, since $q < +\infty$, using a polar change of coordinates,

$$\|F_{\mathbf{x}, h}\|_{L^q(B(\mathbf{x}, h))}^q = 2\pi \int_0^h \rho \ln\left(\frac{h}{\rho}\right)^q d\rho.$$

The function $\rho \mapsto \rho \ln\left(\frac{h}{\rho}\right)^q$ reaches its maximum over $[0, h]$ at $\rho = e^{-q}h$ and thus

$$\|F_{\mathbf{x}, h}\|_{L^q(B(\mathbf{x}, h))}^q \leq 2\pi \int_0^h e^{-q} h q^q d\rho = q^q e^{-q} h^2.$$

This proves (55) with $C_{14} = (2\pi)^{1/q} q e^{-1}$.

CASE $d \geq 3$.

We write

$$F_{\mathbf{x}, h}(\mathbf{z}) = \frac{1}{d-2} \left[\left(\frac{h}{|\mathbf{x}-\mathbf{z}|} \right)^{d-2} - 1 \right] \leq \frac{1}{d-2} \left(\frac{h}{|\mathbf{x}-\mathbf{z}|} \right)^{d-2}$$

and, using again polar coordinates,

$$\|F_{\mathbf{x}, h}\|_{L^q(B(\mathbf{x}, h))}^q \leq \frac{\omega_d}{(d-2)^q} h^{(d-2)q} \int_0^h \rho^{(2-d)q+d-1} d\rho$$

where ω_d is the area of the unit sphere in \mathbb{R}^d . The assumption $q < \frac{d}{d-2}$ ensures that $(2-d)q+d-1 > -1$ and therefore

$$\|F_{\mathbf{x}, h}\|_{L^q(B(\mathbf{x}, h))}^q \leq \frac{\omega_d}{(d-2)^q((2-d)q+d)} h^d.$$

The proof is complete by choosing $C_{14} = \frac{\omega_d^{1/q}}{(d-2)[(2-d)q+d]^{1/q}}$. \blacksquare

Lemma A.10 (First order polynomial approximation of elements of $W^{2,p}$): Assume that $p > \frac{d}{2}$, and let $B \subset V$ be bounded subsets of \mathbb{R}^d , such that B is a ball and V is star-shaped with respect to all points of B . Let $\theta \geq \text{diam}(V)/\text{diam}(B)$.

Take $\varphi \in W^{2,p}(V) \cap C(\bar{V})$. Then there exists $C_{15} > 0$, depending only on d, p and θ , and an affine function $A_\varphi : V \rightarrow \mathbb{R}$ such that

$$\sup_{\mathbf{x} \in \bar{V}} |\varphi(\mathbf{x}) - A_\varphi(\mathbf{x})| \leq C_{15} \text{diam}(V)^{2-\frac{d}{p}} \| |D^2\varphi| \|_{L^p(V)}, \quad (56)$$

$$\| \nabla A_\varphi - \nabla \varphi \|_{L^p(V)^d} \leq C_{15} \text{diam}(V) \| |D^2\varphi| \|_{L^p(V)}. \quad (57)$$

Remark A.11: If V is sufficiently regular, $W^{2,p}(V) \subset C(\bar{V})$ and we only need to assume that $\varphi \in W^{2,p}(V)$.

Proof. To simplify the notations, set $h_B = \text{diam}(B)$ and $h_V = \text{diam}(V)$. Let us first assume that $\varphi \in C_c^2(\mathbb{R}^d)$. For a given $\mathbf{x} \in \bar{V}$ and any $\mathbf{y} \in B$, write the Taylor expansion

$$\varphi(\mathbf{x}) = \varphi(\mathbf{y}) + \nabla \varphi(\mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) + \int_0^1 s D^2 \varphi(\mathbf{x} + s(\mathbf{y} - \mathbf{x})) (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) ds. \quad (58)$$

Denote by $\bar{\mathbf{y}}$ the centre of B , and set $\bar{\varphi} = \frac{1}{|B|} \int_B \varphi(\mathbf{y}) d\mathbf{y}$ and $\bar{\nabla \varphi} = \frac{1}{|B|} \int_B \nabla \varphi(\mathbf{y}) d\mathbf{y}$. Taking the average of (58) over $\mathbf{y} \in B$ gives $\varphi(\mathbf{x}) = A_\varphi(\mathbf{x}) + R_1(\mathbf{x}) + R_2(\mathbf{x})$ with

$$A_\varphi(\mathbf{x}) = \bar{\varphi} + \bar{\nabla \varphi} \cdot (\mathbf{x} - \bar{\mathbf{y}}),$$

$$R_1(\mathbf{x}) = \frac{1}{|B|} \int_B \int_0^1 s D^2 \varphi(\mathbf{x} + s(\mathbf{y} - \mathbf{x})) (\mathbf{x} - \mathbf{y}) \cdot (\mathbf{x} - \mathbf{y}) ds d\mathbf{y},$$

and

$$R_2(\mathbf{x}) = \frac{1}{|B|} \int_B (\nabla \varphi(\mathbf{y}) - \bar{\nabla \varphi}) \cdot (\mathbf{x} - \mathbf{y}) d\mathbf{y}.$$

Hence,

$$|\varphi(\mathbf{x}) - A_\varphi(\mathbf{x})| \leq |R_1(\mathbf{x})| + |R_2(\mathbf{x})|. \quad (59)$$

It remains to bound R_1 and R_2 .

BOUND ON R_1 .

The change of variable $\mathbf{y} \in B \rightarrow \mathbf{z} = \mathbf{x} + s(\mathbf{y} - \mathbf{x})$ has values in V , since V is star-shaped with respect to all points in B . This gives

$$|R_1(\mathbf{x})| \leq \frac{h_V^2}{|B|} \int_V \int_{I(\mathbf{x}, \mathbf{z})} s^{1-d} |D^2 \varphi(\mathbf{z})| ds d\mathbf{z},$$

where $I(\mathbf{x}, \mathbf{z}) = \{s \in (0, 1) : \exists \mathbf{y} \in B, \mathbf{z} = \mathbf{x} + s(\mathbf{y} - \mathbf{x})\}$. If $s \in I(\mathbf{x}, \mathbf{z})$ then $|\mathbf{z} - \mathbf{x}| = s|\mathbf{y} - \mathbf{x}| \leq sh_V$ for some $\mathbf{y} \in B$, and thus $s \geq \frac{|\mathbf{z} - \mathbf{x}|}{h_V}$. Hence,

$$|R_1(\mathbf{x})| \leq \frac{h_V^2}{|B|} \int_V |D^2 \varphi(\mathbf{z})| \int_{\frac{|\mathbf{z} - \mathbf{x}|}{h_V}}^1 s^{1-d} ds d\mathbf{z} = \frac{h_V^2}{|B|} \int_V |D^2 \varphi(\mathbf{z})| F_{\mathbf{x}, h_V}(\mathbf{z}) d\mathbf{z}$$

where $F_{\mathbf{x}, h_V}$ is defined by (54). Using Hölder's inequality, the inclusion $V \subset B(\mathbf{x}, h_V)$ and Lemma A.9 we infer

$$|R_1(\mathbf{x})| \leq \frac{h_V^2}{|B|} \| |D^2 \varphi| \|_{L^p(V)} \| F_{\mathbf{x}, h_V} \|_{L^{p'}(B(\mathbf{x}, h_V))} \leq C_{16} \frac{h_V^{2+\frac{d}{p'}}}{|B|} \| |D^2 \varphi| \|_{L^p(V)}$$

where C_{16} depends only on d and p (notice that $p > d/2$ implies $p' < \frac{d}{d-2}$ if $d \geq 2$). Since $\frac{d}{p'} = d - \frac{d}{p}$ and $|B| = |B(0, 1)| h_B^d \geq |B(0, 1)| \theta^{-d} h_V^d$, this gives the existence of C_{17} depending only on θ, p and d such that

$$|R_1(\mathbf{x})| \leq C_{17} h_V^{2-\frac{d}{p}} \| |D^2 \varphi| \|_{L^p(V)}. \quad (60)$$

BOUND ON R_2 .

By Hölder's inequality and $|B| = |B(0, 1)|h_B^d$,

$$|R_2(\mathbf{x})| \leq h_B |B|^{\frac{1}{p'}-1} \|\nabla\varphi - \overline{\nabla\varphi}\|_{L^p(B)^d} \leq |B(0, 1)|^{-\frac{1}{p}} h_B^{1-\frac{d}{p}} \|\nabla\varphi - \overline{\nabla\varphi}\|_{L^p(B)^d}.$$

Apply Lemma A.8 with $V = B$ and φ replaced by $\partial_i\varphi$ (for $i = 1, \dots, d$). This gives C_{18} depending only on d and p such that

$$|R_2(\mathbf{x})| \leq C_{18} h_B^{2-\frac{d}{p}} \| |D^2\varphi| \|_{L^p(B)}. \quad (61)$$

CONCLUSION.

Combining (59), (60) and (61) gives (56). To prove (57), notice that

$$\nabla A_\varphi = \overline{\nabla\varphi} = \frac{1}{|B|} \int_B \nabla\varphi(\mathbf{y}) d\mathbf{y}$$

and apply Lemma A.8 with φ replaced by $\partial_i\varphi$, for all $i = 1, \dots, d$. This gives C_{19} depending only on d and p such that

$$\|\nabla A_\varphi - \nabla\varphi\|_{L^p(V)^d} \leq C_{19} \frac{h_V^{d/p+1}}{h_B^{d/p}} \| |D^2\varphi| \|_{L^p(V)}.$$

Since $h_B \geq \theta^{-1}h_V$, this completes the proof of (57) if $\varphi \in C_c^2(\mathbb{R}^d)$.

All quantities and norms involved in (56) and (57) are continuous with respect to φ for the norm of $W^{2,p}(V) \cap C(\overline{V})$ (sum of the norms in both spaces). Since V is star-shaped, a classical dilatation and regularisation argument shows that the restrictions of $C_c^2(\mathbb{R}^d)$ functions to V are dense in $W^{2,p}(V) \cap C(\overline{V})$. This density ensures that (56) and (57) are still valid for $\varphi \in W^{2,p}(V) \cap C(\overline{V})$. ■

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