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# Common agency dilemma with information asymmetry in continuous time\*

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## Abstract

In this paper, we consider a problem of contract theory in which several Principals hire a common Agent and we study the model in the continuous time setting. We show that optimal contracts should satisfy some equilibrium conditions and we reduce the optimisation problem of the Principals to a system of coupled Hamilton-Jacobi-Bellman (HJB) equations. Further, in a more specific linear-quadratic model where two interacting Principals hire one common Agent, we are able to calculate the optimal effort by the Agent for both Principals. In this continuous time model, we extend the result of [BW86] in which the authors compare the optimal effort of the Agent in a non-cooperative Principals model and that in the aggregate model, and give the condition under which these two optimisations coincide.

**Key words:** Moral hazard models, common agency, system of HJB equations, BSDEs.

**AMS 2000 subject classifications:** 93E20, 91A06, 91A15, 91B16, 91B70, 60H30.

**JEL subject classifications:** C73, D21, D82, D86, J41.

## 1 Introduction

In 2016, the Intergovernmental Science-Policy Platform on Biodiversity and Ecosystem Services (IPBES for short) published an assessment report<sup>1</sup> deeply investigating pollinators and pollination processes in food production issues. One of the key messages pointed in this report is the following "*given that pollinator-dependent crops rely on animal pollination to varying degrees, it is estimated that 5-8 per cent of current global crop production, with an annual market value of 235 billion-577 billion (in 2015, United States dollars) worldwide, is directly attributable to animal pollination*" and emphasized that pollination processes lead to a lot of benefit for the world economy. To illustrate the motivation of our study in the light of the report of the IPBES, consider for instance one field divided for planting various crops (biofuels, cotton, linen, orchard, vineyard...). Each crop is

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<sup>1</sup>see the report [PIFN<sup>+</sup>16] approved by the Plenary of IPBES at its fourth session in Kuala Lumpur.

managed by an owner (*she*). It interests the owners to collaborate with a beekeeper (*he*) to ensure a better yield. In this case, the beekeeper distributes his beehives in the fields for the different crops, and manage to improve, through the pollination, the production of the crop-owners. Therefore, each crop-owner has to provide the beekeeper sufficiently good incentives to take care of her crop. The situation in this example typically illustrates a common agency dilemma with information asymmetry, that is, the crop-owners need to design contracts to motivate the beekeeper, each hoping of improving her own production, without observing directly how the beekeeper and his bees impact the crops. The crop-owners play thus the role of the Principals and the beekeeper can be seen as an Agent, in the so-called multi-Principals/ Agent problem and more generally in the theory of incentives.

The theory of incentives or contract theory appeared in the 70's with the work of Mirrlees [Mir76] among others, by studying optimal payments schedule between two economical entities under imperfect observations on the performances of the employee who manages the employer's wealth. This has been investigated later in the textbook of Laffont and Tirole [LT93] with illustrations in regulation systems and the underlying game played between the two parties, and we refer to the survey book of Laffont and Martimort [LM09] for a lot of relevant examples related to this theory. More precisely, the situation with moral hazard considered in contract theory can be described as follows. The Principal (*she*) want to hire another entity, namely the Agent (*he*), to manage a project. In most of the situations, the Principal has no access to observe the effort of her agent, but only observe the outcome of his work. Also, the Agent has a reservation utility which allows him to accept or reject the contract proposed by the Principal. The problem of the Principal is thus to design an optimal contract maximizing her utility among all the contracts satisfying the Agent's reservation utility, without observing, not to say controlling, his effort. In practice, we identify this game between the Agent and the Principal with a Stackelberg equilibrium, *i.e.* given any fixed contract, the Agent has the corresponding best reaction, and based on these predictable reactions the Principal choose the best contract in order to maximize her utility.

The researches made in the 70's related to contract theory mainly treat the problem in a discrete time framework with one or more periods, *i.e.* the effort of the Agent, the outcome, and all other relevant quantity take values in a finite space. Hölmstrom and Milgrom studied a continuous time model in the seminal paper [HM87] in which they consider a Brownian model for the wealth process, and assume the Principal does the optimization with the exponential utility. This work has then been extended by many other authors. Schättler and Sung in [SS93] provided first-order sufficient conditions to solve the problem. Sung in [Sun95] investigated this issue when the Agent can control the diffusion of the output. Then, Hellwig and Schmidt in [HS01] made a link between the discrete time and continuous time models. All these papers used sophisticated tools of stochastic control theory such as dynamic programming and martingale approach. For more details, we refer to the survey paper of Sung [Sun01] or the book of Cvitanic and Zhang [CZ13].

Recently, new ingredients have been introduced to the contract theory, which allows us to study more general models and obtain tractable solutions. Sannikov studied in [San08] a model in which the Principal provides continuous payments to the Agent and chooses a random retiring time. As a remarkable contribution of the paper, Sannikov observed that the value of the Agent's problem (the continuation utility) should be considered as a state variable for the problem of

the Principal. Later, Cvitanić, Possamaï and Touzi proposed in [CPT14, CPT15] a very general procedure to solve Principal/Agent problems with lump-sum payments in which the Agent can control both the drift and the volatility of the output. They observe that as an utility maximization problem, the Agent's problem can be reduced<sup>2</sup> to solving a backward stochastic differential equation (BSDE) introduced in Pardoux and Peng [PP90] (see also El Karoui, Peng and Quenez [EKPQ97]), or its fully-nonlinear generalization, namely second-order BSDE (2BSDE) introduced in Soner, Touzi and Zhang [STZ12]. The problem of the Principal, given the best reaction of the Agent, is thus a standard stochastic controlled problem with the two state variables: the output process and the continuation utility of the Agent, as Sannikov suggested.

Extensions of the Hölmstrom and Milgrom problem are nevertheless not restricted to the single Principal/single Agent model. It was extended to the single Principal/multi-Agents model by Koo, Shim and Sung in [KKSS08] then by Elie and Possamaï in [EP16]. In these works, the Agents aim at finding a Nash equilibrium given contracts proposed by the Principal. Elie and Possamaï proved that the Nash equilibrium among the Agents can be characterized the solution to a multi-dimensional BSDE, and that the problem of the Principal can be reduced to solving a stochastic control problem which takes the continuation utilities of all the Agents as state variables. Elie and Possamaï, in particular, studied a single Principal/2-Agents example in which the agents compete with each other. It is curious to see in the example that the less ambitious agent may make less effort and leave the job to the other agent. Recently, it was studied in [Mas17] that a Planner aims at acting for the welfare of the agents by finding Pareto optima, and the result is compared with that in [EP16] in which the agents reach Nash equilibria. In particular, the sufficient and necessary conditions are provided in [Mas17] so that the Pareto optima coincide with the Nash equilibria or that the Planner can have a higher return than in the classical multi-agents second best case.

In this paper, we focus on a multi-Principals/single Agent problem in the continuous setting. More precisely, we assume in this paper that several Principals aim at hiring one common agent who works simultaneously for all the Principals. This common agency problem echoes the example of the beekeeper and crop-owners presented above, and as far as we know, it has not been studied in the continuous setting. Nevertheless, in the discrete case, there are several works having investigated this problem. In the 80's, Baron in [Bar85] studied a common agency problem involving regulators facing non-localized externalities. He illustrated his study with the example of the Environmental Protection Agency (EPA) and the public utility commission for the control of a non-localized pollution externality. Each entity has its own goal in preventing risks, so the conflicting interests can appear and a cooperation between the regulators does not hold. Baron thus implement cooperative and non-cooperative equilibrium in his particular regulation model. Another interesting application of common agency problem was shown in Braverman and Stiglitz [BS82], dealing with the sharecropping of the farmers and landlords. Later, Bernheim and Whinston (see [BW85, BW86]) proposed a general approach for analysing the problem with common agency. They provided a sufficient and necessary condition which characterizes the equilibrium, and proved that a non-cooperative equilibrium between the Principals is efficient (that is, it attains the classical second best<sup>3</sup> level of effort and outcome) only for the first

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<sup>2</sup>We refer to the works of El Karoui and Rouge [REK00] and Hu, Imkeller and Müller [HIM<sup>+</sup>05].

<sup>3</sup>*i.e.* in the single-Principal/Agent model, the Principal does not control directly the effort of the Agent

best<sup>4</sup> level of effort. Dixit, Grossman and Helpman have then extended in [DGH97] the works of Bernheim and Whinston to more general cases, by considering non-quasilinear utility functions and by characterizing equilibria for the general common agency problem in terms of Nash equilibria. They also discussed the efficiency in the Pareto sense.

As mentioned above, all the previous works studied the common agency problem in the discrete time model. As far as we know, this topic has not been explored yet with a diffusion model in the continuous time setting, while such setting-up apparently simplifies the technical argument and produces tractable solutions (as demonstrated in [San08, CPT14, CPT15]). Our work can be treated as an extension of the works of Bernheim and Whinston and that of Dixit, Grossman and Helpman in this direction, and we also try to characterize the Nash equilibria among the contracts. Our study remains part of the analysis of Nash equilibrium with asymmetry of information (see for instance [ÇD16]), by showing that the problems of each Principal to find such equilibrium is reduced to find a solution to a fully coupled HJB equation. This relation was well investigated in the books [Car16, DJrLS00] and was also applied by Carmona and Yang to a predator trading with looser time constraints in [CY11]. Here, we register our model in the continuation of all the mentioned works by assuming that each Principal is rational and aims at finding his best reaction remuneration facing to the actions of other Principals which naturally leads to the investigation of Nash equilibria.

The general structure of our paper is the following, in Section 2 we state the common agency model that we study. In particular, we introduce the definition of equilibrium. We then solve the problem of the common agent in Section 3 by reducing it as usual to the solution of a BSDE. At the heart of this work, we propose in Section 4 a general procedure to solve the common agency dilemma of the Principals, by providing a characterization of the Nash equilibria in a particular set of contracts and deriving the corresponding fully coupled system of HJB equations. As an application of our work, we study in Section 5 a model with two Principals hiring a common Agent by comparing a competitive model in which the principals are not cooperative to a situation in which the offers of the principals are aggregated and the aggregated offer fits the classical bilateral Principal/Agent model. All along this study, we try to give economical interpretations of our mathematical model and results.

Finally, by going beyond the example of beekeepers remunerated by crop-owners introduced at the beginning, this paper could be a good starting point to treat the applications such as government management in which different level of a same Minister compensated a firm, to recall the example in [BW85] or foreign borrowing issues as those exposed in the work of Tirole [Tir03].

To simplify the reading of the paper, the technical notations are postponed to Appendix A.1.

## 2 The multi-Principal/Agent model

In this paper we consider the Principals-Agent problem in which  $N$  Principals hire an Agent in order to manage  $N$  risky projects. In this model, we assume that the projects are possibly

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<sup>4</sup>*i.e.* when in the single-Principal/Agent model, the Principal controls the effort of the Agent, called risk-sharing problem

correlated among each others and the action of the hired Agent impacts on every project. This section introduces the mathematical model of the problem. We first define the dynamics of the  $N$  projects, and then set the utility maximization problems of both the Agent and the Principals as a Stackelberg equilibrium.

## 2.1 Agent problem

As mentioned before, the hired agent can make effort to influence the dynamic of the  $N$  projects. In this section, we describe the stochastic control problem with the weak formulation.

### 2.1.1 The dynamic and the coefficients

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. We denote  $W$  an  $N$ -dimensional Brownian motion on this space and  $\mathbb{F} := (\mathcal{F}_t)_{t \in [0, T]}$  its natural (completed) filtration. We define the general project  $X := (X_i)_{1 \leq i \leq N}$  which is an  $\mathbb{F}$ -adapted process  $\mathbb{R}^N$ -valued satisfying the following dynamic

$$X_t := x + \int_0^t \Sigma_s dW_s, \quad (2.1)$$

where  $x \in \mathbb{R}$  and  $\Sigma$  is  $\mathbb{F}$ -progressively measurable process taking values in  $\mathbb{R}^{N \times N}$ .

**Assumption ( $\Sigma$ ).**  $\Sigma$  is bounded and for any  $t \in [0, T]$ , the matrix  $\Sigma_t$  is invertible with bounded inverse.

The model that we will consider involves the following coefficients:

- the drift function  $b : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \longrightarrow \mathbb{R}^N$ , with components denoted by  $b^i$  such that  $b(t, x, e) = (b^i(t, x, e^i))_{1 \leq i \leq N}^\top$ ;
- the discount factor  $k : [0, T] \times \mathbb{R}^N \longrightarrow \mathbb{R}$ ;
- the cost function  $c : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \longmapsto \mathbb{R}_+$ .

In this paper, the following assumption holds for some fixed  $(m, \underline{m}) \in [1, +\infty) \times (0, m]$ .

**Assumption ( $\mathbf{H}^{m, \underline{m}}$ ).** There exists  $0 < \underline{\kappa} < \kappa$  such that for any  $(t, x, e, \eta) \in [0, T] \times \Omega \times A \times N$

- (i) For any  $\mathbb{F}$ -progressive process  $\nu$ ,  $b(t, X_t, \nu_t)$ ,  $c(t, X_t, \nu_t)$  and  $k(t, X_t)$  are  $\mathbb{F}$ -progressive.
- (ii) For any  $i \in \{1, \dots, N\}$  and for every  $(t, x) \in [0, T] \times \mathbb{R}^N$ , the map  $e \in \mathbb{R}^N \mapsto b^i(t, x, e^i)$  is continuously differentiable and there exists a positive constant  $C$  such that for any  $(t, x, e) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N$

$$|b^i(t, x, e^i)| \leq C(1 + \|x\| + |e^i|), \quad \|\nabla_e b(t, x, e)\| \leq C.$$

- (iii) The discount factor  $k$  is bounded.

- (iv) For any  $(t, x) \in [0, T] \times \mathbb{R}^N$ , the map  $a \mapsto c(t, x, a)$  is continuously differentiable, increasing, convexe satisfying

$$0 \leq c(t, x, e) \leq C(1 + \|x\| + \|e\|^{1+m}),$$

$$\underline{\kappa} \|e\|^{\underline{m}} \leq \|\nabla_e c(t, x, e)\| \leq \kappa \left(1 + \|e\|^m\right) \quad \text{and} \quad \overline{\lim}_{\|e\| \rightarrow \infty} \frac{c(t, x, e)}{\|e\|} = +\infty.$$

Before going further, let us explain the meaning of the characteristics  $b, c, k$  and the previous assumption. The drift function  $b$  depends on both the effort  $e$  of the Agent and the value of the outcome  $X$  itself. The dependency with respect to the action of the Agent is very classical since it directly drives the outcome through its drift. Concerning the dependency with respect to  $X$ , notice that the dynamic of the  $i$ th project can depend on the value of the other projects. An interpretation of it should be that the different projects are possibly strongly correlated and each of them is driven by its own value and possibly the value of the others, leading to some synergy effects. The discount factor is a classical term used to model a discounting mechanism. Finally, the cost function  $c$  depends obviously on the effort of the Agent and the fact that  $c$  is increasing with it emphasizes that making a bigger effort leads to a bigger cost for the Agent. The convexity assumption can be seen as an exhaustion effect, *i.e.* higher is the work of the Agent, higher is his sensitivity to increase or decrease his effort. The cost  $c$  can also depend on the outcome itself. For instance, if the value of the outcome is low, the Agent could be somehow depressed and his action could be directly impacted by it.

We now define the set of admissible effort:

**Definition 2.1** (Admissible effort). *We denote by  $\mathcal{A}$  the set of admissible effort  $\nu$  such that the following stochastic exponential*

$$(M_t)_{t \in [0, T]} := \left( \mathcal{E} \left( \int_0^t b(s, X_s, \nu_s) \cdot \Sigma_s^{-1} dW_s \right) \right)_{t \in [0, T]},$$

is an  $\mathbb{F}$ -martingale.

In view of the previous definition, for any  $\nu \in \mathcal{A}$ , one can define a new probability  $\mathbb{P}^\nu$  which is equivalent to  $\mathbb{P}$  given by

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}} := M_T.$$

According to Girsanov's theorem, the process

$$W^\nu := W - \int_0^\cdot \Sigma_s^{-1} b(s, X_s, \nu_s) ds,$$

is a  $\mathbb{P}^\nu$ -Brownian motion under  $\mathbb{P}^\nu$  and the dynamic (2.1) of  $X$  thus becomes for any  $\nu \in \mathcal{A}$

$$X_t = x + \int_0^t b(s, X_s, \nu_s) ds + \int_0^t \Sigma_s dW_s^\nu. \quad (2.2)$$

**Remark 2.1.** *Notice that we work under the weak formulation of the problem in this paper by considering any weak solution of SDE (2.2).*

Before stating the model, we introduce also the notion of admissibility for contracts proposed to the Agent through the following definition

**Definition 2.2** (Admissible contracts). *We define*

$$\mathcal{C}_N := \{ \xi, \mathcal{F}_T\text{-measurable and } \mathbb{R}^N\text{-valued with } \mathbb{E} [\exp(p \|\xi\|)] < +\infty, \forall p \geq 0 \}.$$

$\mathcal{C}_N$  is called the set of admissible contracts. Similarly, we denote by  $\mathcal{C}$  the set of admissible contracts proposed by one Principal, such that

$$\mathcal{C} := \{ \xi, \mathcal{F}_T\text{-measurable and } \mathbb{R}\text{-valued with } \mathbb{E} [\exp(p|\xi|)] < +\infty, \forall p \geq 0 \}.$$

## 2.1.2 The optimisation of the Agent

The Agent derives a utility given the salaries from the  $N$  Principals at time  $T$ , and his effort diminishes his general payoff through the cost function  $c$ . More exactly, each  $i$ th Principal proposes to the Agent a contract with the salary denoted by  $\xi^i$ , which is an  $\mathbb{R}$  valued  $\mathcal{F}_T$ -measurable random variable. The total salary of the Agent is thus  $\xi \cdot \mathbf{1}_N$ , where  $\xi = (\xi^1, \dots, \xi^N)^\top$ . We consider exponential utility for the Agent with risk aversion parameter  $R_A > 0$  such that, the utility of the Agent at time  $t = 0$  given a set of salary  $\xi$  and an admissible effort  $\nu \in \mathcal{A}$  is

$$u_0^A(\xi, \nu) := \mathbb{E}^{\mathbb{P}^\nu} \left[ -\exp \left( -R_A \left( \xi \cdot \mathbf{1}_N - \int_0^T c(t, X_t, \nu_t) dt \right) \right) \right].$$

The aim of the Agent is thus to maximize his utility given a panel of salaries  $\xi$  with admissible effort. Therefore, given a panel of contracts  $\xi$  proposed by the  $N$  Principals, the weak formulation of the Agent's problem is

$$U_0^A(\xi) := \sup_{\nu \in \mathcal{A}} u_0^A(\xi, \nu). \quad (2.3)$$

We denote by  $\mathcal{A}^*(\xi)$  the set of best response efforts of the Agent, given a vector of salaries  $\xi$ . As usual in this kind of problem, the set  $\mathcal{A}^*(\xi)$  is not necessarily reduced to a singleton. We can admit that the Agent has a criterion denoted by  $\preceq$  inside the set  $\mathcal{A}^*(\xi)$  to select the best effort. For instance, one can think that for two efforts  $\nu_1, \nu_2 \in \mathcal{A}^*(\xi)$ , we have  $\nu_1 \preceq \nu_2$  if the variance of his utility is higher with the effort  $\nu_1$  than with the effort  $\nu_2$ . We denote by  $\mathcal{A}_{\preceq}^*(\xi)$  the subspace of  $\mathcal{A}^*(\xi)$  endowed with this selection criterion  $\preceq$ . However, it is not necessary that this subspace of best response effort is reduced to a singleton. In this case, we assume that the Principals choose the effort of the Agent among the effort of the Agent in  $\mathcal{A}_{\preceq}^*(\xi)$ .

For the sake of simplicity here, we assume in the general model that  $\mathcal{A}_{\preceq}^*(\xi)$  will be reduced to a unique element  $\nu^*(\xi)$ .

This problem is a classical utility maximization problem with control  $\nu$  which can be reduced to solve a standard BSDE with the following form:

$$Y_t = \xi \cdot \mathbf{1}_N + \int_t^T \left( -\frac{R_A}{2} \left\| \Sigma_s^\top Z_s \right\|^2 + f^*(s, X_s, Z_s) \right) ds - \int_t^T Z_s \cdot \Sigma_s dW_s, \quad (2.4)$$

where  $f^*$  will be defined in Section 3 in which we solve the Agent's problem. In particular, by solving the Agent problem, we will find out that the sum of the salaries  $\xi \cdot \mathbf{1}_N$  admits a semi-martingale decomposition.

## 2.2 The $N$ Principals problem

Our Principals-Agent problem consists of a Stackelberg equilibrium between the Agent and the  $N$  Principals, and a Nash equilibrium among the  $N$ -Principals. We will solve the problem in two main steps. First, as we have seen, the Agent provides a best response effort given the sum of the salaries offered by the  $N$ -Principals. Then, the  $N$ -Principals compute a Nash equilibrium to optimize their own utility taking into account the best response of the Agent.



### 2.2.1 The Principals problem and the Nash equilibrium

Consider an Agent with a reservation utility  $R_0 \in \mathbb{R}$ , that is to say, the  $N$ -Principals have to solve their utility maximization problem by ensuring that any contract  $\xi \in \mathcal{C}_N$  proposed to the Agent satisfies

$$U_0^A(\xi) \geq R_0. \quad (2.5)$$

Focus now on the problem of the  $i$ th Principal knowing that other  $N - 1$  Principals propose to the Agent a contract  $\xi^{-i} \in \mathcal{C}_{N-1}$ . We assume that her payoff is given through a map  $\mathcal{U}_{P^i} : \mathbb{R} \rightarrow \mathbb{R}$  increasing and concave. The problem of Principal  $i$  is thus

$$U_0^{P^i} := \sup_{\substack{\xi^i \in \mathcal{C} \\ U_0^A(\xi^i \otimes_i \xi^{(-i)}) \geq R_0}} \mathbb{E}^{\nu^*} [\mathcal{K}_0^T U_{P^i}(\ell_i(X_T) - \xi^i)], \quad (2.6)$$

with  $\mathcal{K}_0^T := e^{-\int_0^T k_r dr}$ , and where  $\ell_i : \mathbb{R}^N \rightarrow \mathbb{R}$  is a liquidation function of the  $i$ th Agent with a linear growth.

In the case of  $N = 1$ , one may use the semimartingale decomposition provided by the BSDE (2.4), which solves the Agent problem, to replace the r.v.  $\xi$  by the forward semimartingale  $Y$ . This transfers the Principal problem to a standard stochastic control problem with two states variables: the output  $X$  and the process  $Y$ , and with two control variables:  $Y_0$  and  $Z$ . The stochastic control problem leads to the corresponding HJB equation, and we may solve it with a verification argument under mild conditions. We refer to [San08, CPT14, CPT15] for more details on the relation between the Principal-Agent problem and the HJB equation. Obviously, we can no longer follow the same approach when  $N > 1$ , because by solving the Agent problem, we only obtain a semimartingale decomposition for the sum of the salaries  $\xi \cdot \mathbf{1}_N$ .

We can provide two interesting examples which have been investigated in the discrete case by [BW85].

**Competitive Principals.** *If one considers now that the  $i$ th Principal receives  $X_T^i$  at time  $T$  and possibly gets a higher utility if her project is higher than the empirical mean of the others, we have*

$$\ell_i(x) = x^i + \gamma^i \left( x^i - \frac{1}{N-1} \sum_{j \neq i} x^j \right), \quad x \in \mathbb{R}^N,$$

where  $\gamma^i \geq 0$  is typically her appetite parameter toward project  $i$ . This situation fits exactly with noncooperative Principals' behaviors investigated in [BW85]. We refer to Sections 5.3 for an example of the impact of the different parameters in this situation.

**Aggregated offer.** *If one considers now that one can aggregate the  $N$ -Principals, then we reduce this problem to a single Principal-Agent model in the classical second-best case in which each  $i$ th component of the output has an efficiency parameter  $\gamma^i$  compared to the other components. In this case, the payoff of the aggregated Principal, called the parent firm is*

$$\sum_{i=1}^N X_T^i \left( 1 + \gamma^i - \frac{1}{N-1} \sum_{j \neq i} \gamma^j \right) - \xi$$

where  $\xi$  is the aggregation of the  $N$  salaries. This case coincides with the cooperative model of Section 2 in [BW85], and it will be studied more deeply in Section 5.2.

Every Principal aims at solving her problem (2.6) simultaneously, this leads to find a Nash equilibrium for our model, defined as follow.

**Definition 2.3** (Nash equilibrium). *A contract  $\xi^* \in \mathcal{C}_N$  is a Nash equilibrium for the  $N$  Principals if it satisfies*

$$\sup_{\xi^i \in \mathcal{C}} \mathbb{E}^{\nu^*(\xi^i \otimes \xi^{(-i),*})} [\mathcal{K}_0^T U_{P_i} (\ell_i(X_T) - \xi^i)] = \mathbb{E}^{\nu^*(\xi^*)} [\mathcal{K}_0^T U_{P_i} (\ell_i(X_T) - \xi^{i,*})], \quad 1 \leq i \leq N$$

with

$$U_0^A(\xi^*) \geq R_0.$$

We will show how to solve the Principals' problem in Section 4.

### 3 Solving the Agent problem

We fix a contract  $\xi \in \mathcal{C}_N$  proposed by the  $N$  Principals to the Agent. We introduce the dynamic version of the Agent problem (2.3) at time  $t \in [0, T]$  :

$$U_t^A(\xi) = \sup_{\nu \in \mathcal{A}} u_t^A(\xi, \nu),$$

with

$$u_t^A(\xi, \nu) := \mathbb{E}^{\mathbb{P}^\nu} \left[ -\exp \left( -R_A \left( \xi \cdot \mathbf{1}_N - \int_t^T c(s, X_s, \nu_s) ds \right) \right) \middle| \mathcal{F}_t \right].$$

Note that

$$\left( M_t e^{R_A \int_0^t c(s, X_s, \nu_s) ds} u_t^A(\xi, \nu) \right)_{t \in [0, T]}$$

is an  $(\mathbb{F}, \mathbb{P})$ -martingale. Therefore, by the martingale representation theorem, there exists an  $\mathbb{F}$ -predictable process  $\tilde{Z}$  such that

$$M_t e^{R_A \int_0^t c(s, X_s, \nu_s) ds} u_t^A(\xi, \nu) = -M_T e^{-R_A (\xi \cdot \mathbf{1}_N - \int_0^T c(s, X_s, \nu_s) ds)} - \int_t^T \tilde{Z}_s \cdot \Sigma_s dW_s.$$

By setting

$$Y_t^\nu := -\frac{\log(-u_t^A(\xi, \nu))}{R_A},$$

we can deduce from Ito's Formula that there exists an  $\mathbb{F}$ -predictable process  $Z^\nu$  such that

$$Y_t^\nu = \xi \cdot \mathbf{1}_N + \int_t^T \left( -\frac{R_A}{2} \left\| \Sigma_s^\top Z_s^\nu \right\|^2 + f(s, X_s, Z_s^\nu, \nu_s) \right) ds - \int_t^T Z_s^\nu \cdot \Sigma_s dW_s, \quad (3.1)$$

with  $f(t, x, z, \nu) := b(t, x, \nu) \cdot z - c(t, x, \nu)$  for  $(t, x, z, \nu) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$ . We now define

$$\mathcal{A}_{x,z}^* := \left\{ (\nu_s(x, z))_{s \in [0, T]} : \nu_s(x, z) \in \operatorname{argmax}_{\nu \in \mathbb{R}^N} f(s, x, z, \nu), \text{ for a.e. } s \in [0, T] \right\}. \quad (3.2)$$

We also define for any  $(t, x, z) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N$  and  $\nu^*(x, z) \in \mathcal{A}_{x,z}^*$

$$f^*(t, x, z) := f(t, x, z, \nu_t^*(x, z)).$$

**Lemma 3.1.** *Let Assumptions  $(\Sigma)$  and  $(\mathbf{H}^{m, \underline{m}})$  be true for some  $(m, \underline{m}) \in [1, +\infty) \times (0, m]$ . Then, the set  $\mathcal{A}_{x,z}^*$  is non empty and any  $\nu^*(x, z) \in \mathcal{A}_{x,z}^*$  has the following growth*

$$\|\nu_t^*(x, z)\| \leq C \left(1 + \|z\|^{\frac{1}{\underline{m}}}\right),$$

and

$$|f^*(t, x, z)| \leq C \left(1 + \|z\|^{\frac{1+m}{\underline{m}}} + \|z\| \|x\|\right).$$

*Proof.* The proof follows the same line that the proofs of [EP16] or [EMP16].  $\square$

We thus have the following theorem which reduces the problem of the Agent to the solution of the following BSDE

$$Y_t = \xi \cdot \mathbf{1}_N + \int_t^T \left( -\frac{R_A}{2} \|\Sigma_s^\top Z_s\|^2 + f^*(s, X_s, Z_s) \right) ds - \int_t^T Z_s \cdot \Sigma_s dW_s. \quad (3.3)$$

and we refer to Appendix A.2 for its proof.

**Theorem 3.1.** *Let Assumptions  $(\Sigma)$  and  $(\mathbf{H}^{m, \underline{m}})$  be true for some  $(m, \underline{m}) \in [1, +\infty) \times (0, m]$  such that  $\frac{1+m}{\underline{m}} \leq 2$ . There exists a unique solution  $(Y^\xi, Z^\xi)$  in the spaces<sup>5</sup>  $\mathbb{S}(\mathbb{R}) \times \mathbb{H}_{\text{BMO}}^2(\mathbb{R}^N)$  to BSDE (3.3). Moreover, we have  $U_0^A(\xi) = -e^{-R_A Y_0^\xi}$ . Besides, if  $\mathcal{A}_{X, Z^\xi}^* \cap \mathcal{A}$  is non empty, any  $\nu^*(X, Z^\xi) \in \mathcal{A}_{X, Z^\xi}^* \cap \mathcal{A}$  is a best effort for the Agent.*

**Remark 3.1.** *At this stage, one has to justify the assumption  $\mathcal{A}_{X, Z^\xi}^* \cap \mathcal{A} \neq \emptyset$ . This section as to be seen as a general setting to solve an  $N$ -Principals-Agent problem and we aim at providing a general framework to solve it. Even if this assumption seems to be restrictive, a simple case in which this assumption is easily satisfied is  $b(t, x, \nu) = \nu$  and  $c(t, x, \nu) = \frac{\|\nu\|^2}{2}$ . Thus,  $\nu^*(X, Z^\xi) = Z^\xi \in \mathbb{H}_{\text{BMO}}^2 \cap \mathcal{A}^*(X, Z^\xi)$  since  $\Sigma^{-1}$  is bounded under the standing assumption  $(\Sigma)$ . From Proposition A.1, we directly deduce that  $\nu^* \in \mathcal{A}$ .*

## 4 Solving the Principals problem

Similarly to [BW85, (i')], the Problem of the  $i$ th Principal can be rewritten given the contracts  $\xi^j$  provides by the other Principals  $j \neq i$

$$\sup_{\substack{\xi^i \in \mathcal{C} \\ U_0^A(\xi^i \otimes_i \xi^{(-i)}) \geq R_0}} \mathbb{E}^{\nu^*(\xi)} \left[ \mathcal{K}_0^T U_{P_i} \left( \ell_i(X_T) - \xi \cdot \mathbf{1}_N + \sum_{j \neq i} \xi^j \right) \right]. \quad (4.1)$$

The main difficulty here is that we do not have a semimartingale decomposition for the component  $\xi^i$ , from solving the Agent problem. Instead, we only have a such decomposition for the sum of the salaries,  $\xi \cdot \mathbf{1}_N$ . In order to overcome the difficulty, we will address the problem in steps:

<sup>5</sup>We refer to Appendix A.1 for the definitions of these spaces.

- We study the problem of the  $i$ th Principal by assuming that the other Principals all propose  $\xi^j$  ( $j \neq i$ ) with semimartingale decompositions. More precisely, we restrict the study to the following subclass of admissible contracts

$$\tilde{\mathcal{C}}^{(-i)} := \left\{ \xi^i \in \mathcal{C} : \xi^i \otimes_i (\xi^j)_{j \neq i} \in \mathcal{C}_N, \exists (y_j, \alpha^j, \beta^j) \in \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{H}_{BMO}^2(\mathbb{R}^N), \right. \\ \left. \xi^j = y_j + \int_0^T \alpha_s^j ds + \int_0^T \beta_s^j \cdot dX_s, \forall j \neq i \right\}.$$

By solving the Agent problem, we know that the sum of the salaries  $\sum_{j=1}^N \xi^j$  admits a semimartingale decomposition. Therefore, the optimal contract  $\xi^i \in \tilde{\mathcal{C}}^{(-i)}$  provided by the  $i$ th Principal also has a semimartingale decomposition. This decomposition will help us to transfer the problem to a standard stochastic control problem.

- For each principal, the standard stochastic control problem leads to a HJB equation. Therefore, we naturally obtain a system of  $N$  coupled HJB equations associated with the problems of Principals.
- We show that this system of HJB equations provides a Nash equilibrium for the Principals' problems in the class of contracts

$$\tilde{\mathcal{C}}_N := \left\{ \xi \in \mathcal{C}_N, \exists (y, \alpha, \beta) \in \mathbb{D}^0 \times \mathcal{P}(\mathbb{R}^N) \times \mathbb{H}_{BMO}^2(\mathbb{R}^{N,N}), \right. \\ \left. \xi^i = y^i + \int_0^T \alpha_s^i ds + \int_0^T \beta_s^{:,i} \cdot dX_s, \forall 1 \leq i \leq N \right\},$$

where

$$\mathbb{D}^0 := \left\{ y \in \mathbb{R}^N, y \cdot \mathbf{1}_N \geq R := -\frac{\log(-R_0)}{R_A} \right\}.$$

**Remark 4.1.** We would like to comment and explain why this class of contract seems to be reasonable in our model. First of all, this class of contract is non-Markovian and it is well known that in the single Principal-Agent model, as soon as one assume that the dynamic of the output is boosted by some parameter  $\alpha$ , i.e.  $b(t, x, a) := \alpha x + a_t$ , one gets non-Markovian contracts. Thus, there is no reason to consider a subset of  $\tilde{\mathcal{C}}_N$  only containing functions of  $X_T$ .

The second remark is that one could think that since any contract  $\xi^i$  is an  $\mathcal{F}_T$ -measurable random variable with nice integrability property, by considering  $Y_t := \mathbb{E}[\xi^i | \mathcal{F}_t]$ , we get from the martingale representation theorem that there exists some process  $Z$  such that  $\xi^i = \mathbb{E}[\xi^i] + \int_0^T Z_t \cdot dW_s$ . Thus,  $\xi^i$  can be seen as the terminal value of a stochastic integral. However, we will show below that due to the Stackelberg game between the Agent and the Principals, this class of contracts is not robust and does not contain a Nash equilibrium (see Lemma 4.1 and Corollary 4.1 below).

Before going further let us explain more the meaning if the class of contracts  $\tilde{\mathcal{C}}_N$ . Any salary  $\xi^i$  given by the  $i$ th Principal depends on the projects of other Principal  $X^j$ ,  $j \neq i$ . We provide some interpretations of this phenomenon.

**Interpretation of the model** *First, we can assume that the  $i$ th Principal promises to the Agent a part of the value of her firm but also a part of the outcome of an other Principal. Second, some externalities can appear as a network effect (see for instance [Laf89, BS62, LM94] for more explanations with definition of this phenomenon). This fits typically with the introductory example with the beekeeper and crop-owners, since each crop-owner can benefit from the beehives in the other crops through the pollinisation process. An other interpretation is to consider a parent firm and some subsidiaries (the  $N$  Principals) who employed the same Agent. For instance and in concrete terms, this model is quite suitable in the example of the German firm Sonnen GmbH in which Agents are connected to produce, use and share energy.*

**Remark 4.2.** *There is absolutely no reason to get the uniqueness of a Nash equilibrium and some Nash equilibria of our model could be somehow pathologic. This kind of problems was emphasized in [BW86] and remedies had been proposed to avoid some pathologic equilibrium. One can imagine that some great authority (government, NGO, regulation institute) does not allow some Nash equilibria, for instance a government who does not allow a firm to employ an Agent without paying him. This seems to be directly link to the tragedy of commons since a balance effects appears in the different characteristic of optimal contract containing a situation in which a Principal takes all the benefit of the work of the Agent in accordance only with her own self-interest behave.*

#### 4.1 Best response for the $i$ th Principal.

We first focus on the problem of the  $i$ th Principal for any  $i \in \{1, \dots, N\}$ , given that all the others have proposed contracts to the Agent. The main difficulty here is that the results of Section 3 does not provide a suitable decomposition for the contract proposed by only one Principal, but for the sum of every contracts. This differs deeply from the classical case and we would like to emphasize that we will unfortunately not be able to solve the general problem, since in that case the control variable for the problem of only one Principal does not appear clearly. This lead us to restrict our study to the following class of contracts

$$\widetilde{\mathcal{C}}_N := \left\{ \xi \in \mathcal{C}_N, \exists (y, \alpha, \beta) \in \mathbb{D}^0 \times \mathcal{P}(\mathbb{R}^N) \times \mathbb{H}_{BMO}^2(\mathbb{R}^{N,N}), \xi^i = y^i + \int_0^T \alpha_s^i ds + \int_0^T \beta_s^{:,i} \cdot dX_s \right\}.$$

Although this class of contract is strictly include in  $\mathcal{C}_N$ , we will show that there exists in fact a Nash equilibrium in it explicitly.

##### 4.1.1 Equilibrium conditions

We begin with the following lemma ensuring that the considered class of contracts is well-posed. the proof of the lemma is postponed to Appendix A.2

**Lemma 4.1** (Robustness of  $\widetilde{\mathcal{C}}_N$ ). *Let  $\xi^i \in \widetilde{\mathcal{C}}^{(-i)}$ , with fixed  $(y^j, \alpha^j, \beta^j) \in \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathbb{H}_{BMO}^2(\mathbb{R}^N)$ ,  $j \neq i$ . Then,  $\xi^i$  has the semimartingale decomposition:*

$$\xi^i = y_i + \int_0^T \alpha_s^i ds + \int_0^T \beta_s^i \cdot dX_s, \quad (4.2)$$

such that

$$\begin{cases} y_i := Y_0 - \sum_{j \neq i} y_j, \\ \alpha_s^i := G(s, X_s, Z_s) - \sum_{j \neq i} \alpha_s^j, \\ \beta_s^i := Z_s - \sum_{j \neq i} \beta_s^j. \end{cases} \quad (4.3)$$

where  $G(s, x, z) := \frac{R_A}{2} \|\Sigma_s^\top z\|^2 - f^*(s, x, z)$ , and  $(Y, Z)$  is the solution to the BSDE (3.3).

**Remark 4.3.** In view of Lemma 4.1 and in other words, if  $\xi^i \in \tilde{\mathcal{C}}^{(-i)}$  is associated with a sequence of contracts  $(\xi^j)_{j \neq i}$  with semimartingale decompositions, then  $\xi^i \otimes_i (\xi^j)_{j \neq i} \in \tilde{\mathcal{C}}_N$  such that the components of the corresponding  $\alpha$  parameter are the parameters  $\alpha^i$  of each  $\xi^i$ , and the columns of the corresponding parameter  $\beta$  are the parameters  $\beta^i$  of each  $\xi^i$ .

The previous lemma suggests that if we restrict our study to the contracts in  $\tilde{\mathcal{C}}_N$ , we obtain the following equilibrium conditions from (4.3):

$$\begin{cases} \sum_{i=1}^N y_i = Y_0, \\ \sum_{i=1}^N \alpha_s^i = G(s, X_s, \sum_{i=1}^N \beta_s^i) \\ \sum_{i=1}^N \beta_s^i = Z_s. \end{cases} \quad (4.4)$$

We immediately have the following corollary which provides a necessary condition of a Nash equilibrium in the class  $\tilde{\mathcal{C}}_N$ .

**Corollary 4.1** (Necessary condition of Nash equilibrium). *Let  $\xi^* = (\xi^{i,*})_{1 \leq i \leq N}$  be a Nash equilibrium in  $\tilde{\mathcal{C}}_N$  such that  $\xi^{i,*}$  is characterized by the triplet  $(y_0^i, \alpha^{i,*}, \beta^{i,*})$ . Then, the equilibrium condition (4.4) holds for  $(y_0^i, \alpha^{i,*}, \beta^{i,*})_{1 \leq i \leq N}$*

We denote by  $\mathcal{D}$  the set of  $(y, \alpha, \beta) \in \mathbb{D}^0 \times \mathcal{P}(\mathbb{R}^N) \times \mathbb{H}_{BMO}^2(\mathbb{R}^{N,N})$  satisfying the equilibrium condition (4.4). Thus, by setting

$$\mathfrak{C} := \left\{ \xi \in \tilde{\mathcal{C}}_N, \exists (y, \alpha, \beta) \in \mathcal{D}, \xi^i = y^i + \int_0^T \alpha_s^i ds + \int_0^T \beta_s^{i,i} \cdot dX_s, 1 \leq i \leq N \right\},$$

we have the following proposition as a direct consequence of Lemma 4.1.

**Proposition 4.1** (Equilibrium characterization). *The optimal contract in the set  $\mathfrak{C}$  coincides with that in  $\tilde{\mathcal{C}}_N$ .*

#### 4.1.2 Problem of the $i$ th Principal and HJB equation

We now would like to fully characterize a Nash equilibrium. In this heuristic section, we study the problem of the  $i$ th Principal in the set of contracts  $\tilde{\mathcal{C}}^{(-i)}$ . The problem for the  $i$ th Principal is thus given by

$$U_0^{P_i}(x) := \sup_{\substack{\xi^i \in \tilde{\mathcal{C}}^{(-i)} \\ U_0^A(\xi^i \otimes_i (\xi^j)_{j \neq i}) \geq R_0}} \mathbb{E}^{\nu^*} [\mathcal{K}_0^T U_{P_i}(\ell_i(X_T) - \xi^i)]. \quad (4.5)$$

It follows from Theorem 3.1 that the problem (4.5) can be rewritten by

$$U_0^{P_i}(x) = \sup_{\substack{\xi^i \in \tilde{\mathcal{C}}^{(-i)} \\ Y_0^{\xi^i \otimes_i (\xi^j)_{j \neq i}} \geq R_0}} \mathbb{E}^{\nu^*} [\mathcal{K}_0^T U_{P_i}(\ell(X_T) - \xi^i)].$$

Further, it follows from Proposition 4.1 that

$$\begin{aligned} U_0^{P_i}(x) &= \sup_{y_0^i \geq R - \sum_{j \neq i} y_0^j} \sup_{\substack{(\alpha^i, \beta^i) \in \mathcal{P}(\mathbb{R}) \times \mathcal{P}_r(\mathbb{R}^N) \\ \sum_{s=1}^N \alpha_s^i = G(s, X_s, \sum_{i=1}^N \beta_s^i)}} \mathbb{E}^{\nu^*} [\mathcal{K}_0^T U_{P_i}(\ell_i(X_T) - Y_T^{y_0^i, \alpha^i, \beta^i})] \\ &= \sup_{y_0^i \geq R - \sum_{j \neq i} y_0^j} U_0^{P_i}(x, y_0^i), \end{aligned} \quad (4.6)$$

where

$$U_0^{P_i}(x, y_0^i) := \sup_{\beta^i \in \mathcal{P}_r(\mathbb{R}^N)} \mathbb{E}^{\nu^*} [\mathcal{K}_0^T U_{P_i}(\ell(X_T) - Y_T^{y_0^i, \alpha^i, \beta^i})], \quad (4.7)$$

and

$$Y_T^{y_0^i, \alpha^i, \beta^i} = y_0^i + \int_0^T \left( G(s, X_s, S_{\beta_s^{(-i)}} + \beta_s^i) - S_{\alpha_s^{(-i)}} \right) ds + \int_0^T \beta_s^i \cdot dX_s, \quad (4.8)$$

where  $S_{\alpha^{(-i)}} := \sum_{j \neq i} \alpha^j$  and  $S_{\beta^{(-i)}} := \sum_{j \neq i} \beta^j$  and with fixed  $(\alpha_j, \beta_j)_{j \neq i}$ .

The problem of the Principal (4.7) coincides with a stochastic control problem with the following characteristics:

- two state variables: the output  $X$  and the value process  $Y^{y_0^i, \alpha^i, \beta^i}$ ;
- one control variable: the coefficient  $\beta^i$ .

This suggests to introduce the following HJB equation for the  $i$ th Principal with  $\alpha^j, \beta^j$  fixed for all  $j \neq i$ ,

$$\begin{cases} -(\partial_t u^i - k u^i)(t, x, y) - \sup_{\beta^i \in \mathbb{R}^N} H(t, x, y, \nabla_x u^i, \partial_y u^i, \Delta u^i, \partial_{yy}^2 u^i, \partial_{x,y}^2 u^i, S_{\alpha_s^{(-i)}}, S_{\beta_s^{(-i)}}, \beta^i) = 0, \\ u^i(T, x, y) = U_{P_i}(\ell_i(x) - y), \quad (t, x, y) \in [0, T) \times \mathbb{R}^N \times \mathbb{R}. \end{cases} \quad (4.9)$$

with Hamiltonian  $H$  defined for any  $(t, x, p, \tilde{p}, q, \tilde{q}, r, s_a, s_b, \mathbf{b}) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^{N,N} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N$  by

$$\begin{aligned} H(t, x, p, \tilde{p}, q, \tilde{q}, r, s_a, s_b, \mathbf{b}) &:= p \cdot b(t, x, \nu^*(x, \mathbf{b} + s_b)) + \tilde{p} \left( G(s, x, \mathbf{b} + s_b) - s_a + \mathbf{b} \cdot b(t, x, \nu^*(x, \mathbf{b} + s_b)) \right) \\ &\quad + \frac{1}{2} \text{Tr}(\Sigma_t \Sigma_t^\top q) + \frac{1}{2} \text{Tr}(\mathbf{b}^\top \Sigma_t \Sigma_t^\top \mathbf{b} \tilde{q}) + \text{Tr}(\Sigma_t \Sigma_t^\top \mathbf{b} r) \end{aligned}$$

We now consider the following assumption.

**Assumption 4.1.** *There exists a maximizer  $\beta^{i,*}$  for  $\beta \mapsto H(t, x, p, \tilde{p}, q, \tilde{q}, r, s_a, s_b, \beta)$  satisfying some first order condition given by the relation*

$$\lambda^i(t, x, y, p, \tilde{p}, \tilde{q}, r, s_a, s_b, \beta_t^{i,*}) = 0. \quad (4.10)$$

The problem of the Principal is then reduced to solve HJB equation (4.9) as showed in the following proposition, which can be proved as usual with a verification theorem.

**Proposition 4.2 (Verification).** *Let  $\xi \in \tilde{\mathcal{C}}^{(-i)}$  and denote by  $(y^j, \alpha^j, \beta^j)$  the triplet associated with  $\xi^j$ ,  $j \neq i$ . Assume that there exists a solution  $u^i$  to HJB equation (4.9) in  $C^{1,2,2}([0, T] \times \mathbb{R}^N \times \mathbb{R})$ . Then, by denoting  $\beta_s^{i,*}$  the corresponding maximizer, we deduce that  $U_0^{P_i}(x) = u^i(0, x, R - \sum_{j \neq i} y_0^j)$  and the optimal contract  $\xi^{i,*}$  proposed by the  $i$ th Principal is*

$$\xi^{i,*} = R - \sum_{j \neq i} y_0^j + \int_0^T \alpha_s^{i,*} ds + \int_0^T \beta_s^{i,*} \cdot dX_s,$$

with

$$\alpha_s^{i,*} = G \left( s, X_s, \sum_{j \neq i} \beta_s^j + \beta_s^{i,*} \right) - \sum_{j \neq i} \alpha_s^j.$$

## 4.2 System of coupled HJB equations and Nash equilibrium

Similarly to [DJrLS00, Theorems 8.4 and 8.5], one can derive from a fully coupled system of HJB equations a Nash equilibrium in the sense of Definition 2.3.

**Assumption 4.2.** *For any  $i \in \{1, \dots, N\}$  and  $(t, x, y, p^i, \tilde{p}^i, \tilde{q}^i, r^i) \in [0, T] \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$  there exists  $(\alpha^{i,*}(t, x, y, (p^i, \tilde{p}^i, \tilde{q}^i, r^i)_{1 \leq i \leq N}))_{1 \leq i \leq N}$ ,  $\beta^{i,*}(t, x, y, (p^i, \tilde{p}^i, \tilde{q}^i, r^i)_{1 \leq i \leq N})_{1 \leq i \leq N}$  such that for any  $1 \leq i \leq N$*

$$\begin{cases} \lambda^i \left( t, x, y, p^i, \tilde{p}^i, \tilde{q}^i, r^i, \sum_{j \neq i} \alpha^{j,*}, \sum_{j \neq i} \beta^{j,*}, \beta^{i,*} \right) = 0, \\ \sum_{i=1}^N \alpha_t^{i,*} = G \left( t, x, \sum_{i=1}^N \beta_t^{i,*} \right). \end{cases} \quad (4.11)$$

Moreover, we assume that the following system of fully coupled PDEs admits a solution  $u = (u^i)_{1 \leq i \leq N}$  such that  $u^i \in C^{1,2,2}$  for all  $1 \leq i \leq N$ :

$$\begin{cases} \mathcal{L} \left( t, x, y, u^i, \nabla_x u^i, \partial_y u^i, \Delta u^i, \partial_{yy}^2 u^i, \partial_{x,y}^2 u^i, S_\alpha^{(-i)}, S_\beta^{(-i)}, \beta^{i,*} \right) = k_t u^i, \\ u^i(T, x, y) = U_{P_i}(\ell_i(x) - y), \quad (t, x, y) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}. \end{cases} \quad (4.12)$$

with  $\mathcal{L} := \partial_t + H$ ,

$$\begin{aligned} \beta^{i,*} &:= \beta^{i,*} \left( t, x, y, (\nabla_x u^i, \partial_y u^i, \partial_{yy}^2 u^i, \partial_{x,y}^2 u^i)_{1 \leq i \leq N} \right) \\ S_\alpha^{(-i)} &:= \sum_{j \neq i} \alpha^{j,*} \left( t, x, y, (\nabla_x u^i, \partial_y u^i, \partial_{yy}^2 u^i, \partial_{x,y}^2 u^i)_{1 \leq i \leq N} \right) \end{aligned}$$

and

$$S_\beta^{(-i)} := \sum_{j \neq i} \beta^{j,*} \left( t, x, y, (\nabla_x u^i, \partial_y u^i, \partial_{yy}^2 u^i, \partial_{x,y}^2 u^i)_{1 \leq i \leq N} \right).$$



**Theorem 4.1.** *Let Assumption 4.2 be true. Further, define for any  $1 \leq i \leq N$*

$$\xi^{i,*} = y_i + \int_0^T \alpha_s^{i,*} ds + \int_0^T \beta_s^{i,*} \cdot dX_s,$$

with

$$\alpha_t^{i,*} := \alpha^{i,*}(t, x, y, (\nabla_x u^i, \partial_y u^i, \partial_{yy}^2 u^i, \partial_{x,y}^2 u^i)_{1 \leq i \leq N})$$

and

$$\beta_t^{i,*} := \beta^{i,*}(t, x, y, (\nabla_x u^i, \partial_y u^i, \partial_{yy}^2 u^i, \partial_{x,y}^2 u^i)_{1 \leq i \leq N}).$$

Then, the set of contract  $(\xi^{i,*})_{1 \leq i \leq N}$  is an admissible Nash equilibrium.

*Proof of Theorem 4.1.* The proof follows the proof of [DJrLS00, Theorems 8.4 and 8.5] as a direct consequence of both Lemma 4.1 and Proposition 4.1 together with Theorem 4.1 (optimal response in  $\mathfrak{C}$  derives from HJB equation (4.9)).  $\square$

**Remark 4.4.** *For the readers familiar with the BSDE literature, we need to point out that the Principals' problem, as a classical stochastic control problem, can be characterized by the BSDEs. In the general framework proposed in this section, we have the state variable  $Y$  of the dynamic (4.8) in which the control variable  $\beta$  appears in the volatility. Therefore, it is necessary to apply the second order BSDEs. However, in some particular cases, it can be less technically involved. For example, if the Principals are all risk-neutral, then we do not need to consider the  $Y$  component of the state variable. Thus, the Principals' problem can be characterized by a system of BSDEs. Further, one can apply the verification argument related to the BSDEs to obtain the equilibrium result in Theorem 4.1 under (indeed) more general conditions. However, in this paper, we are more interested in explaining our approach through typical examples (see in the next section) than proposing the most general conditions in the abstract context.*

## 5 Application to a model with two correlated Principals with appetite parameters

### 5.1 The bi-Principals model

Let  $\nu := (\nu^0, \nu^1)^\top$  an admissible effort. Let  $W$  be a 2-dimensional Brownian motion and  $\Sigma \in \mathbb{R}^{2 \times 2}$  be an invertible matrix. We assume that

- The drift  $b$  is a linear function of the effort such that  $b(t, x, \nu) := K\nu$ , where  $K$  is a diagonal matrix with coefficients  $k_1, k_2$  on the diagonal which represents the efficiency of the Agent with project 1 and 2.
- There is no discount factor, i.e.  $k = 0$ .
- The map  $c$  is the classical quadratic cost function defined by  $c(t, x, \nu) = \frac{\|\nu_t\|^2}{2}$

Under the probability  $\mathbb{P}^\nu$  we thus have

$$dX_t = K\nu_t dt + \Sigma dW_t^\nu.$$

In this model, by (3.2) and the third line of (4.3), we have the following relation between the optimal effort of the Agent and the volatility coefficients of the contracts:

$$\nu^* = K(\beta^1 + \beta^2). \quad (5.1)$$

We consider in this section that each Principal has a utility depending on her wealth controlled by the Agent and on her performance compared to the wealth of the other. More explicitly, let  $x = (x_1, x_2)^\top \in \mathbb{R}^2$  and  $y \in \mathbb{R}$ , and take  $\ell_i(x) := (1 + \gamma_i)x_i - \gamma_i x_j$ ,  $1 \leq i \neq j \leq 2$  with appetite parameters  $\gamma_1, \gamma_2 \in [0, 1]$ . Let  $\Gamma := (1 + \gamma_1 - \gamma_2, 1 + \gamma_2 - \gamma_1)^\top$ . Further, we assume in this example that both Principals are risk-neutral, that is,  $U_{P_1} = U_{P_2} = I$ , the identity function. Therefore, it follows from (4.7) and (4.8) that the value functions read:

$$\begin{aligned} U_0^{P_i}(x) &= \sup_{\beta^i} \mathbb{E}^{\nu^*} \left[ \ell(X_T) - Y_T^{0, \alpha^i, \beta^i} \right] \\ &= \sup_{\beta^i} \mathbb{E}^{\nu^*} \left[ \ell(X_T) - \int_0^T \left( G(s, X_s, \beta_s^1 + \beta_s^2) - S_{\alpha_s^{(-i)}} + K\nu_s^* \cdot \beta_s^i \right) ds \right], \end{aligned}$$

so that by using (5.1)

$$\begin{aligned} U_0^{P_i}(x) &= \sup_{\beta^i} \mathbb{E}^{\nu^*} \left[ \ell(X_T) - \int_0^T \left( \frac{R_A}{2} \|\Sigma^\top(\beta_s^1 + \beta_s^2)\|^2 - \frac{\|K(\beta_s^1 + \beta_s^2)\|^2}{2} - S_{\alpha_s^{(-i)}} + K^2(\beta_s^1 + \beta_s^2) \cdot \beta_s^i \right) ds \right] \end{aligned}$$

A crucial step to obtain the equilibrium result in Theorem 4.1 is to check whether Assumption 4.2 holds true. In this model, we verify the conditions through the following lemma.

**Lemma 5.1.** *There exist  $(\alpha^{1,*}, \beta^{1,*})$  and  $(\alpha^{2,*}, \beta^{2,*})$  such that the following system of PDEs admits a unique solution  $(v^1, v^2)$  in  $\mathcal{C}^{1,2}([0, T] \times \mathbb{R}^2)$*

$$\begin{cases} -\partial_t v^1(t, x) - \left\{ \nabla_x v^1 \cdot K^2(\beta_t^{1,*} + \beta_t^{2,*}) + \frac{1}{2} \text{Tr} \left( \Delta_x v^1 \Sigma \Sigma^\top \right) - \frac{R_A}{2} \|\Sigma^\top(\beta_t^{1,*} + \beta_t^{2,*})\|^2 \right. \\ \quad \left. + \frac{\|K(\beta_t^{1,*} + \beta_t^{2,*})\|^2}{2} + \alpha_t^{2,*} - K^2(\beta_t^{1,*} + \beta_t^{2,*}) \cdot \beta_t^{1,*} \right\} = 0 \\ v^1(T, x) = (1 + \gamma_1)x_1 - \gamma_1 x_2, \end{cases} \quad (5.2)$$

and

$$\begin{cases} -\partial_t v^2(t, x) - \left\{ \nabla_x v^2 \cdot K^2(\beta_t^{1,*} + \beta_t^{2,*}) + \frac{1}{2} \text{Tr} \left( \Delta_x v^2 \Sigma \Sigma^\top \right) - \frac{R_A}{2} \|\Sigma^\top(\beta_t^{1,*} + \beta_t^{2,*})\|^2 \right. \\ \quad \left. + \frac{\|K(\beta_t^{1,*} + \beta_t^{2,*})\|^2}{2} + \alpha_t^{1,*} - K^2(\beta_t^{1,*} + \beta_t^{2,*}) \cdot \beta_t^{2,*} \right\} = 0 \\ v^2(T, x) = (1 + \gamma_2)x_2 - \gamma_2 x_1, \end{cases} \quad (5.3)$$

and that the first order conditions (4.11) hold true, that is,

$$\begin{cases} \beta^{1,*} = (K^2 + R_A \Sigma \Sigma^\top)^{-1} (K^2 \nabla_x v^1 - R_A \Sigma \Sigma^\top \beta^{2,*}) \\ \beta^{2,*} = (K^2 + R_A \Sigma \Sigma^\top)^{-1} (K^2 \nabla_x v^2 - R_A \Sigma \Sigma^\top \beta^{1,*}) \\ \alpha^{1,*} + \alpha^{2,*} = \frac{R_A}{2} \|\Sigma^\top(\beta_t^{1,*} + \beta_t^{2,*})\|^2 - \frac{K^2 \|\beta_t^{1,*} + \beta_t^{2,*}\|^2}{2}. \end{cases} \quad (5.4)$$

Obviously, the equations (5.2) and (5.3) are the HJB equations corresponding to the control problem of Principal 1 and 2, respectively. Before giving the proof of the lemma, we first admit the result and show the main equilibrium result.

**Proposition 5.1.** *Optimal contracts in the class  $\tilde{\mathcal{C}}_2$  are given by*

$$\begin{aligned}\xi^{1,*} &= y_0^1 + \int_0^T \alpha_s^{1,*} ds + \int_0^T \beta_s^{1,*} \cdot dX_s \\ \xi^{2,*} &= y_0^2 + \int_0^T \alpha_s^{2,*} ds + \int_0^T \beta_s^{2,*} \cdot dX_s,\end{aligned}$$

where the parameters satisfy the following Nash-equilibrium conditions

$$\mathbf{(Eq-Na)} \begin{cases} y_0^1 + y_0^2 = R_0 \\ \beta_t^{1,*} + \beta_t^{2,*} = M\Gamma, \\ \alpha_t^{1,*} + \alpha_t^{2,*} = \frac{R_A}{2} \|\Sigma_s^\top M\Gamma\|^2 - \frac{\|KM\Gamma\|^2}{2}, \end{cases}$$

with  $M := (2R_A \Sigma \Sigma^\top + K^2)^{-1} K^2$ . In particular, the optimal effort  $\nu^*$  of the Agent is given by

$$\nu^* = KM\Gamma.$$

*Proof.* We have verified the conditions in Assumption 4.2 through Lemma 5.1. Therefore, according to Theorem 4.1, we know that the optimal contracts have the forms of the corresponding semimartingales. Now, define a function  $V = v^1 + v^2$ . It follows from (5.2), (5.3) and the third line of (5.4) that the function  $V$  satisfies the following PDE:

$$\begin{cases} -\partial_t V(t, x) - \frac{1}{2} \text{Tr} \left( \Sigma \Sigma^\top \Delta_x V \right) - \nabla_x V \cdot K^2 (\beta_t^{1,*} + \beta_t^{2,*}) \\ \quad + \frac{R_A}{2} \|\Sigma^\top (\beta^{1,*} + \beta^{2,*})\|^2 + \frac{\|K(\beta^{1,*} + \beta^{2,*})\|^2}{2} = 0 \\ V(T, x) = x \cdot \Gamma, \end{cases} \quad (5.5)$$

Further, by (5.4) we may calculate that

$$\begin{cases} \beta^{1,*} + \beta^{2,*} = M \nabla_x V, \\ \alpha^{1,*} + \alpha^{2,*} = \frac{R_A}{2} \|\Sigma_s^\top M \nabla_x V\|^2 - \frac{\|KM \nabla_x V\|^2}{2} \end{cases} \quad (5.6)$$

Replacing  $\beta^{1,*} + \beta^{2,*}$  by  $M \nabla_x V$  in (5.5), we obtain

$$\begin{cases} -\partial_t V(t, x) - \frac{1}{2} \text{Tr} \left( \Sigma \Sigma^\top \Delta_x V \right) - \nabla_x V \cdot K^2 M \nabla_x V + \frac{R_A}{2} \|\Sigma^\top M \nabla_x V\|^2 + \frac{\|KM \nabla_x V\|^2}{2} = 0 \\ V(T, x) = x \cdot \Gamma, \end{cases} \quad (5.7)$$

It is easy to observe that PDE (5.7) has the unique smooth solution

$$V(t, x) = x \cdot \Gamma + \lambda(T - t), \quad \text{with} \quad \lambda = \Gamma \cdot K^2 M \Gamma - \frac{R_A}{2} \|\Sigma^\top M \Gamma\|^2 - \frac{\|KM \Gamma\|^2}{2} \quad (5.8)$$

where  $\lambda$  is a constant depending on  $R_A$  and  $M$ . In particular,  $\nabla_x V = \Gamma$ . Finally, the equilibrium result **(Eq-Na)** follows from (5.6), and by (5.1) we have  $\nu^* = KM\Gamma$ .  $\square$

We observe in the previous proof that the optimal pair  $(\beta^{1,*}, \beta^{2,*})$  should satisfy

$$\beta^{1,*} + \beta^{2,*} = M\Gamma.$$

Together with the desired first order condition (5.4), we should have

$$\begin{cases} \beta^{1,*} = \nabla_x v^1 - R_A(K^2)^{-1}\Sigma\Sigma^\top M\Gamma, \\ \beta^{2,*} = \nabla_x v^2 - R_A(K^2)^{-1}\Sigma\Sigma^\top M\Gamma. \end{cases} \quad (5.9)$$

In order to prove Lemma 5.1, it remains to show that the system of PDEs (5.2), (5.3) with such coefficients  $(\beta^{1,*}, \beta^{2,*})$  admits unique smooth solutions.

**Proof of Lemma 5.1** Let  $(\beta^{1,*}, \beta^{2,*})$  be as in (5.9) and  $\alpha^{1,*} = \alpha^{2,*} := \frac{R_A}{4}\|\Sigma_s^\top M\Gamma\|^2 - \frac{\|KM\Gamma\|^2}{4}$ , and plug them into the equations (5.2) and (5.3). Denote

$$\tilde{\lambda} := \Gamma \cdot K^2 M\Gamma - \frac{5R_A}{4}\|\Sigma_s^\top M\Gamma\|^2 - \frac{3}{4}\|KM\Gamma\|^2,$$

one may check that we have in fact  $\tilde{\lambda} = \frac{1}{2}\lambda$ . Define

$$v^1(t, x) := \tilde{\lambda}(T - t) + (1 + \gamma_1, -\gamma_1)x, \quad v^2(t, x) := \tilde{\lambda}(T - t) + (-\gamma_2, 1 + \gamma_2)x.$$

Then, it is easy to verify that  $v^1$  and  $v^2$  defined as above are the unique smooth solutions to (5.2) and (5.3), respectively.  $\square$

**Remark 5.1.** Notice that the choice of  $\alpha^{1,*}$  and  $\alpha^{2,*}$  in the previous proof is somehow arbitrary. Indeed, any constant process  $\alpha^{i,*}$  and  $\alpha^{j,*} := \frac{R_A}{2}\|\Sigma_s^\top M\Gamma\|^2 - \frac{\|KM\Gamma\|^2}{2} - \alpha^{i,*}$  with  $1 \leq i \neq j \leq 2$  is acceptable for the proof of Lemma 5.1. This can be easily explained by the fact that we have a degree of freedom on the choice of on parameter  $\alpha$  which provided infinitely many combinations of optimal contracts proposes by the two Principals.

## 5.2 Comparison with the model with the aggregated offer

In this section, we compare the competitive common agency example studied in the previous section with the model in which the Principals can be aggregated, that is, the problem can be reduced to one single Principal-Agent model as mentioned at the end of Section 2.2.1. In this case, the Agent problem can still be solved through the same BSDE, that is, there exists  $(Y_0, Z) \in [R_0, +\infty) \times \mathbb{H}_{BMO}^2(\mathbb{R}^2)$  such that

$$\begin{aligned} \xi &= Y_0 + \int_0^T Z_s \cdot \Sigma dW_s + \int_0^T \left( \frac{R_A}{2}\|\Sigma_s^\top Z_s\|^2 + \frac{\|KZ_s\|^2}{2} - KZ_s \cdot Z_s \right) ds \\ &= Y_0 + \int_0^T Z_s \cdot \Sigma dW_s^* + \int_0^T \left( \frac{R_A}{2}\|\Sigma_s^\top Z_s\|^2 + \frac{\|KZ_s\|^2}{2} \right) ds, \end{aligned}$$

where  $W^*$  is a  $\mathbb{P}^{\nu^*}$ -Brownian motion. The Principal manages to do the following optimization:

$$U_0 = \sup_Z \mathbb{E}^* [(1 + \gamma_1 - \gamma_2)X_T^1 + (1 + \gamma_2 - \gamma_1)X_T^2 - \xi]$$

As before, we can calculate the optimal control:

$$Z^* = M_a \Gamma, \quad \text{with} \quad M_a := \left( R_A \Sigma \Sigma^\top + K^2 \right)^{-1} K^2.$$

Further, the optimal effort of the Agent should be

$$\nu_{P_f}^* = KM_a\Gamma.$$

By comparing the value of  $\nu_{P_f}^*$  and that of  $\nu^*$  in Proposition 5.1, we immediately have the following conclusion.

**Proposition 5.2.** *The effort of the common Agent in the competitive model coincides with the effort of the common Agent in the aggregated model if and only if the Agent is risk-neutral, that is,  $R_A = 0$ .*

**Link with the results of [BW86] for the discrete-case model.** Recall that if the Agent is risk-neutral, it is well-known that in the single Principal-Agent problem, the effort in the first best case coincides with that in the second best case (see for instance [LM09, Proposition 4.1]). Indeed, as an extension of it in our particular model, if one computes the first best effort, one has to solve

$$U_0^{FB} = \sup_{\nu, \xi} \mathbb{E}^\nu \left[ X_T \cdot \Gamma - \xi \cdot \mathbf{1}_2 - \rho e^{-R_A(\xi \cdot \mathbf{1}_2 - \int_0^T \frac{\|\nu_s\|^2}{2} ds)} \right],$$

with a Lagrange multiplier  $\rho > 0$  ensuring that the Agent receives his utility reservation  $R_0 < 0$ . In this case, by using Gâteaux derivative to characterize the optimal  $\xi$  (see for instance the method used in [EP16]) one gets after an easy computation the following optimizers

$$\begin{aligned} \nu_t^{*,FB} &:= K\Gamma, \\ \xi^* \cdot \mathbf{1}_2 &= K\Gamma - \frac{1}{R_A} \log(-R_0). \end{aligned}$$

Therefore, we have proved that the 'second best' effort of a risk-neutral Agent in the competitive model coincides with the 'first best' effort. This is exactly an extension of the result in [BW86] in the discrete-case.

**Comparison of optimal remunerations.** In the common agency model we have seen that the remuneration given to the Agent is

$$\xi_1^* + \xi_2^* = R_0 + T \underbrace{\left( \frac{R_A}{2} \|\Sigma^\top M\Gamma\|^2 + \frac{\|KM\Gamma\|^2}{2} \right)}_{=: \delta} + \int_0^T M\Gamma \cdot \Sigma dW_s^*.$$

In the aggregated model, we recall that

$$\xi^* := R_0 + T \underbrace{\left( \frac{R_A}{2} \|\Sigma^\top M_a\Gamma\|^2 + \frac{\|KM_a\Gamma\|^2}{2} \right)}_{=: \delta_a} + \int_0^T M_a\Gamma \cdot \Sigma dW_s^*$$

Let  $\Sigma := I_2$ , then, we get

$$M = \begin{pmatrix} \frac{k_1^2}{2R_A + k_1^2} & 0 \\ 0 & \frac{k_2^2}{2R_A + k_2^2} \end{pmatrix}$$

and

$$M_a = \begin{pmatrix} \frac{k_1^2}{R_A + k_1^2} & 0 \\ 0 & \frac{k_2^2}{R_A + k_2^2} \end{pmatrix}$$

In this particular model, we have  $\delta_a \geq \delta$ , *i.e.* the non-risk part of the remuneration is higher for the Agent if he is employed by the aggregated firm (the parent firm), comparing to the case he is hired by two different firms. This is curious, because intuitively the Agent should receive higher return if the firms compete with each other. However, one must note that the Agent works more ( $\|\nu_{Pf}^*\|_1 \geq \|\nu^*\|_1$ ) in the aggregated model which could explained this effect.

### 5.3 Impact of the appetite, efficiency and correlation parameters

Let  $\rho \in [-1, 1]$  be a correlation parameter such that

$$\Sigma := \begin{pmatrix} 1 & 0 \\ \rho & \sqrt{1 - \rho^2} \end{pmatrix}, \quad K := \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$$

In this case, we get after a (tedious but easy) computation

$$\begin{aligned} \nu^{1,*} &= \frac{2R_A k_1 ((1 + \gamma_2 - \gamma_1)k_2^2 \rho - (1 + \gamma_1 - \gamma_2)k_1^2) - k_1^3 k_2^2 (1 + \gamma_1 - \gamma_2)}{2R_A^2 (\rho^2 - 1) - 2R_A (k_1^2 + k_2^2) - k_1^2 k_2^2} \\ \nu^{2,*} &= \frac{2R_A k_2 ((1 + \gamma_1 - \gamma_2)k_1^2 \rho - (1 + \gamma_2 - \gamma_1)k_2^2) - k_1^2 k_2^3 (1 + \gamma_2 - \gamma_1)}{2R_A^2 (\rho^2 - 1) - 2R_A (k_1^2 + k_2^2) - k_1^2 k_2^2} \end{aligned}$$

**Impact of the correlation and the appetite parameters.** Assume that  $k := k^1 = k^2$ . In this case

$$\begin{aligned} \nu^{1,*}(\rho) &:= \frac{2R_A k^3 ((1 + \gamma_2 - \gamma_1)\rho - (1 + \gamma_1 - \gamma_2)) - k^5 (1 + \gamma_1 - \gamma_2)}{2R_A^2 (\rho^2 - 1) - 4R_A k^2 - k^4} \\ \nu^{2,*}(\rho) &:= \frac{2R_A k^3 ((1 + \gamma_1 - \gamma_2)\rho - (1 + \gamma_2 - \gamma_1)) - k^5 (1 + \gamma_2 - \gamma_1)}{2R_A^2 (\rho^2 - 1) - 4R_A k^2 - k^4} \end{aligned}$$

Note that

$$\begin{aligned} \nu^{1,*}(\rho) - \nu^{2,*}(\rho) &= \frac{4R_A k^3 ((\gamma_2 - \gamma_1)(\rho + 1)) - 2k^5 (\gamma_1 - \gamma_2)}{2R_A^2 (\rho^2 - 1) - 4R_A k^2 - k^4} \\ &= (\gamma_1 - \gamma_2) \frac{-4R_A k^3 (\rho + 1) - 2k^5}{2R_A^2 (\rho^2 - 1) - 4R_A k^2 - k^4}. \end{aligned}$$

Thus as soon as  $\gamma_1 > \gamma_2$ , we get  $\nu^{1,*}(\rho) > \nu^{2,*}(\rho)$  so that the Agent works more for the more ambitious principal. Moreover, we have

$$d(\rho) := \frac{\nu^{1,*}}{\overline{\nu^*}}(\rho) - \frac{\nu^{2,*}}{\overline{\nu^*}}(\rho) = (\gamma_1 - \gamma_2) \frac{2R_A k^3 (1 + \rho) + k^5}{2R_A k^3 (1 - \rho) + k^5},$$

with  $\overline{\nu^*} := \nu^{1,*} + \nu^{2,*}$ .

By noticing that the function  $\rho \in [-1, 1] \mapsto d(\rho)$  is convex and increasing, we deduce that the difference between the proportion of effort given by the Agent with the more ambitious principal and the proportion of effort given by the Agent with the less ambitious principal increases with

the parameter  $\rho$ . In other words, the more the projects are correlated, the more the proportion of effort given by the Agent is impacted by the difference of the ambition parameters of each Principal. Besides, the convexity shows that the sensibility of the proportion of work given by the Agent for each principal increases with the correlation parameter. The more the projects are correlated, the more a little variation of the correlation has a big effect on the difference of proportion of effort given by the Agent.

**Risk-Neutral Agent.** Assume that  $\Sigma = \text{Id}_2$ ,  $R_A = 0$ . In this case,  $M = \text{Id}_2$  and  $\nu^* = K\Gamma$ . In other work we have

$$\begin{aligned}\nu^{1,*} &= k_1(1 + \gamma_1 - \gamma_2) \\ \nu^{2,*} &= k_2(1 + \gamma_2 - \gamma_1).\end{aligned}$$

The Agent works more for the Principal 1 if and only if

$$k_1(1 + \gamma_1 - \gamma_2) > k_2(1 + \gamma_2 - \gamma_1). \quad (5.10)$$

Let now  $x := \gamma_2 - \gamma_1$ .

- First, we note some intuitive results. If the two Principals have the same ambition parameters, (*i.e.*  $\gamma_1 = \gamma_2$ ), the Agent prefers working with the Principal with whom he is more efficient. Similarly, if the two Principals have the efficiency parameters, (*i.e.*  $k_1 = k_2$ ), the Agent works more for the more ambitious Principal.
- Assume now for instance that  $\gamma_1 = 0$ ,  $\gamma_2 = 1$ . Then condition (5.10) is never satisfies and the Agent does not work for Principal 1 whatever is her performance. It means that the Agent does not work for the Principal indifferent to the competition, while the other Principal is every competitive.
- Assume now that  $\gamma_1 \in [0, 1]$  and  $\gamma_2 \in [0, 1)$ . Denote  $x := \gamma_2 - \gamma_1 \in [-1, 1)$ . The domain  $x \leq 0$  coincides with the situation where the Principal 1 is more ambitious that the Principal 2 (and conversely, Principal 1 is less ambitious than Principal 2 when  $x \geq 0$ ). Condition (5.10) can be rewritten

$$\frac{k_1}{k_2} > \frac{1+x}{1-x}, \quad x \in [-1, 1).$$

Let  $f(x) := \frac{1+x}{1-x}$ ,  $x \in [-1, 1)$ , then  $f$  is clearly increasing and convex.

Let us provide an interpretation of the growth of  $f$ . When  $x$  is bigger than 0, we notice that  $f(x)$  is bigger than 1, which shows that a lack of ambition the Principal 1, *i.e.*  $\gamma_1 < \gamma_2$ , can be balanced with the efficiency parameter of her since it exists a domain of  $x \geq 0$ , *i.e.* for which the Principal 1 is less ambitious than the Principal 2 but such that the Agent works for the less ambitious Principal 1.

Turn now to the convexity of  $f$ . This phenomenon is quite interesting since it suggests that when a Principal is clearly more ambitious than the other, for instance  $\gamma_1 \ll \gamma_2$ , a little modification of the ambition parameters leads to a high variation of the quotient between the effort of the Agent for the Principal 1 and for the Principal 2. In other words, if the Principal 1 is clearly less ambitious than the Principal 2, derived lightly from this state leads to

a big difference of the efforts provides by the Agent to manage the project of the Principal 1 and the Principal 2. If for instance the Principal 1 increases a little her ambition, the Agent will manage quite more her project. There is a kind of leverage effect between a deviation of an initial ambition parameter and the quantity of work provides to the Agent, when the two Principal have very different behaviours concerning their relative performances. Besides, a little modification of the ambition parameter of a Principal significantly more indifferent than the other with fixed ambition parameter, increases or decreases meaningfully the range of possible efficiency parameters to have an Agent more devoted to her project.

**Remark 5.2.** *As explained in the section above with Proposition 5.2, this case can be in fact reduced to the classical second best case (with one Principal), by aggregating the Principals. Thus, we recover the classical interpretations by adding efficiency parameter  $\gamma_i$  for project  $i$  with  $i = 1, 2$ .*

## 6 Conclusion

In this paper, we show that in the common agency problem, the value functions of the Principals should be the solutions of a system of HJB equations. Meanwhile, the coefficients of the system should satisfy the Nash equilibrium conditions. In particular, We study a model in which two Principals hire a common Agent, and we obtain one of the main results in [BW86], that is, the outcome with two non-cooperative employers and that with an aggregated employer coincide only when the Agent is risk-neutral.

At the opposite of common agency problem, other works have investigated a competition problem between two principals who want the exclusive service of an agent. In 1976, Rothschild and Stiglitz [RS76] have studied insurance markets in which principals are identical and compete for a single agent. It has then been extended by Biglaiser and Mezzetti in [BM93] in both moral hazard and adverse selection cases. This problem is not considered here, because we allows the agents to be remunerated by all the Principals, and it will be left for future researches.

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# A Notations and Technical details

## A.1 General notations

We fix throughout the paper a time horizon  $T > 0$  and  $N$  a positive integer. We denote by  $\|\cdot\|$  the Euclidian norm in  $\mathbb{R}^N$  and by  $\cdot$  the inner product. For any positive integers  $n$  and  $m$ , we identify  $\mathbb{R}^{n \times m}$  with the space of real matrices with  $n$  rows and  $m$  columns, endowed with the Euclidean norm on  $\mathbb{R}^{n \times m}$ . Let  $M$  be in  $\mathbb{R}^{n \times m}$ ,  $1 \leq j \leq n$  and  $1 \leq \ell \leq m$ , we denote by  $M^{j,\cdot} \in \mathbb{R}^{1 \times m}$  (resp.  $M^{\cdot,\ell} \in \mathbb{R}^{n,1}$ ) its  $j$ -th row (resp. its  $\ell$ -th column). We set  $M^\top \in \mathbb{R}^{m \times n}$  to be the transpose of  $M$ . We also identify  $\mathbb{R}^N$  with  $\mathbb{R}^{1,N}$  and for any element  $x$  of  $\mathbb{R}^N$ , we denote by  $x^i$  its  $i$ th component and  $x^{(-i)}$  an  $\mathbb{R}^{N-1}$  dimensional vector such that  $x^{(-i)} := (x^1, \dots, x^{i-1}, x^{i+1}, \dots, x^N)^\top$ . For any vector  $x^* \in \mathbb{R}^N$  and to allege the notations in the indexation, we denote in all this paper  $x^{i,*}$  the  $i$ th component of  $x^*$  and similarly  $x^{(-i),*} := (x^*)^{(-i)}$ . We set  $\otimes_i$  as the concatenate operator at rank  $i$  such that

$$\begin{aligned} \mathbb{R} \times \mathbb{R}^{N-1} &\longrightarrow \mathbb{R}^N \\ (a, y) &\longmapsto a \otimes_i y := (y_1, \dots, y^{i-1}, a, y^i, \dots, y^{N-1})^\top. \end{aligned}$$

Let  $\mathbf{1}_N$  be the element of  $\mathbb{R}^N$  defined by  $\mathbf{1}_N := (1, \dots, 1)^\top$ .

For any finite dimensional normed space  $(E, \|\cdot\|_E)$ ,  $\mathcal{P}(E)$  (resp.  $\mathcal{P}_r(E)$ ) will denote the set of  $E$ -valued,  $\mathbb{F}$ -adapted integrable processes (resp.  $\mathbb{F}$ -predictable processes) and for any  $p \geq 1$

$$\begin{aligned} \mathbb{S}^p(E) &:= \left\{ Y \in \mathcal{P}(E), \text{ càdlàg, such that } \|Y\|_{\mathbb{S}^p(E)}^p := \mathbb{E} \left[ \sup_{t \in [0, T]} \|Y_t\|_E^p \right] < +\infty \right\}, \\ \mathbb{H}^p(E) &:= \left\{ Z \in \mathcal{P}_r(E), \|Z\|_{\mathbb{H}^p}^p := \mathbb{E} \left[ \left( \int_0^T \|Z_t\|_E^2 dt \right)^{p/2} \right] < +\infty \right\}. \end{aligned}$$

Let  $Z \in \mathcal{P}_r(E)$ , we set  $\mathcal{E} \left( \int_0^\cdot Z_s \cdot dW_s \right)$  as the stochastic exponential of  $\int_0^\cdot Z_s \cdot dW_s$  defined by

$$\mathcal{E} \left( \int_0^\cdot Z_s \cdot dW_s \right) := \exp \left( \int_0^\cdot Z_s \cdot dW_s - \frac{1}{2} \int_0^\cdot \|Z_s\|_E^2 ds \right).$$

We also denote  $\mathbb{S}(E) := \bigcup_{p \geq 1} \mathbb{S}^p(E)$  and  $\mathbb{H}(E) := \bigcup_{p \geq 1} \mathbb{H}^p(E)$ .

We recall the notion of Bounded mean oscillation (BMO for short) martingale.

**Definition A.1** ([Kaz94]). *For any process  $Z \in \mathbb{H}^2(E)$ , we say that  $\int_0^\cdot Z_s \cdot dW_s$  is a BMO martingale if there exists a non negative constant  $C$  such that for any stopping time  $\tau \leq T$  we have*

$$\mathbb{E} \left[ \int_\tau^T \|Z_s\|^2 ds \right] \leq C^2.$$

We denote this space by  $\mathbb{H}_{BMO}^2(E)$ .

In particular, we have the following result

**Proposition A.1.** [Kaz94, Theorem 2.3] *If  $\int_0^\cdot Z_s \cdot dW_s$  is a BMO-martingale, then  $\mathcal{E} \left( \int_0^\cdot Z_s \cdot dW_s \right)$  is a uniformly integrable martingale.*

## A.2 Technical proofs

*Proof of Theorem 3.1.* The fact that BSDE (3.3) admits a unique solution  $(Y^\xi, Z^\xi)$  in  $\mathcal{S}(\mathbb{R}) \times \mathbb{H}_{\text{BMO}}^2(\mathbb{R}^N)$  using Definition 2.2 of admissible contract  $\xi$  is a direct consequence of the results in [BH06] under Assumption  $(\Sigma)$ . By noticing that for any  $\nu^* \in \mathcal{A}_{X, Z^\xi}^*$ , the component  $Y_t^\xi$  coincides with the first component  $Y_t^{\nu^*(X, Z^\xi)}$  of the solution to BSDE (3.1), we directly deduce from comparison theorem (see for instance [BH06]) that  $Y_0^\xi = U_0^A(\xi)$  and any  $\nu^*(X, Z^\xi) \in \mathcal{A}_{X, Z^\xi}^* \cap \mathcal{A}$  is a best reaction effort for the Agent.  $\square$

*Proof of Lemma 4.1.* Let  $\xi \in \tilde{\mathcal{C}}^{(-i)}$ , with fixed  $(y^j, \alpha^j, \beta^j) \in \mathbb{R} \times \mathcal{P}(\mathbb{R}) \times \mathcal{P}_r(\mathbb{R}^N)$ ,  $j \neq i$ . We know that there exists  $(Y_0, Z) \in [R_0, +\infty) \times \mathbb{H}_{\text{BMO}}^2(\mathbb{R}^N)$  such that

$$X_T = x + \int_0^T b(s, X_s, \nu^*(X_s, Z_s)) ds + \int_0^T \Sigma_s dW_s^*,$$

and

$$\xi^i = Y_0 - \sum_{j \neq i} y_j + \int_0^T (Z_s - \sum_{j \neq i} \beta_s^j) \cdot \Sigma_s dW_s + \int_0^T F(s, X_s, Z_s, \sum_{j \neq i} \alpha_s^j) ds,$$

with for any  $(s, x, z, s_a) \in [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}$

$$F(s, x, z, s_a) := \frac{R_A}{2} \|\Sigma_s^\top z\|^2 - f^*(s, x, z) - s_a$$

Thus,

$$\xi^i = y_i + \int_0^T \alpha_s^i ds + \int_0^T \beta_s^i \cdot dX_s,$$

with

$$\begin{cases} y_i := Y_0 - \sum_{j \neq i} y_j, \\ \alpha_s^i := G(s, X_s, Z_s) - \sum_{j \neq i} \alpha_s^j, \\ \beta_s^i := Z_s - \sum_{j \neq i} \beta_s^j. \end{cases}$$

$\square$