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► To cite this version:

Nermin Salepci, Jean-Yves Welschinger. Asymptotic Topology of Random Subcomplexes in a Finite Simplicial Complex. International Mathematics Research Notices, 2019, 2019 (17), pp.5219–5255. 10.1093/imrn/rnx272 . hal-01534296

HAL Id: hal-01534296

<https://hal.science/hal-01534296>

Submitted on 7 Jun 2017

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Asymptotic topology of random subcomplexes in a finite simplicial complex

Nermin Salepci and Jean-Yves Welschinger

June 7, 2017

Abstract

We consider a finite simplicial complex K together with its successive barycentric subdivisions $\text{Sd}^d(K)$, $d \geq 0$, and study the expected topology of a random subcomplex in $\text{Sd}^d(K)$, $d \gg 0$. We get asymptotic upper and lower bounds for the expected Betti numbers of those subcomplexes, together with the average Morse inequalities and expected Euler characteristic.

Keywords : simplicial complex, barycentric subdivisions, Euler characteristic, Betti numbers, triangulations, random variable.

MATHEMATICS SUBJECT CLASSIFICATION 2010: 52C99, 60C05, 60B05

1 Introduction

Let K be a locally finite simplicial complex of dimension n and $\text{Sd}(K)$ be its first barycentric subdivision. Let $\mathcal{C}^{k-1}(K)$ be the group of $(k-1)$ -dimensional simplicial cochains of K with $\mathbb{Z}/2\mathbb{Z}$ -coefficients, $k \in \{1, 2, \dots, n\}$. For every $\epsilon \in \mathcal{C}^{k-1}(K)$, we denote by V_ϵ the subcomplex of $\text{Sd}(K)$ dual to the cocycle $d\epsilon$, where $d : \mathcal{C}^{k-1}(K) \rightarrow \mathcal{C}^k(K)$ denotes the coboundary operator, see [10]. Recall that simplices of $\text{Sd}(K)$ of dimension $i \in \{0, 1, \dots, n\}$ are of the form $[\hat{\sigma}_0, \dots, \hat{\sigma}_i]$, where $\hat{\sigma}_j$ denotes the barycenter of the simplex $\sigma_j \in K$ for all $j \in \{0, \dots, i\}$ and $\sigma_j < \sigma_{j+1}$, that is σ_j is a proper face of σ_{j+1} for all $j \in \{0, \dots, i-1\}$. A simplex thus belongs to V_ϵ if and only if it is a face of a simplex $[\hat{\sigma}_0, \dots, \hat{\sigma}_i]$ such that $\dim \sigma_0 = k$ and $\langle d\epsilon, \sigma_0 \rangle \neq 0$. The latter condition means that ϵ must take value 1 on an odd number of facets of σ_0 , see Figure 1. In other words, V_ϵ is the union of the blocks $\overline{D}(\sigma_0)$ dual to the simplices $\sigma_0 \in K$ such that $\langle d\epsilon, \sigma_0 \rangle \neq 0$, see Section 2 and [10].

When $k = 1$ and K is the moment polytope of some toric manifold equipped with a convex triangulation, the pair (K, V_ϵ) gets homeomorphic to the pair (K, V'_ϵ) , where V'_ϵ is the patchwork (tropical) hypersurface defined by O. Viro, see Proposition 7 and [12, 13].

Of special interest are triangulations of compact (topological) manifolds. However, when $k > 1$ and $\epsilon \in \mathcal{C}^{k-1}(K)$, V_ϵ does not inherit the structure of a triangulated codimension k submanifold, see Remark 13. When $k = 1$, we prove the following theorem (see Corollary 11).

Theorem 1 *Let K be a triangulated homology n -manifold. Then, for every $\epsilon \in \mathcal{C}^0(K)$, V_ϵ is a triangulated homology $(n-1)$ -manifold. Moreover, if K is a PL-triangulation of a topological n -manifold, then for every $\epsilon \in \mathcal{C}^0(K)$, V_ϵ is a PL-triangulated topological $(n-1)$ -manifold.*

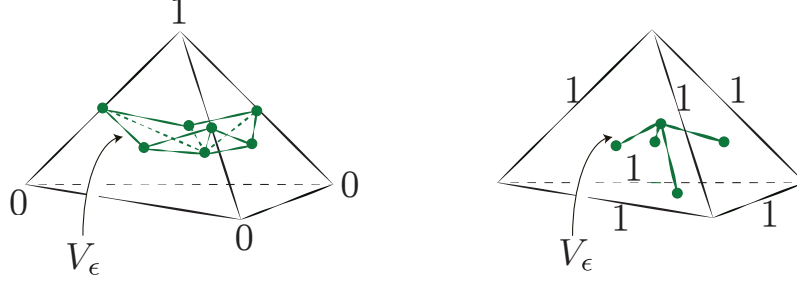


Figure 1: Some examples of simplices of V_ϵ . On the left the case $n = 3, k = 1$; on the right $n = 3, k = 2$.

Recall that a homology n -manifold is a topological space X such that for every point $x \in X$, the relative homology $H_*(X, X \setminus \{x\}; \mathbb{Z})$ is isomorphic to $H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z})$. Any smooth or topological n -dimensional manifold is thus a homology n -manifold.

Poincaré duality holds true in such compact homology manifolds, see [10]. And a triangulation K is called piecewise linear (PL) if for every simplex $\tau \in K$, the link $\text{Lk}(\tau, K) = \{\sigma \in K \mid \exists \eta \in K \text{ such that } \sigma, \tau < \eta \text{ and } \tau \cap \sigma = \emptyset\}$ is homeomorphic to a sphere, see Section 2.

Our main goal is to understand the topology of V_ϵ when $\epsilon \in \mathcal{C}^{k-1}(K)$ is chosen at random. More precisely, for every $d \geq 0$, let us denote by $\text{Sd}^d(K)$ the d^{th} barycentric subdivision of K , with the convention that $\text{Sd}^0(K) = K$. When K is finite, the asymptotic behavior of the number of simplices of $\text{Sd}^d(K)$ in each dimension $p \in \{0, 1, \dots, n\}$ has been studied in [2], [4], see also [11]. This number $f_p^d(K) = f_p(\text{Sd}^d(K))$ is equivalent to $q_{p,n} f_n(K) (n+1)!^d$ as d grows to $+\infty$ for some universal constant $q_{p,n} > 0$, where $f_p(K)$ denotes the number of p -dimensional simplices of K . Let $\nu \in [0, 1]$. For every $d \geq 0$, we equip $\mathcal{C}^{k-1}(\text{Sd}^d(K))$ with the product probability measure μ_ν so that for every $\epsilon \in \mathcal{C}^{k-1}(\text{Sd}^d(K))$, the probability that ϵ takes the value 0 on a $(k-1)$ -simplex of $\text{Sd}^d(K)$ is ν , while the probability that it takes the value 1 is $1 - \nu$. When K is finite, we set $\mathbb{E}_{\nu,d}(\chi) = \int_{\mathcal{C}^{k-1}(\text{Sd}^d(K))} \chi(V_\epsilon) d\mu_\nu(\epsilon)$ and for every $0 \leq i \leq n-k$, $\mathbb{E}_{\nu,d}(b_i) = \int_{\mathcal{C}^{k-1}(\text{Sd}^d(K))} b_i(V_\epsilon) d\mu_\nu(\epsilon)$, where $b_i(V_\epsilon) = \dim H_i(V_\epsilon)$ denotes the i^{th} Betti number of V_ϵ with $\mathbb{Z}/2\mathbb{Z}$ -coefficients and $\chi(V_\epsilon) = \sum_{i=0}^{n-k} (-1)^i b_i(V_\epsilon)$ its Euler characteristic. To simplify the notation, we will write $\mathbb{E}_{\nu,d} = \mathbb{E}_\nu$ since there won't be any ambiguity on the simplicial complex concerned. When $\nu = \frac{1}{2}$, we will moreover omit ν from the notation. Our main result is the following, see Corollaries 35 and 43.

Theorem 2 *Let $k \in \{1, \dots, n\}$. For every $0 \leq i \leq n-k$, there exist universal constants $c_i^+(n, k), c_i^-(n, k) > 0$, such that for every finite n -dimensional simplicial complex K and every $0 \leq i \leq n-k$,*

$$c_i^-(n, k) \leq \liminf_{d \rightarrow +\infty} \frac{\mathbb{E}_\nu(b_i)}{(n+1)!^d f_n(K)} \leq \limsup_{d \rightarrow +\infty} \frac{\mathbb{E}_\nu(b_i)}{(n+1)!^d f_n(K)} \leq c_i^+(n, k).$$

Moreover,

$$\lim_{d \rightarrow +\infty} \frac{\mathbb{E}_\nu(\chi)}{(n+1)!^d f_n(K)} = \sum_{i=0}^{n-k} (-1)^i c_i^+(n, k)$$

and

$$\limsup_{d \rightarrow +\infty} \frac{1}{(n+1)!^d f_n(K)} \left(\sum_{j=0}^i (-1)^{i-j} \mathbb{E}_\nu(b_j) \right) \leq \sum_{j=0}^i (-1)^{i-j} c_j^+(n, k).$$

The universal constants $c_i^+(n, k)$ are linear combinations of the coefficients $q_{p,n}$ above, see Definition 34. The upper estimates in Theorem 2 actually follow from a computation of the expected face polynomial of the subcomplexes. The face polynomial of a finite n -dimensional simplicial complex K is the polynomial $q_K(T) = \sum_{i=0}^n f_i(K)T^i$. We denote by $\mathbb{E}_\nu(q_{V_\epsilon}(T)) = \int_{C^{k-1}(K)} q_{V_\epsilon}(T) d\mu_\nu(\epsilon)$ the expected value of the face polynomial of a random subcomplex. Then, we prove the following (see Theorem 32).

Theorem 3 *Let K be a finite n -dimensional simplicial complex and $k \in \{1, \dots, n\}$. Then, for every $\nu \in [0, 1]$,*

$$\mathbb{E}_\nu(q_{V_\epsilon}(T)) = \int_K q_{D(\sigma)}(T) dm_k(\sigma).$$

Moreover, if K is a compact triangulated homology n -manifold and $k = 1$, then

$$\mathbb{E}_\nu(R_{V_\epsilon}(-1 - T)) = (-1)^n \mathbb{E}_\nu(R_{V_\epsilon}(T)),$$

where $R_{V_\epsilon}(T) = Tq_{V_\epsilon}(T) - \chi(V_\epsilon)T$.

Recall that the first barycentric subdivision $\text{Sd}(K)$ inherits a decomposition into blocks, the block $D(\sigma)$ dual to $\sigma \in K$, see Section 2 and [10]. The right hand side in Theorem 3 is thus the total face polynomial of these blocks with respect to the measure m_k on K . This measure m_k equals to $\sum_{\sigma \in K} \mu_\nu\{\epsilon \in C^{k-1}(K) | \hat{\sigma} \in V_\epsilon\} \delta_{\hat{\sigma}}$, where $\hat{\sigma}$ is the barycenter of σ and $\delta_{\hat{\sigma}}$ is the Dirac measure on it, see Section 3 and [11] for a study of such measures and integrals. This means that the density of m_K at $\sigma \in K$ with respect to the canonical measure $\sum_{\sigma \in K} \delta_{\hat{\sigma}}$ studied in [11] is given by the probability that $\hat{\sigma}$ belongs to V_ϵ . When K is a compact triangulated homology n -manifold, we obtain the analog of Theorem 3 where the face polynomial is replaced by the (simpler) block polynomial, see Theorem 27.

The constants $c_i^-(n, k)$ depend on some combinatorial complexity of the codimension k closed submanifolds Σ of \mathbb{R}^n , see Definition 42. Namely, for every such closed connected codimension k submanifold Σ of \mathbb{R}^n , we define its complexity as the smallest value m such that (\mathbb{R}^n, Σ) gets homeomorphic to $(\mathring{\Delta}_n, V_\epsilon)$ for some $\epsilon \in \mathcal{C}^{k-1}(\text{Sd}^m(\Delta_n))$, where Δ_n denotes the standard simplex of dimension n . We actually estimate from below the asymptotic expected number of connected components of V_ϵ which are homeomorphic to a given codimension k closed connected submanifold Σ of \mathbb{R}^n , see Theorem 41. The complexity of surfaces in \mathbb{R}^3 is studied in Section 5.2, see Theorem 45.

When $k = 1$, the subcomplexes V_ϵ turn out to inherit an additional CW-complex structure, see Corollary 10, which make it possible to improve the upper estimate in Theorem 2. We prove the following, see Corollaries 19 and 20.

Theorem 4 *Let K be a finite n -dimensional simplicial complex, $\nu \in [0, 1]$, $k = 1$ and $i \in \{0, \dots, n-1\}$. Then, $\mathbb{E}_\nu(b_i) \leq f_{i+1}(K)(1 - \nu^{i+2} - (1 - \nu)^{i+2})$. Moreover,*

$$\mathbb{E}_\nu(\chi) + \chi(K) = \sum_{i=0}^n (-1)^i (\nu^{i+1} + (1 - \nu)^{i+1}) f_i(K).$$

Theorem 4 has the following quite surprising corollary which has already been observed by T. Akita [1] with a different (non-probabilistic) method, but also follows from the symmetry property observed by I.G. Macdonald [9], see [11].

Corollary 5 ([1]) *If K is a triangulated compact homology $2n$ -manifold, then*

$$\chi(K) = \sum_{p=0}^{2n} \left(-\frac{1}{2}\right)^p f_p(K).$$

We checked in [11] that -1 together with $-\frac{1}{2}$ in even dimensions are the only universal parameters for which the polynomial $q_K(T) = \sum_{p=0}^n f_p(K)T^p$ equals $\chi(K)$ for every triangulated manifold K . The paper is organized as follows. Section 2 is devoted to a study of the topological structures of V_ϵ . We prove in particular Theorem 1 and exhibit an additional CW-structure on V_ϵ when $k = 1$, see Corollary 10. Section 3 is devoted to a study of the measure m_k and several computations, in particular when $\nu = \frac{1}{2}$, see Corollary 17. In Section 4.1, we take profit of the CW-complex structure to prove Theorem 4 and Corollary 5. Section 4.3 is devoted to the upper estimates in Theorem 2 in the general case ($k > 1$), while Section 4.2 is devoted to the special case of compact homology manifolds.

Finally, we prove in Section 5 the lower estimates of Theorem 2, see Theorem 41, Corollary 43, and study in the second part of Section 5 the combinatorial complexity of surfaces in \mathbb{R}^3 , see Definition 37, Theorem 45.

These results thus provide counterparts in this combinatorial framework to the ones obtained in [7] and [6, 8] for the expected Betti numbers of real algebraic submanifolds of real projective manifolds or nodal domains in smooth manifolds respectively. The paper ends with an appendix devoted to a further study and interpretation of the constants $c_i^+(n, k)$.

Acknowledgement : The second author is partially supported by the ANR project MICROLOCAL (ANR-15CE40-0007-01).

2 Structure of the subcomplexes V_ϵ

The aim of this section is to prove that when $k = 1$, the subcomplexes V_ϵ inherit an additional CW-complex structure. Moreover, they are homology manifold when K itself is a homology manifold, see Corollary 10 and 11.

Let us start with recalling some definitions concerning simplicial complexes. Let K be a finite simplicial complex and $\tau \in K$. The *star* of τ in K , denoted $\text{St}(\tau, K)$, is the union of interiors of all simplices of K having τ as a face. The closure $\overline{\text{St}}(\tau, K)$ of $\text{St}(\tau, K)$ is the union of all simplices of K having τ as a face. The *link* of τ in K , denoted $\text{Lk}(\tau, K)$, is the union of all simplices of K lying in $\overline{\text{St}}(\tau, K)$ that are disjoint from τ .

Let us recall as well that the *join* $K * L$ of the simplicial complexes K and L is the simplicial complex whose simplices are the joins $\sigma_K * \sigma_L$ where σ_K (respectively σ_L) are the simplices of K (respectively L), including \emptyset . If $\sigma_K = [e_0, \dots, e_k]$ and $\sigma_L = [f_0, \dots, f_l]$ then by definition $\sigma_K * \sigma_L = [e_0, \dots, e_k, f_0, \dots, f_l]$. In particular, if L is a point, then $K * L$ is the cone over K centered at L , see [10].

Theorem 6 *Let K be a locally finite n -dimensional simplicial complex. For every $1 \leq k \leq n$, every $\epsilon \in \mathcal{C}^{k-1}(K)$ and every $\sigma = [\hat{\sigma}_0, \dots, \hat{\sigma}_p] \in V_\epsilon \subset \text{Sd}(K)$, there exists a canonical*

isomorphism of simplicial complexes between the link $\text{Lk}(\sigma, V_\epsilon)$ of σ in V_ϵ and $(V_\epsilon \cap \partial\sigma_0) * \text{Sd}(\partial\text{Lk}(\sigma_0, \sigma_1)) * \dots * \text{Sd}(\partial\text{Lk}(\sigma_{p-1}, \sigma_p)) * \text{Sd}(\text{Lk}(\sigma_p, K))$.

Proof. By definition, a simplex τ of $\overline{\text{St}}(\sigma, V_\epsilon)$ is of the form $[\hat{\tau}_0, \dots, \hat{\tau}_q]$ such that there exists a sequence of subindices $0 \leq i_0 < i_1 < \dots < i_p \leq q$ satisfying $\tau_{i_j} = \sigma_j$, where $\tau_0 < \tau_1 < \dots < \tau_q$ and ϵ restricted to τ_0 is not constant. Therefore, a simplex of $\text{Lk}(\sigma, V_\epsilon)$ is of the form $[\hat{\tau}_0, \dots, \hat{\tau}_{i_0-1}, \hat{\tau}_{i_0+1}, \dots, \hat{\tau}_{i_p-1}, \hat{\tau}_{i_p+1}, \dots, \hat{\tau}_q]$, and thus it has a canonical decomposition $[\hat{\tau}_0, \dots, \hat{\tau}_{i_0-1}] * [\hat{\tau}_{i_0+1}, \dots, \hat{\tau}_{i_1-1}] * \dots * [\hat{\tau}_{i_p+1}, \dots, \hat{\tau}_q]$. Note that the first term $[\hat{\tau}_0, \dots, \hat{\tau}_{i_0-1}]$ is an element of $V_\epsilon \cap \partial\sigma_0$ as τ_{i_0-1} is a proper face of σ_0 and ϵ being non-constant on τ_0 is the condition of belonging to V_ϵ . The last term $[\hat{\tau}_{i_p+1}, \dots, \hat{\tau}_q]$ can be identified to $[\text{Lk}(\widehat{\sigma_p, \tau_{i_p+1}}), \dots, \text{Lk}(\widehat{\sigma_p, \tau_q})]$ by replacing each $\hat{\tau}_s$ with $\text{Lk}(\widehat{\sigma_p, \tau_s})$ for every $s \in \{i_p+1, \dots, q\}$, while for every $j \in \{0, \dots, p-1\}$, the intermediate term $[\hat{\tau}_{i_j+1}, \dots, \hat{\tau}_{i_{j+1}-1}]$ can be identified to $[\text{Lk}(\widehat{\sigma_j, \tau_{i_j+1}}), \dots, \text{Lk}(\widehat{\sigma_j, \tau_{i_{j+1}-1}})]$, respectively. Therefore, doing so, we obtain an element of $\text{Sd}(\text{Lk}(\sigma_p, K))$ and respectively an element of $\text{Sd}(\partial\text{Lk}(\sigma_j, \sigma_{j+1}))$ for every $j \in \{0, \dots, p-1\}$.

Let L denote the join $(V_\epsilon \cap \partial\sigma_0) * \text{Sd}(\partial\text{Lk}(\sigma_0, \sigma_1)) * \dots * \text{Sd}(\partial\text{Lk}(\sigma_{p-1}, \sigma_p)) * \text{Sd}(\text{Lk}(\sigma_p, K))$, we then obtain a canonical simplicial map

$$\begin{aligned} \Phi : \text{Lk}(\sigma, V_\epsilon) &\rightarrow L \\ [\hat{\tau}_0, \dots, \hat{\tau}_q] &\rightarrow [\hat{\tau}_0, \dots, \hat{\tau}_{i_0-1}] * [\text{Lk}(\widehat{\sigma_0, \tau_{i_0+1}}), \dots, \text{Lk}(\widehat{\sigma_0, \tau_{i_1-1}})] * \dots * \\ &\dots * [\text{Lk}(\widehat{\sigma_p, \tau_{i_p+1}}), \dots, \text{Lk}(\widehat{\sigma_p, \tau_q})]. \end{aligned}$$

To define the inverse map, we note that an element of $V_\epsilon \cap \partial\sigma_0$ is of the form $\Gamma = [\hat{\gamma}_0, \dots, \hat{\gamma}_{i_0-1}]$ such that $\gamma_0 < \dots < \gamma_{i_0-1} < \sigma_0$ and that $\epsilon|_{\gamma_0}$ is not constant. Meanwhile, for every $j \in \{0, \dots, p-1\}$, an element of $\text{Sd}(\partial\text{Lk}(\sigma_j, \sigma_{j+1}))$ is of the form $\Gamma_j = [\hat{\gamma}_{i_j+1}, \dots, \hat{\gamma}_{i_{j+1}-1}]$ such that $\gamma_{i_j+1} < \dots < \gamma_{i_{j+1}-1} < \sigma_{j+1}$ and that γ_s is disjoint from σ_j for every $s \in \{i_j+1, \dots, i_{j+1}-1\}$. Finally, an element of $\text{Sd}(\text{Lk}(\sigma_p, K))$ is of the form $\Gamma_p = [\hat{\gamma}_{i_p+1}, \dots, \hat{\gamma}_q]$ such that $\gamma_{i_p+1} < \dots < \gamma_q \in \text{St}(\sigma_p, K)$ and that γ_s is disjoint from σ_p for every $s \in \{i_p+1, \dots, q\}$. Therefore, there is a canonical simplicial map $\Psi : L \rightarrow \text{Lk}(\sigma, V_\epsilon)$ such that

$$\begin{aligned} \Psi : L &\rightarrow \text{Lk}(\sigma, V_\epsilon) \\ \Gamma * \Gamma_0 * \dots * \Gamma_p &\rightarrow [\hat{\gamma}_0, \dots, \hat{\gamma}_{i_0-1}, \sigma_0 * \widehat{\gamma_{i_0+1}}, \dots, \sigma_0 * \widehat{\gamma_{i_1-1}}, \dots, \\ &\dots, \sigma_{p-1} * \widehat{\gamma_{i_p-1}}, \sigma_p * \widehat{\gamma_{i_p+1}}, \dots, \sigma_p * \widehat{\gamma_q}]. \end{aligned}$$

The maps Φ and Ψ are inverse to each other, hence the result. \square

The case $k = 1$ is of special interest due to the following proposition and its corollaries.

Proposition 7 *Let K be a locally finite simplicial complex. For every $\epsilon \in \mathcal{C}^0(K)$, the intersection of V_ϵ with a p -simplex σ of K , if not empty, is isotopic to an affine hyperplane section $H_\sigma \subset \sigma$ which separates the vertices of σ labelled 1 from the vertices labelled 0.*

By isotopic in Proposition 7 we mean that there exists a continuous family of homeomorphisms of σ from the identity to a homeomorphism which maps $V_\epsilon \cap \sigma$ to H_σ .

Figure 2 exhibits some examples of hyperplane sections given by Proposition 7.

Remark 8 *Note that Proposition 7 does not hold true for $k > 1$. Indeed, consider for example $\epsilon \in \mathcal{C}^1(T)$ where T is a tetrahedron such that ϵ has constant value 1 on all the edges of K , the*

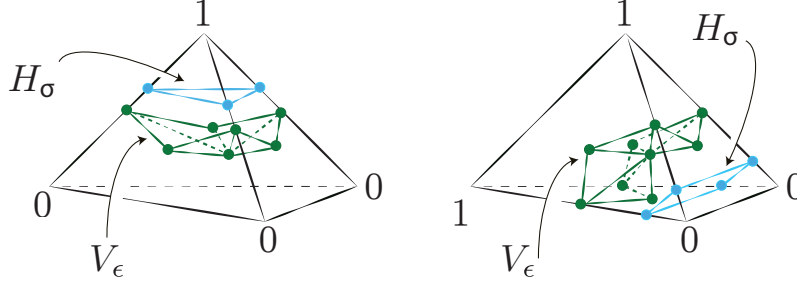


Figure 2: Examples of $V_\epsilon \cap \sigma$ versus H_σ for $p = 3$.

case depicted on the right of Figure 1. Then, V_ϵ is the cone over the barycenters of the four 2-dimensional faces of T centered at the barycenter of T .

Proof. Suppose that ϵ takes the value 1 on $0 < j + 1 < p + 1$ vertices of a p -simplex σ . There is then a j -face τ_1 of σ with all its vertices labelled 1 and a $(p - j - 1)$ -face τ_0 with vertices labelled by 0. Let $H_\sigma \subset \sigma$ be a hyperplane section separating τ_0 from τ_1 intersecting transversally exactly $(j + 1)(p - j)$ -many 1-faces which are neither in τ_0 nor in τ_1 .

Let us prove by induction on the dimension of the skeleton of σ that $V_\epsilon \cap \sigma$ is isotopic to H_σ . As ϵ takes value 1 on exactly $j + 1$ vertices, there are $(j + 1)(p - j)$ -many edges on which $d\epsilon$ is non trivial. Those are the edges which are neither on τ_0 nor on τ_1 . Thus the intersection of V_ϵ with the 1-skeleton of σ is a set of isolated points, one point on each of these $(j + 1)(p - j)$ -many edges. So, on each such edge we have two points, one defined by the intersection of V_ϵ and the other by the intersection of H_σ . We perform an isotopy which takes one set of points to the other. Now, let us suppose that V_ϵ can be isotoped to H_σ on the l -skeleton for $l \geq 1$ and choose such an isotopy. Let η be a $(l + 1)$ -face of σ . The intersection $V_\epsilon \cap \eta$, if not empty, is the cone centered at the barycenter of η over the intersection of V_ϵ with the boundary of η . Besides $H_\sigma \cap \eta$ is an affine hyperplane section. Let b be a point in the hyperplane section so that $H_\sigma \cap \eta$ is the cone over $H_\sigma \cap \partial\eta$ centered at b . By means of choosing a path $(b_t)_{t \in [0,1]}$ in $\overset{\circ}{\sigma}$ from b to the barycenter of η , we can get an isotopy between $H_\sigma \cap \eta$ and $V_\epsilon \cap \eta$ by taking the cone centered at b_t over the isotopy between $H_\sigma \cap \partial\eta$ and $V_\epsilon \cap \partial\eta$. \square

Remark 9 Note that when K is a convex triangulation of the moment polytope of some toric manifold, then $\epsilon \in \mathcal{C}^0(K)$ provides a distribution of signs on every vertex of K and the collection $(H_\sigma)_{\sigma \in K}$ defines on K the (tropical) hypersurface constructed by O. Viro [13] in his patchwork theorem, provided H_σ is chosen to intersect every edge in its middle point. From Proposition 7, we thus deduce that the pair (K, V_ϵ) is homeomorphic to the pair (K, H) defined by O. Viro, where $H = \cup_{\sigma \in K} H_\sigma$.

Corollary 10 Let K be a locally finite n -dimensional simplicial complex and $\epsilon \in \mathcal{C}^0(K)$. Then, V_ϵ inherits the structure of a CW-complex, having a cell of dimension p for every $(p + 1)$ -simplex of K on which ϵ is not constant, $p \in \{0, \dots, n - 1\}$.

Proof. From Proposition 7, we indeed know that the intersection of V_ϵ with any $(p + 1)$ -simplex of K on which ϵ is not constant is homeomorphic to a p -cell. Hence, the result. \square

Corollary 11 *Let K be a triangulated homology n -manifold. Then, for every $\epsilon \in \mathcal{C}^0(K)$, V_ϵ is a triangulated $(n-1)$ -homology manifold. Moreover, if K is a PL-triangulation of a homology n -manifold, then for every $\epsilon \in \mathcal{C}^0(K)$, V_ϵ is a PL-triangulation of a homology $(n-1)$ -manifold.*

Proof. From Theorem 6 it follows that for every $\sigma \in V_\epsilon$, the link $\text{Lk}(\sigma, V_\epsilon)$ is canonically isomorphic to $(V_\epsilon \cap \partial\sigma_0) * \text{Sd}(\partial\text{Lk}(\sigma_0, \sigma_1)) * \dots * \text{Sd}(\partial\text{Lk}(\sigma_{p-1}, \sigma_p)) * \text{Sd}(\text{Lk}(\sigma_p, K))$, where the first term is homeomorphic to a sphere by Proposition 7 and the intermediate terms are by definition homeomorphic to spheres. Finally, the last term is a homology sphere by Lemma 63.1 of [10] in the case where K is a triangulated homology manifold (respectively, it is a sphere in the case where K is a PL-triangulation). Therefore, the link of any simplex of V_ϵ is the join of a homology sphere with spheres, which is a homology sphere (respectively, join of spheres which is a sphere). Hence the result. \square

Example 12 *If K is a triangulation of a closed manifold, then V_ϵ may not be a triangulation of a submanifold. A counterexample can be constructed from the double suspension of the Poincaré sphere. Namely, let K be a triangulation of the Poincaré sphere and let S denote the simplicial complex of a 0-dimensional sphere. We take the double suspension $(S * S * K)$ of K together with a simplicial complex structure obtained by considering successive cones first over the simplexes of K centered at vertices of S and then over $S * K$ centered at the vertices of S . The obtained complex, denoted \tilde{K} , is a triangulation of a 5-dimensional sphere, see [3, 5]. Now, let us consider $\epsilon \in \mathcal{C}^0(\tilde{K})$ such that ϵ takes value 1 on one of the four vertices that corresponds to one of the four centers of suspensions and zero on all other vertices. There is a natural isotopy from V_ϵ to $S * K$. The latter is not a submanifold as the link of the two points corresponding to the center of suspension are Poincaré spheres.*

Remark 13 *Corollary 11 does not hold true for $k > 1$. Indeed, in the case of the tetrahedron discussed in Remark 8 for example, the link $\text{Lk}(\hat{T}, V_\epsilon)$ of the barycenter \hat{T} of T is the set of four vertices which is not a homology sphere. This example can be implemented in any triangulated homology 3-manifold K so that V_ϵ need not be a triangulated homology manifold, although K is.*

Corollary 14 *Let K be an even dimensional compact triangulated homology manifold. Then, for every $\epsilon \in \mathcal{C}^0(K)$, $\chi(V_\epsilon) = 0$.*

Proof. By Corollary 11, V_ϵ is a homology manifold in which case the Poincaré duality with $\mathbb{Z}/2\mathbb{Z}$ -coefficients applies, see Chapter 8 of [10]. When the dimension of K is even, the dimension of V_ϵ is odd, hence the result. \square

3 The induced measures m_k

Let K be a locally finite n -dimensional simplicial complex and $k \in \{1, \dots, n\}$, for every $\sigma \in K$, we set

$$m_k(\sigma) = \mu_\nu\{\epsilon \in C^{k-1}(K, \mathbb{Z}/2\mathbb{Z}) | \hat{\sigma} \in V_\epsilon\}.$$

This is the probability that the barycenter $\hat{\sigma}$ belongs to V_ϵ . It defines a measure on K , namely

$$m_k = \sum_{\sigma \in K} \mu_\nu\{\epsilon \in C^{k-1}(K, \mathbb{Z}/2\mathbb{Z}) | \hat{\sigma} \in V_\epsilon\} \delta_{\hat{\sigma}},$$

where $\delta_{\hat{\sigma}}$ denotes the Dirac measure on $\hat{\sigma}$. We likewise set, for every $p \in \{0, \dots, n\}$,

$$\gamma_{p,K} = \sum_{\sigma \in K^{[p]}} \delta_{\hat{\sigma}},$$

where $K^{[p]}$ denotes the set of p -dimensional simplices of K .

We proved in [11] that $\frac{1}{(n+1)!^d} \gamma_{p, \text{Sd}^d(K)}$ weakly converges to $q_{p,n} d\text{vol}_K$ as d grows to $+\infty$, where $q_{p,n} > 0$ is some universal constant and $d\text{vol}_K = \sum_{\sigma \in K^{[n]}} (f_\sigma)_*(d\text{vol}_{\Delta_n})$. In the latter $d\text{vol}_{\Delta_n}$ is the Lebesgue measure of the standard simplex Δ_n normalized in such a way that it has total measure 1 and $f_\sigma : \Delta_n \rightarrow \sigma$ is some affine isomorphism. Our aim in this section is to study the measure m_k .

Proposition 15 *Let K be a locally finite n -dimensional simplicial complex and $k \in \{1, \dots, n\}$. Then, $m_k = \sum_{p=k}^n (1 - \mu_\nu(Z^{k-1}(\Delta_p))) \gamma_{p,K}$, where $Z^{k-1}(\Delta_p) \subset C^{k-1}(\Delta_p)$ denotes the subspace of $(k-1)$ -cocycles of Δ_p with $\mathbb{Z}/2\mathbb{Z}$ -coefficients.*

Proof. By definition, for every $\sigma \in K$ and every $\epsilon \in C^{k-1}(K)$, $\hat{\sigma}$ belongs to V_ϵ if and only if the restriction of $d\epsilon$ to σ does not vanish, that is $\epsilon|_\sigma \notin Z^{k-1}(\sigma)$. Since μ_ν is a product measure, we deduce that $\mu_\nu\{\epsilon \in C^{k-1}(K) | \hat{\sigma} \in V_\epsilon\} = 1 - \mu_\nu(Z^{k-1}(\sigma))$. Moreover, by definition of μ_ν , $\mu_\nu(Z^{k-1}(\sigma))$ only depends on the dimensions of σ . Finally, $m_k = \sum_{\sigma \in K} (1 - \mu_\nu(Z^{k-1}(\sigma))) \delta_{\hat{\sigma}} = \sum_{p=k}^n (1 - \mu_\nu(Z^{k-1}(\Delta_p))) \gamma_{p,K}$. \square

We may additionally set $m_0 = \sum_{p=0}^n \gamma_{p,K}$ so that all statements involving m_k will make sense when $k = 0$ as well, but this case is of no interest for us.

We have been able to compute explicitly the universal constants $\mu_\nu(Z^{k-1}(\Delta_p))$ appearing in Proposition 15 in several cases, in particular, when $\nu = \frac{1}{2}$, see Corollary 17. These computations are based on the following theorem.

Theorem 16 *For every $1 \leq k \leq p$,*

$$\mu_\nu(Z^{k-1}(\Delta_p)) = \begin{cases} \nu^{p+1} + (1 - \nu)^{p+1} & \text{if } k = 1, \\ \int_{\eta \in C^{k-2}(\Delta_{p-1})} \mu_\nu(d\eta) d\mu_\nu(\eta) & \text{if } k \geq 2. \end{cases}$$

The case of $k \geq 2$ in Theorem 16 is thus the expected value of the random variable $\mu_\nu \circ d$ on the probability space $C^{k-2}(\Delta_{p-1})$.

Proof. When $k = 1$, $Z^0(\Delta_p) = \{0, 1\}$ where 0 (respectively 1) denotes the 0-cochain which is constant and equal to 0 (respectively 1) on $C_0(\Delta_p)$. By definition of μ_ν , $\mu_\nu(0) = \nu^{p+1}$ and $\mu_\nu(1) = (1 - \nu)^{p+1}$ since Δ_p has $p+1$ vertices. Let us assume now that $k \geq 2$ and let s be a vertex of Δ_p . We are going to prove that whatever the value of ϵ on the $(k-1)$ -faces of Δ_p containing s is, there is a unique way to extend ϵ to all $(k-1)$ -faces of Δ_p in such a way that $d\epsilon$ restricted to Δ_p vanishes. Indeed, let us assume that we fix a value of ϵ on the

$(k-1)$ -faces of Δ_p containing s and consider a k -face τ of Δ_p that contains s . In this case, among the $k+1$ many $(k-1)$ -faces of τ there is only one, say η , which does not contain s . By the assumption above, all $(k-1)$ -faces of τ but η are labelled by ϵ . There is a bijection on those labelled $(k-1)$ -faces of τ and the $(k-2)$ -faces of η , since the former are cones over the latter. This bijection thus induces labels on the $(k-2)$ -faces of η . Let us assign to η the value 1 if an odd number of its codimension-1 faces are labelled 1; and 0 otherwise. By doing so, in either case an even number of $(k-1)$ -faces of τ get labelled 1, which results in $\langle d\epsilon, \tau \rangle = 0$. Moreover, this way is the only way to label η for having $\langle d\epsilon, \tau \rangle = 0$. At this point, ϵ has been extended to all $(k-1)$ -faces of Δ_p and we have to check that $d\epsilon$ vanishes on Δ_p .

Let then now τ be a k -face of Δ_p which does not contain s . By definition, the restriction of ϵ to τ equals $d\tilde{\epsilon}$ where $\tilde{\epsilon} \in \mathcal{C}^{k-2}(\tau)$ is inherited by the values of ϵ on cones over the $(k-2)$ -faces of τ , centered at s . Since $d^2 = 0$, we deduce that $\langle d\epsilon, \tau \rangle = \langle d^2\tilde{\epsilon}, \tau \rangle = 0$. Now, we deduce that

$$\mu_\nu(Z^{k-1}(\Delta_p)) = \sum_{\eta \in \mathcal{C}^{k-2}(\text{Lk}(s, \Delta_p))} \mu_\nu(\eta) \mu_\nu(d\eta),$$

since μ_ν is a product measure. Recall that $\text{Lk}(s, \Delta_p)$ denotes the link of s in Δ_p , that is the $(p-1)$ -simplex spanned by all the vertices of Δ_p but s . The coefficient $\mu_\nu(\eta)$ computes the value of μ_ν on cones centered at s over the $(k-2)$ -faces of $\text{Lk}(s, \Delta_p)$, while the coefficients $\mu_\nu(d\eta)$ computes the value of μ_ν on the $(k-1)$ -faces of $\text{Lk}(s, \Delta_p)$. \square

Corollary 17 1. If $\nu = \frac{1}{2}$, then for every $1 \leq k \leq p$,

$$\mu_\nu(Z^{k-1}(\Delta_p)) = \frac{1}{2^{\binom{p}{k}}}.$$

2. For every $\nu \in [0, 1]$,

$$\mu_\nu(Z^{k-1}(\Delta_p)) = \begin{cases} \frac{1}{2}(1 + (2\nu - 1)^{k+1}) & \text{if } 1 \leq k = p, \\ \nu^{\binom{p+1}{2}} \sum_{l=0}^p \binom{p}{l} \left(\frac{1-\nu}{\nu}\right)^{(l+1)(p-l)} & \text{if } 2 \leq k \leq p. \end{cases}$$

The first line remains valid for every $1 \leq k \leq p$ if $\nu \in \{0, 1\}$.

Proof. Let $\nu = \frac{1}{2}$. If $k = 1$, the result directly follows from Theorem 16. If $2 \leq k \leq p$, note that the $(p-1)$ -simplex has $\binom{p}{k-1}$ faces of dimension $k-2$ and $\binom{p}{k}$ faces of dimension $k-1$, so we deduce from Theorem 16 that

$$\begin{aligned} \mu_\nu(Z^{k-1}(\Delta_p)) &= \sum_{\eta \in \mathcal{C}^{k-2}(\Delta_{p-1})} \mu_\nu(\eta) \mu_\nu(d\eta) \\ &= \sum_{\eta \in \mathcal{C}^{k-2}(\Delta_{p-1})} \frac{1}{2} \binom{p}{k-1} + \binom{p}{k} \\ &= \frac{1}{2^{\binom{p}{k}}}. \end{aligned}$$

Now, let $\nu \in [0, 1]$. If $k = 1 = p$, the result directly follows from Theorem 16. If $2 \leq k = p$, then from Theorem 16,

$$\begin{aligned}\mu_\nu(Z^{k-1}(\Delta_p)) &= \sum_{\eta \in \mathcal{C}^{k-2}(\Delta_{p-1})} \mu_\nu(\eta) \mu_\nu(d\eta) \\ &= \sum_{l=0}^{\lfloor \frac{p}{2} \rfloor} \binom{p}{2l} (1-\nu)^{2l} \nu^{p-2l+1} + \sum_{l=0}^{\lfloor \frac{p-1}{2} \rfloor} \binom{p}{2l+1} \nu^{p-2l-1} (1-\nu)^{2l+2}\end{aligned}$$

since the $(p-1)$ -simplex Δ_{p-1} has p facets and the value of $d\epsilon$ on its unique $(p-1)$ -face depends on the parity of the number of facets where $\epsilon = 1$. We deduce

$$\begin{aligned}\mu_\nu(Z^{k-1}(\Delta_p)) &= \nu \sum_{l=0}^p \binom{p}{l} (1-\nu)^l \nu^{p-l} (-1)^l + \sum_{l=0}^p \binom{p}{l} (1-\nu)^l \nu^{p-l} \left(\frac{1-(-1)^l}{2}\right) \\ &= \nu(2\nu-1)^p + \frac{1}{2}(1-(2\nu-1)^p) \\ &= \frac{1}{2}(1+(2\nu-1)^{p+1}).\end{aligned}$$

When $\nu = 1$, for every $2 \leq k \leq p$,

$$\mu_\nu(Z^{k-1}(\Delta_p)) = \mu_\nu(0) \mu_\nu(d0) = 1,$$

since μ_ν is then a Dirac on $0 \in \mathcal{C}^{k-2}(\Delta_{p-1})$, while when $\nu = 0$,

$$\mu_\nu(Z^{k-1}(\Delta_p)) = \mu_\nu(1) \mu_\nu(d1) = \begin{cases} 0 & \text{if } k \text{ is even,} \\ 1 & \text{if } k \text{ is odd.} \end{cases}$$

The latter remains valid for $k = 1$, from Theorem 16.

Now if $2 = k \leq p$, then from Theorem 16,

$$\begin{aligned}\mu_\nu(Z^{k-1}(\Delta_p)) &= \sum_{\eta \in \mathcal{C}^0(\Delta_{p-1})} \mu_\nu(\eta) \mu_\nu(d\eta) \\ &= \sum_{l=0}^p \binom{p}{l} \nu^l (1-\nu)^{p-l} \nu^{\binom{l}{2} + \binom{p-l}{2}} (1-\nu)^{l(p-l)},\end{aligned}$$

where $\binom{l}{2} = 0$ (respectively $\binom{p-l}{2} = 0$) if $l < 2$ (respectively $p-l < 2$). Indeed, the standard $(p-1)$ -simplex has p vertices and if ϵ takes the value 0 on l of them, it has $\binom{l}{2} + \binom{p-l}{2}$ edges where $d\epsilon = 0$ and $l(p-l)$ edges where $d\epsilon = 1$. The result follows from the relation $\binom{l}{2} + \binom{p-l}{2} = \binom{p}{2} - l(p-l)$, valid for every $l \in \{0, \dots, p\}$. \square

4 Asymptotic topology of V_ϵ , upper estimates

4.1 The case $k = 1$

Let K be a finite n -dimensional simplicial complex. When $k = 1$, V_ϵ admits a CW-complex structure given by Corollary 10 whatever $\epsilon \in \mathcal{C}^{k-1}(K)$ is. Let $\tilde{f}_p(V_\epsilon)$ be the number of p -cells of V_ϵ , $p \in \{0, \dots, n\}$, for this CW structure and $\tilde{q}_{V_\epsilon}(T) = \sum_{p=0}^n \tilde{f}_p(V_\epsilon) T^p$.

The expected value of this polynomial is given by the following theorem.

Theorem 18 *Let K be a finite n -dimensional simplicial complex and $\nu \in [0, 1]$. Then, when $k = 1$,*

$$\mathbb{E}_\nu(T\tilde{q}_{V_\epsilon}(T)) = \int_K T^{\dim \sigma} dm_1(\sigma) = q_K(T) - \nu q_K(\nu T) - (1 - \nu)q_K((1 - \nu)T),$$

where for every $\epsilon \in \mathcal{C}^0(K)$, V_ϵ is equipped with the CW-complex structure given by Corollary 10. In particular, for every $p \in \{0, \dots, n - 1\}$.

$$\mathbb{E}_\nu(\tilde{f}_p(V_\epsilon)) = m_1(K^{[p+1]}) = f_{p+1}(K)(1 - \nu^{p+2} - (1 - \nu)^{p+2}).$$

Proof. For every $\nu \in [0, 1]$, $p \in \{0, \dots, n - 1\}$,

$$\begin{aligned} \mathbb{E}_\nu(\tilde{f}_p(V_\epsilon)) &= \int_{\mathcal{C}^0(K)} \tilde{f}_p(V_\epsilon) d\mu_\nu(\epsilon) \\ &= \sum_{\epsilon \in \mathcal{C}^0(K)} \#\{\sigma \in K^{[p+1]} | V_\epsilon \cap \overset{\circ}{\sigma} \neq \emptyset\} \mu_\nu(\epsilon) \\ &= \sum_{\sigma \in K^{[p+1]}} \mu_\nu\{\epsilon \in \mathcal{C}^0(K) | V_\epsilon \cap \overset{\circ}{\sigma} \neq \emptyset\} \\ &= m_1(K^{[p+1]}) \\ &= f_{p+1}(K)(1 - \nu^{p+2} - (1 - \nu)^{p+2}), \end{aligned}$$

since μ_ν is a product measure and the probability that ϵ is identically 0 (respectively 1) on the $p + 2$ vertices of σ is ν^{p+2} (respectively $(1 - \nu)^{p+2}$). This proves the second part together with the first equality of the theorem. Then,

$$\begin{aligned} \mathbb{E}_\nu(T\tilde{q}_{V_\epsilon}(T)) &= \sum_{p=0}^{n-1} f_{p+1}(K)(1 - \nu^{p+2} - (1 - \nu)^{p+2})T^{p+1} \\ &= \sum_{p=0}^n f_p(K)(1 - \nu^{p+1} - (1 - \nu)^{p+1})T^p \\ &= q_K(T) - \nu q_K(\nu T) - (1 - \nu)q_K((1 - \nu)T). \end{aligned}$$

□

Corollary 19 *Let $k = 1$. For every finite n -dimensional simplicial complex K , every $\nu \in [0, 1]$ and every $0 \leq i \leq n - 1$,*

$$\mathbb{E}_\nu(b_i) \leq m_1(K^{[i+1]}) = f_{i+1}(K)(1 - \nu^{i+2} - (1 - \nu)^{i+2})$$

and

$$\sum_{j=0}^i (-1)^{i-j} \mathbb{E}_\nu(b_j) \leq \sum_{j=0}^i (-1)^{i-j} f_{j+1}(K)(1 - \nu^{j+2} - (1 - \nu)^{j+2}),$$

where the latter is an equality if $i = n - 1$.

Proof. By Corollary 10, V_ϵ inherits a CW-complex structure for all $\epsilon \in \mathcal{C}^0(K)$. From cellular homology theory, we deduce that $b_i(V_\epsilon) \leq \tilde{f}_i(V_\epsilon)$ for all $0 \leq i \leq n-1$ and the Morse inequalities $\sum_{j=0}^i (-1)^{i-j} b_j(V_\epsilon) \leq \sum_{j=0}^i (-1)^{i-j} \tilde{f}_j(V_\epsilon)$. The result follows by integrating over $\mathcal{C}^0(K)$ and applying the second part of Theorem 18. \square

Corollary 20 *Let $k = 1$. For every finite simplicial complex K and every $\nu \in [0, 1]$,*

$$\mathbb{E}_\nu(\chi) + \chi(K) = \nu q_K(-\nu) + (1 - \nu) q_K(\nu - 1).$$

In particular, $\mathbb{E}(\chi) + \chi(K) = q_K(-\frac{1}{2})$.

Proof. The first part of the result follows from Theorem 18, by letting $T = -1$ and the second part by letting further $\nu = \frac{1}{2}$. \square

Corollary 21 *Let $k = 1$ and K be a triangulation of a compact homology n -manifold. For every $\nu \in [0, 1]$,*

$$\mathbb{E}_\nu(\chi) = ((-1)^n - 1) R_K(-\nu).$$

In particular, if n is odd, the values of the polynomial $q_n^\infty(T) = \sum_{p=0}^n q_{p,n} T^p$ on the interval $[-1, 0[$ are given by

$$\forall \nu \in]0, 1], \quad q_n^\infty(-\nu) = \frac{1}{2\nu} \lim_{d \rightarrow +\infty} \frac{\mathbb{E}_\nu(\chi)}{(n+1)!^d f_n(K)}.$$

Proof. From Theorem 18, we get

$$\begin{aligned} \mathbb{E}_\nu(T \tilde{q}_{V_\epsilon}(T)) &= q_K(T) - \nu q_K(\nu T) - (1 - \nu) q_K((1 - \nu)T) \\ &= \frac{1}{T} [R_K(T) - R_K(\nu T) - R_K((1 - \nu)T)]. \end{aligned}$$

The first part of the result follows by setting $T = -1$ and applying the property $R_K(-1 + \nu) = (-1)^{n+1} R_K(-\nu)$ given by [9] (see also [11]).

The second part is then obtained after performing d barycentric subdivisions to K , dividing the both sides by $f_n(K)(n+1)!^d$ and letting d go to $+\infty$. Indeed, the Euler characteristic of K is invariant under barycentric subdivisions, so that by [4], the right hand side $\frac{-2R_{\text{Sd}^d(K)}(-\nu)}{f_n(K)(n+1)!^d}$ converges to $2\nu q_n^\infty(-\nu)$ as d goes to $+\infty$. Hence the result. \square

Corollary 22 *Let $k = 1$. For every finite n -dimensional simplicial complex K , every $\nu \in [0, 1]$ and every $0 \leq i \leq n-1$,*

$$\lim_{d \rightarrow +\infty} \frac{\mathbb{E}(\chi)}{(n+1)!^d f_n(K)} = q_n^\infty(-\frac{1}{2}) = \begin{cases} 0 & \text{if } n \equiv 0[2] \\ > 0 & \text{if } n \equiv 1[4] \\ < 0 & \text{if } n \equiv 3[4]. \end{cases}$$

Moreover,

$$\limsup_{d \rightarrow +\infty} \frac{\mathbb{E}_\nu(b_i)}{(n+1)!^d f_n(K)} \leq q_{i+1,n}(1 - \nu^{i+2} - (1 - \nu)^{i+2})$$

and

$$\limsup_{d \rightarrow +\infty} \sum_{j=0}^i (-1)^{i-j} \frac{\mathbb{E}_\nu(b_j)}{(n+1)!^d f_n(K)} \leq \sum_{j=0}^i (-1)^{i-j} q_{j+1,n} (1 - \nu^{j+2} - (1 - \nu)^{j+2}).$$

Recall that $q_{i,n}, 0 \leq i \leq n$, are the coefficients of the polynomial $q_n^\infty(T)$.

Proof. The computation $\lim_{d \rightarrow +\infty} \frac{\mathbb{E}(\chi)}{(n+1)!^d f_n(K)} = q_n^\infty(-\frac{1}{2})$ is obtained from the second part of Corollary 20 after performing d barycentric subdivisions to K , dividing both sides by $(n+1)!^d f_n(K)$ and taking the limit as d tends to $+\infty$, since the Euler characteristic of K is invariant under barycentric subdivisions and $\frac{q_K}{(n+1)!^d f_n(K)}$ converges to q_n^∞ by [4].

From [4], we know that the roots of $Tq_n^\infty(T)$ are symmetric with respect to $T \rightarrow -1 - T$. This implies that $q_n^\infty(-\frac{1}{2}) = 0$ when n is even since $Tq_n^\infty(T)$ has then an odd number of roots. By [2], we know that all the roots of $Tq_n^\infty(T)$ are simple and lie in the interval $[-1, 0]$. Hence, when $n = 4s + 1$ (respectively $n = 4s + 3$) for some $s \in \mathbb{N}$, the polynomial $Tq_n^\infty(T)$ has $2s + 1$ (respectively $2s + 2$) roots between $-\frac{1}{2}$ and 0. Since $Tq_n^\infty(T)$ is positive for $T > 0$, $q_n^\infty(-\frac{1}{2})$ is positive if there is an odd number of roots in $]-\frac{1}{2}, 0]$, negative otherwise. Hence the first part of Corollary 22.

Now, by the second part of Theorem 18, for every $0 \leq i \leq n - 1$,

$$\mathbb{E}_\nu(\tilde{f}_i(V_\epsilon)) = f_{i+1}(K)(1 - \nu^{i+2} - (1 - \nu)^{i+2})$$

so that performing d barycentric subdivisions on K we get

$$\frac{\mathbb{E}_\nu(b_i(V_\epsilon))}{(n+1)!^d f_n(K)} \leq \frac{\mathbb{E}_\nu(\tilde{f}_i(V_\epsilon))}{(n+1)!^d f_n(K)} = \frac{f_{i+1}^d(K)(1 - \nu^{i+2} - (1 - \nu)^{i+2})}{(n+1)!^d f_n(K)}.$$

By [4], (see also [11]) $q_{i+1,n} = \lim_{d \rightarrow +\infty} \frac{f_{i+1}^d(K)}{(n+1)!^d f_n(K)}$ so that the second part follows by taking the limit.

Finally, the last part follows along the same lines from Corollary 19. Hence the result. \square

Remark 23 Unlike $q_n^\infty(T)$, the polynomial $q_K(T)$ might have in general complex or non simple roots. However, it has real coefficients and when K is a compact triangulated homology n -manifold, it is symmetric with respect to $T \rightarrow -1 - T$, so that provided it has no complex root with real part $-\frac{1}{2}$, we deduce as in the proof of Corollary 22 the weaker equalities:

$$q_K(-\frac{1}{2}) = \begin{cases} 0 & \text{if } n \equiv 0[2] \\ \geq 0 & \text{if } n \equiv 1[4] \\ \leq 0 & \text{if } n \equiv 3[4]. \end{cases}$$

We finally deduce a probabilistic proof of the following theorem.

Theorem 24 *If K is a compact triangulated homology $2n$ -manifold, then*

$$\chi(K) = \sum_{p=0}^{2n} (-\frac{1}{2})^p f_p(K).$$

Theorem 24 has already been proved in [1], see also [11].

Proof. It follows from Corollary 11 that for every $\epsilon \in \mathcal{C}^0(K)$, the hypersurface V_ϵ is the triangulation of an odd-dimensional homology manifold, so that $\chi(V_\epsilon)$ vanishes from Poincaré duality, see [10]. We deduce that $\mathbb{E}(\chi)$ vanishes as well and so the result follows from Corollary 20. \square

Remark 25 Note that Theorem 24 implies that $q_n^\infty(-\frac{1}{2}) = 0$ when $n > 0$ is even and $q_n^\infty(-1) = 0$ for every $n > 0$. This fact can be deduced from the symmetry property of $R_K(T) = Tq_K(T) - \chi(K)T$ see [9] and also [11].

Note also that from Corollary 20 we more generally deduce that under the hypothesis of Theorem 24, $\chi(K) = \nu q_K(-\nu) + (1 - \nu)q_K(\nu - 1)$ for every $\nu \in [0, 1]$. This formula has been observed by A. Kassel right after the first author gave an informal talk for students on Theorem 24.

4.2 The case of compact homology manifolds

When K is a triangulated compact homology manifold, it inherits a decomposition into blocks which is dual to the triangulation and is useful to prove Poincaré duality, see [10]. In particular, these blocks span a chain complex which computes the homology of K exactly as if it were a CW-complex (from the homology point of view, there is no difference).

By definition, the block $D(\sigma)$ dual to a simplex $\sigma \in K$ is the union of all open simplices $[\hat{\sigma}_0, \dots, \hat{\sigma}_p]$ of $\text{Sd}(K)$ such that $\sigma_0 = \sigma$. The union of closed such simplices is denoted by $\overline{D}(\sigma)$.

Lemma 26 Let K be a triangulated homology manifold of dimension $n > 0$. Then, for every $1 \leq k \leq n$ and every $\epsilon \in \mathcal{C}^{k-1}(K)$, V_ϵ is the union of the blocks $D(\sigma)$ dual to the simplices σ of K such that the restriction of $d\epsilon$ to σ does not vanish.

Proof. This follows from the definitions of V_ϵ and the dual block decompositions of K . \square

For every $\epsilon \in \mathcal{C}^{k-1}(K)$, every $i \in \{0, \dots, n-k\}$ and every $\nu \in [0, 1]$, let us denote by $\check{f}_i(V_\epsilon)$ the number of i -dimensional blocks of K that are in V_ϵ and by $\mathbb{E}_\nu(\check{f}_i) = \int_{\epsilon \in \mathcal{C}^{k-1}(K)} \check{f}_i(V_\epsilon) d\mu_\nu(\epsilon)$ its mathematical expectation.

Theorem 27 Let K be a compact triangulated homology manifold of dimension $n > 0$. Then, for every $1 \leq k \leq n$, every $0 \leq i \leq n-k$ and every $\nu \in [0, 1]$, $\mathbb{E}_\nu(\check{f}_i) = m_k(K^{[n-i]})$, so that $\mathbb{E}_\nu(\check{q}_{V_\epsilon}(T)) = \int_K \check{q}_{D(\sigma)}(T) dm_k(\sigma)$.

Note that for every $\sigma \in K$, $D(\sigma)$ is made of a single block, so that $\check{q}_{D(\sigma)}(T) = T^{n-\dim \sigma}$.

Proof. By definition,

$$\begin{aligned} \mathbb{E}_\nu(\check{f}_i) &= \int_{\epsilon \in \mathcal{C}^{k-1}(K)} \sum_{\substack{\sigma \in K^{[n-i]} \\ D(\sigma) \in V_\epsilon}} 1 d\mu_\nu(\epsilon) \\ &= \sum_{\sigma \in K^{[n-i]}} \mu_\nu\{\epsilon \in \mathcal{C}^{k-1}(K) | \hat{\sigma} \in V_\epsilon\}. \end{aligned}$$

We thus deduce from Lemma 26 and the definition of m_k that

$$\mathbb{E}_\nu(\check{f}_i) = m_k(K^{[n-i]}).$$

Summing over all $i \in \{0, \dots, n-k\}$ we get $\mathbb{E}_\nu(\check{q}_{V_\epsilon}(T)) = \int_K \check{q}_{D(\sigma)} dm_k(\sigma)$, since by definition $\check{q}_{D(\sigma)}(T) = T^{n-\dim \sigma}$. \square

Corollary 28 *Under the hypotheses of Theorem 27, the following average Morse inequalities hold true:*

$$\mathbb{E}_\nu(b_i) \leq m_k(K^{[n-i]})$$

and

$$\sum_{j=0}^i (-1)^{i-j} \mathbb{E}_\nu(b_j) \leq \sum_{j=0}^i (-1)^{i-j} m_k(K^{[n-j]}),$$

where the latter is an equality if $i = n-k$.

Proof. It follows from Lemma 26 that for every $\epsilon \in \mathcal{C}^{k-1}(K)$ the blocks in V_ϵ provide a filtration of V_ϵ so that the hypotheses of Theorem 39.5 and Theorem 64.1 of [10] are satisfied. As a consequence, the chain complex spanned by the blocks of V_ϵ compute the homology of V_ϵ with $\mathbb{Z}/2\mathbb{Z}$ -coefficients and thus the result follows from Theorem 27 and the Morse inequalities. \square

Corollary 29 *Let K be a compact triangulated homology manifold of dimension $n > 0$. Then, for every $1 \leq k \leq n$ and every $\nu \in [0, 1]$,*

$$\mathbb{E}_\nu(\chi) = \int_K (-1)^{n-\dim \sigma} dm_k(\sigma).$$

In particular,

$$(-1)^{n+1} \mathbb{E}_\nu(\chi) + \chi(K) = \sum_{p=0}^n (-1)^p \mu_\nu(Z^{k-1}(\Delta_p)) f_p(K),$$

where $\mu_\nu(Z^{k-1}(\Delta_p)) = 1$, if $p < k$.

Proof. As in Corollary 19, the case of equality $i = n-k$ in Corollary 28 computes the expected Euler characteristic of V_ϵ , hence the first part. By Proposition 15, we then deduce $(-1)^n \mathbb{E}_\nu(\chi) = \sum_{p=k}^n (-1)^p (1 - \mu_\nu(Z^{k-1}(\Delta_p))) f_p(K)$ and the second part. \square

Remark 30 *Corollary 28 holds true for every k provided K is a compact triangulated homology manifold while Corollary 19 holds true for every finite simplicial complex provided $k = 1$. When both conditions are satisfied, that is $k = 1$ and K is a triangulated compact homology n -manifold, then from Corollary 11, V_ϵ is a compact triangulated homology $(n-1)$ -manifold for every $\epsilon \in \mathcal{C}^{k-1}(K)$ so that Poincaré duality holds true for V_ϵ by Theorem 65.1 of [10]. Then, for every $0 \leq i \leq n-1$, $\mathbb{E}_\nu(b_i) = \mathbb{E}_\nu(b_{n-1-i})$, so that the upper estimates of Corollary 19 and Corollary 28 coincide after the change $i \mapsto n-1-i$.*

Corollary 31 *Under the hypotheses of Theorem 27,*

$$\limsup_{d \rightarrow +\infty} \frac{\mathbb{E}_\nu(b_i)}{(n+1)!^d f_n(K)} \leq q_{n-i,n}(1 - \mu_\nu(Z^{k-1}(\Delta_{n-i})))$$

and

$$\limsup_{d \rightarrow +\infty} \sum_{j=0}^i \frac{(-1)^{i-j} \mathbb{E}_\nu(b_i)}{(n+1)!^d f_n(K)} \leq \sum_{j=0}^i (-1)^{i-j} q_{n-j,n}(1 - \mu_\nu(Z^{k-1}(\Delta_{n-j})))$$

where the latter is an equality if $i = n - k$.

Proof. It follows from Corollary 28, along the same lines as Corollary 22. \square

4.3 The general case

When K is a general finite simplicial complex, the dual block decomposition of K does not span a chain complex which computes the homology of K so that the results of Section 4.2 do not apply. Likewise, Proposition 7 does not extend to $k > 1$, so that in this case the results of Section 4.1 do not apply. However, for every $\epsilon \in C^{k-1}(K)$, V_ϵ is a subcomplex of $\text{Sd}(K)$ so that its homology can be computed with the help of the simplicial homology theory, providing weaker upper estimates which we are going to obtain now.

Recall that every simplex $\alpha \in \text{Sd}(K)$ is of the form $\alpha = [\hat{\sigma}_0, \dots, \hat{\sigma}_p]$, where $\sigma_0 < \dots < \sigma_p$ are simplices of K . We set, following [10], $\text{in}(\alpha) = \sigma_0$ and $\text{fin}(\alpha) = \sigma_p$.

Theorem 32 *Let K be a finite n -dimensional simplicial complex and $k \in \{1, \dots, n\}$. Then, for every $\nu \in [0, 1]$,*

$$\mathbb{E}_\nu(q_{V_\epsilon}(T)) = \int_K q_{D(\sigma)}(T) dm_k(\sigma).$$

Moreover, If K is a compact triangulated homology n -manifold and $k = 1$, then

$$\mathbb{E}_\nu(R_{V_\epsilon}(-1 - T)) = (-1)^n \mathbb{E}_\nu(R_{V_\epsilon}(T)),$$

where $R_{V_\epsilon}(T) = T q_{V_\epsilon}(T) - \chi(V_\epsilon)T$.

Proof. We first observe that for every $0 \leq i \leq n - k$ and every $\nu \in [0, 1]$,

$$\begin{aligned} \mathbb{E}_\nu(f_i) &= \sum_{\epsilon \in C^{k-1}(K)} \sum_{\substack{\alpha \in \text{Sd}(K)^{[i]} \\ \alpha \in V_\epsilon}} \mu_\nu(\epsilon) \\ &= \sum_{\alpha \in \text{Sd}(K)^{[i]}} \mu_\nu\{\epsilon \in C^{k-1}(K) | \alpha \in V_\epsilon\}. \end{aligned}$$

We then deduce

$$\begin{aligned} \mathbb{E}_\nu(q_{V_\epsilon}(T)) &= \sum_{i=0}^{n-k} T^i \sum_{\alpha \in \text{Sd}(K)^{[i]}} \mu_\nu\{\epsilon \in C^{k-1}(K) | \alpha \in V_\epsilon\} \\ &= \sum_{\sigma \in K} \sum_{\substack{\alpha \in \text{Sd}(K) \\ \text{in}(\alpha) = \sigma}} T^{\dim \alpha} \mu_\nu\{\epsilon \in C^{k-1}(K) | \alpha \in V_\epsilon\} \\ &= \int_K q_{D(\sigma)}(T) dm_k(\sigma). \end{aligned}$$

Finally, when K is a compact triangulated homology manifold and $k = 1$, we know from Corollary 11 that for every $\epsilon \in \mathcal{C}^{k-1}(K)$, V_ϵ is itself a compact triangulated homology manifold. Then, from Theorem 2.1 of [9] (see also [11]), it follows that $R_{V_\epsilon}(-1-T) = (-1)^n R_{V_\epsilon}(T)$. The result thus follows after integration over $\mathcal{C}^{k-1}(K)$. \square

Corollary 33 *Let K be a finite n -dimensional simplicial complex and $k \in \{1, \dots, n\}$, $\nu \in [0, 1]$. Then,*

$$\mathbb{E}_\nu(\chi) = \int_K \left(\chi(D(\sigma)) - \chi(\dot{D}(\sigma)) \right) dm_k(\sigma).$$

Moreover, for every $0 \leq i \leq n - k$, the following average Morse inequalities hold:

$$\mathbb{E}_\nu(b_i) \leq \mathbb{E}_\nu(f_i) = \int_K f_i(D(\sigma)) dm_k(\sigma)$$

and

$$\sum_{j=0}^i (-1)^{i-j} \mathbb{E}_\nu(b_j) \leq \int_K \sum_{j=0}^i (-1)^{i-j} f_j(D(\sigma)) dm_k(\sigma),$$

where the latter is an equality if $i = n - k$.

Proof. The first part follows from Theorem 32 after evaluation at $T = -1$, since by definition, for every $\epsilon \in \mathcal{C}^{k-1}(K)$, $\chi(V_\epsilon) = q_{V_\epsilon}(-1)$. Then, for every $\epsilon \in \mathcal{C}^{k-1}(K)$, the Morse inequalities applied to the simplicial chain complex of V_ϵ with $\mathbb{Z}/2\mathbb{Z}$ -coefficients read $b_i(V_\epsilon) \leq f_i(V_\epsilon)$ and $\sum_{j=0}^i (-1)^{i-j} b_j(V_\epsilon) \leq \sum_{j=0}^i (-1)^{i-j} f_j(V_\epsilon)$, the latter being an equality when $i = n - k$. The last part of Corollary 33 thus follows after integration over $\mathcal{C}^{k-1}(K)$. \square

Let us denote by $\lambda_{i,j}$ the number of interior $(j - 1)$ -faces of the subdivided standard simplex $\text{Sd}(\Delta_{i-1})$, with the convention that $\lambda_{i,0} = 0$ if $i > 0$ and $\lambda_{0,0} = 1$.

Definition 34 *For every $0 \leq i \leq n - k$, we set*

$$c_i^+(n, k) = \sum_{p=k+i}^n \delta_i^{p,k} q_{p,n}$$

where $\delta_i^{p,k} = \sum_{l=k}^{p-i} (1 - \mu_\nu(Z^{k-1}(\Delta_l))) f_l(\Delta_p) \lambda_{p-l,i}$.

The constant $\delta_i^{p,k}$ appearing in Definition 34 has actually a probabilistic interpretation, namely $\delta_i^{p,k} = \int_{C^{k-1}(\Delta_p)} f_i(V_\epsilon \cap \overset{\circ}{\Delta_p}) d\mu_\nu(\epsilon)$. We develop this interpretation in Appendix A (see (2)).

Corollary 35 *Let K be a finite n -dimensional simplicial complex. Then, for every $k \in \{1, \dots, n\}$ and every $0 \leq i \leq n - k$,*

$$\limsup_{d \rightarrow +\infty} \frac{\mathbb{E}_\nu(b_i)}{(n+1)^d f_n(K)} \leq c_i^+(n, k),$$

$$\limsup_{d \rightarrow +\infty} \frac{1}{(n+1)!^d f_n(K)} \left(\sum_{j=0}^i (-1)^{i-j} \mathbb{E}_\nu(b_j) \right) \leq \sum_{j=0}^i (-1)^{i-j} c_j^+(n, k)$$

and

$$\lim_{d \rightarrow +\infty} \frac{\mathbb{E}_\nu(\chi)}{(n+1)!^d f_n(K)} = \sum_{i=0}^{n-k} (-1)^i c_i^+(n, k).$$

Proof. By Proposition 15, $m_k = \sum_{p=k}^n (1 - \mu_\nu(Z^{k-1}(\Delta_p))) \gamma_{p,K}$ while $\gamma_{p,K}^d = \frac{\gamma_{p, \text{Sd}^d(K)}}{(n+1)!^d}$ weakly converges to $d\text{vol}_K$ as d grows to $+\infty$, see [11]. By Corollary 33, for every $0 \leq i \leq n-k$, $\frac{\mathbb{E}_\nu(b_i)}{(n+1)!^d f_n(K)} = \frac{1}{f_n(K)} \sum_{p=k}^n (1 - \mu_\nu(Z^{k-1}(\Delta_p))) \int_K f_i(D(\sigma)) d\gamma_{p,K}^d(\sigma)$. Now, by [11], $\frac{1}{f_n(K)} \int_K f_i(D(\sigma)) d\gamma_{p,K}^d(\sigma)$ converges to $\sum_{h=i}^{n-p} q_{p+h,n} f_p(\Delta_{p+h}) \lambda_{h,i}$ as d grows to $+\infty$. We deduce that

$$\limsup_{d \rightarrow +\infty} \frac{\mathbb{E}_\nu(b_i)}{(n+1)!^d f_n(K)} \leq c_i^+(n, k)$$

with

$$\begin{aligned} c_i^+(n, k) &= \sum_{p=k}^{n-i} (1 - \mu_\nu(Z^{k-1}(\Delta_p))) \sum_{h=p+i}^n q_{h,n} f_p(\Delta_h) \lambda_{h-p,i} \\ &= \sum_{h=k+i}^n q_{h,n} \sum_{p=k}^{h-i} (1 - \mu_\nu(Z^{k-1}(\Delta_p))) f_p(\Delta_h) \lambda_{h-p,i}. \end{aligned}$$

Hence the first part of the result.

The second part just follows from the Morse inequalities in Corollary 33. As for the last part, it follows from Corollary 33 and what we have just done, since $\mathbb{E}_\nu(\chi) = \sum_{i=0}^{n-k} (-1)^i \int_K f_i(D(\sigma)) dm_k(\sigma)$. \square

5 Asymptotic topology of V_ϵ , lower estimates

5.1 Lower estimates for the expected Betti numbers

Let us start with a key proposition, the proof of which is inspired by H. Whitney's proof of the existence of triangulation on smooth manifolds [14].

Proposition 36 *For every closed (not necessarily connected) submanifold $\Sigma \subset \mathbb{R}^n$ of codimension $k \geq 1$, there exists $m_n > 0$ such that for every $m \geq m_n$ the pair $(\overset{\circ}{\Delta}_n, V_\epsilon)$ gets homeomorphic to (\mathbb{R}^n, Σ) for some $\epsilon \in \mathcal{C}^{k-1}(\text{Sd}^m(\Delta_n))$, where Δ_n denotes the standard n -simplex.*

In the light of Proposition 36, let us set the following.

Definition 37 *The complexity of a closed (not necessarily connected) submanifold Σ of \mathbb{R}^n is the smallest value m_n given by Proposition 36. It is denoted by $m_n(\Sigma)$. Likewise, the n -dimensional complexity of a closed connected manifold Σ which embeds into \mathbb{R}^n is the infimum of $m_n(\Sigma)$ over all embeddings $\Sigma \hookrightarrow \mathbb{R}^n$.*

Remark 38 *Recall that from H. Whitney's embedding theorem, every manifold Σ of dimension n embeds in \mathbb{R}^{2n+1} . Proposition 36 and Definition 37 thus provide a combinatorial complexity of such closed n -dimensional manifold, namely the infimum of $m_{2n+1}(\Sigma)$ over all embeddings $\Sigma \hookrightarrow \mathbb{R}^{2n+1}$.*

Proof of Proposition 36. Consider a diffeomorphism $\phi : \mathbb{R}^n \rightarrow \mathring{\Delta}_n$ and set $\tilde{\Sigma} = \phi(\Sigma)$. There is a positive integer m_n such that for every $m \geq m_n$, possibly after a small perturbation of ϕ by an isotopy of $\mathring{\Delta}_n$ with compact support, we have the following properties for $\tilde{\Sigma}$ (conditions that appear in Section 13 of [14]) :

1. $\tilde{\Sigma}$ does not intersect the $(k-1)$ -skeleton of $\text{Sd}^m(\Delta_n)$,
2. the intersection, $\widehat{\Sigma}_k$, of $\tilde{\Sigma}$ with each k -simplex of $\text{Sd}^m(\Delta_n)$, if not empty, is transversal at one point,
3. the intersection of $\tilde{\Sigma}$ with each l -simplex σ of dimension $l > k$ is isotopic to the cone over the intersection of $\widehat{\Sigma}_{l-1}$ with the $(l-1)$ -skeleton of σ centered at the barycenter $\hat{\sigma}$ of σ .

Let $\widehat{\Sigma} = \widehat{\Sigma}_n$ denote the cone described above. The pair $(\mathring{\Delta}_n, \tilde{\Sigma})$ is homeomorphic to $(\mathring{\Delta}_n, \widehat{\Sigma})$.

Let $\hat{\epsilon} \in \mathcal{C}^k(\text{Sd}^m(\Delta_n))$ be the k -cochain defined by the relation $\langle \hat{\epsilon}, \sigma \rangle = \widehat{\Sigma} \circ \sigma$ which is either 0 or 1 for every k -simplex $\sigma \in \text{Sd}^m(\Delta_n)$ (here \circ denotes the intersection index of Σ and σ). By definition, $\hat{\epsilon}$ is a cocycle. Indeed, for all $(k+1)$ -simplex τ of $\text{Sd}^m(\Delta_n)$, the intersection of $\widehat{\Sigma}$ and τ is either empty or isotopic to an interval. Thus, we have $\langle d\hat{\epsilon}, \tau \rangle = \langle \hat{\epsilon}, \partial\tau \rangle = 0$ for all $(k+1)$ -simplex $\tau \in \text{Sd}^m(\Delta_n)$. Therefore, as the simplex Δ_n is acyclic, there exists $\epsilon \in \mathcal{C}^{k-1}(\text{Sd}^m(\Delta_n))$ such that $\hat{\epsilon} = d\epsilon$.

To construct the isotopy between V_ϵ and $\widehat{\Sigma}$, we proceed by induction on the dimension of the skeleton as in the proof of Proposition 7. Let us recall that the intersection of a k -simplex σ with V_ϵ is either empty or a single point, as in the case of the intersection of σ with $\widehat{\Sigma}$. Moreover, by definition of ϵ , $V_\epsilon \cap \sigma$ is non empty if and only if $\widehat{\Sigma} \cap \sigma$ is non empty. When the intersection is non-empty, we consider the two points on σ determined as the intersection with V_ϵ and with $\widehat{\Sigma}$ and isotope V_ϵ so that these two points match. Now let us suppose that V_ϵ can be isotoped to $\widehat{\Sigma}$ up to the level of l -skeleton for $l \geq k$. Let τ be a $(l+1)$ -simplex. If V_ϵ meets τ , then the intersection $V_\epsilon \cap \tau$ is the cone centered at the barycenter $\hat{\tau}$ of τ over the intersections of V_ϵ with the l -skeleton of τ . Besides $\widehat{\Sigma} \cap \tau$, if not empty, is the cone defined at $\hat{\tau}$ over the intersection of $\widehat{\Sigma}$ with the l -skeleton of τ .

We can extend the isotopy between $\Sigma \cap \partial\tau$ and $V_\epsilon \cap \partial\tau$ to an isotopy between $\Sigma \cap \tau$ and $V_\epsilon \cap \tau$ by taking the cone over $\hat{\tau}$. \square

Remark 39 Note that the cochain $\epsilon \in \mathcal{C}^{k-1}(\text{Sd}^m(\Delta_n))$ given by Proposition 36 is not unique since we may add to ϵ any $(k-1)$ -cocycle. For example, when k is odd, another ϵ is obtained by switching the labelings of 0 and 1, that is replacing ϵ by $1 - \epsilon$. When k is even, consider for example the cochain $\tilde{\epsilon} \in \mathcal{C}^{k-2}(\text{Sd}^m(\Delta_n))$ which labels 1 only one $(k-2)$ -dimensional simplex. Then $d\tilde{\epsilon}$ is a $(k-1)$ -cocycle and we may replace ϵ by $\epsilon + d\tilde{\epsilon}$.

Let $\Sigma \subset \mathbb{R}^n$ and $m_n(\Sigma)$ be given by Proposition 36. Let $m_n(\Sigma) \leq m \leq d$ and $\sigma \in \text{Sd}^{d-m}(K)$ be an n -simplex. We set

$$\text{prob}(\sigma, \Sigma) = \mu_\nu\{\epsilon \in \mathcal{C}^{k-1}(\text{Sd}^d(K)) | (\mathring{\sigma}, V_\epsilon \cap \mathring{\sigma}) \text{ is homeomorphic to } (\mathbb{R}^n, \Sigma)\}.$$

Theorem 40 For every n -dimensional finite simplicial complex K , every closed codimension $k \geq 1$ submanifold Σ of \mathbb{R}^n , every $m_n(\Sigma) \leq m \leq d$ and every n -simplex $\sigma \in \text{Sd}^{d-m}(K)$,

$$\text{prob}(\sigma, \Sigma) \geq \frac{1}{2^{f_{k-1}^m(\Delta_n)-1}}$$

where $f_{k-1}^m(\Delta_n) = f_{k-1}(\text{Sd}^m(\Delta_n))$.

Proof. Observe that $f_{k-1}^d(K) = f_{k-1}^m(\sigma) + R$ where R is the number of $(k-1)$ -simplices of $\text{Sd}^d(K)$ which are not in $\text{Sd}^m(\sigma)$. By Remark 39, we know that there are at least two choices of $\epsilon_\sigma \in \mathcal{C}^{k-1}(\text{Sd}^m(\sigma))$ with the property that the pair $(\overset{\circ}{\sigma}, V_{\epsilon_\sigma})$ is homeomorphic to (\mathbb{R}^n, Σ) . Any of these two ϵ_σ extend to $\epsilon \in \mathcal{C}^{k-1}(\text{Sd}^d(K))$ such that $(\overset{\circ}{\sigma}, V_\epsilon \cap \overset{\circ}{\sigma})$ is homeomorphic to (\mathbb{R}^n, Σ) . The restriction of ϵ on the R many $(k-1)$ -simplices of $\text{Sd}^d(K) \setminus \text{Sd}^m(\sigma)$ is arbitrary and so we deduce that $\text{prob}(\sigma, \Sigma)$ is at least $\frac{2 \times 2^R}{2^{f_{k-1}^d(K)}} = \frac{2 \times 2^R}{2^{f_{k-1}^m(\sigma) + R}} = \frac{1}{2^{f_{k-1}^m(\sigma) - 1}} \cdot \square$

For every $\epsilon \in \mathcal{C}^{k-1}(\text{Sd}^d(K))$, let $N_\Sigma(\epsilon)$ be the maximum number of disjoint open simplices $(\overset{\circ}{\sigma}_j)_{j \in J}$ which can be packed in K in such a way that the pair $(\overset{\circ}{\sigma}_j, V_\epsilon \cap \overset{\circ}{\sigma}_j)$ is homeomorphic to (\mathbb{R}^n, Σ) , where for every $j \in J$, σ_j is a simplex of $\text{Sd}^{d-m_j}(K)$ for some $m_j \in \{m_n(\Sigma), \dots, d\}$. We now set $p_\Sigma = \frac{1}{2^{f_{k-1}^{m_n(\Sigma)}(\Delta_n)-1}}$, the right hand side of the inequality in Theorem 40 for $m = m_n(\Sigma)$ and $c_\Sigma = \frac{p_\Sigma}{(n+1)!^{m_n(\Sigma)}} > 0$ and $\mathbb{E}_\nu(N_\Sigma) = \int_{\mathcal{C}^{k-1}(\text{Sd}^d(K))} N_\Sigma(\epsilon) d\mu_\nu(\epsilon)$.

Theorem 41 For every n -dimensional finite simplicial complex K and every closed codimension $k \geq 1$ submanifold Σ of \mathbb{R}^n ,

$$c_\Sigma \leq \liminf_{d \rightarrow +\infty} \frac{\mathbb{E}_\nu(N_\Sigma)}{(n+1)!^d f_n(K)}.$$

Proof. Let $d \geq m_n(\Sigma)$. For every n -simplex $\sigma \in \text{Sd}^{d-m_n(\Sigma)}(K)$ and every $\epsilon \in \mathcal{C}^{k-1}(\text{Sd}^d(K))$, let $N_{\Sigma, \sigma}(\epsilon)$ be equal to 1 if $(\overset{\circ}{\sigma}, V_\epsilon \cap \overset{\circ}{\sigma})$ is homeomorphic to (\mathbb{R}^n, Σ) and 0 otherwise. Then

$$\begin{aligned} \mathbb{E}_\nu(N_\Sigma) &= \int_{\mathcal{C}^{k-1}(\text{Sd}^d(K))} N_\Sigma(\epsilon) d\mu_\nu(\epsilon) \\ &\geq \int_{\mathcal{C}^{k-1}(\text{Sd}^d(K))} \sum_{\sigma \in \text{Sd}^{d-m_n(\Sigma)}(K)} N_{\Sigma, \sigma}(\epsilon) d\mu_\nu(\epsilon) \\ &= \sum_{\sigma \in \text{Sd}^{d-m_n(\Sigma)}(K)} \int_{\mathcal{C}^{k-1}(\text{Sd}^d(K))} N_{\Sigma, \sigma}(\epsilon) d\mu_\nu(\epsilon) \\ &\geq (n+1)!^{d-m_n(\Sigma)} f_n(K) p_\Sigma. \end{aligned}$$

The last line follows from Theorem 40 and the fact that the number of n -simplices of an n -dimensional simplicial complex gets multiplied by $\lambda_{n+1, n+1} = (n+1)!$ after a barycentric subdivision, see [4] or also [11]. Thus, we get

$$\frac{\mathbb{E}_\nu(N_\Sigma)}{(n+1)!^d f_n(K)} \geq \frac{p_\Sigma}{(n+1)!^{m_n(\Sigma)}} = c_\Sigma.$$

Hence the result. \square

Now, for every $m \geq 1$, let $\mathbf{C}(m)$ be the finite set of homeomorphism classes of pairs (\mathbb{R}^n, Σ) , where Σ is a closed connected $(n-k)$ -dimensional manifold embedded in \mathbb{R}^n by an embedding of complexity $m = m_n(\Sigma)$, see Definition 37. For every $\epsilon \in \mathcal{C}^{k-1}(\text{Sd}^d(K))$ and every $i \in \{0, 1, \dots, n-k\}$, we set $b_i(V_\epsilon) = \dim H_i(V_\epsilon)$ and $\mathbb{E}_\nu(b_i) = \int_{\mathcal{C}^{k-1}(\text{Sd}^d(K))} b_i(V_\epsilon) d\mu_\nu(\epsilon)$.

Definition 42 For every $1 \leq k \leq n$ and $0 \leq i \leq n - k$, we set

$$c_i^-(n, k) = \sum_{m=1}^{+\infty} \frac{1}{2^{f_{k-1}^m(\Delta_n)-1} (n+1)!^m} \sum_{(\mathbb{R}^n, \Sigma) \in \mathbf{C}(m)} b_i(\Sigma).$$

Corollary 43 For every finite n -dimensional complex K , every $1 \leq k \leq n$ and every $i \in \{0, 1, \dots, n - k\}$,

$$\liminf_{d \rightarrow +\infty} \frac{\mathbb{E}_\nu(b_i)}{(n+1)!^d f_n(K)} \geq c_i^-(n, k)$$

Proof. Let $\epsilon \in \mathcal{C}^{k-1}(\text{Sd}^d(K))$. For every connected component of V_ϵ which is contained in the interior of an n -simplex $\sigma \in \text{Sd}^{d-m}(K)$ in such a way that $(\overset{\circ}{\sigma}, V_\epsilon \cap \overset{\circ}{\sigma})$ is homeomorphic to (\mathbb{R}^n, Σ) for some codimension k submanifold Σ of \mathbb{R}^n , where $m_n(\Sigma) \leq m \leq d$, the homeomorphism type of the pair (\mathbb{R}^n, Σ) does not depend on the choice of σ and m in the case it is not unique. We deduce that for every $M > 0$,

$$b_i(V_\epsilon) \geq \sum_{m=1}^M \sum_{(\mathbb{R}^n, \Sigma) \in \mathbf{C}(m)} b_i(\Sigma) N_\Sigma(\epsilon).$$

After integration we get, $\mathbb{E}_\nu(b_i) \geq \sum_{m=1}^M \sum_{(\mathbb{R}^n, \Sigma) \in \mathbf{C}(m)} b_i(\Sigma) \mathbb{E}_\nu(N_\Sigma)$. Theorem 41 then implies that

$$\liminf_{d \rightarrow +\infty} \frac{\mathbb{E}_\nu(b_i)}{(n+1)!^d f_n(K)} \geq \sum_{m=1}^M \sum_{(\mathbb{R}^n, \Sigma) \in \mathbf{C}(m)} \frac{b_i(\Sigma)}{2^{f_{k-1}^m(\Delta_n)-1} (n+1)!^m}.$$

The result follows by letting M grow to $+\infty$. □

5.2 Complexity of surfaces in \mathbb{R}^3

Let us now study the 3-dimensional complexity of surfaces in the sense of Definition 37. We first observe that there exists $\epsilon \in \mathcal{C}^0(\text{Sd}(\Delta_3))$ such that V_ϵ is homeomorphic to a 2-sphere. Indeed, let ϵ take the value 0 on the barycenter of Δ_3 and 1 on all the other vertices of $\text{Sd}(\Delta_3)$, see Figure 3. The complexity of the 2-sphere is thus 1.

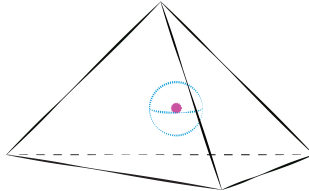


Figure 3: In $\text{Sd}(\Delta_3)$, V_ϵ may be a topological sphere.

More generally,

Lemma 44 *For every $r \in \{0, 1, 2, 3, 4\}$, there exists $\epsilon \in \mathcal{C}^0(\text{Sd}(\Delta_3))$ such that V_ϵ is homeomorphic to a sphere with r holes.*

Proof. Let ϵ take the value 1 on each vertex of Δ_3 and each barycenter of an edge of Δ_3 and let ϵ take the value 0 on the barycenter of Δ_3 itself. Now, depending on whether ϵ takes the value 0 or 1 on each barycenter of the codimension-1 faces of Δ_3 , V_ϵ becomes homeomorphic to a sphere with up to four holes, see Figures 3, 4, and 5. \square

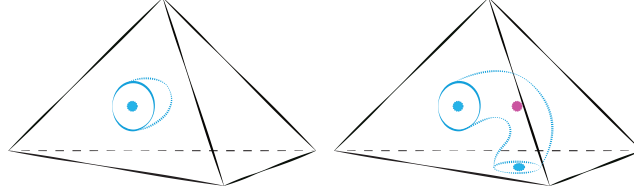


Figure 4: In $\text{Sd}(\Delta_3)$, V_ϵ can be a disc or a cylinder.

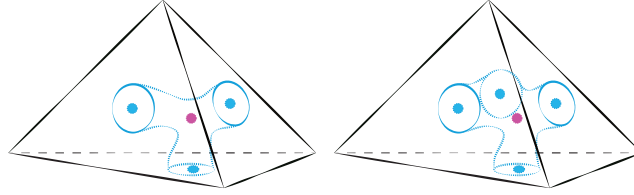


Figure 5: In $\text{Sd}(\Delta_3)$, V_ϵ a sphere with 3 or 4 holes.

Theorem 45 *Let Σ be a compact connected orientable surface of Euler characteristic $\chi(\Sigma) \geq 4(3!)^{d-1} - 2(4!)^{d-1}$ with $d \geq 1$. Then, there exists $\epsilon \in \mathcal{C}^0(\text{Sd}^d(\Delta_3))$ such that V_ϵ is homeomorphic to Σ .*

For example, a compact connected orientable surface of genus $0 < g \leq 13$ (respectively $13 < g \leq 505$) has embeddings of complexity two (respectively three) in \mathbb{R}^3 , in the sense of Definition 37.

Proof. We proceed as in Lemma 44. Let ϵ take the value 1 on each vertex and on the barycenter of each edge of $\text{Sd}^{d-1}(\Delta_3)$ and take the value 0 on the barycenter of each 3-simplex of $\text{Sd}^{d-1}(\Delta_3)$. The number of such 3-simplices is $(4!)^{d-1}$. If we let ϵ be 1 on the barycenter of each codimension-2 face of $\text{Sd}^{d-1}(\Delta_3)$, then V_ϵ becomes homeomorphic to the disjoint union of $(4!)^{d-1}$ copies of the 2-sphere. Changing this value to 0 on the barycenter of one interior triangle of $\text{Sd}^{d-1}(\Delta_3)$ results in a connected sum of the two corresponding spheres, which gives rise to a decrease in the Euler characteristic by two. Now the number of 2-dimensional faces of $\text{Sd}^{d-1}(\Delta_3)$ which lie in the interior of Δ_3 is $\frac{1}{2}(4(4!)^{d-1} - 4(3!)^{d-1}) = 2((4!)^{d-1} - (3!)^{d-1})$, since every such face bounds two 3-simplices and each such 3-simplex has 4 codimension 1 faces. By letting ϵ be 1 on the barycenter of each 2-simplex on the boundary of $\text{Sd}(\Delta_3)$ and 0 or 1 on the barycenter of interior ones, we may thus connect sum together the disjoint union of $(4!)^{d-1}$ copies of S^2 using up to $2((4!)^{d-1} - (3!)^{d-1})$ cylinders. The first $(4!)^d - 1$ connected

sums can be made to connect together the $(4!)^{d-1}$ copies of S^2 to get a single S^2 . The result follows. \square

Remark 46 *The proof can be carried out in higher dimensions as well, to produce hypersurfaces which are connected sums of spheres S^{n-1} with handles $S^1 \times S^{n-2}$.*

A More on the universal constants $c_i^+(n, k)$

A.1 Section 4.3 revisited

For every $1 \leq k \leq p$, we set

$$\delta^{p,k}(T) = \mathbb{E}_\nu(q_{V_\epsilon \cap \overset{\circ}{\Delta}_p}(T)) = \sum_{i=0}^{p-k} \delta_i^{p,k} T^i,$$

where $\delta_i^{p,k} = \int_{\epsilon \in \mathcal{C}^{k-1}(\Delta_p)} f_i(V_\epsilon \cap \overset{\circ}{\Delta}_p) d\mu_\nu(\epsilon)$.

These universal polynomials are associated to the standard simplices $(\Delta_p)_{p \geq 1}$. Recall that for every $\epsilon \in \mathcal{C}^{k-1}(\Delta_p)$, V_ϵ is a subcomplex of $\text{Sd}(\Delta_p)$ so that by $V_\epsilon \cap \overset{\circ}{\Delta}_p$, we mean the simplices of V_ϵ , the interior of which lie in $\overset{\circ}{\Delta}_p$. These are the simplices of the form $[\hat{\sigma}_0, \dots, \hat{\sigma}_i]$ where $\sigma_i = \Delta_p$.

The first part of Theorem 32 can be formulated in terms of those polynomials as follows.

Theorem 47 *Let K be a finite n -dimensional simplicial complex and $k \in \{1, \dots, n\}$. Then, for every $\nu \in [0, 1]$,*

$$\mathbb{E}_\nu(q_{V_\epsilon}(T)) = \sum_{p=k}^n f_p(K) \delta^{p,k}(T).$$

In order to prove Theorem 47, we need first the following lemma.

Lemma 48 *Let K be a finite n -dimensional simplicial complex and L be a union of simplices of K . Then, for every $\nu \in [0, 1]$, every $0 \leq i \leq n - k$ and every $k \in \{1, \dots, n\}$,*

$$\mathbb{E}_\nu(f_i(V_\epsilon \cap \overset{\circ}{L})) = \sum_{\sigma \in \text{Sd}(K)^{[i]} \cap \overset{\circ}{L}} \mu_\nu\{\epsilon \in \mathcal{C}^{k-1}(K) | \sigma \in V_\epsilon\}.$$

Proof. We observe that

$$\begin{aligned}
\mathbb{E}_\nu(f_i(V_\epsilon \cap \overset{\circ}{L})) &= \int_{\mathcal{C}^{k-1}(K)} f_i(V_\epsilon \cap \overset{\circ}{L}) d\mu_\nu(\epsilon) \\
&= \sum_{\epsilon \in \mathcal{C}^{k-1}(K)} \sum_{\substack{\sigma \in \text{Sd}(K)^{[i]} \cap \overset{\circ}{L} \\ \sigma \in V_\epsilon}} \mu_\nu(\epsilon) \\
&= \sum_{\sigma \in \text{Sd}(K)^{[i]} \cap \overset{\circ}{L}} \sum_{\substack{\epsilon \in \mathcal{C}^{k-1}(K) \\ \sigma \in V_\epsilon}} \mu_\nu(\epsilon) \\
&= \sum_{\sigma \in \text{Sd}(K)^{[i]} \cap \overset{\circ}{L}} \mu_\nu\{\epsilon \in \mathcal{C}^{k-1}(K) | \sigma \in V_\epsilon\},
\end{aligned}$$

□

In case $L = K$, Lemma 48 gives

$$\mathbb{E}_\nu(f_i) = \sum_{\sigma \in \text{Sd}(K)^{[i]}} \mu_\nu\{\epsilon \in \mathcal{C}^{k-1}(K) | \sigma \in V_\epsilon\}.$$

Proof of Theorem 47. From Lemma 48, we know that for every $0 \leq i \leq n - k$,

$$\mathbb{E}_\nu(f_i) = \sum_{\sigma \in \text{Sd}(K)^{[i]}} \mu_\nu\{\epsilon \in \mathcal{C}^{k-1}(K) | \sigma \in V_\epsilon\}.$$

Recall that the i -simplex $\sigma = [\hat{\sigma}_0, \dots, \hat{\sigma}_i] \in \text{Sd}(K)$ belongs to V_ϵ if and only if $\dim \sigma_0 \geq k$ and there exists a k -face τ of σ_0 such that $\langle d\epsilon, \tau \rangle \neq 0$.

In particular, we deduce that $p = \dim \sigma_i \geq k + i$ and

$$\begin{aligned}
\mathbb{E}_\nu(f_i) &= \sum_{p=k+i}^n \sum_{\sigma_i \in K^{[p]}} \sum_{\sigma \in \text{Sd}(\sigma_i)^{[i]} \cap \overset{\circ}{\sigma_i}} \mu_\nu\{\epsilon \in \mathcal{C}^{k-1}(K) | \sigma \in V_\epsilon\} \\
&= \sum_{p=k+i}^n f_p(K) \delta_i^{p,k}
\end{aligned} \tag{1}$$

since from Lemma 48 it follows that the i^{th} coefficient $\delta_i^{p,k}$ of $\delta^{p,k}(T)$ equals $\sum_{\sigma \in \text{Sd}(\Delta_p)^{[i]} \cap \overset{\circ}{\Delta_p}} \mu_\nu\{\epsilon \in \mathcal{C}^{k-1}(\Delta_p) | \sigma \in V_\epsilon\}$, while μ_ν is a product measure.

Now,

$$\begin{aligned}
\mathbb{E}_\nu(q_{V_\epsilon}(T)) &= \sum_{i=0}^{n-k} \mathbb{E}_\nu(f_i) T^i \\
&= \sum_{i=0}^{n-k} T^i \sum_{p=k+i}^n f_p(K) \delta_i^{p,k} \\
&= \sum_{i=0}^{n-k} T^i \sum_{p=i}^{n-k} f_{p+k}(K) \delta_i^{p+k,k} \\
&= \sum_{p=0}^{n-k} f_{p+k}(K) \sum_{i=0}^p \delta_i^{p+k,k} T^i \\
&= \sum_{p=k}^n f_p(K) \delta^{p,k}(T).
\end{aligned}$$

□

Let us set, for every $\nu \in [0, 1]$ and $k \leq p \leq n$, $\tau_p = \delta^{p,k}(-1)$.

Corollary 49 *Let K be a finite n -dimensional simplicial complex and $k \in \{1, \dots, n\}$, $\nu \in [0, 1]$. Then,*

$$\mathbb{E}_\nu(\chi) = \sum_{p=k}^n f_p(K) \tau_p.$$

Moreover, for every $0 \leq i \leq n - k$, the following average Morse inequalities hold

$$\mathbb{E}_\nu(b_i) \leq \mathbb{E}_\nu(f_i) = \sum_{p=k+i}^n f_p(K) \delta_i^{p,k}$$

and

$$\sum_{j=0}^i (-1)^{i-j} \mathbb{E}_\nu(b_j) \leq \sum_{j=0}^i (-1)^{i-j} \mathbb{E}_\nu(f_j),$$

where the latter is an equality if $i = n - k$.

From Corollary 49 we deduce another formulation of the last part of Corollary 35, since we deduce that $\sum_{i=0}^{n-k} (-1)^i c_i^+(n, k) = \sum_{p=k}^n \tau_p q_{p,n}$.

Proof. The first part follows from Theorem 47 after evaluation at $T = -1$, since by definition, for every $\epsilon \in \mathcal{C}^{k-1}(K)$, $\chi(V_\epsilon) = q_{V_\epsilon}(-1)$. Then, for every $\epsilon \in \mathcal{C}^{k-1}(K)$, the Morse inequalities applied to the simplicial chain complex of V_ϵ with $\mathbb{Z}/2\mathbb{Z}$ -coefficients read $b_i(V_\epsilon) \leq f_i(V_\epsilon)$ and $\sum_{j=0}^i (-1)^{i-j} b_j(V_\epsilon) \leq \sum_{j=0}^i (-1)^{i-j} f_j(V_\epsilon)$, the latter being an equality when $i = n - k$. The last part of Corollary 49 thus follows from (1) after integration over $\mathcal{C}^{k-1}(K)$. □

Examples :

1. When $k = 1$ the first part of Corollary 49 gives back Corollary 20, as follows from Corollary 52.

2. When $k = n$ and $\nu = \frac{1}{2}$, V_ϵ is a finite set of points for every $\epsilon \in \mathcal{C}^{n-1}(K)$ and Corollary 49 combined with Corollary 52 gives $\mathbb{E}(\chi) = \frac{1}{2}f_n(K)$.
3. When $k = n - 1$ and $\nu = \frac{1}{2}$, V_ϵ is a graph for every $\epsilon \in \mathcal{C}^{n-2}(K)$ and the second part of Corollary 49 combined with Corollary 52 gives

$$\mathbb{E}(b_0) \leq \frac{f_{n-1}(K)}{2} + (1 - \frac{1}{2^n})f_n(K),$$

$$\mathbb{E}(b_1) \leq \frac{n+1}{2}f_n(K),$$

$$\mathbb{E}(\chi) = \frac{f_{n-1}(K)}{2} + (\frac{1-n}{2} - \frac{1}{2^n})f_n(K).$$

A.2 Computations of the universal polynomials $\delta^{p,k}(T)$

The coefficients of the universal polynomials $\delta^{p,k}(T)$ introduced in Section A.1 are given by the following theorem.

Theorem 50 *For every $k \in \{1, \dots, n\}$, every $i \in \{0, \dots, p - k\}$ and every $\nu \in [0, 1]$,*

$$\delta_i^{p,k} = \sum_{j=0}^i \binom{i}{j} (-1)^{i-j} \sum_{l=i}^{p-k} \binom{p+1}{l} (1 - \mu_\nu(Z^{k-1}(\Delta_{p-l}))) j^l$$

and

$$\tau_p = \sum_{l=k}^p \binom{p+1}{l+1} (-1)^{p-l} (1 - \mu_\nu(Z^{k-1}(\Delta_l))).$$

In particular, we deduce from Corollary 17 and Theorem 50 that for $\nu = \frac{1}{2}$,

$$\delta_i^{p,k} = \sum_{j=0}^i \binom{i}{j} (-1)^{i-j} \sum_{l=i}^{p-k} \binom{p+1}{l} (1 - \frac{1}{2^{\binom{p-l}{k}}}) j^l$$

and

$$\tau_p = \sum_{l=k}^p \binom{p+1}{l+1} (-1)^{p-l} (1 - \frac{1}{2^{\binom{l}{k}}}).$$

In order to prove Theorem 50, we need first the following Lemma 51.

For every $l < p$ and every $0 < i \leq p - l$, let us denote by $\text{ind}(l, p, i)$ the number of i -simplices of $\text{Sd}(\Delta_p)$ which are of the form $[\hat{\sigma}_0, \dots, \hat{\sigma}_i]$ with $\dim \sigma_0 = l$ and $\dim \sigma_i = p$.

Lemma 51 *For every $l < p$ and every $0 < i \leq p - l$, $\text{ind}(l, p, i) = \binom{p+1}{l+1} \lambda_{p-l,i}$.*

Recall that $\lambda_{p-l,i}$ is the number of interior $(i - 1)$ -faces of $\text{Sd}(\Delta_{p-l-1})$.

Proof. We observe that

1. there are $\binom{p+1}{l+1}$ choices for a l -simplex σ_0 of Δ_p .
2. there is a bijection between the $(i + 1)$ -flags $(\sigma_0 < \dots < \sigma_i = \Delta_p)$ and the i -flags $(\text{Lk}(\sigma_0, \sigma_1) < \dots < \text{Lk}(\sigma_0, \sigma_i) = \Delta_{p-l-1})$.

Now, the i -flags $(\text{Lk}(\sigma_0, \sigma_1) < \dots < \text{Lk}(\sigma_0, \sigma_i))$ exactly define the $(i-1)$ -simplices interior to $\text{Sd}(\Delta_{p-l-1})$. By definition there are $\lambda_{p-l,i}$ many such simplices, hence the result. \square

Proof of Theorem 50. By definition and Lemma 48,

$$\delta_i^{p,k} = \sum_{\sigma \in \text{Sd}(\Delta_p)^{[i]} \cap \overset{\circ}{\Delta}_p} \mu_\nu\{\epsilon \in \mathcal{C}^{k-1}(\Delta_p) | \sigma \in V_\epsilon\}.$$

If $i = 0$, we deduce that $\delta_0^{p,k} = 1 - \mu_\nu(Z^{k-1}(\Delta_p))$. If $i > 0$, a simplex $\sigma \in \text{Sd}(\Delta_p)^{[i]} \cap \overset{\circ}{\Delta}_p$ is of the form $[\hat{\sigma}_0, \dots, \hat{\sigma}_i]$ where $\sigma_i = \Delta_p$ and $\sigma_0 < \sigma_i$ is a face of dimension $l \in \{0, \dots, p-i\}$. Moreover, such a simplex belongs to V_ϵ if and only if $l \geq k$ and the restriction of $d\epsilon$ to σ_0 does not vanish. We thus deduce

$$\begin{aligned} \delta_i^{p,k} &= \sum_{l=k}^{p-i} \text{ind}(l, p, i) (1 - \mu_\nu(Z^{k-1}(\Delta_l))) \\ &= \sum_{l=k}^{p-i} \binom{p+1}{l+1} \lambda_{p-l,i} (1 - \mu_\nu(Z^{k-1}(\Delta_l))) \\ &= \sum_{l=i}^{p-k} \binom{p+1}{l} \lambda_{l,i} (1 - \mu_\nu(Z^{k-1}(\Delta_{p-l}))), \end{aligned} \tag{2}$$

where the second line follows from Lemma 51. From [4], (see also [11]) we now deduce,

$$\begin{aligned} \delta_i^{p,k} &= \sum_{l=i}^{p-k} \binom{p+1}{l} \sum_{j=0}^i \binom{i}{j} (-1)^{i-j} j^l (1 - \mu_\nu(Z^{k-1}(\Delta_{p-l}))) \\ &= \sum_{j=0}^i \binom{i}{j} (-1)^{i-j} \sum_{l=i}^{p-k} \binom{p+1}{l} (1 - \mu_\nu(Z^{k-1}(\Delta_{p-l}))) j^l, \end{aligned}$$

where the formula remains valid for $i = 0$ with the convention $0^0 = 1$. Then,

$$\begin{aligned} \tau_p &= \sum_{i=0}^{p-k} (-1)^i \delta_i^{p,k} = \sum_{i=0}^{p-k} (-1)^i \sum_{l=i}^{p-k} \binom{p+1}{l} \lambda_{l,i} (1 - \mu_\nu(Z^{k-1}(\Delta_{p-l}))) \\ &= \sum_{l=0}^{p-k} \binom{p+1}{l} (1 - \mu_\nu(Z^{k-1}(\Delta_{p-l}))) \sum_{i=0}^l (-1)^i \lambda_{l,i} \\ &= 1 - \mu_\nu(Z^{k-1}(\Delta_p)) + \sum_{l=1}^{p-k} \binom{p+1}{l} (1 - \mu_\nu(Z^{k-1}(\Delta_{p-l}))) \sum_{i=0}^l (-1)^i \lambda_{l,i} \end{aligned}$$

Since by definition $\lambda_{l,0} = 0$ for every $l > 0$. Moreover, $\sum_{i=1}^l (-1)^i \lambda_{l,i} = -(\chi(\Delta_{l-1}) - \chi(\partial\Delta_{l-1})) = (-1)^l$. We thus deduce

$$\begin{aligned}
\tau_p &= \sum_{l=0}^{p-k} \binom{p+1}{l} (1 - \mu_\nu(Z^{k-1}(\Delta_{p-l}))) (-1)^l \\
&= \sum_{l=k}^p \binom{p+1}{l+1} (-1)^{p-l} (1 - \mu_\nu(Z^{k-1}(\Delta_l))).
\end{aligned}$$

□

Corollary 52 1. Let $\nu = \frac{1}{2}$. Then, for every $0 \leq i \leq n - k$, $\delta_i^{k+i,k} = \frac{i!}{2} \binom{k+i+1}{i}$.

2. Let $\nu = \frac{1}{2}$. Then, for every $1 \leq k \leq p \leq n$, $\delta_0^{p,k} = 1 - \frac{1}{2 \binom{p}{k}}$, and $\tau_k = \frac{1}{2}$.

3. If $k = 1$, $\tau_p = (-1)^{p+1} (1 - \nu^{p+1} - (1 - \nu)^{p+1})$ for every $1 \leq p \leq n$ and every $\nu \in [0, 1]$.

Proof.

1. From Corollary 17 and Theorem 50, $\delta_i^{k+i,k} = \sum_{l=i}^i \binom{k+i+1}{l} \lambda_{l,i} (1 - \frac{1}{2 \binom{k+i-l}{k}}) = \binom{k+i+1}{i} \lambda_{i,i} (1 - \frac{1}{2})$. The result then follows from [11].
2. Since $\lambda_{l,0} = 0$ for every $l > 0$ and $\lambda_{0,0} = 1$, Corollary 17 and Theorem 50 imply that $\delta_0^{p,k} = 1 - \frac{1}{2 \binom{p}{k}}$, and $\tau_k = (1 - \frac{1}{2}) = \frac{1}{2}$.
3. When $k = 1$, we know from Theorem 50 that for every $\nu \in [0, 1]$,

$$\tau_p = \sum_{l=0}^{p-1} (-1)^l \binom{p+1}{l} (1 - \nu^{p-l+1} - (1 - \nu)^{p-l+1}).$$

Thus, τ_p is written as the sum of three terms $A_1 = \sum_{l=0}^{p-1} (-1)^l \binom{p+1}{l}$, $A_2 = -\nu^{p+1} \sum_{l=0}^{p-1} (\frac{-1}{\nu})^l \binom{p+1}{l}$ and $A_3 = -(1 - \nu)^{p+1} \sum_{l=0}^{p-1} (\frac{-1}{1-\nu})^l \binom{p+1}{l}$.

Using binomial expansion we get

$$\begin{aligned}
A_1 &= (1 - 1)^{p+1} - (-1)^p p - (-1)^{p+1}, \\
A_2 &= -\nu^{p+1} ((1 - \frac{1}{\nu})^{p+1} - (\frac{-1}{\nu})^p p - (\frac{-1}{\nu})^{p+1}), \\
A_3 &= -(1 - \nu)^{p+1} ((1 - \frac{1}{1-\nu})^{p+1} - (\frac{-1}{1-\nu})^p p - (\frac{-1}{1-\nu})^{p+1}).
\end{aligned}$$

Hence the result.

□

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