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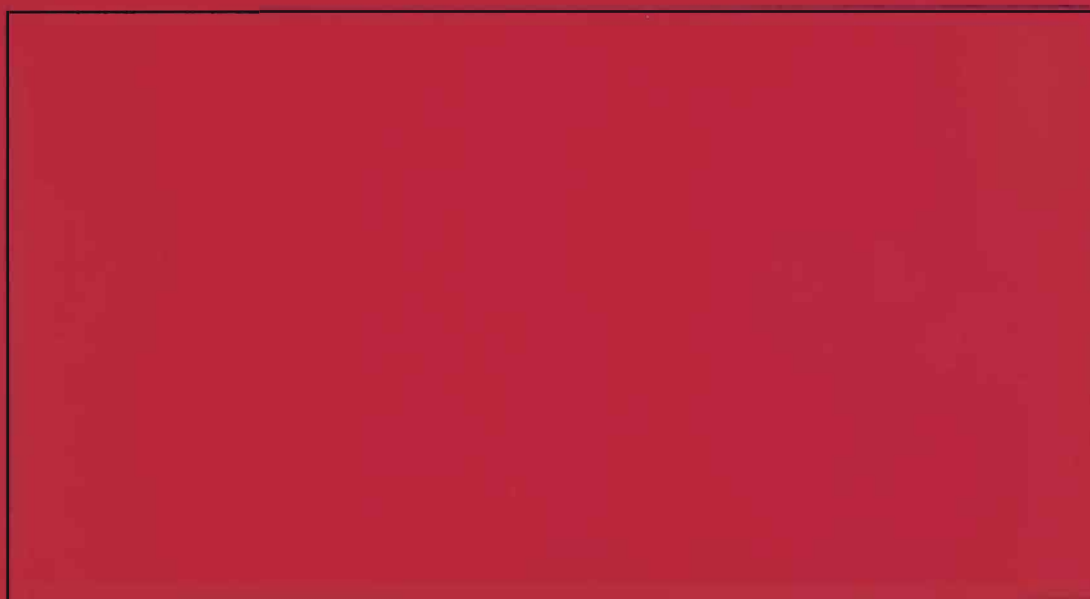
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**NOTE ON THE RANKING OF FUZZY NUMBERS :
CONDITIONS FOR A TOTAL ORDER RELATION**

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*Paper dedicated to
Professor Jerzy Albrycht*

Abstract : It is well-known that the set of fuzzy numbers is not ordered. The author states the conditions which have to be satisfied so that the problem of ranking fuzzy numbers has a solution. In fact, the subset of fuzzy numbers, the elements of which are strictly defined and continuous, is endowed with a total order relation.

Keywords : fuzzy numbers, ranking of fuzzy numbers, total order relation.

From pioneer-work on fuzzy numbers theory made by Dubois and Prade (1978, 1980), the problem of their ranking has been studied by many authors.

Usual methods are applied together to the problem of ranking fuzzy numbers and to the more general problem of ranking whatever fuzzy subsets (for a review, see Bortolan and Degani, 1985). However, by definition, a fuzzy number is a normalized and convex fuzzy subset of the real line \mathbb{R} . These conditions are often understood in a broad sense : the normality property is not imposed in one and only one point (flat fuzzy numbers) ; the convexity property is replaced by a less restrictive piecewise convexity property.

The scope of this paper is to state that the subset of the fuzzy numbers which are strictly defined is endowed with a total order relation if its elements are continuous.

Definition : strict fuzzy number. A strict fuzzy number, denoted by n , is defined by a membership function μ_n from \mathbb{R} to $[0, 1]$ such that, $\forall x_i \in \mathbb{R}, \mu_n(x_i) \in [0, 1]$ with the three following conditions :

(1) $\exists ! x_0 \in \mathbb{R}$ such that $\mu_n(x_0) = 1$: strict normality.

(2) $\forall (x_i, x_j) \in \mathbb{R}^2, \forall \lambda > 0, \forall \lambda' > 0, \lambda + \lambda' = 1,$

$\mu_n(\lambda x_i + \lambda' x_j) \geq \text{Min}(\mu_n(x_i), \mu_n(x_j))$: convexity (quasi-concavity of the membership function).

(5) $\text{supp } n$ is bounded and closed, where $\text{supp } n$ is the support of n .

By definition, $\text{supp } n = \{x_i / \mu_n(x_i) > 0\}$.

According to the theorem on the decomposition of a fuzzy subset by its α -cuts, we have :

$$n = \bigvee_{\alpha} (\alpha \cdot (n)_{\alpha}) \text{ where } (n)_{\alpha} = \{x_i / \mu_n(x_i) \geq \alpha\}, \forall \alpha \in]0, 1].$$

As $\{x_i / \mu_n(x_i) \geq \alpha\} = [a_i, a_j]_{\alpha}$ where $[a_i, a_j]_{\alpha}$ are the closed intervals of \mathbb{R} associated with each value of α , we have $(n)_{\alpha} = [a_i, a_j]_{\alpha}$, $\forall \alpha \in]0, 1]$.

Theorem 1. A strict fuzzy number is continuous iff the intervals which are defined on \mathbb{R} by its α -cuts are nested.

Proof : Consider \mathbb{R} endowed with its natural metric structure and let n be a strict fuzzy number. If the closed intervals which are defined by the α -cuts of n are nested, the α -cuts are got by a mapping σ from $]0, 1]$ to $\mathcal{P}([a_i, a_j])$ the power-set of the closed intervals of \mathbb{R} , such that : $(\alpha' > \alpha) \Rightarrow \sigma(\alpha') \subset \sigma(\alpha)$, $\forall \alpha' > \alpha$, i.e. $[a_i, a_j]_{\alpha'} \subset [a_i, a_j]_{\alpha}$.

By virtue of the normality and convexity properties of a strict fuzzy number, the series of its α -cuts is monotonous and decreasing for increasing values of α . We have :

$$[a_i, a_j]_{\alpha'} = \bigcap_{\alpha} [a_i, a_j]_{\alpha}, \forall \alpha' > \alpha.$$

Thus, the mapping σ is continuous as the intersection of any family of closed subsets of a metric space is closed.

Reciprocally, as n is strictly normalized, we have :

$[a_i, a_j]_{\alpha_1} \subset [a_i, a_j]_{\alpha}$, $\forall \alpha < \alpha_1$. Consider the support of n : we have $\text{supp } n = [a_i, a_j]_{\alpha_0} \supset [a_i, a_j]_{\alpha}$, $\forall \alpha > \alpha_0$.

As the function μ_n is quasi-concave and continuous, $\forall \alpha \in]0, 1]$, $\forall \alpha' \in]0, 1]$, with $\alpha_1 \geq \alpha' > \alpha > \alpha_0$, we have : $\{x_i / \mu_n(x_i) \geq \alpha'\} \subset \{x_i / \mu_n(x_i) \geq \alpha\}$. In other words, $[a_i, a_j]_{\alpha'} \subset [a_i, a_j]_{\alpha}$ and the intervals which are defined by the α -cuts of n are nested. \square

Corollary. A triangular fuzzy number is a strict fuzzy number such that the intervals defined on \mathbb{R} by its α -cuts are nested.

Evident. A triangular fuzzy number is continuous on the right and on the left. \square

Now, we name \mathcal{N}_c^p the subset of strict and continuous fuzzy numbers. Let n_i and n_j be two elements of \mathcal{N}_c^p and $[a_1, a_2]_{\alpha}$ and $[b_1, b_2]_{\alpha}$, $\alpha \in]0, 1]$, their α -cuts respectively.

Definitions : Inclusion : $n_i \leq n_j \Leftrightarrow [a_1, a_2]_{\alpha} \ll [b_1, b_2]_{\alpha}$, $\forall \alpha \in]0, 1]$.

Equality : $n_i = n_j \Leftrightarrow n_i \leq n_j$ and $n_j \leq n_i$.

Theorem 2. The set \mathcal{N}_c^p , endowed with the inclusion and equality relations, is totally ordered.

Proof : According to theorem 1, the intervals defined on \mathbb{R} by the α -cuts of the elements of \mathcal{N}_c^p are nested. So, it suffices to argue about their respective supports. To simplify the notation, α_0 will be omitted.

Reflexivity : $n_i \subseteq n_i \Leftrightarrow [a_1, a_2] \subseteq [a_1, a_2]$. Evident.

Antisymmetry : $n_i \subseteq n_j \Leftrightarrow a_2 \leq b_2$ and $a_1 > b_1$

or $a_1 = b_1$ and $a_2 < b_2$

$n_j \subseteq n_i \Leftrightarrow b_2 \leq a_2$ and $b_1 > a_1$

or $a_1 = b_1$ and $b_2 < a_2$

As it is impossible to have both : $a_2 < b_2$ and $b_2 < a_2$, only one of the two inclusion relations holds. In the particular case where $a_2 = b_2$, we have either $a_1 > b_1$ or $b_1 > a_1$.

Transitivity : consider a third element of \mathcal{N}_c^p , denoted by n_k , and its support, $\text{supp } n_k = [c_1, c_2]$.

We have : $n_i \subseteq n_j \Leftrightarrow a_2 \leq b_2$ and $a_1 > b_1$

or $a_1 = b_1$ and $a_2 < b_2$

$n_j \subseteq n_k \Leftrightarrow b_2 \leq c_2$ and $b_1 > c_1$

or $b_1 = c_1$ and $b_2 < c_2$

Six cases are possible :

(1) $(a_2 < b_2$ and $a_1 > b_1)$ and $(b_2 \leq c_2$ and $b_1 > c_1)$

$$(2) (a_2 < b_2 \text{ and } a_1 > b_1) \text{ and } (b_1 = c_1 \text{ and } b_2 < c_2)$$

$$(3) (a_2 = b_2 \text{ and } a_1 > b_1) \text{ and } (b_2 < c_2 \text{ and } b_1 > c_1)$$

$$(4) (a_2 = b_2 \text{ and } a_1 > b_1) \text{ and } (b_1 = c_1 \text{ and } b_2 < c_2)$$

$$(5) (a_1 = b_1 \text{ and } a_2 < b_2) \text{ and } (b_2 < c_2 \text{ and } b_1 > c_1)$$

$$(6) (a_1 = b_1 \text{ and } a_2 < b_2) \text{ and } (b_1 = c_1 \text{ and } b_2 < c_2)$$

In all the cases, we have $n_i \subseteq n_k$.

Totality : $\forall (n_i, n_j) \in \mathcal{P}_c^2$, we have either : $a_2 < b_2$

or : $b_2 < a_2$. If $a_2 = b_2$, we can compare a_1 to b_1 . \square

Finally, the set of fuzzy numbers is not ordered when at least one of its elements is not strictly defined and not continuous. In applications, the best thing we can do is to make use of strict and continuous fuzzy numbers as often as the representation of an empirical model enables it and when the existence of a total order relation on the set of fuzzy numbers is important for the mathematical modelling. If not, we find again the more general problem of ranking any fuzzy numbers (e.g. Kaufmann, 1983) and ranking whatever fuzzy sets (e.g. Ovchinnikov and Migdal, 1987). But, the solution is dependent on the rules which are chosen in the framework of each method and so, it is not unique.

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