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Dynamics of Mechanical Systems

Part 1: Concepts, Geometry and Kinematic Considerations

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Classical Mechanics is a field of physical science, which deals with the motion of objects in our three-dimensional physical space, and the interaction among these objects. This field characterizes the phenomena observed at our usual scale of space and time, ranging from the scale of molecules to planetary objects. It is also important to understand that just like any other field of physics it only provides approximations for the descriptions of physical phenomena; it relies on a set of principles and axioms that cannot be exactly proven, but have not been contradicted by physical experiments. These principles and axioms allow for the development of models that capture certain features of observed physical phenomena, and also make it possible to introduce mathematical representations for analysis.

1 Fundamental Observations About Motion

Geometry is a pre-requisite to Mechanics. The understanding of the geometry related to our three-dimensional physical space is necessary to interpret the motion of physical objects and the interaction between them. We intend to build up a framework that relies on basic everyday observations understandable to everybody, and these observations can be reproduced by simple experiments in common everyday physical settings.

The way how the motion of an object is seen in physical space depends on the position and motion of the observer. This requires the interpretation of the observer's *frame of reference*; this is a key notion to describe motion. Each observer can be attached to a different frame of reference.

A frame of reference is an infinite dimensional geometric object, with no particular shape.¹ We will use notation \mathcal{F} with appropriate index to represent a frame of reference.² To parameterize the frame of reference a coordinate system can be defined with origin at an arbitrary point of the frame. Such a coordinate system can be used to identify the location of any point of the frame of reference and also its orientation. An infinite number of coordinate systems can be defined for the parameterization of a frame of reference. These coordinate systems can all be transformed to each other via constant transformations. We will see this later in more detail.

The notion of a coordinate system is not essential to the physical meaning of a frame of reference. It is only needed to parameterize the original geometric object and make it possible to establish algebraic representations to describe the geometric concepts in a “processable” form. But, often it is easier to think about a coordinate system than the more abstract geometric concept of the frame of reference.

¹For example, four non-coplanar points that are always constant distance of each other can interpret such frame of reference.

²In the following we will often refer to a frame of reference simply as a frame.

There are two important types of motion we normally identify in everyday life: the motion of a point relative to a reference frame; and the motion of two reference frames relative to each other.

To observe the position and motion of a point with respect an observer two types of information are needed in general: magnitude and direction. For example, in the case of the position, the distance and the direction along which the distance is measured represent these two pieces of information. In the case of velocity and acceleration, the two pieces of information needed to completely define them are their magnitudes and directions. Therefore, such physical quantities cannot mathematically be represented by scalars alone. They can be represented by physical vectors; such a quantity has two parts: a scalar magnitude and a unit vector representing its direction.

A *physical vector* can be considered as a geometric object and can be denoted as \vec{v} . It can be decomposed in different ways. One way is to directly represent the magnitude and the direction as

$$\vec{v} = v\vec{e}_v \quad (1)$$

where v represents the magnitude, which is a scalar value, and \vec{e}_v indicates the *geometric unit vector* giving the direction of the physical vector. This is illustrated in Fig. 1, the orientation of the unit vector also defines the *line of action* of the vector. There is also a third part of a physical

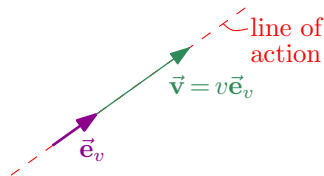


Figure 1: *Representation of a physical vector*

vector: the physical unit. For now we assume that if not indicated separately then the physical unit is part of the magnitude. Later on we will revisit this. The concepts of physical and geometric vectors are only different in that a physical vector carries a physical unit while a geometric vector may not need to carry unit and may only indicate a direction in space.

2 Physical Vectors

Here we will use arbitrary physical vectors, \vec{a} , \vec{b} , \vec{c} , \vec{d} for illustration. In each subsections these may carry different meaning.

2.1 Addition

Let us think of a displacement of a point as a typical representation for physical vectors. We consider two displacements represented by vectors \vec{a} and \vec{b} as shown in Fig. 2. The addition of these two vectors, $\vec{c} = \vec{a} + \vec{b}$ is naturally, geometrically, defined as shown in the figure. Based on this \vec{a} and \vec{b} can also be called the *vector components* of \vec{c} .

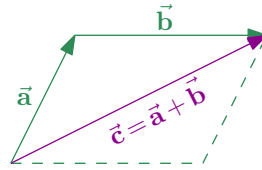


Figure 2: Representation of vector addition

2.2 Concepts of Product

Let us consider again two such physical vectors \vec{a} and \vec{b} , which may also be represented with displaying the magnitude and direction explicitly as $\vec{a} = a\vec{e}_a$ and $\vec{b} = b\vec{e}_b$. The concept of product between geometric vectors can be interpreted in different ways. Let us look at the geometric representations shown in Figs. 3-5. The *scalar product* results in a scalar value, *cross (or vector) product* gives a vector, and the *dyadic product* produces a higher order geometric object that will be referred to as a *dyadic* or *second order tensor*.

2.2.1 Scalar Product

Let us go back to the geometric representation, Fig. 3, and consider \vec{a} and unit vector \vec{e}_b . Vector \vec{a} can be decomposed to two vector components, one along \vec{e}_b and another perpendicular to \vec{e}_b . Based on the geometry of triangles we can immediately see that the magnitude of the component along \vec{e}_b equals $a \cos \varphi$ where φ is the angle between \vec{a} and \vec{e}_b . This gives the possibility to

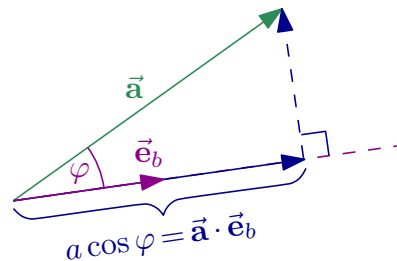


Figure 3: Scalar product

define a product concept, the *scalar product*, between \vec{a} and \vec{e}_b where the result is $a \cos \varphi$, i.e., the magnitude of the vector component of \vec{a} along \vec{e}_b , as

$$\vec{a} \cdot \vec{e}_b = a\vec{e}_a \cdot \vec{e}_b = a \cos \varphi \quad (2)$$

where \cdot indicates the scalar product operator. This product is also often called the *inner* or *dot* product. We can also see that the scalar product operation essentially comes down to operations between the geometric unit vectors and the angle defined by them,

$$\vec{e}_a \cdot \vec{e}_b = \cos \varphi$$

and based on simple geometric interpretation, Fig. 3, the order can be changed, i.e., the product is commutative

$$\vec{e}_b \cdot \vec{e}_a = \cos \varphi$$

Based on this the scalar product between two arbitrary vectors can be interpreted as

$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a} = ab\vec{e}_a \cdot \vec{e}_b = ab \cos \varphi \quad (3)$$

Using this interpretation the scalar product of two vectors is zero if they are perpendicular, or orthogonal in other words, to each other, and it equals to the product of their magnitudes if they are collinear.

2.2.2 Cross Product

Let us again consider two physical vectors \vec{a} and \vec{b} as shown in Fig. 4. The two vectors define a plane. Based on this the concept of the cross, or vector, product

$$\vec{a} \times \vec{b} = \vec{c} \quad (4)$$

can be interpreted to result in a third vector, \vec{c} that is perpendicular to the plane so that the three vectors \vec{a} , \vec{b} , \vec{c} form a right-handed vector triad, and the magnitude of \vec{c} equals the area of the parallelogram \vec{a} and \vec{b} span

$$\vec{c} = ab \sin \varphi \vec{e}_c \quad (5)$$

where $c = ab \sin \varphi$ is the magnitude of the vector \vec{c} and \vec{e}_c is the unit vector normal to the plane defined by \vec{a} and \vec{b} . The cross product of two collinear vectors is zero due to that the area

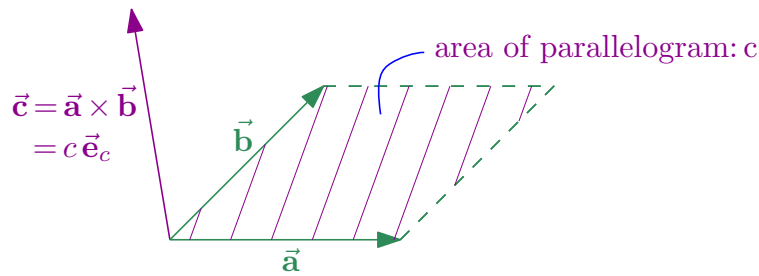


Figure 4: *Cross product*

spanned by the two vectors is zero. Using the right-hand rule we can see that the cross product is anti-commutative

$$\vec{b} \times \vec{a} = -\vec{c} \quad (6)$$

Orthogonal Unit Vectors: A particularly important case when we consider the cross product of two perpendicular unit vectors, \vec{g}_1 and \vec{g}_2 . Based on the above interpretation and considering that the two unit vectors form a right angle we obtain

$$\vec{g}_1 \times \vec{g}_2 = \vec{g}_3 \quad (7)$$

where \vec{g}_3 is also a unit vector and the three unit vectors form an right-handed orthonormal vector triad that is often used to define an orthogonal Cartesian coordinate system. Considering \vec{g}_1 , \vec{g}_2 , \vec{g}_3 it can be easily shown that

$$\vec{g}_2 \times \vec{g}_3 = \vec{g}_1, \quad \vec{g}_3 \times \vec{g}_1 = \vec{g}_2, \quad \vec{g}_2 \times \vec{g}_1 = -\vec{g}_3, \quad \vec{g}_3 \times \vec{g}_2 = -\vec{g}_1, \quad \vec{g}_1 \times \vec{g}_3 = -\vec{g}_2 \quad (8)$$

These relations can serve as basis to evaluate the cross product between two vectors when the vectors are decomposed with respect to an orthonormal set of base vectors.

We note that in the interpretation of the above operations and vector quantities we have completely followed geometric ideas motivated by our simple observations in our physical space to interpret these concepts of addition and products.

2.2.3 Mixed and Triple Products

Based on the previous considerations it can be shown that the following triple products also hold. Scalar triple product:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b}) \quad (9)$$

Vector triple product:

$$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b}) \quad (10)$$

and

$$(\vec{a} \times \vec{b}) \times \vec{c} = -\vec{c} \times (\vec{a} \times \vec{b}) = -(\vec{c} \cdot \vec{b})\vec{a} + (\vec{c} \cdot \vec{a})\vec{b} \quad (11)$$

2.2.4 Dyadic Product

Let us now consider a third important concept of product for physical vectors, the dyadic product. This is not very intuitive geometrically compared to the previous two product concepts; it more relates to the case when a physical vector is defined as a linear function of another physical vector. The dyadic product is also often called general product.

Let us consider again two vectors, \vec{a} and \vec{b} and also a third one \vec{c} . We consider the case when \vec{c} is transformed into a vector \vec{d} the direction of which is defined by \vec{a} and the magnitude is given by the scalar product of \vec{c} with \vec{b} scaled with the magnitude of \vec{a} . This result can be given as $\vec{d} = (\vec{b} \cdot \vec{c})\vec{a}$. This is a so-called linear transformation, that is essentially represented by the dyadic product, $\vec{a} \otimes \vec{b}$, and illustrated in Fig. 5

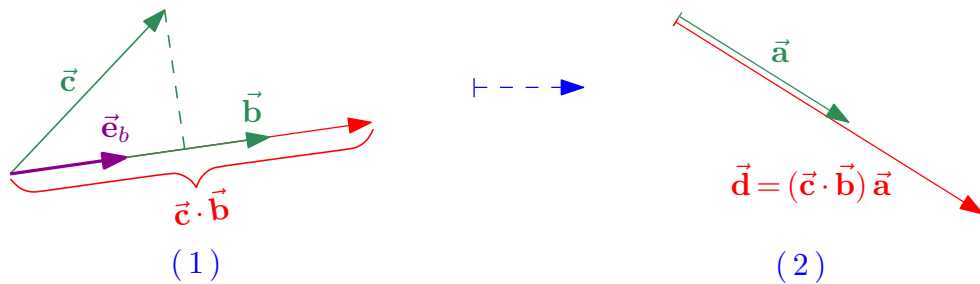


Figure 5: *Illustration of the dyadic product*

The dyadic product of vectors \vec{a} and \vec{b} can be defined as

$$\vec{a} \otimes \vec{b} = \hat{\mathbf{D}} \quad (12)$$

where $\hat{\mathbf{D}}$ is a geometric object on its own representing the dyadic, a second order tensor, that can act on/transform any vector \vec{c} as

$$(\vec{a} \otimes \vec{b}) \cdot \vec{c} = \hat{\mathbf{D}} \cdot \vec{c} = (\vec{b} \cdot \vec{c}) \vec{a} = \vec{d} \quad (13)$$

which operation is described above and represented in Fig. 5. We have already seen that the vector product operations essentially come down to operations among the unit vectors as the magnitudes act only as simple multipliers. Let us now consider a unit dyadic $\vec{e}_1 \otimes \vec{e}_2$, the dyadic product of two unit vectors (generally not perpendicular to each other) in the above formula as it operates on vector \vec{c}

$$(\vec{e}_1 \otimes \vec{e}_2) \cdot \vec{c} = (\vec{e}_2 \cdot \vec{c}) \vec{e}_1 \quad (14)$$

We can see that the geometric meaning can be interpreted as taking the scalar component of vector \vec{c} along \vec{e}_2 via the scalar product, and determining a new vector that has this scalar component as magnitude but its direction is along \vec{e}_1 .

Transpose of the Dyadic: The transpose of the dyadic defined in Eq. (12) can be interpreted as

$$(\vec{a} \otimes \vec{b})^T = \vec{b} \otimes \vec{a} = \hat{D}^T \quad (15)$$

and we can also see based on the interpretation of the operation of the dyadic on a vector, Eq. (13), that

$$(\vec{a} \otimes \vec{b}) \cdot \vec{c} = \vec{c} \cdot (\vec{a} \otimes \vec{b})^T = \vec{c} \cdot (\vec{b} \otimes \vec{a}) = \vec{c} \cdot \hat{D} = (\vec{b} \cdot \vec{c}) \vec{a} = \vec{d} \quad (16)$$

3 Representation of Physical Vectors and Their Transformations

As physical vectors and their addition are fundamentally geometric concepts the analysis of the geometry and related pictures, as shown in Fig. 6, can tell us that the geometric decomposition of an arbitrary vector \vec{a} with respect to a set of three base vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ defining three directions in space³

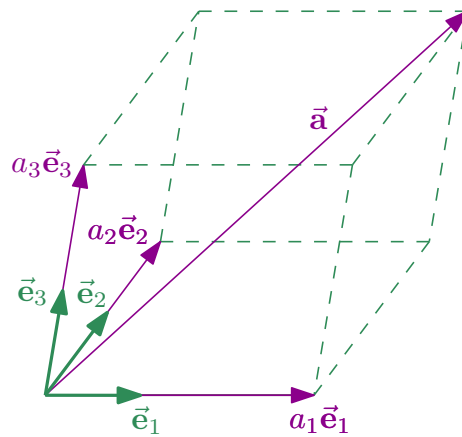


Figure 6: *Representation/decomposition of a physical vector*

$$\vec{a} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + a_3 \vec{e}_3 \quad (17)$$

³These vectors are unit vectors most of the time. But, they do not necessarily have to be unit vectors.

should hold as long as the volume of the parallelepiped defined by $\vec{e}_1, \vec{e}_2, \vec{e}_3$ is not zero. In this decomposition $\vec{a}_1 = a_1\vec{e}_1, \vec{a}_2 = a_2\vec{e}_2, \vec{a}_3 = a_3\vec{e}_3$ are the *vector components* and a_1, a_2, a_3 are the so-called *scalar components* that are also often called *coordinates*. Given \vec{a} and $\vec{e}_1, \vec{e}_2, \vec{e}_3$ the decomposition of Eq. (17) can always be determined by geometric methods such as direct geometric construction. To investigate the decomposition in another way, more quantitatively, the scalar components need to be determined. Let us consider the scalar products of Eq. (17) by \vec{e}_1, \vec{e}_2 and \vec{e}_3 as

$$\vec{a} \cdot \vec{e}_1 = a_1\vec{e}_1 \cdot \vec{e}_1 + a_2\vec{e}_2 \cdot \vec{e}_1 + a_3\vec{e}_3 \cdot \vec{e}_1 \quad (18)$$

$$\vec{a} \cdot \vec{e}_2 = a_1\vec{e}_1 \cdot \vec{e}_2 + a_2\vec{e}_2 \cdot \vec{e}_2 + a_3\vec{e}_3 \cdot \vec{e}_2 \quad (19)$$

$$\vec{a} \cdot \vec{e}_3 = a_1\vec{e}_1 \cdot \vec{e}_3 + a_2\vec{e}_2 \cdot \vec{e}_3 + a_3\vec{e}_3 \cdot \vec{e}_3 \quad (20)$$

then Eqs. (18)-(20) form a system of linear equations for a_1, a_2 and a_3 and can be written as

$$\underbrace{\begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_3 \cdot \vec{e}_1 \\ \vec{e}_1 \cdot \vec{e}_2 & \vec{e}_2 \cdot \vec{e}_2 & \vec{e}_3 \cdot \vec{e}_2 \\ \vec{e}_1 \cdot \vec{e}_3 & \vec{e}_2 \cdot \vec{e}_3 & \vec{e}_3 \cdot \vec{e}_3 \end{bmatrix}}_{\mathbf{H}} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} \vec{a} \cdot \vec{e}_1 \\ \vec{a} \cdot \vec{e}_2 \\ \vec{a} \cdot \vec{e}_3 \end{bmatrix} \quad (21)$$

where \mathbf{H} is the 3×3 coefficient matrix, the inner product matrix; this system of equations can be solved for the scalar components as long as \mathbf{H} is an invertible, i.e., the rows of \mathbf{H} are independent of each other. This means, based on Eqs. (17)-(21), that \vec{e}_1, \vec{e}_2 and \vec{e}_3 are *linearly independent* of each other, i.e., if $\vec{a} = \vec{0}$ is the zero vector then the composition of that using \vec{e}_1, \vec{e}_2 and \vec{e}_3 is only possible if all scalar components in Eq. (17) are equal to zero. The geometric interpretation of linear independence, as mentioned above, is that the parallelepiped defined by \vec{e}_1, \vec{e}_2 and \vec{e}_3 has nonzero volume. Based on the above reasoning we also note that orthogonality among the base vectors is not required for the above representation and decomposition. We can see from Eq. (21) and using the concept of the scalar product that \mathbf{H} is a symmetric matrix.

The above considerations also imply that the decomposition given by Eq. (17) is always possible for any triads of linearly independent vectors; they do not have to be unit vectors. Such linearly independent set of three vectors can form a *basis* for the representation of quantities in space, and they are called *base vectors*. Often, the more typical case when the base vectors are also unit vectors.

Orthogonal Unit Vectors - Orthonormal Basis: A special, but particularly important case when the base vectors are unit vectors and they are also mutually perpendicular to each other, which represents an orthonormal basis. For that case matrix \mathbf{H} in Eq. (21) becomes the 3×3 identity matrix, and the scalar components a_1, a_2 and a_3 can simply be determined as

$$a_1 = \vec{a} \cdot \vec{e}_1, \quad a_2 = \vec{a} \cdot \vec{e}_2, \quad a_3 = \vec{a} \cdot \vec{e}_3 \quad (22)$$

3.1 Representation of Vector Operations

In the following we will represent the various vector operations when the vectors are decomposed/represented with respect to the same base unit vectors.

3.1.1 Vector Addition

We consider $\vec{c} = \vec{a} + \vec{b}$ as was discussed in section 2 and illustrated in Fig. 1. Vectors \vec{a} and \vec{b} are given with respect to base unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ as

$$\vec{a} = a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 \quad (23)$$

and

$$\vec{b} = b_1\vec{e}_1 + b_2\vec{e}_2 + b_3\vec{e}_3 \quad (24)$$

then the result of the vector addition can be written as

$$\vec{c} = \vec{a} + \vec{b} = \underbrace{(a_1 + b_1)}_{c_1}\vec{e}_1 + \underbrace{(a_2 + b_2)}_{c_2}\vec{e}_2 + \underbrace{(a_3 + b_3)}_{c_3}\vec{e}_3 \quad (25)$$

where c_1, c_2, c_3 are the scalar components of the resultant vector \vec{c} .

3.1.2 Scalar Product

Let us consider again these two arbitrary vectors, \vec{a} and \vec{b} . Using Eqs. (23) and (24) and the concept of scalar product introduced in the previous section

$$\begin{aligned} \vec{a} \cdot \vec{b} &= a_1b_1\vec{e}_1 \cdot \vec{e}_1 + a_1b_2\vec{e}_1 \cdot \vec{e}_2 + a_1b_3\vec{e}_1 \cdot \vec{e}_3 + a_2b_1\vec{e}_2 \cdot \vec{e}_1 + a_2b_2\vec{e}_2 \cdot \vec{e}_2 \\ &\quad + a_2b_3\vec{e}_2 \cdot \vec{e}_3 + a_3b_1\vec{e}_3 \cdot \vec{e}_1 + a_3b_2\vec{e}_3 \cdot \vec{e}_2 + a_3b_3\vec{e}_3 \cdot \vec{e}_3 \end{aligned} \quad (26)$$

which can be arranged as

$$\vec{a} \cdot \vec{b} = \underbrace{\begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}}_{\mathbf{a}^T} \underbrace{\begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_1 \cdot \vec{e}_2 & \vec{e}_1 \cdot \vec{e}_3 \\ \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_2 & \vec{e}_2 \cdot \vec{e}_3 \\ \vec{e}_3 \cdot \vec{e}_1 & \vec{e}_3 \cdot \vec{e}_2 & \vec{e}_3 \cdot \vec{e}_3 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}_{\mathbf{b}} = \mathbf{a}^T \mathbf{H} \mathbf{b} \quad (27)$$

where matrix \mathbf{H} is the so-called *inner product matrix* associated with base vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, this matrix already appeared earlier in Eq. (21), 3×3 arrays, \mathbf{a} and \mathbf{b} contain the scalar components of \vec{a} and \vec{b} with respect to the selected base vectors; we will term such arrays as *algebraic vectors*. We will often work with algebraic vectors after we transform the geometric/physical representation of a system to an algebraic form. However, it is important to understand that an algebraic vector do not completely represent a physical vector; it only contains its scalar components with respect to a set of base vectors, but from the algebraic vector alone it is not possible to know what the base vectors are. Therefore, it is important to indicate the difference between physical/geometric quantities and algebraic representations in the notations used. Writing something such as

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad (28)$$

is incorrect; this is often forgotten and concepts are discussed this way incorrectly, significantly limiting the potential generality and applicability of formulations.

Orthogonal Base Vectors: If $\vec{e}_1, \vec{e}_2, \vec{e}_3$ represent an orthogonal system of unit vectors then \mathbf{H} in Eq. (27) becomes the identity matrix and for such a case the representation of scalar product with scalar components reduces to

$$\vec{\mathbf{a}} \cdot \vec{\mathbf{b}} = \mathbf{a}^T \mathbf{b} \quad (29)$$

We can also see here comparing Eq. (27) and Eq. (29) that just by considering algebraic vectors \mathbf{a} and \mathbf{b} alone, only limited information is available about the complete physical vectors, and, for example, we would not know how to perform the vector operations and what inner product matrix to use.

3.1.3 Cross Product

We consider the two arbitrary vectors, $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$. Using Eqs. (23) and (24) and the concept of cross product discussed in section 2.2.2, we can write

$$\begin{aligned} \vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{c}} = & a_1 b_1 \vec{e}_1 \times \vec{e}_1 + a_1 b_2 \vec{e}_1 \times \vec{e}_2 + a_1 b_3 \vec{e}_1 \times \vec{e}_3 + a_2 b_1 \vec{e}_2 \times \vec{e}_1 + a_2 b_2 \vec{e}_2 \times \vec{e}_2 \\ & + a_2 b_3 \vec{e}_2 \times \vec{e}_3 + a_3 b_1 \vec{e}_3 \times \vec{e}_1 + a_3 b_2 \vec{e}_3 \times \vec{e}_2 + a_3 b_3 \vec{e}_3 \times \vec{e}_3 \end{aligned} \quad (30)$$

Considering that $\vec{e}_1 \times \vec{e}_1 = \vec{e}_2 \times \vec{e}_2 = \vec{e}_3 \times \vec{e}_3 = \vec{\mathbf{0}}$, this can be further reduced as

$$\vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{c}} = a_1 b_2 \vec{e}_1 \times \vec{e}_2 + a_1 b_3 \vec{e}_1 \times \vec{e}_3 + a_2 b_1 \vec{e}_2 \times \vec{e}_1 + a_2 b_3 \vec{e}_2 \times \vec{e}_3 + a_3 b_1 \vec{e}_3 \times \vec{e}_1 + a_3 b_2 \vec{e}_3 \times \vec{e}_2 \quad (31)$$

For the general case when the base vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are linearly independent but not orthogonal to each other the cross products among the base vectors have to be evaluated based on the angles formed by the base vectors and the general definition of the cross product as described in section 2.2.2.

Orthogonal Base Vectors: However, the most important case for the cross product when the base vectors are orthogonal to each other, then based on Eq. (8) the above result can be further simplified as

$$\begin{aligned} \vec{\mathbf{a}} \times \vec{\mathbf{b}} = \vec{\mathbf{c}} = & a_1 b_2 \vec{e}_3 - a_1 b_3 \vec{e}_2 - a_2 b_1 \vec{e}_3 + a_2 b_3 \vec{e}_1 + a_3 b_1 \vec{e}_2 - a_3 b_2 \vec{e}_1 = \\ = & \underbrace{(a_2 b_3 - a_3 b_2)}_{c_1} \vec{e}_1 + \underbrace{(a_3 b_1 - a_1 b_3)}_{c_2} \vec{e}_2 + \underbrace{(a_1 b_2 - a_2 b_1)}_{c_3} \vec{e}_3 \end{aligned} \quad (32)$$

where c_1, c_2 and c_3 are the scalar components of the resultant vector $\vec{\mathbf{c}}$ with respect to the base vectors, and they can also be collected in an algebraic vector $\mathbf{c} = [c_1 \ c_2 \ c_3]^T$.

If we express the above operation only with the scalar components collected in algebraic vectors \mathbf{a}, \mathbf{b} and \mathbf{c} then based on Eq. (32) we can write

$$\mathbf{c} = \tilde{\mathbf{A}} \mathbf{b} \quad (33)$$

where

$$\tilde{\mathbf{A}} = \begin{bmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{bmatrix} \quad (34)$$

is a skew-symmetric matrix, sometimes called the cross-product matrix, formed from the scalar components of $\vec{\mathbf{a}}$. Again, we can see that if we only work with algebraic vectors, i.e., scalar components, then we also have to know what base vectors were used to determine the scalar components.

3.1.4 Dyadic Product and Second-Order Tensors

We use again three arbitrary physical vectors, \vec{a} , \vec{b} and \vec{c} to illustrate this operation. Using Eqs. (23) and (24) and the concept of the dyadic product discussed in section 2.2.4, we obtain

$$\begin{aligned} \vec{a} \otimes \vec{b} = \widehat{\mathbf{D}} = & a_1 b_1 \vec{e}_1 \otimes \vec{e}_1 + a_1 b_2 \vec{e}_1 \otimes \vec{e}_2 + a_1 b_3 \vec{e}_1 \otimes \vec{e}_3 + a_2 b_1 \vec{e}_2 \otimes \vec{e}_1 + a_2 b_2 \vec{e}_2 \otimes \vec{e}_2 \\ & + a_2 b_3 \vec{e}_2 \otimes \vec{e}_3 + a_3 b_1 \vec{e}_3 \otimes \vec{e}_1 + a_3 b_2 \vec{e}_3 \otimes \vec{e}_2 + a_3 b_3 \vec{e}_3 \otimes \vec{e}_3 \end{aligned} \quad (35)$$

This can also be written as

$$\vec{a} \otimes \vec{b} = \widehat{\mathbf{D}} = \sum_{i=1}^3 \sum_{j=1}^3 a_i b_j \vec{e}_i \otimes \vec{e}_j = \sum_{i=1}^3 \sum_{j=1}^3 D_{ij} \vec{e}_i \otimes \vec{e}_j \quad (36)$$

where $D_{ij} = a_i b_j$, and the transpose of the dyadic can be interpreted as

$$(\vec{a} \otimes \vec{b})^T = \vec{b} \otimes \vec{a} = \widehat{\mathbf{D}}^T = \sum_{i=1}^3 \sum_{j=1}^3 D_{ji} \vec{e}_i \otimes \vec{e}_j \quad (37)$$

Considering now the scalar product of the dyadic $\vec{a} \otimes \vec{b}$ with vector $\vec{c} = c_1 \vec{e}_1 + c_2 \vec{e}_2 + c_3 \vec{e}_3$ as in Eq. (13), we can expand the expression as

$$\begin{aligned} (\vec{a} \otimes \vec{b}) \cdot \vec{c} = \widehat{\mathbf{D}} \cdot \vec{c} = \vec{d} = & \\ = & a_1 b_1 (\vec{e}_1 \otimes \vec{e}_1) \cdot \vec{e}_1 c_1 + a_1 b_2 (\vec{e}_1 \otimes \vec{e}_2) \cdot \vec{e}_1 c_1 + a_1 b_3 (\vec{e}_1 \otimes \vec{e}_3) \cdot \vec{e}_1 c_1 + a_2 b_1 (\vec{e}_2 \otimes \vec{e}_1) \cdot \vec{e}_1 c_1 + \\ + & a_2 b_2 (\vec{e}_2 \otimes \vec{e}_2) \cdot \vec{e}_1 c_1 + a_2 b_3 (\vec{e}_2 \otimes \vec{e}_3) \cdot \vec{e}_1 c_1 + a_3 b_1 (\vec{e}_3 \otimes \vec{e}_1) \cdot \vec{e}_1 c_1 + a_3 b_2 (\vec{e}_3 \otimes \vec{e}_2) \cdot \vec{e}_1 c_1 + a_3 b_3 (\vec{e}_3 \otimes \vec{e}_3) \cdot \vec{e}_1 c_1 + \\ & + a_1 b_1 (\vec{e}_1 \otimes \vec{e}_1) \cdot \vec{e}_2 c_2 + a_1 b_2 (\vec{e}_1 \otimes \vec{e}_2) \cdot \vec{e}_2 c_2 + a_1 b_3 (\vec{e}_1 \otimes \vec{e}_3) \cdot \vec{e}_2 c_2 + a_2 b_1 (\vec{e}_2 \otimes \vec{e}_1) \cdot \vec{e}_2 c_2 + \\ + & a_2 b_2 (\vec{e}_2 \otimes \vec{e}_2) \cdot \vec{e}_2 c_2 + a_2 b_3 (\vec{e}_2 \otimes \vec{e}_3) \cdot \vec{e}_2 c_2 + a_3 b_1 (\vec{e}_3 \otimes \vec{e}_1) \cdot \vec{e}_2 c_2 + a_3 b_2 (\vec{e}_3 \otimes \vec{e}_2) \cdot \vec{e}_2 c_2 + a_3 b_3 (\vec{e}_3 \otimes \vec{e}_3) \cdot \vec{e}_2 c_2 + \\ & + a_1 b_1 (\vec{e}_1 \otimes \vec{e}_1) \cdot \vec{e}_3 c_3 + a_1 b_2 (\vec{e}_1 \otimes \vec{e}_2) \cdot \vec{e}_3 c_3 + a_1 b_3 (\vec{e}_1 \otimes \vec{e}_3) \cdot \vec{e}_3 c_3 + a_2 b_1 (\vec{e}_2 \otimes \vec{e}_1) \cdot \vec{e}_3 c_3 + \\ + & a_2 b_2 (\vec{e}_2 \otimes \vec{e}_2) \cdot \vec{e}_3 c_3 + a_2 b_3 (\vec{e}_2 \otimes \vec{e}_3) \cdot \vec{e}_3 c_3 + a_3 b_1 (\vec{e}_3 \otimes \vec{e}_1) \cdot \vec{e}_3 c_3 + a_3 b_2 (\vec{e}_3 \otimes \vec{e}_2) \cdot \vec{e}_3 c_3 + a_3 b_3 (\vec{e}_3 \otimes \vec{e}_3) \cdot \vec{e}_3 c_3 \end{aligned} \quad (38)$$

Based on the definition of the dyadic product as described in section 2.2.4 and forming the scalar products with the unit dyadics in the above expression we obtain

$$\begin{aligned} (\vec{a} \otimes \vec{b}) \cdot \vec{c} = \widehat{\mathbf{D}} \cdot \vec{c} = \vec{d} = & \\ = & a_1 b_1 (\vec{e}_1 \cdot \vec{e}_1) \vec{e}_1 c_1 + a_1 b_2 (\vec{e}_2 \cdot \vec{e}_1) \vec{e}_1 c_1 + a_1 b_3 (\vec{e}_3 \cdot \vec{e}_1) \vec{e}_1 c_1 + a_2 b_1 (\vec{e}_1 \cdot \vec{e}_1) \vec{e}_2 c_1 + \\ + & a_2 b_2 (\vec{e}_2 \cdot \vec{e}_1) \vec{e}_2 c_1 + a_2 b_3 (\vec{e}_3 \cdot \vec{e}_1) \vec{e}_2 c_1 + a_3 b_1 (\vec{e}_1 \cdot \vec{e}_1) \vec{e}_3 c_1 + a_3 b_2 (\vec{e}_2 \cdot \vec{e}_1) \vec{e}_3 c_1 + a_3 b_3 (\vec{e}_3 \cdot \vec{e}_1) \vec{e}_3 c_1 + \\ & + a_1 b_1 (\vec{e}_1 \cdot \vec{e}_2) \vec{e}_1 c_2 + a_1 b_2 (\vec{e}_2 \cdot \vec{e}_2) \vec{e}_1 c_2 + a_1 b_3 (\vec{e}_3 \cdot \vec{e}_2) \vec{e}_1 c_2 + a_2 b_1 (\vec{e}_1 \cdot \vec{e}_2) \vec{e}_2 c_2 + \\ + & a_2 b_2 (\vec{e}_2 \cdot \vec{e}_2) \vec{e}_2 c_2 + a_2 b_3 (\vec{e}_3 \cdot \vec{e}_2) \vec{e}_2 c_2 + a_3 b_1 (\vec{e}_1 \cdot \vec{e}_2) \vec{e}_3 c_2 + a_3 b_2 (\vec{e}_2 \cdot \vec{e}_2) \vec{e}_3 c_2 + a_3 b_3 (\vec{e}_3 \cdot \vec{e}_2) \vec{e}_3 c_2 + \\ & + a_1 b_1 (\vec{e}_1 \cdot \vec{e}_3) \vec{e}_1 c_3 + a_1 b_2 (\vec{e}_2 \cdot \vec{e}_3) \vec{e}_1 c_3 + a_1 b_3 (\vec{e}_3 \cdot \vec{e}_3) \vec{e}_1 c_3 + a_2 b_1 (\vec{e}_1 \cdot \vec{e}_3) \vec{e}_2 c_3 + \\ + & a_2 b_2 (\vec{e}_2 \cdot \vec{e}_3) \vec{e}_2 c_3 + a_2 b_3 (\vec{e}_3 \cdot \vec{e}_3) \vec{e}_2 c_3 + a_3 b_1 (\vec{e}_1 \cdot \vec{e}_3) \vec{e}_3 c_3 + a_3 b_2 (\vec{e}_2 \cdot \vec{e}_3) \vec{e}_3 c_3 + a_3 b_3 (\vec{e}_3 \cdot \vec{e}_3) \vec{e}_3 c_3 \end{aligned} \quad (39)$$

which can then be grouped to result in the final, detailed expression

$$\begin{aligned}
(\vec{\mathbf{a}} \otimes \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}} &= \widehat{\mathbf{D}} \cdot \vec{\mathbf{c}} = \vec{\mathbf{d}} = d_1 \vec{\mathbf{e}}_1 + d_2 \vec{\mathbf{e}}_2 + d_3 \vec{\mathbf{e}}_3 = \\
&\left[a_1 b_1 (\vec{\mathbf{e}}_1 \cdot \vec{\mathbf{e}}_1) c_1 + a_1 b_2 (\vec{\mathbf{e}}_2 \cdot \vec{\mathbf{e}}_1) c_1 + a_1 b_3 (\vec{\mathbf{e}}_3 \cdot \vec{\mathbf{e}}_1) c_1 + a_1 b_1 (\vec{\mathbf{e}}_1 \cdot \vec{\mathbf{e}}_2) c_2 + a_1 b_2 (\vec{\mathbf{e}}_2 \cdot \vec{\mathbf{e}}_2) c_2 + a_1 b_3 (\vec{\mathbf{e}}_3 \cdot \vec{\mathbf{e}}_2) c_2 + \right. \\
&\quad \left. + a_1 b_1 (\vec{\mathbf{e}}_1 \cdot \vec{\mathbf{e}}_3) c_3 + a_1 b_2 (\vec{\mathbf{e}}_2 \cdot \vec{\mathbf{e}}_3) c_3 + a_1 b_3 (\vec{\mathbf{e}}_3 \cdot \vec{\mathbf{e}}_3) c_3 \right] \vec{\mathbf{e}}_1 + \\
&\left[a_2 b_1 (\vec{\mathbf{e}}_1 \cdot \vec{\mathbf{e}}_1) c_1 + a_2 b_2 (\vec{\mathbf{e}}_2 \cdot \vec{\mathbf{e}}_1) c_1 + a_2 b_3 (\vec{\mathbf{e}}_3 \cdot \vec{\mathbf{e}}_1) c_1 + a_2 b_1 (\vec{\mathbf{e}}_1 \cdot \vec{\mathbf{e}}_2) c_2 + a_2 b_2 (\vec{\mathbf{e}}_2 \cdot \vec{\mathbf{e}}_2) c_2 + a_2 b_3 (\vec{\mathbf{e}}_3 \cdot \vec{\mathbf{e}}_2) c_2 + \right. \\
&\quad \left. + a_2 b_1 (\vec{\mathbf{e}}_1 \cdot \vec{\mathbf{e}}_3) c_3 + a_2 b_2 (\vec{\mathbf{e}}_2 \cdot \vec{\mathbf{e}}_3) c_3 + a_2 b_3 (\vec{\mathbf{e}}_3 \cdot \vec{\mathbf{e}}_3) c_3 \right] \vec{\mathbf{e}}_2 + \\
&\left[a_3 b_1 (\vec{\mathbf{e}}_1 \cdot \vec{\mathbf{e}}_1) c_1 + a_3 b_2 (\vec{\mathbf{e}}_2 \cdot \vec{\mathbf{e}}_1) c_1 + a_3 b_3 (\vec{\mathbf{e}}_3 \cdot \vec{\mathbf{e}}_1) c_1 + a_3 b_1 (\vec{\mathbf{e}}_1 \cdot \vec{\mathbf{e}}_2) c_2 + a_3 b_2 (\vec{\mathbf{e}}_2 \cdot \vec{\mathbf{e}}_2) c_2 + a_3 b_3 (\vec{\mathbf{e}}_3 \cdot \vec{\mathbf{e}}_2) c_2 + \right. \\
&\quad \left. + a_3 b_1 (\vec{\mathbf{e}}_1 \cdot \vec{\mathbf{e}}_3) c_3 + a_3 b_2 (\vec{\mathbf{e}}_2 \cdot \vec{\mathbf{e}}_3) c_3 + a_3 b_3 (\vec{\mathbf{e}}_3 \cdot \vec{\mathbf{e}}_3) c_3 \right] \vec{\mathbf{e}}_3 \tag{40}
\end{aligned}$$

where the detailed expressions of the scalar components, d_1 , d_2 , d_3 , are shown in the square brackets. Based on this detailed expansion it can be shown that the scalar components of $\vec{\mathbf{d}}$, d_1 , d_2 , d_3 , arranged in an algebraic vector $\mathbf{d} = [d_1 \ d_2 \ d_3]^T$ can be expressed as

$$\mathbf{D}\mathbf{H}\mathbf{c} = \mathbf{d} \tag{41}$$

where

$$\mathbf{D} = \mathbf{a}\mathbf{b}^T = \begin{bmatrix} a_1 b_1 & a_1 b_2 & a_1 b_3 \\ a_2 b_1 & a_2 b_2 & a_2 b_3 \\ a_3 b_1 & a_3 b_2 & a_3 b_3 \end{bmatrix} \tag{42}$$

and \mathbf{H} is the inner product matrix for base vectors $\vec{\mathbf{e}}_1$, $\vec{\mathbf{e}}_2$, $\vec{\mathbf{e}}_3$ as was already introduced earlier for example in Eq. (27), and $\mathbf{c} = [c_1 \ c_2 \ c_3]^T$ contains the scalar components of vector $\vec{\mathbf{c}}$.

Symmetric and Skew-Symmetric Tensors: Considering Eqs. (36) and (37) a symmetric tensor (dyadic) can be interpreted as

$$\widehat{\mathbf{D}} = \widehat{\mathbf{D}}^T = \sum_{i=1}^3 \sum_{j=1}^3 D_{ij} \vec{\mathbf{e}}_i \otimes \vec{\mathbf{e}}_j = \sum_{i=1}^3 \sum_{j=1}^3 D_{ji} \vec{\mathbf{e}}_i \otimes \vec{\mathbf{e}}_j \tag{43}$$

where we can see that the D_{ij} scalar components can form a symmetric 3×3 matrix. A skew-symmetric tensor is defined as

$$\widehat{\mathbf{D}} = -\widehat{\mathbf{D}}^T = \sum_{i=1}^3 \sum_{j=1}^3 D_{ij} \vec{\mathbf{e}}_i \otimes \vec{\mathbf{e}}_j = -\sum_{i=1}^3 \sum_{j=1}^3 D_{ji} \vec{\mathbf{e}}_i \otimes \vec{\mathbf{e}}_j \tag{44}$$

Based on this we can see for this case the D_{ii} components must vanish $D_{ii} = 0$ ($i = 1, 2, 3$), only three nonzero scalar components exist in the matrix formed from the scalar components, and this matrix will look like the cross-product matrix introduced in Eq. (34). Therefore, the operation of a skew-symmetric tensor on a vector can be represented with a simple cross product operation as

$$\widehat{\mathbf{D}}_s \cdot \vec{\mathbf{c}} = \vec{\mathbf{w}} \times \vec{\mathbf{c}} \tag{45}$$

where $\vec{\mathbf{w}} = w_1\vec{\mathbf{e}}_1 + w_2\vec{\mathbf{e}}_2 + w_3\vec{\mathbf{e}}_3$ is the so-called vector invariant (or axial vector) of skew-symmetric tensor $\widehat{\mathbf{D}}_s$ where subscript “s” emphasizes that the tensor is skew-symmetric. The nonzero scalar components of the tensor are formed from the scalar components of $\vec{\mathbf{w}}$, w_1 , w_2 , w_3 , and their negatives.

Orthogonal Base Vectors: Again, the special, but very important case is when the base vectors are unit vectors and they are orthogonal to each other. In that case the inner product matrix \mathbf{H} becomes the identity matrix and the dyadic product of $(\vec{\mathbf{a}} \otimes \vec{\mathbf{b}}) \cdot \vec{\mathbf{c}} = \widehat{\mathbf{D}} \cdot \vec{\mathbf{c}} = \vec{\mathbf{d}}$ using only scalar components can be written based on Eq. (41) as

$$(\mathbf{ab}^T)\mathbf{c} = \mathbf{Dc} = \mathbf{d} \quad (46)$$

We can see that in such a case the dyadic product $\vec{\mathbf{a}} \otimes \vec{\mathbf{b}}$ can be fully represented with a 3×3 matrix \mathbf{D} with respect to the base vectors. This matrix contains essentially the scalar components of the dyadic product that is an entity on its own and the scalar components represent all possible product combinations of the scalar components of the two vectors that were used to compose the dyadic product. We note again that matrix \mathbf{D} itself does not carry information about the base vectors; so formulas such as Eq. (46) can only be used if we have an understanding of what the base vectors are and how the different geometric quantities are decomposed.

Note:

We note that all of the above considerations rely on geometric concepts at the first place, and the assumption that a decomposition for a physical vector, e.g. $\vec{\mathbf{a}}$, is available in the form of Eq. (17). Therefore, if $\vec{\mathbf{a}}$ is only given geometrically without any prior decomposition such as Eq. (17), then the only way to determine the scalar products $\vec{\mathbf{a}} \cdot \vec{\mathbf{e}}_1$, $\vec{\mathbf{a}} \cdot \vec{\mathbf{e}}_2$, $\vec{\mathbf{a}} \cdot \vec{\mathbf{e}}_3$ is via geometric construction,⁴ as illustrated in Fig. 2 for example. Most of the time, however, a geometric construction is not necessary to perform as the vector in question is already given in a decomposed form of Eq. (17) with respect to a set of base vectors, a_1 , a_2 , a_3 are known, and the problem is to transform the representation into another set of base vectors.

3.2 Change of Basis and Transformation of Scalar Components

Let us then consider now that the decomposition Eq. (17) is already known and we would like to change the base vectors of the representation to $\vec{\mathbf{e}}'_1$, $\vec{\mathbf{e}}'_2$, $\vec{\mathbf{e}}'_3$, which is another set of independent unit vectors and the decomposition of $\vec{\mathbf{a}}$ can be expressed in that basis as

$$\vec{\mathbf{a}} = a'_1\vec{\mathbf{e}}'_1 + a'_2\vec{\mathbf{e}}'_2 + a'_3\vec{\mathbf{e}}'_3 \quad (47)$$

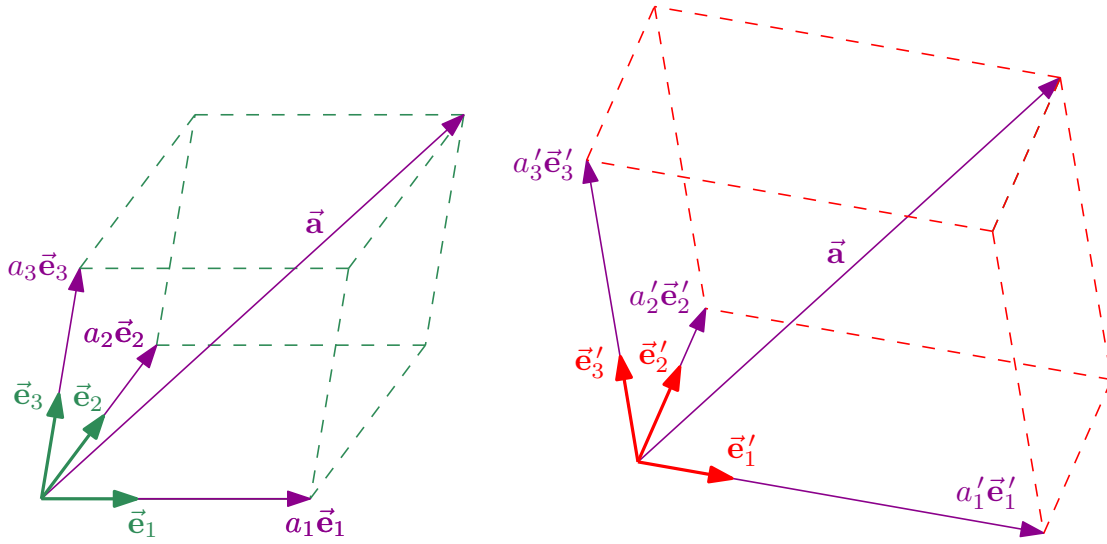
and illustrated in Fig. 7. Therefore, based on this and Eq. (17) we can write

$$\vec{\mathbf{a}} = a_1\vec{\mathbf{e}}_1 + a_2\vec{\mathbf{e}}_2 + a_3\vec{\mathbf{e}}_3 = a'_1\vec{\mathbf{e}}'_1 + a'_2\vec{\mathbf{e}}'_2 + a'_3\vec{\mathbf{e}}'_3 \quad (48)$$

Let us form the scalar products of Eq. (48) with the three new base vectors, $\vec{\mathbf{e}}'_1$, $\vec{\mathbf{e}}'_2$, $\vec{\mathbf{e}}'_3$, as

$$a_1\vec{\mathbf{e}}_1 \cdot \vec{\mathbf{e}}'_1 + a_2\vec{\mathbf{e}}_2 \cdot \vec{\mathbf{e}}'_1 + a_3\vec{\mathbf{e}}_3 \cdot \vec{\mathbf{e}}'_1 = a'_1\vec{\mathbf{e}}'_1 \cdot \vec{\mathbf{e}}'_1 + a'_2\vec{\mathbf{e}}'_2 \cdot \vec{\mathbf{e}}'_1 + a'_3\vec{\mathbf{e}}'_3 \cdot \vec{\mathbf{e}}'_1 \quad (49)$$

⁴This is an important concept, which again shows that all of these considerations stem from simple geometric notions.

Figure 7: *Illustration of change of basis*

$$a_1 \vec{e}_1 \cdot \vec{e}'_2 + a_2 \vec{e}_2 \cdot \vec{e}'_2 + a_3 \vec{e}_3 \cdot \vec{e}'_2 = a_1 \vec{e}'_1 \cdot \vec{e}'_2 + a_2 \vec{e}'_2 \cdot \vec{e}'_2 + a_3 \vec{e}'_3 \cdot \vec{e}'_2 \quad (50)$$

$$a_1 \vec{e}_1 \cdot \vec{e}'_3 + a_2 \vec{e}_2 \cdot \vec{e}'_3 + a_3 \vec{e}_3 \cdot \vec{e}'_3 = a_1 \vec{e}'_1 \cdot \vec{e}'_3 + a_2 \vec{e}'_2 \cdot \vec{e}'_3 + a_3 \vec{e}'_3 \cdot \vec{e}'_3 \quad (51)$$

These equations can be expressed for the scalar components as

$$\underbrace{\begin{bmatrix} \vec{e}_1 \cdot \vec{e}'_1 & \vec{e}_2 \cdot \vec{e}'_1 & \vec{e}_3 \cdot \vec{e}'_1 \\ \vec{e}_1 \cdot \vec{e}'_2 & \vec{e}_2 \cdot \vec{e}'_2 & \vec{e}_3 \cdot \vec{e}'_2 \\ \vec{e}_1 \cdot \vec{e}'_3 & \vec{e}_2 \cdot \vec{e}'_3 & \vec{e}_3 \cdot \vec{e}'_3 \end{bmatrix}}_{\mathbf{C}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} \vec{e}'_1 \cdot \vec{e}'_1 & \vec{e}'_2 \cdot \vec{e}'_1 & \vec{e}'_3 \cdot \vec{e}'_1 \\ \vec{e}'_1 \cdot \vec{e}'_2 & \vec{e}'_2 \cdot \vec{e}'_2 & \vec{e}'_3 \cdot \vec{e}'_2 \\ \vec{e}'_1 \cdot \vec{e}'_3 & \vec{e}'_2 \cdot \vec{e}'_3 & \vec{e}'_3 \cdot \vec{e}'_3 \end{bmatrix}}_{\mathbf{H}'} \underbrace{\begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix}}_{\mathbf{a}'} \quad (52)$$

where \mathbf{H}' is the inner product matrix associated with the new set of base vectors, and \mathbf{C} is the so-called *direction cosine* matrix formed from the scalar products among the two sets of base vectors. The inner product matrix is always invertible as long as the set of base vectors are linearly independent; it is in fact a positive definite matrix. Therefore, the scalar components $\mathbf{a}' = [a'_1 \ a'_2 \ a'_3]^T$ for the new basis can be expressed as

$$(\mathbf{H}')^{-1} \mathbf{C} \mathbf{a} = \mathbf{a}' \quad (53)$$

Considering that the base vectors are unit vectors, the elements of the direction cosine matrix can be written as

$$c_{ij} = \vec{e}_i \cdot \vec{e}'_j = \vec{e}'_j \cdot \vec{e}_i = \cos \varphi_{ij} \quad i, j = 1, 2, 3 \quad (54)$$

where φ_{ij} is the angle between base vectors \vec{e}_i and \vec{e}'_j and

$$\mathbf{C} = \begin{bmatrix} \vec{e}_1 \cdot \vec{e}'_1 & \vec{e}_2 \cdot \vec{e}'_1 & \vec{e}_3 \cdot \vec{e}'_1 \\ \vec{e}_1 \cdot \vec{e}'_2 & \vec{e}_2 \cdot \vec{e}'_2 & \vec{e}_3 \cdot \vec{e}'_2 \\ \vec{e}_1 \cdot \vec{e}'_3 & \vec{e}_2 \cdot \vec{e}'_3 & \vec{e}_3 \cdot \vec{e}'_3 \end{bmatrix} = \begin{bmatrix} c_{11} & c_{21} & c_{31} \\ c_{12} & c_{22} & c_{32} \\ c_{13} & c_{23} & c_{33} \end{bmatrix} \quad (55)$$

Let us now consider again Eq. (48) but now analyze the transformation from base vectors $\vec{e}'_1, \vec{e}'_2, \vec{e}'_3$ to set $\vec{e}_1, \vec{e}_2, \vec{e}_3$. Consider the scalar products of Eq. (48) with $\vec{e}_1, \vec{e}_2, \vec{e}_3$, respectively, as

$$a_1 \vec{e}_1 \cdot \vec{e}_1 + a_2 \vec{e}_2 \cdot \vec{e}_1 + a_3 \vec{e}_3 \cdot \vec{e}_1 = a_1 \vec{e}'_1 \cdot \vec{e}_1 + a_2 \vec{e}'_2 \cdot \vec{e}_1 + a_3 \vec{e}'_3 \cdot \vec{e}_1 \quad (56)$$

$$a_1 \vec{e}_1 \cdot \vec{e}_2 + a_2 \vec{e}_2 \cdot \vec{e}_2 + a_3 \vec{e}_3 \cdot \vec{e}_2 = a_1 \vec{e}'_1 \cdot \vec{e}_2 + a_2 \vec{e}'_2 \cdot \vec{e}_2 + a_3 \vec{e}'_3 \cdot \vec{e}_2 \quad (57)$$

$$a_1 \vec{e}_1 \cdot \vec{e}_3 + a_2 \vec{e}_2 \cdot \vec{e}_3 + a_3 \vec{e}_3 \cdot \vec{e}_3 = a_1 \vec{e}'_1 \cdot \vec{e}_3 + a_2 \vec{e}'_2 \cdot \vec{e}_3 + a_3 \vec{e}'_3 \cdot \vec{e}_3 \quad (58)$$

These equations can again be expressed for the scalar components as

$$\underbrace{\begin{bmatrix} \vec{e}_1 \cdot \vec{e}_1 & \vec{e}_2 \cdot \vec{e}_1 & \vec{e}_3 \cdot \vec{e}_1 \\ \vec{e}_1 \cdot \vec{e}_2 & \vec{e}_2 \cdot \vec{e}_2 & \vec{e}_3 \cdot \vec{e}_2 \\ \vec{e}_1 \cdot \vec{e}_3 & \vec{e}_2 \cdot \vec{e}_3 & \vec{e}_3 \cdot \vec{e}_3 \end{bmatrix}}_{\mathbf{H}} \underbrace{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}}_{\mathbf{a}} = \underbrace{\begin{bmatrix} \vec{e}'_1 \cdot \vec{e}_1 & \vec{e}'_2 \cdot \vec{e}_1 & \vec{e}'_3 \cdot \vec{e}_1 \\ \vec{e}'_1 \cdot \vec{e}_2 & \vec{e}'_2 \cdot \vec{e}_2 & \vec{e}'_3 \cdot \vec{e}_2 \\ \vec{e}'_1 \cdot \vec{e}_3 & \vec{e}'_2 \cdot \vec{e}_3 & \vec{e}'_3 \cdot \vec{e}_3 \end{bmatrix}}_{\mathbf{C}^T} \underbrace{\begin{bmatrix} a'_1 \\ a'_2 \\ a'_3 \end{bmatrix}}_{\mathbf{a}'} \quad (59)$$

where \mathbf{H} is the inner product matrix for base vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$, and we can see using Eqs. (52), (54) and (55) that the direction cosine matrix for this representation is simply the transpose of what was obtained previously in Eq. (52). The scalar components $\mathbf{a} = [a_1 \ a_2 \ a_3]^T$ can then be expressed as

$$\mathbf{H}^{-1} \mathbf{C}^T \mathbf{a}' = \mathbf{a} \quad (60)$$

3.2.1 Orthogonal Base Vectors and Canonical Cases

Let us consider the special, but very important case when each set of base vectors consists of three unit vectors that are orthogonal to each other. In that case both inner product matrices, \mathbf{H} and \mathbf{H}' , become the identity matrix, and Eqs. (53) and (60) for the transformation of the scalar components become

$$\mathbf{C} \mathbf{a} = \mathbf{a}' \quad (61)$$

and

$$\mathbf{C}^T \mathbf{a}' = \mathbf{a} \quad (62)$$

which clearly show that for this case $\mathbf{C}^{-1} = \mathbf{C}^T$, i.e., the direction cosine matrix becomes an *orthogonal matrix*, this alone will represent the transformation between the two sets of scalar components. We can also see based on Eq. (55) that in this case the columns of \mathbf{C} contain the scalar components of base vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ in terms of $\vec{e}'_1, \vec{e}'_2, \vec{e}'_3$, and the columns of \mathbf{C}^T contain the scalar components of $\vec{e}'_1, \vec{e}'_2, \vec{e}'_3$ with respect to $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

We can consider three canonical cases for orthogonal base vectors, when two base vectors, one in each set, are the same. The three cases are: (1) $\vec{e}_1 = \vec{e}'_1$, (2) $\vec{e}_2 = \vec{e}'_2$, (3) $\vec{e}_3 = \vec{e}'_3$.

Case 1: This is illustrated in Fig. 8 In that case any change only happens in the plane spanned by \vec{e}_2 and \vec{e}_3 , and also \vec{e}'_2 and \vec{e}'_3 . Based on the figure we can see that $\varphi_{22} = \varphi_{33} = \varphi$ and $\varphi_{32} = 90^\circ - \varphi$, and $\varphi_{23} = 90^\circ + \varphi$. Using these, matrix \mathbf{C} , detailed expression is in Eq. (52), can be expanded as

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \vec{e}_2 \cdot \vec{e}'_2 & \vec{e}_3 \cdot \vec{e}'_2 \\ 0 & \vec{e}_2 \cdot \vec{e}'_3 & \vec{e}_3 \cdot \vec{e}'_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & \cos(90^\circ - \varphi) \\ 0 & \cos(90^\circ + \varphi) & \cos \varphi \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & \sin \varphi \\ 0 & -\sin \varphi & \cos \varphi \end{bmatrix} \quad (63)$$

where trigonometric identities $\cos(90^\circ - \varphi) = \sin \varphi$ and $\cos(90^\circ + \varphi) = -\sin \varphi$ were also used.

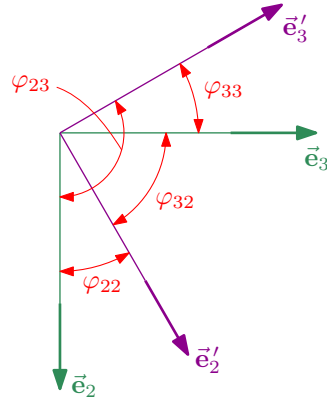


Figure 8: Transformation for canonical case 1

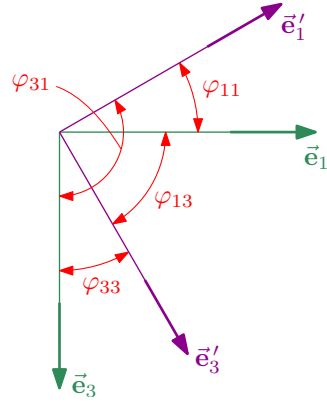


Figure 9: Transformation for canonical case 2

Case 2: This case is illustrated in Fig. 9 Here the plane spanned by \vec{e}_1 and \vec{e}_3 , and also \vec{e}'_1 and \vec{e}'_3 is of primary interest. Based on the figure we can determine that $\varphi_{11} = \varphi_{33} = \varphi$ and $\varphi_{13} = 90^\circ - \varphi$, and $\varphi_{31} = 90^\circ + \varphi$. Matrix \mathbf{C} can then be expanded as

$$\mathbf{C} = \begin{bmatrix} \vec{e}_1 \cdot \vec{e}'_1 & 0 & \vec{e}_3 \cdot \vec{e}'_1 \\ 0 & 1 & 0 \\ \vec{e}_1 \cdot \vec{e}'_3 & 0 & \vec{e}_3 \cdot \vec{e}'_3 \end{bmatrix} = \begin{bmatrix} \cos \varphi & 0 & \cos(90^\circ + \varphi) \\ 0 & 1 & 0 \\ \cos(90^\circ - \varphi) & 0 & \cos \varphi \end{bmatrix} = \begin{bmatrix} \cos \varphi & 0 & -\sin \varphi \\ 0 & 1 & 0 \\ \sin \varphi & 0 & \cos \varphi \end{bmatrix} \quad (64)$$

where again the above trigonometric identities were used.

Case 3: This is shown in Fig. 10 illustrating the plane spanned by \vec{e}_1 and \vec{e}_2 , and also \vec{e}'_1 and \vec{e}'_2 . Based on the figure we can determine that $\varphi_{11} = \varphi_{22} = \varphi$ and $\varphi_{12} = 90^\circ + \varphi$, and $\varphi_{21} = 90^\circ - \varphi$. Matrix \mathbf{C} can then be expanded as

$$\mathbf{C} = \begin{bmatrix} \vec{e}_1 \cdot \vec{e}'_1 & \vec{e}_2 \cdot \vec{e}'_1 & 0 \\ \vec{e}_1 \cdot \vec{e}'_2 & \vec{e}_2 \cdot \vec{e}'_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \varphi & \cos(90^\circ - \varphi) & 0 \\ \cos(90^\circ + \varphi) & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \cos \varphi & \sin \varphi & 0 \\ -\sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (65)$$

which gives the form of the direction cosine matrix for the third important canonical case.

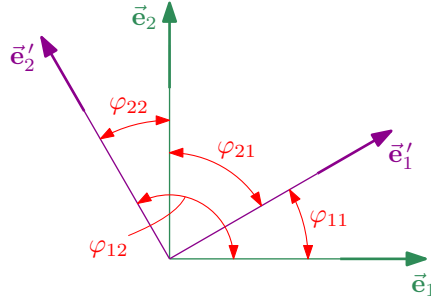


Figure 10: Transformation for canonical case 3

4 Vector Projections and Decompositions

Consider the standard three-dimensional representation shown in Fig. 11.

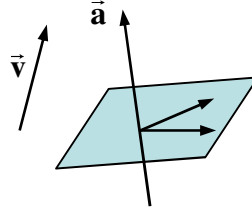


Figure 11: Definition of two complementary subspaces

We have a vector \vec{v} and a direction in space defined by another vector \vec{a} , which is used to define a subspace, in this case, a single line. We can also define another subspace orthogonal to this direction, as shown in the figure. The unit vector along the direction defined by \vec{a} is

$$\vec{e}_a = \frac{\vec{a}}{|\vec{a}|} \quad (66)$$

where $|\vec{a}| = a$ is the magnitude of the vector. Using \vec{e}_a , we can decompose \vec{v} as the sum of two vector components, one of them along the direction defined by \vec{a} and the other one perpendicular to it as

$$\vec{v} = \vec{v}_a + \vec{v}_{a\perp} = v_a \vec{e}_a + \vec{v}_{a\perp} \quad (67)$$

Multiplying Eq. (67) with \vec{e}_a

$$\vec{v} \cdot \vec{e}_a = (v_a \vec{e}_a) \cdot \vec{e}_a = v_a \quad (68)$$

because $\vec{v}_{a\perp} \cdot \vec{e}_a = 0$. We can express the two vector components as

$$\vec{v}_a = (\vec{v} \cdot \vec{e}_a) \vec{e}_a ; \quad \vec{v}_{a\perp} = \vec{v} - \vec{v}_a \quad (69)$$

and considering the dyadic product, \vec{v}_a can also be expressed as

$$\vec{v}_a = (\vec{v} \cdot \vec{e}_a) \vec{e}_a = \underbrace{(\vec{e}_a \otimes \vec{e}_a)}_{\hat{\mathbf{P}}} \cdot \vec{v} \quad (70)$$

where dyadic product $\widehat{\mathbf{P}}$ can be called projector tensor, and $\vec{v}_{a\perp}$ as

$$\vec{v}_{a\perp} = \vec{v} - (\vec{v}_a \cdot \vec{e}_a) \vec{e}_a = \vec{v} - (\vec{e}_a \otimes \vec{e}_a) \cdot \vec{v} = \underbrace{(\widehat{\mathbf{I}} - \vec{e}_a \otimes \vec{e}_a)}_{\widehat{\mathbf{I}} - \widehat{\mathbf{P}}} \cdot \vec{v} \quad (71)$$

where $\widehat{\mathbf{I}}$ represents the identity tensor (dyadic), i.e., $\widehat{\mathbf{I}} \cdot \vec{v} = \vec{v}$, and $\widehat{\mathbf{I}} - \widehat{\mathbf{P}}$ is the projector tensor to the plane perpendicular to the direction of \vec{a} , Fig. 11.

5 Change of Orientation and Finite Rotations

For the study of motion we have introduced two basic geometric objects: the point, and the vector. The displacements of points can be represented with vectors and can be tracked by the addition of such vectors as was discussed previously. On the other hand, everyday observations involve the perception of another type of displacement: the change of orientation, which cannot be represented with the transformation of a point; it requires the object that also includes information about orientation, a vector. For example, here we can think about physical vectors that are interpreted at the position level, e.g., to describe the relative position of two points.

To represent the change in orientation of a vector we can consider, without loss of generality that one end point, the starting point, of the vector remains fixed, and the magnitude of the vector does not change either. The change in orientation can then be considered as a rotation about that point.

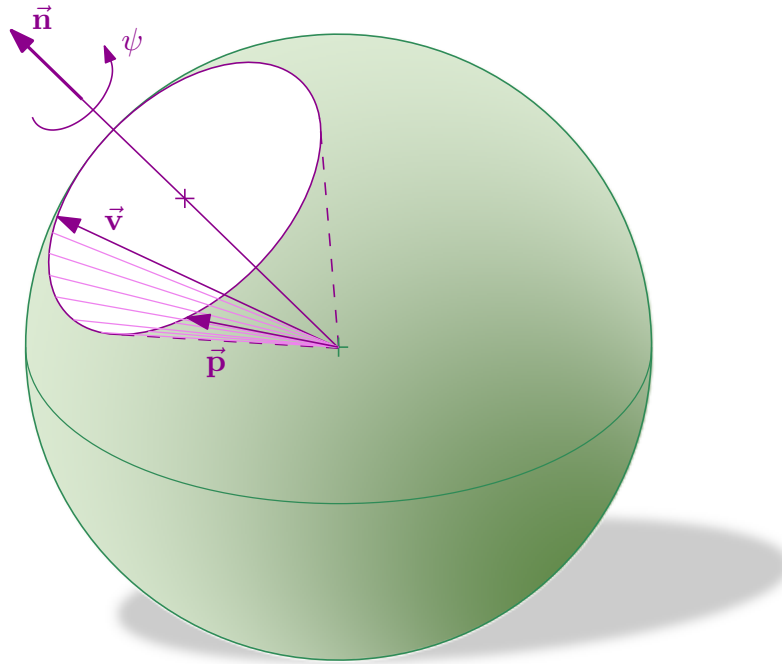


Figure 12: *General change of orientation*

For any change in orientation the end point of the vector will move on the surface of a sphere that has the radius equal to the magnitude of the vector and the centre of the sphere is at the

starting, fixed point of the vector, as illustrated in Fig. 12. We can consider that we are dealing with two different vectors that represent the initial configuration, \vec{v} and final configuration \vec{p} , respectively. To characterize the change in orientation we are interested in how these two vectors are related to each other, i.e., how the first vector can be transformed into the second one. This is not related to any frame of reference, passage of time, nor the actual path the end point of the vector \vec{v} would trace to arrive to the second configuration. We simply look at the static picture of the two vectors with different orientations and attempt to decide what transformation \vec{v} would need to undergo to become \vec{p} .

The end points of the two vectors are on the surface of the sphere, Fig. 12. Based on the spherical geometry, we can see that it is always possible to define a circle on the sphere, which will contain the two end points and it is the base of a right circular cone having its vertex at the centre of the sphere. The vectors define lines that are two generators of the cone. This shows that the first vector \vec{v} can always be transformed into the new vector \vec{p} via a single rotation about the axis of the cone with a certain angle.

This represents the general case of displacement difference between two vectors due to pure change in orientation. Hence, the most general change in orientation of a vector can be represented as a rotation about an axis with an angle.⁵ The representation of such a displacement is called a pure *finite rotation* that transforms one vector into another. The axis of the cone interpreted is the axis of rotation can be indicated by a unit vector \vec{n} pointing from the centre of the sphere towards the centre of the circle, and the angle of rotation can be denoted by ψ (Fig. 12).

Let us now consider this transformation in more detail. Consider that vector $\vec{v} = \overrightarrow{OA}$ is transformed into vector $\vec{p} = \overrightarrow{OB}$ via a pure finite rotation about axis indicated by unit vector \vec{n} (Fig. 13). The angle of the rotation is ψ . Based on the figure we can write that

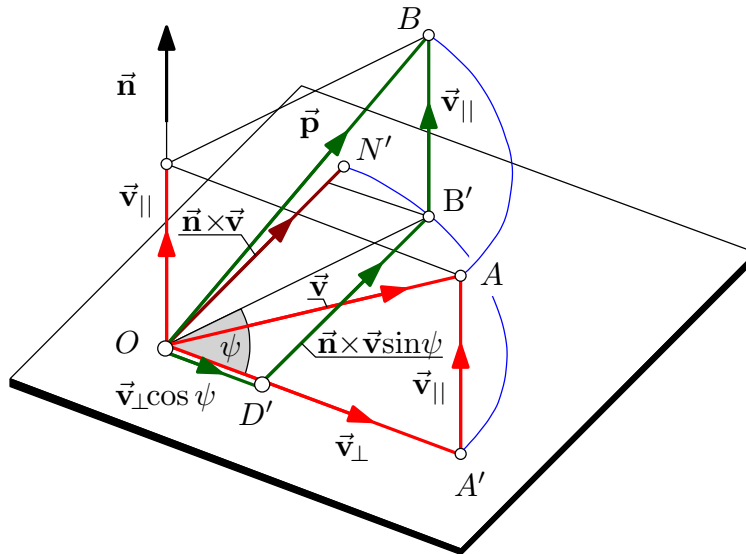


Figure 13: *Representation of finite rotation*

⁵This does not completely define the change of orientation of an arbitrary extended object as a rotation can also be possible with the vector itself being the axis. However, it generally proves the notion of how a general change in orientation, a finite rotation, can be represented with a rotation about an axis.

$$\vec{\mathbf{p}} = \vec{\mathbf{v}}_{\parallel} + \overrightarrow{OB'} = \vec{\mathbf{v}}_{\parallel} + \overrightarrow{OD'} + \overrightarrow{D'B'} \quad (72)$$

$$\vec{\mathbf{v}}_{\parallel} = \vec{\mathbf{n}}(\vec{\mathbf{n}} \cdot \vec{\mathbf{v}}) = (\vec{\mathbf{n}} \otimes \vec{\mathbf{n}}) \cdot \vec{\mathbf{v}} \quad (73)$$

and using the properties of vector operations and dyadic products discussed in the previous sections we can write

$$\vec{\mathbf{v}}_{\perp} = \vec{\mathbf{v}} - \vec{\mathbf{v}}_{\parallel} = \left(\widehat{\mathbf{I}} - \vec{\mathbf{n}} \otimes \vec{\mathbf{n}} \right) \cdot \vec{\mathbf{v}} \quad (74)$$

where $\widehat{\mathbf{I}}$ is the identity tensor as was introduced for Eq. (71). Considering the $OD'B'$ right-angled triangle we can write that

$$\overrightarrow{OD'} = \vec{\mathbf{v}}_{\perp} \cos \psi = \cos \psi \left(\widehat{\mathbf{I}} - \vec{\mathbf{n}} \otimes \vec{\mathbf{n}} \right) \cdot \vec{\mathbf{v}} \quad (75)$$

Based on the figure we can also see that rotating the $\vec{\mathbf{v}}_{\perp} = \overrightarrow{OA'}$ by $\pi/2$ about the $\vec{\mathbf{n}}$ axis results in the $\overrightarrow{ON'}$ vector. Using this we can establish that

$$\overrightarrow{ON'} = \vec{\mathbf{n}} \times \vec{\mathbf{v}}_{\perp} = \vec{\mathbf{n}} \times (\vec{\mathbf{v}}_{\parallel} + \vec{\mathbf{v}}_{\perp}) = \vec{\mathbf{n}} \times \vec{\mathbf{v}} \quad (76)$$

We can also notice using the figure that $\overline{ON'} = \overline{OB'}$ and $\overline{D'B'} = \overline{OB'} \sin \psi = \overline{ON'} \sin \psi$. Lines $\overline{ON'}$ and $\overline{D'B'}$ are parallel. Considering this and Eq. (76) we can obtain

$$\overrightarrow{D'B'} = \sin \psi \vec{\mathbf{n}} \times \vec{\mathbf{v}} = \sin \psi \widehat{\mathbf{N}} \cdot \vec{\mathbf{v}} \quad (77)$$

where $\widehat{\mathbf{N}}$ is a skew-symmetric cross product tensor associated with vector $\vec{\mathbf{n}}$. (Vector $\vec{\mathbf{n}}$ is the so-called vector invariant of $\widehat{\mathbf{N}}$.) Based on the above equations and Fig. 13, the transformation relationship between $\vec{\mathbf{p}}$ and $\vec{\mathbf{v}}$ can be established as

$$\vec{\mathbf{p}} = \left[\vec{\mathbf{n}} \otimes \vec{\mathbf{n}} + \cos \psi (\widehat{\mathbf{I}} - \vec{\mathbf{n}} \otimes \vec{\mathbf{n}}) + \sin \psi \widehat{\mathbf{N}} \right] \cdot \vec{\mathbf{v}} = \left[\cos \psi \widehat{\mathbf{I}} + (1 - \cos \psi) \vec{\mathbf{n}} \otimes \vec{\mathbf{n}} + \sin \psi \widehat{\mathbf{N}} \right] \cdot \vec{\mathbf{v}} \quad (78)$$

which can also be written as

$$\vec{\mathbf{p}} = \widehat{\mathbf{R}} \cdot \vec{\mathbf{v}} \quad (79)$$

where

$$\widehat{\mathbf{R}} = \cos \psi \widehat{\mathbf{I}} + (1 - \cos \psi) \vec{\mathbf{n}} \otimes \vec{\mathbf{n}} + \sin \psi \widehat{\mathbf{N}} \quad (80)$$

is the *rotation tensor* transforming vector $\vec{\mathbf{v}}$ into vector $\vec{\mathbf{p}}$. It is interpreted for the general case of a finite rotation.

A Property of the Rotation Tensor: Let us consider two arbitrary vectors $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$, that are both transformed by the same finite rotation tensor $\widehat{\mathbf{R}}$ as $\widehat{\mathbf{R}} \cdot \vec{\mathbf{r}}_1$ and $\widehat{\mathbf{R}} \cdot \vec{\mathbf{r}}_2$. The rotation keeps the magnitudes and the relative orientation between the two vectors constant. Therefore, the scalar products before and after the rotation have to be the same, and we can write

$$\vec{\mathbf{r}}_1 \cdot \vec{\mathbf{r}}_2 = \left(\widehat{\mathbf{R}} \cdot \vec{\mathbf{r}}_1 \right) \cdot \left(\widehat{\mathbf{R}} \cdot \vec{\mathbf{r}}_2 \right) = \vec{\mathbf{r}}_1 \cdot \widehat{\mathbf{R}}^T \cdot \widehat{\mathbf{R}} \cdot \vec{\mathbf{r}}_2 \quad (81)$$

from this with the introduction of the identity tensor we can obtain

$$\vec{\mathbf{r}}_1 \cdot \widehat{\mathbf{R}}^T \cdot \widehat{\mathbf{R}} \cdot \vec{\mathbf{r}}_2 - \vec{\mathbf{r}}_1 \cdot \vec{\mathbf{r}}_2 = \vec{\mathbf{r}}_1 \cdot \left(\widehat{\mathbf{R}}^T \cdot \widehat{\mathbf{R}} - \widehat{\mathbf{I}} \right) \cdot \vec{\mathbf{r}}_2 = 0 \quad (82)$$

As $\vec{\mathbf{r}}_1$ and $\vec{\mathbf{r}}_2$ are two arbitrary vectors Eq. (82) can only hold if

$$\widehat{\mathbf{R}}^T \cdot \widehat{\mathbf{R}} = \widehat{\mathbf{I}} \quad (83)$$

which means that $\widehat{\mathbf{R}}^T = \widehat{\mathbf{R}}^{-1}$, i.e., the inverse of the rotation tensor is its transpose. Due to this property the tensor is called an *orthogonal tensor*.

Linearization: Expanding the expression of the rotation tensor into Taylor series in terms of ψ for $\psi_0 = 0$ we obtain

$$\widehat{\mathbf{R}} = \widehat{\mathbf{I}} + \widehat{\mathbf{N}}\psi + \frac{1}{2}(\widehat{\mathbf{n}} \otimes \widehat{\mathbf{n}} - \widehat{\mathbf{I}})\psi^2 + \text{HOT} \quad (84)$$

where HOT stands for terms whose order in ψ is cubic or higher. If the angle of rotation is small, so that the HOT and the ψ^2 terms are negligible compared to the first two terms, then the rotation tensor can be approximated as

$$\overline{\widehat{\mathbf{R}}} = \widehat{\mathbf{I}} + \widehat{\mathbf{N}}\psi \quad (85)$$

where the $\overline{}$ represents that this is valid only for small rotations.

5.1 Composition of Two Finite Rotations

As we saw the transformation of physical vectors via a finite rotation can be represented by $\widehat{\mathbf{n}}$ and ψ . This finite rotation transforms vector $\vec{\mathbf{v}}$ into vector $\vec{\mathbf{p}}$ as

$$\vec{\mathbf{p}} = \widehat{\mathbf{R}} \cdot \vec{\mathbf{v}} \quad (86)$$

Consider now that the resulting vector undergoes another finite rotation represented by $\widehat{\mathbf{n}}_a$ and ψ_a . This transformation can be interpreted as

$$\vec{\mathbf{p}}_a = \widehat{\mathbf{R}}_a \cdot \vec{\mathbf{p}} \quad (87)$$

where $\widehat{\mathbf{R}}_a$ is the rotation tensor associated with $\widehat{\mathbf{n}}_a$ and ψ_a

We can write that

$$\vec{\mathbf{p}}_a = \widehat{\mathbf{R}}_a \cdot (\widehat{\mathbf{R}} \cdot \vec{\mathbf{v}}) = \widehat{\mathbf{R}}_a \cdot \widehat{\mathbf{R}} \cdot \vec{\mathbf{v}} \quad (88)$$

which shows that finite rotations are multiplicative, not additive like the displacements of points.

5.2 Time Rate of Change

Consider again a physical vector of constant magnitude, $\vec{\mathbf{r}}$, that is subjected to a finite rotation $\widehat{\mathbf{R}}$.⁶ The change in the orientation of this vector via the finite rotation can be understood if we consider another vector, $\vec{\mathbf{p}}$ that is fixed in the absolute reference frame and in the initial instant it is coincident with $\vec{\mathbf{r}}$. Then the finite rotation for $\vec{\mathbf{r}}$ can be interpreted as

$$\vec{\mathbf{r}} = \widehat{\mathbf{R}} \cdot \vec{\mathbf{p}} \quad (89)$$

⁶For example, a vector describing the position difference between two points of a reference frame.

The time rate of change of \vec{r} can then be expressed

$$\dot{\vec{r}} = \frac{d\widehat{\mathbf{R}}}{dt} \cdot \vec{p} = \dot{\widehat{\mathbf{R}}} \cdot \vec{p} \quad \rightarrow \quad \dot{\vec{r}} = \dot{\widehat{\mathbf{R}}} \cdot \widehat{\mathbf{R}}^T \cdot \vec{r} \quad (90)$$

where it was used that \vec{p} is fixed so that $\dot{\vec{p}} = \vec{0}$, and the rotation tensor is orthogonal, i.e., $\vec{p} = \widehat{\mathbf{R}}^T \cdot \vec{r}$. Therefore

$$\dot{\vec{r}} = \dot{\widehat{\mathbf{R}}} \cdot \widehat{\mathbf{R}}^T \cdot \vec{r} \quad \rightarrow \quad \dot{\vec{r}} = \widehat{\mathbf{\Omega}} \cdot \vec{r} \quad \widehat{\mathbf{\Omega}} = \dot{\widehat{\mathbf{R}}} \cdot \widehat{\mathbf{R}}^T \quad (91)$$

where $\widehat{\mathbf{\Omega}}$ is the *angular velocity tensor*.

Consider the orthogonality condition $\widehat{\mathbf{R}} \cdot \widehat{\mathbf{R}}^T = \widehat{\mathbf{I}}$ and based on that

$$\dot{\widehat{\mathbf{R}}} \cdot \widehat{\mathbf{R}}^T + \widehat{\mathbf{R}} \cdot \dot{\widehat{\mathbf{R}}^T} = \widehat{\mathbf{0}} \quad \rightarrow \quad \widehat{\mathbf{\Omega}} = -\widehat{\mathbf{\Omega}}^T \quad (92)$$

shows that the angular velocity tensor is skew-symmetric and as such, based on the discussion in section 3.1.4, it has a vector invariant, or axial vector, $\vec{\omega}$ so that

$$\dot{\vec{r}} = \widehat{\mathbf{\Omega}} \cdot \vec{r} = \vec{\omega} \times \vec{r} \quad (93)$$

holds where $\vec{\omega}$ is the *angular velocity vector*. This also illustrates that the angular velocity is not the derivative of a vector. It is a so-called nonholonomic quantity.

5.3 Euler Parameters

It can be seen from Eq. (80) that the rotation tensor is essentially defined by the unit vector of the axis of rotation \vec{n} and the rotation angle ψ . However, other parameterizations are also possible. We can consider for example

$$\vec{s} = \sin \frac{\psi}{2} \vec{n} \quad \text{and} \quad s_0 = \cos \frac{\psi}{2} \quad (94)$$

and using Eq. (80) the rotation tensor with these definitions can take the form

$$\widehat{\mathbf{R}} = (s_0^2 - \vec{s} \cdot \vec{s})\widehat{\mathbf{I}} + 2\vec{s} \otimes \vec{s} + 2s_0\widehat{\mathbf{S}} \quad (95)$$

where $\widehat{\mathbf{S}}$ is the skew-symmetric cross product tensor associated with vector \vec{s} . The three scalar components of \vec{s} with respect to a basis plus s_0 give the so-called Euler, or Euler-Rodrigues, parameters.

6 Frames of Reference and Their Parameterization

For the material discussed above the concept of the frame of reference was not needed. The concept of vectors and their representation do not directly require the interpretation of frames. However, as already discussed earlier, to describe typical modes of observed motion and generally interpret physical quantities frames of reference are necessary, and these frames need to be parameterized so that vectors in them can be properly represented.

Let us now consider that the base unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ are attached to a frame of reference \mathcal{F} , and can define a coordinate system/parameterization for that frame. The same way $\vec{e}'_1, \vec{e}'_2, \vec{e}'_3$ can be considered to be attached to frame \mathcal{F}' .

We can consider two fundamental geometric objects related to a frame: an arbitrary point of the frame, and a vector that defines the position of a point of the frame relative to another. This vector may be represented with respect to the base vectors, but that is not always necessary.

6.1 Rotation Matrix

The rotation matrix is the coordinate-level representation of the rotation tensor, i.e., it contains the scalar components of the rotations tensor in a given coordinate system. Based on Eq. (80), we can write the direct form of the rotation matrix as

$$\mathbf{R} = \cos \psi \mathbf{I} + (1 - \cos \psi) \mathbf{nn}^T + \sin \psi \mathbf{N} \quad (96)$$

where $\mathbf{n} = [n_1 \ n_2 \ n_3]^T$ represents the 3×1 column matrix of the three scalar components of the unit vector of the axis of rotation $\vec{\mathbf{n}}$ with respect to a selected coordinate system defined by a set of orthonormal base vectors, \mathbf{I} is the 3×3 identity matrix, and \mathbf{N} is the 3×3 skew-symmetric, cross-product matrix formed from the elements of \mathbf{n} .

A modified expression for the rotation matrix can be obtained if the Euler parameters are introduced. For example, the Euler parameters can then be defined via expressing $\vec{\mathbf{s}}$ from Eq. (94) in the given coordinate system: $\mathbf{s} = [s_1 \ s_2 \ s_3]^T$ and s_0 gives the fourth parameter. With these and using Eq. (95) the rotation matrix, \mathbf{R} can be expressed in the selected coordinate system as

$$\mathbf{R} = (s_0^2 - \mathbf{s}^T \mathbf{s}) \mathbf{I} + 2 \mathbf{s} \mathbf{s}^T + 2 s_0 \mathbf{S} \quad (97)$$

where \mathbf{S} is the skew-symmetric cross-product matrix.

6.2 Reference Frames and Transformations of Scalar Components

We will investigate it now further that how the rotation tensor can be represented with scalar components in a coordinate system attached to a frame. Let us consider that an orthogonal triad of unit vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ with a common origin, attached to frame \mathcal{F} , Fig. . These vectors and frame \mathcal{F} are transformed to $\vec{e}'_1, \vec{e}'_2, \vec{e}'_3$ and frame \mathcal{F}' via a rotation about axis $\vec{\mathbf{n}}$. This transformation is represented by tensor $\widehat{\mathbf{R}}$ and we can write

$$\vec{e}'_1 = \widehat{\mathbf{R}} \cdot \vec{e}_1 \quad \vec{e}'_2 = \widehat{\mathbf{R}} \cdot \vec{e}_2 \quad \vec{e}'_3 = \widehat{\mathbf{R}} \cdot \vec{e}_3 \quad (98)$$

We will consider the decomposition of all these quantities, their representation with scalar components, relative to an arbitrary orthogonal coordinate system attached to frame \mathcal{F}_a . This makes it possible to give the following matrix representation of Eq. (98)

$$\mathbf{E}' = \mathbf{R} \mathbf{E} \quad (99)$$

where

$$\mathbf{E}' = [\mathbf{e}'_1 \ \mathbf{e}'_2 \ \mathbf{e}'_3] \quad \mathbf{E} = [\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3] \quad (100)$$

and \mathbf{R} is the 3×3 rotation matrix containing the scalar components of $\widehat{\mathbf{R}}$, and \mathbf{E}' and \mathbf{E} are 3×3 matrices whose columns, $\mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3$ and $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$, contain the scalar components of the $\vec{\mathbf{e}}'_1, \vec{\mathbf{e}}'_2, \vec{\mathbf{e}}'_3$ and $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3$ vector triads, respectively, \mathbf{e}'_i and \mathbf{e}_i are 3×1 algebraic vectors of scalar components, $i = 1, 2, 3$. As mentioned above, all the components are expressed in the coordinate system associated with \mathcal{F}_a .

Based on (99)

$$\mathbf{R} = \mathbf{E}'\mathbf{E}^{-1} \quad (101)$$

which also gives a general expression to represent the scalar components of the rotation tensor in any coordinate system (in this case in \mathcal{F}_a).

If this frame \mathcal{F}_a is selected more specifically to be the orthogonal frame defined by the vector triad $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3$, i.e., $\mathcal{F}_a \equiv \mathcal{F}$ then $\mathbf{E} = \mathbf{I}$ is the 3×3 identity matrix, and consequently, the rotation matrix representation will be $\mathbf{R} = \mathbf{E}'$. This means that the columns of the rotation matrix will contain the scalar components of the unit vectors of the \mathcal{F}' coordinate frame expressed in terms of the unit vectors of frame \mathcal{F} . These quantities are also often referred to as direction cosines as we already discussed previously in section 3.2, and $\mathbf{R} = \mathbf{C}^T$ is an orthogonal matrix. This gives one possibility to express the rotation matrix and establish transformation relationships between two frames and the attached coordinate systems.

Let us now consider a vector $\vec{\mathbf{r}}$ which is decomposed in an arbitrary orthogonal coordinate system attached to \mathcal{F}_a as

$$\mathbf{r}_a = r_1\mathbf{e}_1 + r_2\mathbf{e}_2 + r_3\mathbf{e}_3 = [\mathbf{e}_1 \ \mathbf{e}_2 \ \mathbf{e}_3] \mathbf{r} = \mathbf{E}\mathbf{r} \quad (102)$$

and

$$\mathbf{r}_a = r'_1\mathbf{e}'_1 + r'_2\mathbf{e}'_2 + r'_3\mathbf{e}'_3 = [\mathbf{e}'_1 \ \mathbf{e}'_2 \ \mathbf{e}'_3] \mathbf{r}' = \mathbf{E}'\mathbf{r}' \quad (103)$$

where \mathbf{r} and \mathbf{r}' are 3×1 arrays that contain the scalar components of $\vec{\mathbf{r}}$ with respect to the coordinate systems attached to \mathcal{F} and \mathcal{F}' , respectively, $\mathbf{r} = [r_1 \ r_2 \ r_3]^T$ and $\mathbf{r}' = [r'_1 \ r'_2 \ r'_3]^T$. If now again we consider the special selection for the coordinate system the one attached to \mathcal{F} , i.e., $\mathcal{F}_a \equiv \mathcal{F}$, then considering Eq. (100), we can also write for Eq. (103) that

$$\mathbf{r}_a = \mathbf{r} = \mathbf{E}'\mathbf{r}' = \mathbf{R}\mathbf{r}' \quad (104)$$

and we arrive to the result that in the most common case the scalar components of vectors are transforming as

$$\mathbf{r} = \mathbf{R}\mathbf{r}' \quad \mathbf{r}' = \mathbf{R}^T\mathbf{r} \quad (105)$$

which describes the transformation of the scalar components of a vector the same way as done employing the direction cosine matrix if orthogonal coordinate systems are used. Therefore, the two main applications of the rotation matrix can be summarized as:

- According to the original interpretation of the rotation tensor, the rotation matrix can be used for the component-level representation of the transformation of a physical vector into another physical vector, where all scalar components are expressed in the same coordinate system.
- It can also be used for the transformation of the scalar components of a vector between two different orthogonal coordinate systems. In that case, for these transformations the use of the rotation matrix becomes equivalent to that of the direction cosine matrix (and its transpose).

6.3 Algebraic form of the composition of two finite rotations

Let us consider again two finite rotations as was discussed in section 5.1. The two rotations are represented by tensors $\widehat{\mathbf{R}}$ and $\widehat{\mathbf{R}}_a$ and the transformation of vectors as described by Eqs. (86)-(88), and the final transformation relationship can be established as

$$\vec{\mathbf{p}}_a = \widehat{\mathbf{R}}_a \cdot (\widehat{\mathbf{R}} \cdot \vec{\mathbf{v}}) = \widehat{\mathbf{R}}_a \cdot \widehat{\mathbf{R}} \cdot \vec{\mathbf{v}} \quad (106)$$

In terms of frames of reference we can consider that $\widehat{\mathbf{R}}$ transforms frame \mathcal{F} to \mathcal{F}_A , and $\widehat{\mathbf{R}}_a$ transforms frame \mathcal{F}_A to \mathcal{F}' . We consider that each of these frames are parameterized by a coordinate system represented by an orthogonal set of unit vectors. These sets of unit vectors are transformed into one another by the two rotations described above.

We are interested to see what the decomposed, algebraic form of the general transformation of Eq. (106) is. If we consider that each vector and tensor is decomposed in the starting \mathcal{F} frame then we can simply write

$$\mathbf{p}_a = \mathbf{R}_a \mathbf{R} \mathbf{v} \quad (107)$$

where $\mathbf{R}_a \mathbf{R} = \mathbf{R}_{res}$ is the resultant rotation matrix, and for the individual rotations according to Eqs. (86)-(87) as $\mathbf{p} = \mathbf{R} \mathbf{v}$ and $\mathbf{p}_a = \mathbf{R}_a \mathbf{p}$.

However, decomposing the vectors and tensors can often be simpler in certain frames. For example, according to section 6.2, $\widehat{\mathbf{R}}$ may be simplest to decompose in the coordinate system attached to \mathcal{F} , and the decomposition for $\widehat{\mathbf{R}}_a$ can be simple in the coordinate system of \mathcal{F}_A .

Let us analyze the individual steps. For the rotation represented by Eq. (86) we can write that

$$\begin{aligned} \mathbf{p} &= \mathbf{R} \mathbf{v} \\ \mathbf{p}^{(A)} &= \mathbf{R}^T \mathbf{p} \\ \mathbf{p}^{(A)} &= \mathbf{v} \end{aligned} \quad (108)$$

where 3×3 matrix and 3×1 algebraic vectors \mathbf{R} , \mathbf{p} , \mathbf{v} represent scalar components in the coordinate system of frame \mathcal{F} , and $\mathbf{p}^{(A)}$ is the 3×1 algebraic vector containing the scalar components of vector $\vec{\mathbf{p}}$ expressed in the coordinate system of frame \mathcal{F}_A .

Then for the rotation represented by Eq. (87)

$$\begin{aligned} \mathbf{p}_a^{(A)} &= \mathbf{R}_a^{(A)} \mathbf{p}^{(A)} \\ \mathbf{p}'_a &= \mathbf{R}_a^{(A)T} \mathbf{p}_a^{(A)} \\ \mathbf{p}'_a &= \mathbf{p}^{(A)} \end{aligned} \quad (109)$$

where $\mathbf{R}_a^{(A)}$ represents the 3×3 rotation matrix containing the scalar components of $\widehat{\mathbf{R}}_a$ expressed in the coordinate system of frame \mathcal{F}_A , 3×1 algebraic vector $\mathbf{p}_a^{(A)}$ contains the scalar components of $\vec{\mathbf{p}}_a$ in the same coordinate system, and \mathbf{p}'_a contains the scalar components of $\vec{\mathbf{p}}_a$ expressed in the coordinate system of frame \mathcal{F}' .

Based on these relationships we can see that $\mathbf{p}_a^{(A)} = \mathbf{R}_a^{(A)} \mathbf{v}$ and $\mathbf{p}_a^{(A)} = \mathbf{R}^T \mathbf{p}_a$ leading to $\mathbf{R}_a^{(A)} \mathbf{v} = \mathbf{R}^T \mathbf{p}_a$ and

$$\mathbf{p}_a = \mathbf{R} \mathbf{R}_a^{(A)} \mathbf{v} \quad (110)$$

which can be compared to Eq. (107), and gives an alternative form of the algebraic expression of the composition of the transformation of two rotations. This results in the alternative form for the resultant rotation matrix as $\mathbf{R}_{res} = \mathbf{R}\mathbf{R}_a^{(A)}$. This is a useful relationship as it makes it possible to determine a resultant rotation matrix of multiple rotations in a way that each individual rotation is always decomposed only in the “local” frame it rotates.

6.4 Rate of change of a vector

Let us consider two frames of reference \mathcal{F} and \mathcal{F}' , and that \mathcal{F} is our primary frame for observation and \mathcal{F}' is moving relative to that, Fig. 14. Consider an arbitrary physical vector $\vec{\mathbf{u}}$ expressed in a

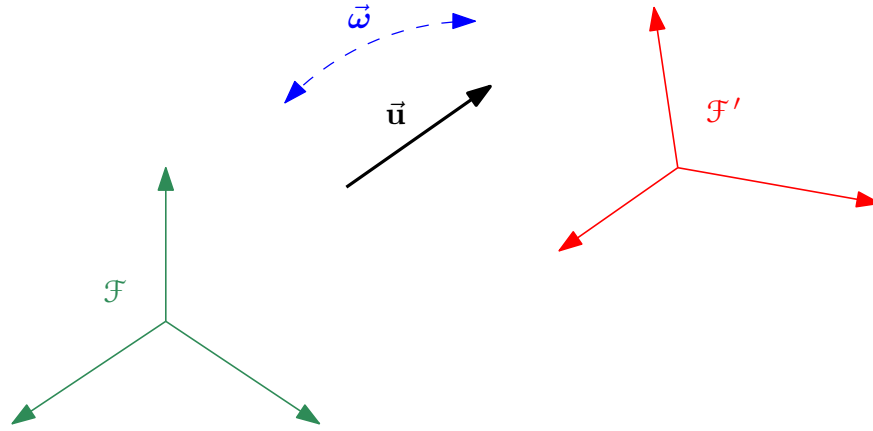


Figure 14: Reference frames in relative motion

moving frame, \mathcal{F}' , parameterized with unit vectors, $\vec{e}'_1, \vec{e}'_2, \vec{e}'_3$ defining coordinate system $x'_1x'_2x'_3$. The vector can then be expressed as

$$\vec{\mathbf{u}} = u'_1\vec{e}'_1 + u'_2\vec{e}'_2 + u'_3\vec{e}'_3 \quad (111)$$

Let us consider that frame \mathcal{F}' is rotating with angular velocity $\vec{\omega}$. The time derivative of $\vec{\mathbf{u}}$ with respect to frame \mathcal{F} can be formed based on Eq. (111) as

$$\dot{\vec{\mathbf{u}}} = \dot{u}'_1\vec{e}'_1 + \dot{u}'_2\vec{e}'_2 + \dot{u}'_3\vec{e}'_3 + u'_1\dot{\vec{e}}'_1 + u'_2\dot{\vec{e}}'_2 + u'_3\dot{\vec{e}}'_3 \quad (112)$$

where

$$\left(\frac{d\vec{\mathbf{u}}}{dt}\right)_{\mathcal{F}'} = \dot{u}'_1\vec{e}'_1 + \dot{u}'_2\vec{e}'_2 + \dot{u}'_3\vec{e}'_3 \quad (113)$$

is the “local” time derivative of $\vec{\mathbf{u}}$ with respect to \mathcal{F}' , i.e., what an observer moving together with \mathcal{F}' would see for the rate of change of the vector. Based on section 5.2

$$u'_1\dot{\vec{e}}'_1 + u'_2\dot{\vec{e}}'_2 + u'_3\dot{\vec{e}}'_3 = u'_1\vec{\omega} \times \vec{e}'_1 + u'_2\vec{\omega} \times \vec{e}'_2 + u'_3\vec{\omega} \times \vec{e}'_3 = \vec{\omega} \times \vec{\mathbf{u}} \quad (114)$$

then the “total” time derivative with respect to \mathcal{F} can be expressed as

$$\dot{\vec{\mathbf{u}}} = \frac{d\vec{\mathbf{u}}}{dt} = \left(\frac{d\vec{\mathbf{u}}}{dt}\right)_{\mathcal{F}'} + \vec{\omega} \times \vec{\mathbf{u}} \quad (115)$$

6.5 Angular acceleration

Let us consider again the angular velocity $\vec{\omega}$, defined for the rotation of frame \mathcal{F}' with respect to frame \mathcal{F} . This is the physical meaning of this angular velocity. It does not depend on what coordinate system is used to decompose it. The the rate of change of the angular velocity $\vec{\omega}$ can be expressed with respect to frame \mathcal{F} as

$$\vec{\alpha} = \frac{d}{dt}\vec{\omega} = \dot{\vec{\omega}} = \left(\frac{d\vec{\omega}}{dt}\right)_{\mathcal{F}} \quad (116)$$

which is the angular acceleration. Based on Eq. (115) we can see that

$$\dot{\vec{\omega}} = \frac{d\vec{\omega}}{dt} = \left(\frac{d\vec{\omega}}{dt}\right)_{\mathcal{F}'} + \vec{\omega} \times \vec{\omega} = \left(\frac{d\vec{\omega}}{dt}\right)_{\mathcal{F}'} \quad (117)$$

i.e., the rate of change of the angular velocity is the same in both frames. However, the same time derivative of the same angular velocity can also be formed using another moving frame, \mathcal{F}_1 , where this frame is rotating with angular velocity $\vec{\omega}_1$ with respect to of our primary frame of observation \mathcal{F} so that $\dot{\vec{\omega}}$ can also be formed as

$$\vec{\alpha} = \dot{\vec{\omega}} = \left(\frac{d\vec{\omega}}{dt}\right)_{\mathcal{F}_1} + \vec{\omega}_1 \times \vec{\omega} \quad (118)$$

this is the general formula for the interpretation of the angular acceleration of frame \mathcal{F}' with respect to \mathcal{F} . It reduces to the simpler form of Eq. (117) if $\mathcal{F}_1 \equiv \mathcal{F}'$.

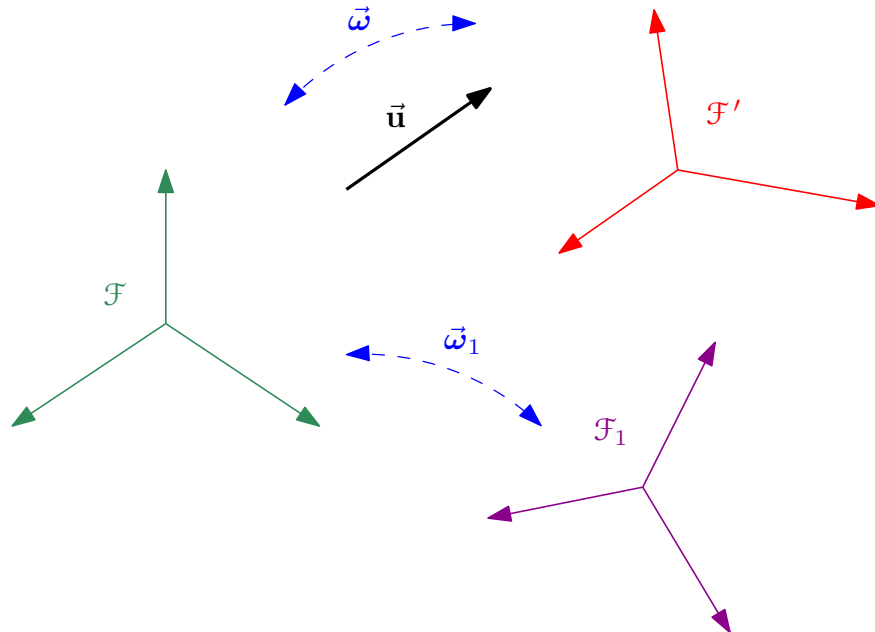


Figure 15: *Reference frames in relative motion: interpretation with respect to third frame*

6.6 Relative velocity and acceleration

Consider two reference frames again, \mathcal{F} and \mathcal{F}' ; the position of an arbitrary point P can be expressed by

$$\vec{\mathbf{r}}_P = \vec{\mathbf{r}}_B + \vec{\mathbf{r}}_{BP} \quad (119)$$

and is illustrated in Fig. 16. Forming the time derivative with respect to frame \mathcal{F} we obtain

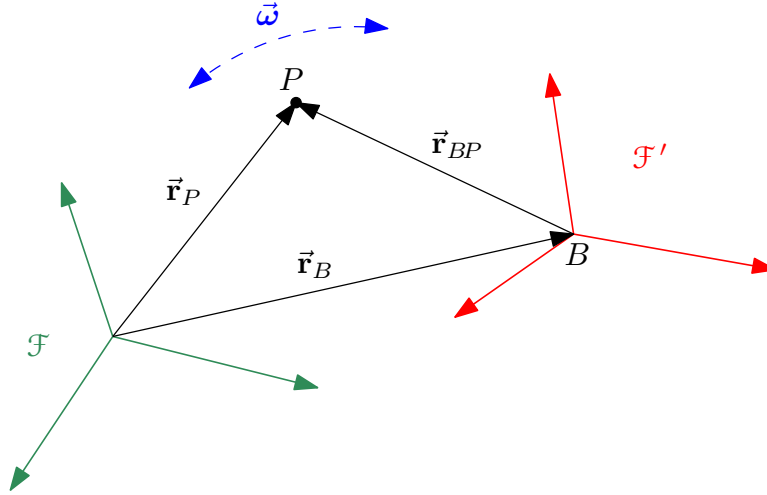


Figure 16: *Relative motion of a point*

$$\vec{\mathbf{v}}_P = \vec{\mathbf{v}}_B + \vec{\mathbf{v}}_{BP} \quad (120)$$

where based on section 6.4 we can write

$$\vec{\mathbf{v}}_{BP} = \dot{\vec{\mathbf{r}}}_{BP} = \vec{\mathbf{v}}_{BP_{\mathcal{F}'}} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_{BP} \quad (121)$$

and then Eq. (120) can be expanded as

$$\vec{\mathbf{v}}_P = \vec{\mathbf{v}}_B + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_{BP} + \vec{\mathbf{v}}_{BP_{\mathcal{F}'}} \quad (122)$$

where $\vec{\mathbf{v}}_B + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_{BP}$ is the transport velocity, and $\vec{\mathbf{v}}_{BP_{\mathcal{F}'}}$ is the relative velocity. The physical interpretation of the terms of this expression can also be given as: $\vec{\mathbf{v}}_B + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_{BP}$ represents the velocity of the point of frame \mathcal{F}' with respect to frame \mathcal{F} , which point is instantaneously coincident with point P , and $\vec{\mathbf{v}}_{BP_{\mathcal{F}'}}$ is the velocity of point P with respect to frame \mathcal{F}' , i.e., an observer attached to frame \mathcal{F}' would see this velocity for point P .

The expressions at the acceleration level can be established forming the time derivatives of the terms of Eq. (122) as

$$\vec{\mathbf{a}}_P = \vec{\mathbf{a}}_B + \frac{d}{dt}(\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_{BP}) + \frac{d}{dt}(\vec{\mathbf{v}}_{BP_{\mathcal{F}'}}) \quad (123)$$

where

$$\frac{d}{dt}(\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_{BP}) = \vec{\boldsymbol{\alpha}} \times \vec{\mathbf{r}}_{BP} + \vec{\boldsymbol{\omega}} \times \vec{\mathbf{v}}_{BP_{\mathcal{F}'}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\mathbf{r}}_{BP}) \quad (124)$$

and

$$\frac{d}{dt}(\vec{v}_{BP_{\mathcal{F}'}}) = \vec{a}_{BP_{\mathcal{F}'}} + \vec{\omega} \times \vec{v}_{BP_{\mathcal{F}'}} \quad (125)$$

then Eq. (123) can be expanded as

$$\vec{a}_P = \vec{a}_B + \vec{\alpha} \times \vec{r}_{BP} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{BP}) + \vec{a}_{BP_{\mathcal{F}'}} + 2\vec{\omega} \times \vec{v}_{BP_{\mathcal{F}'}} \quad (126)$$

where $\vec{a}_B + \vec{\alpha} \times \vec{r}_{BP} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{BP})$ is the transport acceleration giving the acceleration of the point of frame \mathcal{F}' that is instantaneously coincident with point P , $\vec{a}_{BP_{\mathcal{F}'}}$ is the relative acceleration, and $2\vec{\omega} \times \vec{v}_{BP_{\mathcal{F}'}}$ is the Coriolis acceleration.

A special case is when point P is fixed in reference frame \mathcal{F}' . In that case both $\vec{v}_{BP_{\mathcal{F}'}}$ and $\vec{a}_{BP_{\mathcal{F}'}}$ vanish, $\vec{v}_{BP_{\mathcal{F}'}} = \vec{0}$, $\vec{a}_{BP_{\mathcal{F}'}} = \vec{0}$, and Eqs. (122) and (126) reduce to

$$\vec{v}_P = \vec{v}_B + \vec{\omega} \times \vec{r}_{BP} \quad (127)$$

and

$$\vec{a}_P = \vec{a}_B + \vec{\alpha} \times \vec{r}_{BP} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{BP}) \quad (128)$$

This special case will particularly be important for the case of the concept of a rigid body that will be introduced later.

6.7 Additivity of Angular Velocities

Let us consider the relations of three reference frames, \mathcal{F} , \mathcal{F}_A , \mathcal{F}' , and the time rates of change of vector \vec{r} relative to these frames. The frames are illustrated in Fig. 17 with a coordinate system attached to each, and the transformation relationships are also illustrated. Let us again indicate

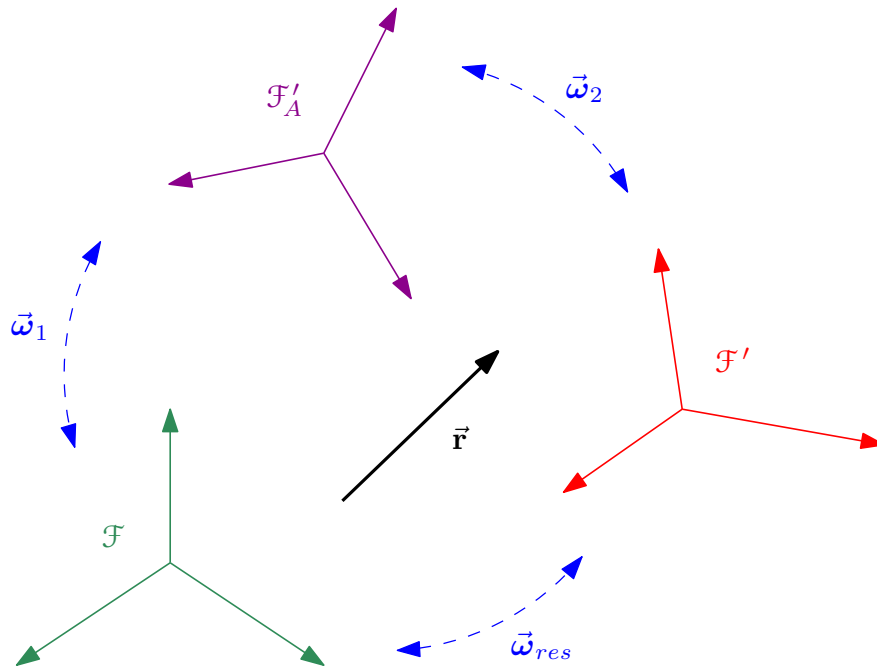


Figure 17: Three frames and angular velocity relations

the rate of change relative to \mathcal{F} as $\frac{d\vec{r}}{dt} = \dot{\vec{r}}$. Based on section 6.4 we can then write

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}}{dt}\right)_{\mathcal{F}_A} + \vec{\omega}_1 \times \vec{r} \quad (129)$$

$$\left(\frac{d\vec{r}}{dt}\right)_{\mathcal{F}_A} = \left(\frac{d\vec{r}}{dt}\right)_{\mathcal{F}'} + \vec{\omega}_2 \times \vec{r} \quad (130)$$

and

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}}{dt}\right)_{\mathcal{F}'} + \vec{\omega}_{res} \times \vec{r} \quad (131)$$

By substituting Eq. (130) into Eq. (129) we obtain

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt} = \left(\frac{d\vec{r}}{dt}\right)_{\mathcal{F}'} + (\vec{\omega}_1 + \vec{\omega}_2) \times \vec{r} \quad (132)$$

from which we can conclude that $\vec{\omega}_{res} = \vec{\omega}_1 + \vec{\omega}_2$. This proves that angular velocities are additive, while the related finite rotations are multiplicative, i.e., $\widehat{\mathbf{R}}_{res} = \widehat{\mathbf{R}}_2 \cdot \widehat{\mathbf{R}}_1$, as was discussed in section 5.1.

6.8 Parameterizations of Finite Rotations

As was discussed in section 5 a general finite rotation connecting two vectors or two frames of reference can be given as a rotation about an axis with an angle. This made it possible to derive the concept of the rotation tensor that interprets the geometric operation of finite rotation. In the algebraic sense the rotation tensor can be represented with the rotation matrix with respect to a system of base vectors, i.e., with respect to a coordinate system.

We can then consider that how many scalar variables we need to give to define the rotation matrix. We can consider that the vector defining the axis of rotation is generally representable with three scalar components relative to any coordinate system, plus we have the angle of rotation ψ . This gives four variables (parameters) to represent a general rotation. However, we have to notice that the axis of rotation is interpreted with a unit vector, hence, this condition exists among the elements of \mathbf{n} , i.e.,

$$\mathbf{n}^T \mathbf{n} = \mathbf{n}^2 = 1 \quad (133)$$

Therefore, only two elements of \mathbf{n} are independent giving the conclusion that a general finite rotation can be algebraically represented with a minimum of three independent variables (parameters). However, the direct four-parameter representation together with the algebraic condition Eq. (133) can be generally beneficial.

6.8.1 Four-parameter representation

This is the representation directly coming from the interpretation of the rotation tensor and the concept of rotation. The related rotation matrices were described in Section 6.1. Let us consider here the expressions in a little more detail when Euler parameters are used, they are defined as

$$\vec{s} = \sin \frac{\psi}{2} \vec{\mathbf{n}} \quad \text{and} \quad s_0 = \cos \frac{\psi}{2} \quad (134)$$

Expressing \vec{s} in a coordinate system gives $\mathbf{s} = [s_1 \ s_2 \ s_3]^T$ and s_0 gives the fourth parameter. They can also be grouped in one single algebraic vector $\mathbf{g} = [s_0 \ \mathbf{s}^T]^T$. With these the rotation matrix, \mathbf{R} can be expressed in the specified orthonormal coordinate system as

$$\mathbf{R} = (s_0^2 - \mathbf{s}^T \mathbf{s})\mathbf{I} + 2\mathbf{s}\mathbf{s}^T + 2s_0\mathbf{S} \quad (135)$$

Based on Eqs. (133) and (134) the four Euler parameters are subjected to

$$s^2 + s_0^2 = 1 \quad \text{or} \quad \mathbf{g}^2 = 1 \quad \mathbf{g} = [s_0 \ \mathbf{s}^T]^T \quad (136)$$

This can also be reduced to a three-parameter representation as we will discuss later.

Expanding Eq. (135) the detailed expression of the rotation matrix can be obtained as

$$\mathbf{R} = \begin{bmatrix} s_0^2 + s_1^2 - s_2^2 - s_3^2 & 2(s_1s_2 - s_0s_3) & 2(s_1s_3 + s_0s_2) \\ 2(s_2s_1 + s_0s_3) & s_0^2 - s_1^2 + s_2^2 - s_3^2 & 2(s_2s_3 - s_0s_1) \\ 2(s_3s_1 - s_0s_2) & 2(s_3s_2 + s_0s_1) & s_0^2 - s_1^2 - s_2^2 + s_3^2 \end{bmatrix} \quad (137)$$

It can also be shown that

$$\mathbf{R} = (s_0^2 - \mathbf{s}^T \mathbf{s})\mathbf{I} + 2\mathbf{s}\mathbf{s}^T + 2s_0\mathbf{S} = \mathbf{G}\mathbf{L}^T \quad (138)$$

where

$$\mathbf{G} = [-\mathbf{s} \ \mathbf{S} + s_0\mathbf{I}] = \begin{bmatrix} -s_1 & s_0 & -s_3 & s_2 \\ -s_2 & s_3 & s_0 & -s_1 \\ -s_3 & -s_2 & s_1 & s_0 \end{bmatrix} \quad (139)$$

$$\mathbf{L} = [-\mathbf{s} \ -\mathbf{S} + s_0\mathbf{I}] = \begin{bmatrix} -s_1 & s_0 & s_3 & -s_2 \\ -s_2 & -s_3 & s_0 & s_1 \\ -s_3 & s_2 & -s_1 & s_0 \end{bmatrix} \quad (140)$$

and

$$\mathbf{G}\mathbf{G}^T = \mathbf{I} \quad \mathbf{L}\mathbf{L}^T = \mathbf{I} \quad (141)$$

where \mathbf{I} is the 3×3 identity matrix.

6.8.2 Three-parameter representation

Let us consider again Fig. 12 interpreting a general change in orientation due to a rotation. The dependency in the four-parameter representation is caused by the representation of the axis of rotation. A unit vector indicating a direction in space can be represented with two independent parameters, therefore any third parameter will not be independent. This gives rise to the constraint given in Eq. (133). Therefore, if the axis of rotation is represented with only two independent parameters then we can obtain a three-parameter representation where all parameters are independent of each other.

One such possibility can be to represent $\vec{\mathbf{n}}$ with spherical coordinates with respect to the selected coordinate system as shown in Figure 18.

In that case the unit vector of the axis of rotation can be represented as

$$\vec{\mathbf{n}} = \sin \Theta \cos \phi \vec{\mathbf{e}}_x + \sin \Theta \sin \phi \vec{\mathbf{e}}_y + \cos \Theta \vec{\mathbf{e}}_z \quad (142)$$

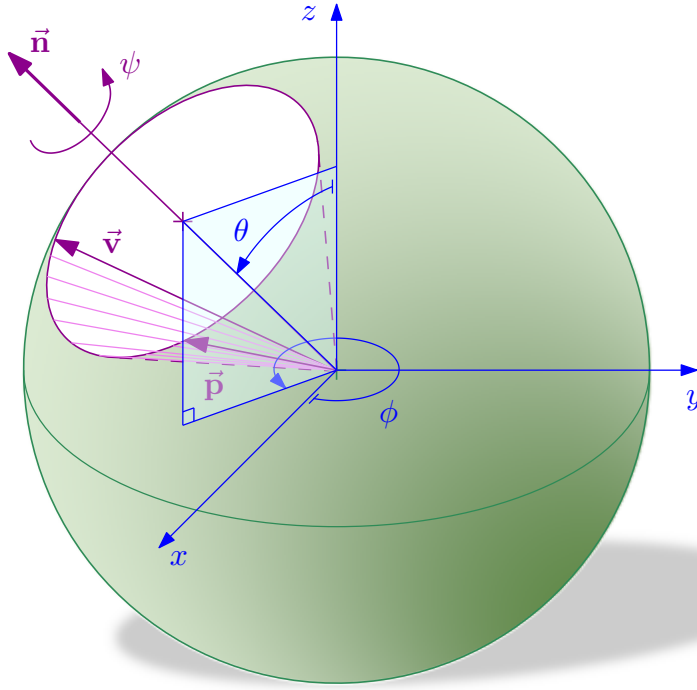


Figure 18: *Three parameter representation of finite rotation*

where Θ and ϕ represent the spherical coordinates of $\vec{\mathbf{n}}$, and the array of the Cartesian components can be given as $\mathbf{n} = [\sin \Theta \cos \phi, \sin \Theta \sin \phi, \cos \Theta]^T$. Substituting this back to formula Eq. (96) we can obtain the detailed expansion of the rotation matrix as

$$\mathbf{R} = \begin{bmatrix} C_\psi + C_\phi^2(1 - C_\psi)S_\theta^2 & C_\phi(1 - C_\psi)S_\theta^2S_\phi - C_\theta S_\psi & C_\theta C_\phi(1 - C_\psi)S_\theta + S_\theta S_\phi S_\psi \\ C_\phi(1 - C_\psi)S_\theta^2S_\phi + C_\theta S_\psi & C_\psi + (1 - C_\psi)S_\theta^2S_\phi^2 & C_\theta(1 - C_\psi)S_\theta S_\phi - C_\phi S_\theta S_\psi \\ C_\theta C_\phi(1 - C_\psi)S_\theta - S_\theta S_\phi S_\psi & C_\theta(1 - C_\psi)S_\theta S_\phi + C_\phi S_\theta S_\psi & C_\theta^2(1 - C_\psi) + C_\psi \end{bmatrix} \quad (143)$$

where S_* and C_* represent the sin and cos of the angles. We note that the orientation and selection of coordinate axis for the coordinate system for frame \mathcal{F} can also be different. Therefore, the interpretation of the angles in Fig. 18, i.e., they are interpreted relative to what coordinate axes, can change depending on the definition of the coordinate system. The minimum parameterization of the axis of rotation can also be accomplished in other ways. If the Euler parameters are use then the dependence among them and the constraint, Eq. (136), can also be handled in similar ways.

Canonical rotations: The concept of the canonical cases for the change of basis and the direction cosine matrix was discussed in section 3.2.1. We also saw that if we use orthonormal base vectors to define the coordinate systems for our frames of reference then there is a close relationship between the form of the direction cosine matrix and the rotation matrix. We can also interpret canonical cases for the rotation of the base vectors, and coordinate systems, of two frames into each other, frame \mathcal{F} with base vectors and related coordinate system $\vec{\mathbf{e}}_1, \vec{\mathbf{e}}_2, \vec{\mathbf{e}}_3$ and frame \mathcal{F}' with base vectors, and coordinate system, $\vec{\mathbf{e}}'_1, \vec{\mathbf{e}}'_2, \vec{\mathbf{e}}'_3$.

We can consider the three canonical cases for orthogonal base vectors, when the axis of rotation

is always identical with one of the base vectors, i.e., the coordinate axis defined by the base vector. The three cases are: (1) $\vec{n} = \vec{e}_1 = \vec{e}'_1$, (2) $\vec{n} = \vec{e}_2 = \vec{e}'_2$, (3) $\vec{n} = \vec{e}_3 = \vec{e}'_3$. For these cases the expressions given below interpret the canonical rotation matrices in the starting frame \mathcal{F} .

Case 1: This is illustrated in Fig. 19. Based on the previous formulas and results we can write

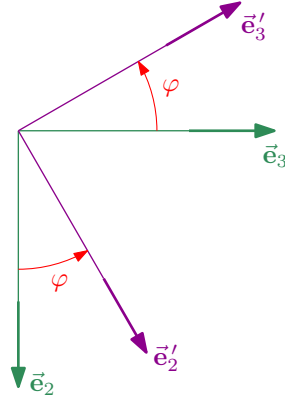


Figure 19: *Rotation: canonical case 1*

for this case that

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{bmatrix} \quad (144)$$

where the index in \mathbf{R}_1 indicates that this is the first canonical case, i.e., the axis of rotation is the first coordinate axis.

Case 2: This case is illustrated in Fig. 20. For this case the second canonical rotation matrix can

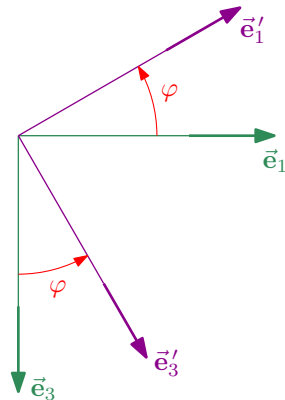


Figure 20: *Rotation: canonical case 2*

be expressed as

$$\mathbf{R}_2 = \begin{bmatrix} \cos \varphi & 0 & \sin \varphi \\ 0 & 1 & 0 \\ -\sin \varphi & 0 & \cos \varphi \end{bmatrix} \quad (145)$$

Case 3: This is shown in Fig. 21. The third canonical rotation matrix can then be obtained as

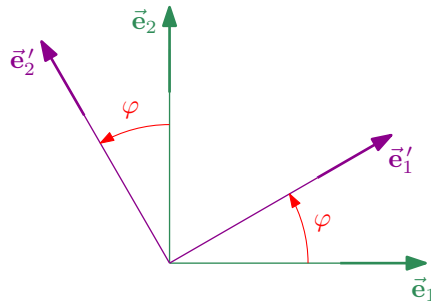


Figure 21: *Rotation: canonical case 3*

$$\mathbf{R}_3 = \begin{bmatrix} \cos \varphi & -\sin \varphi & 0 \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (146)$$

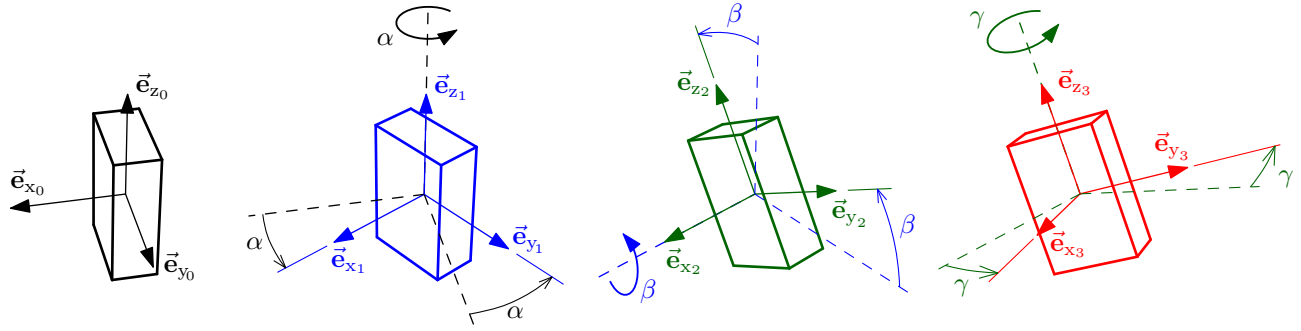
General rotation interpreted as a sequence of rotations: As we saw the result of a general change in orientation, a general rotation, can be described by considering one single rotation about an axis. We also discussed that three independent variables (parameters) describe such a movement. Based on this, on the other hand, an alternative view of the result of a general rotation may also be to consider it as a composition of a series of three individual subsequent rotations in a way that each of these component rotations has only one independent variable. We will call such a rotation a *one-parameter rotation*. This means that for each rotation two of the related variables need to be fixed, i.e., pre-selected. This can be accomplished, for example, if we consider that the axis of rotation for each individual component rotation is pre-selected, e.g., to be a canonical axis, and the only variable that remains is the rotation angle. If we consider the three rotation angles as α , β and γ , and that the individual rotations also take place in this order then the resultant rotation tensor can be written as

$$\widehat{\mathbf{R}} = \widehat{\mathbf{R}}_\gamma \cdot \widehat{\mathbf{R}}_\beta \cdot \widehat{\mathbf{R}}_\alpha \quad (147)$$

where $\widehat{\mathbf{R}}_\gamma$, $\widehat{\mathbf{R}}_\beta$ and $\widehat{\mathbf{R}}_\alpha$ represent the related one-parameter rotation tensors, respectively. We note that the pre-selected rotation axes do not have to be canonical axes. However, the representation is simplified if the rotation axes of the component rotations are defined to be canonical axes. This, however, also requires the pre-selection of coordinate systems for the analysis. Angles α , β and γ are called *Euler angles* if the component rotations are defined about canonical axes.

There are two possible ways to define the axes of rotations: (1) along lines of the intermediate frames that enter through the individual one-parameter rotations, also called the intrinsic view, or body-fixed sequence and (2) rotation axes fixed in the starting frame of reference, also called the extrinsic view, or space-fixed sequence.

Based on these let us now interpret the overall transformation of the finite rotation as a sequence of three individual canonical rotations, each taking place around a coordinate axis. For example, the Fig. 22 shows a 3-1-3 rotation sequence, meaning that we can envision the transformation as first a rotation about the third coordinate axis, with angle α that rotates the frame to an $x_1y_1z_1$ orientation the rotation matrix for this is canonical case 3, Eq. (146). Then, $x_1y_1z_1$ is

Figure 22: *General rotation: sequence of one-parameter rotations*

rotated about the second coordinate axes, y_1 , with angle β to arrive to a x_2, y_2, z_2 orientation. The rotation matrix for this case is canonical case 2, Eq. (145). Then, the final canonical rotation takes place again about the third coordinate axis, z_2 , with angle γ to obtain the final orientation x_3, y_3, z_3 . The form of the canonical rotation matrix for this case again is the one in Eq. (146).

This represents a sequence of three independent rotations that can also completely define the transformation between two reference frames by introducing additional intermediate coordinate systems that correspond to the individual steps. Depending on the coordinate system selections there can be multiple, more precisely 12, combinations for such a sequence. Considering the canonical forms above, if we want to preserve the simple structure of the canonical rotation matrices then these need to be always represented in the starting coordinate system of the individual step. This means that the final form of the resultant rotation matrix can be composed as derived in section 6.3. For this sample case, this results in

$$\mathbf{R} = \mathbf{R}_3(\alpha)\mathbf{R}_1(\beta)\mathbf{R}_3(\gamma) \quad (148)$$

The angles associated with this 3-1-3 rotation sequence are also often called the “classical” Euler angles. Expanding the matrix products in Eq. (148) the rotation matrix can be written for this case as

$$\mathbf{R}_{3-1-3} = \mathbf{R}_3(\alpha)\mathbf{R}_1(\beta)\mathbf{R}_3(\gamma) = \begin{bmatrix} \cos \alpha \cos \gamma - \sin \alpha \cos \beta \sin \gamma & -\cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma & \sin \alpha \sin \beta \\ \sin \alpha \cos \gamma + \cos \alpha \cos \beta \sin \gamma & -\sin \alpha \sin \gamma + \cos \alpha \cos \beta \cos \gamma & -\cos \alpha \sin \beta \\ \sin \beta \sin \gamma & \sin \beta \cos \gamma & \cos \beta \end{bmatrix} \quad (149)$$

and the related direction cosine matrix is

$$\mathbf{R}_{3-1-3}^T = \begin{bmatrix} \cos \alpha \cos \gamma - \sin \alpha \cos \beta \sin \gamma & \sin \alpha \cos \gamma + \cos \alpha \cos \beta \sin \gamma & \sin \beta \sin \gamma \\ -\cos \alpha \sin \gamma - \sin \alpha \cos \beta \cos \gamma & -\sin \alpha \sin \gamma + \cos \alpha \cos \beta \cos \gamma & \sin \beta \cos \gamma \\ \sin \alpha \sin \beta & -\cos \alpha \sin \beta & \cos \beta \end{bmatrix} \quad (150)$$

6.9 Angular velocity representations

6.9.1 Representation with the time rates of change of Euler parameters

If we again consider that $\vec{\omega}$ is the angular velocity of frame \mathcal{F}' relative to \mathcal{F} then it can be shown that

$$\boldsymbol{\omega} = 2\mathbf{G}\dot{\mathbf{g}} \quad (151)$$

and

$$\boldsymbol{\omega}' = 2\mathbf{L}\dot{\mathbf{g}} \quad (152)$$

where $\boldsymbol{\omega}$ is the 3×1 algebraic vector of the scalar components of the angular velocity expressed in the coordinate system of \mathcal{F} , and $\boldsymbol{\omega}'$ is the 3×1 algebraic vector of the scalar components of the angular velocity expressed in the coordinate system of \mathcal{F}' , and \mathbf{G} , \mathbf{L} and \mathbf{g} were interpreted in section 6.8.1.

6.9.2 Relation between the time derivatives of Euler angles and the angular velocity components

As an example we can consider the classical 3-1-3 rotation sequence for describing the finite rotations between two coordinate systems, angles: α , β , ψ . Then the transformations for the scalar components can be described as

$$\begin{aligned} \mathcal{F} \ x_0 y_0 z_0 &\rightarrow \text{rotation about } z_0, \mathbf{R}_3(\alpha) \rightarrow x_1 y_1 z_1 (z_0 = z_1) \ \mathcal{F}_1 \\ &\rightarrow \text{rotation about } x_1, \mathbf{R}_1(\beta) \rightarrow x_2 y_2 z_2 (x_1 = x_2) \ \mathcal{F}_2 \\ &\rightarrow \text{rotation about } z_2, \mathbf{R}_3(\gamma) \Rightarrow x_3 y_3 z_3 \ \mathcal{F}' \end{aligned}$$

where $x_1 y_1 z_1$ and $x_2 y_2 z_2$ represent the two intermediate coordinate systems. If we consider the transformation relationships between the scalar components of a vector $\vec{\mathbf{r}}$ then we can write

$$\mathbf{r}' = \mathbf{R}_{3-1-3}^T \mathbf{r}$$

The angular velocity has a component due to each rotation as

$$\vec{\boldsymbol{\omega}} = \dot{\alpha} \vec{\mathbf{e}}_{z_0} + \dot{\beta} \vec{\mathbf{e}}_{x_1} + \dot{\gamma} \vec{\mathbf{e}}_{z_2}$$

$$\vec{\mathbf{e}}_{z_0} = \sin \beta \sin \gamma \vec{\mathbf{e}}_{x_3} + \sin \beta \cos \gamma \vec{\mathbf{e}}_{y_3} + \cos \beta \vec{\mathbf{e}}_{z_3}, \quad \vec{\mathbf{e}}_{x_1} = \cos \gamma \vec{\mathbf{e}}_{x_3} - \sin \gamma \vec{\mathbf{e}}_{y_3}, \quad \vec{\mathbf{e}}_{z_2} = \vec{\mathbf{e}}_{z_3}$$

and from the interpretation of the columns of the matrix, Eq. (150), we can conclude that

$$\vec{\boldsymbol{\omega}} = (\dot{\alpha} \sin \beta \sin \gamma + \dot{\beta} \cos \gamma) \vec{\mathbf{e}}_{x_3} + (\dot{\alpha} \sin \beta \cos \gamma - \dot{\beta} \sin \gamma) \vec{\mathbf{e}}_{y_3} + (\dot{\alpha} \cos \beta + \dot{\gamma}) \vec{\mathbf{e}}_{z_3}$$

and in algebraic form

$$\begin{bmatrix} \omega'_1 \\ \omega'_2 \\ \omega'_3 \end{bmatrix} = \begin{bmatrix} \dot{\alpha} \sin \beta \sin \gamma + \dot{\beta} \cos \gamma \\ \dot{\alpha} \sin \beta \cos \gamma - \dot{\beta} \sin \gamma \\ \dot{\alpha} \cos \beta + \dot{\gamma} \end{bmatrix} = \begin{bmatrix} \sin \beta \sin \gamma & \cos \gamma & 0 \\ \sin \beta \cos \gamma & -\sin \gamma & 0 \\ \cos \beta & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \boldsymbol{\omega}' = \mathbf{B}\dot{\boldsymbol{\kappa}}$$

where $\boldsymbol{\omega}'$ contains the scalar components of the angular velocity in the coordinate system of \mathcal{F}' ($x_3 y_3 z_3$).