

# Asymptotic measures and links in simplicial complexes

Nermin Salepci, Jean-Yves Welschinger

## ▶ To cite this version:

Nermin Salepci, Jean-Yves Welschinger. Asymptotic measures and links in simplicial complexes. Discrete and Computational Geometry, 2019, 62 (1), pp.164-179. 10.1007/s00454-019-00091-0. hal-01533792

# HAL Id: hal-01533792 https://hal.science/hal-01533792

Submitted on 6 Jun 2017

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers. L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

## Asymptotic measures and links in simplicial complexes

Nermin Salepci and Jean-Yves Welschinger

June 6, 2017

#### Abstract

We introduce canonical measures on a locally finite simplicial complex K and study their asymptotic behavior under infinitely many barycentric subdivisions. We also compute the face polynomial of the asymptotic link and dual block of a simplex in the  $d^{th}$ barycentric subdivision  $\mathrm{Sd}^d(K)$  of K,  $d \gg 0$ . It is almost everywhere constant. When K is finite, we study the limit face polynomial of  $\mathrm{Sd}^d(K)$  after F. Brenti-V. Welker and E. Delucchi-A. Pixton-L. Sabalka.

Keywords : simplicial complex, barycentric subdivisions, face vector, face polynomial, link of a simplex, dual block, measure.

MATHEMATICS SUBJECT CLASSIFICATION 2010: 52C99, 28C15, 28A33

## 1 Introduction

Let K be a finite n-dimensional simplicial complex and  $\operatorname{Sd}^d(K), d \ge 0$ , be its successive barycentric subdivisions, see [7]. We denote by  $f_p(K), p \in \{0, \ldots, n\}$ , the face number of K, that is the number of p-dimensional simplices of K and by  $q_K(T) = \sum_{p=0}^n f_p(K)T^p$  its face polynomial. The asymptotic of  $f_p^d(K) = f_p(\operatorname{Sd}^d(K))$  has been studied in [2] and [3], it is equivalent to  $q_{p,n}f_n(K)(n+1)!^d$  as d grows to  $+\infty$ , where  $q_{p,n} > 0$ . Moreover, it has been proved in [2] that the roots of the limit face polynomial  $q_n^{\infty}(T) = \sum_{p=0}^n q_{p,n}T^p$  are all simple and real in [-1, 0], and in [3] that this polynomial is symmetric with respect to the involution  $T \to -T - 1$ , see Theorem 11. We first observe that this symmetry actually follows from a general symmetry phenomenon obtained by I. G. Macdonald in [6] which can be formulated as follows, see Theorem 8. We set  $R_K(T) = Tq_K(T) - \chi(K)T$ .

**Theorem 1 (Theorem 2.1, [6])** Let K be a triangulated compact homology n-manifold. Then,  $R_K(-1-T) = (-1)^{n+1}R_K(T)$ .

Recall that a homology *n*-manifold is a topological space X such that for every  $x \in X$ , the relative homology  $H_*(X, X \setminus \{x\}; \mathbb{Z})$  is isomorphic to  $H_*(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}; \mathbb{Z})$ . Any smooth or topological manifold is thus a homology manifold and Poincaré duality holds true in such compact homology manifolds, see [7].

We also observe the following theorem (see Corollary 9 and Theorem 12), the first part of which is a corollary of Theorem 1 which has been independently (not as a corollary of Theorem 1) observed by T. Akita [1].

**Theorem 2** Let K be a compact triangulated homology manifold of even dimension. Then  $\chi(K) = q_K(-\frac{1}{2}).$ 

Moreover, t = -1 in odd dimensions and t = -1 together with  $t = -\frac{1}{2}$  in even dimensions are the only complex values t on which  $q_K(T)$  equals  $\chi(K)$  for every compact triangulated homology manifold of the given dimension.

Having spheres in mind for instance, Theorem 1 and Theorem 2 exhibit a striking behavior of simplicial structures compared to cellular structures. In [8], we also provide a probabilistic proof of the first part of Theorem 2.

The limit face polynomial  $q_n^{\infty}(T)$  remains puzzling, but we have been able to prove the following result, see Proposition 14 and Corollaries 17 and 27.

Let 
$$L_j(T) = \frac{1}{j!} \prod_{i=0}^{j-1} (T-i) \in \mathbb{R}[T], j \ge 1$$
, be the  $j^{th}$  Lagrange polynomial and set  $L_0 = 1$ .

**Theorem 3** Let  $\Lambda^t = (\lambda_{j,i})$  be the upper triangular matrix of the vector  $(T^j)_{j\geq 0}$  in the base  $(L_i)_{i\geq 0}$ . Then,  $(q_{p,n})_{0\leq p\leq n}$  is the eigenvectors of  $\Lambda^t$  associated to the eigenvalue (n+1)! normalized in such a way that  $q_{n,n} = 1$  and  $q_{p,n} = 0$  if p > n. Moreover for every  $0 \leq p < n$ ,

$$q_{p,n} = \sum_{(p_1,...,p_j)\in\mathcal{P}_{p,n}} \frac{\lambda_{n+1,p_j}\lambda_{p_j,p_{j-1}}\dots\lambda_{p_2,p_1}}{(\lambda_{n+1,n+1}-\lambda_{p_j,p_j})\dots(\lambda_{n+1,n+1}-\lambda_{p_1,p_1})},$$

where  $\mathcal{P}_{p,n} = \{(p_1, \dots, p_j) \in \mathbb{N}^j | j \ge 1 \text{ and } p + 1 = p_1 < \dots < p_j < n + 1\}.$ 

Our main purpose in this paper is to refine this asymptotic study of the face polynomial by introducing a canonical measure on  $\mathrm{Sd}^d(K)$  and study the density of links in  $\mathrm{Sd}^d(K)$  with respect to these measures. For every  $0 \leq p \leq n$ , set  $\gamma_{p,K} = \sum_{\sigma \in K^{[p]}} \delta_{\hat{\sigma}}$ , where  $\delta_{\hat{\sigma}}$  denotes the Dirac measure on the barycenter  $\hat{\sigma}$  of  $\sigma$  and  $K^{[p]}$  the set of *p*-dimensional simplices of *K*. Likewise, for every  $d \geq 0$ , we set  $\gamma_{p,K}^d = \frac{1}{(n+1)!^d} \gamma_{p,\mathrm{Sd}^d(K)}$ , which provides a canonical sequence of Radon measures on the underlying topological space |K|. The latter is also equipped with the measure  $d\mathrm{vol}_K = \sum_{\sigma \in K^{[n]}} (f_{\sigma})_* d\mathrm{vol}_{\Delta_n}$ , where  $f_{\sigma} : \Delta_n \to \sigma$  denotes a simplicial isomorphism between the standard *n*-simplex  $\Delta_n$  and the simplex  $\sigma$ , and  $d\mathrm{vol}_{\Delta_n}$  denotes the Lebesgue measure normalized in such a way that  $\Delta_n$  has volume 1, see Section 3. We prove the following, see Theorem 19.

**Theorem 4** For every n-dimensional locally finite simplicial complex K and every  $0 \le p \le n$ , the measure  $\gamma_{p,K}^d$  weakly converges to  $q_{p,n}d\mathrm{vol}_K$  as d grows to  $+\infty$ .

When K is finite, Theorem 4 recovers the asymptotic of  $f_p^d(K)$  as d grows to  $+\infty$ , by integration of the constant function 1. Recall that the *link of a simplex*  $\sigma$  in K is by definition  $Lk(\sigma, K) = \{\tau \in K | \sigma \text{ and } \tau \text{ are disjoint and both are faces of a simplex in K}\}$ . Likewise, the *block dual* to  $\sigma$  is the set  $D(\sigma) = \{[\hat{\sigma}_0, \ldots, \hat{\sigma}_p] \in Sd(K) | p \in \{0, \ldots, n\} \text{ and } \sigma_0 = \sigma\}$ , see [7]. Recall that the simplices of Sd(K) are by definition of the form  $[\hat{\sigma}_0, \ldots, \hat{\sigma}_p]$ , where  $\sigma_0 < \ldots < \sigma_p$  are simplices of K with < meaning being a proper face. The dual blocks form a partition of Sd(K), see [7], and the links  $Lk(\sigma, K)$  encodes in a sense the local complexity of K near  $\sigma$ . We finally prove the following, see Theorem 24 and Theorem 25.

**Theorem 5** For every n-dimensional locally finite simplicial complex K and every  $0 \leq p < n$ , the measure  $q_{\text{Lk}(\sigma, \text{Sd}^d(K))}(T)d\gamma_{p,K}^d(\sigma)$  (with value in  $\mathbb{R}_{n-p-1}[T]$ ) weakly converges to  $\left(\sum_{l=0}^{n-p-1} q_{p+l+1,n}f_p(\Delta_{p+l+1})T^l\right) d\text{vol}_K$  as d grows to  $+\infty$ .

And likewise

**Theorem 6** For every n-dimensional locally finite simplicial complex K and every  $0 \le p \le n$ , the measure  $q_{D(\sigma)}(T)d\gamma_{p,K}^d(\sigma)$  weakly converges to  $\sum_{l=0}^{n-p} \left(\sum_{h=l}^{n-p} q_{p+h,n}f_p(\Delta_{p+h})\lambda_{h,l}\right)T^l d\mathrm{vol}_K$ as d grows to  $+\infty$ .

From these theorems we see that asymptotically, the complexity of the link and the dual block is almost everywhere constant with respect to  $dvol_K$ . In [8], we study the asymptotic topology of a random subcomplex in a finite simplicial complex K and its successive barycentric subdivisions. It turns out that the Betti numbers of such a subcomplex get controlled by the measures given in Theorem 6.

**Acknowledgement :** The second author is partially supported by the ANR project MICROLOCAL (ANR-15CE40-0007-01).

### 2 The face polynomial of a simplicial complex

#### 2.1 The symmetry property

Let K be a finite n-dimensional simplicial complex. We set  $R_K(T) = Tq_K(T) - \chi(K)T$ , where  $q_K(T) = \sum_{p=0}^n f_p(K)T^p$  and  $\chi(K)$  is the Euler characteristic of K, so that  $R_K(0) = R_K(-1) = 0$ .

**Example 7** 1. If  $K = \partial \Delta_{n+1}$ , then  $Tq_K(T) = (1+T)^{n+2} - 1 - T^{n+2}$ .

2. If 
$$K = S^0 * \ldots * S^0$$
 is the  $n^{th}$  iterated suspension of the 0-dimensional sphere, then  

$$R_K(T) = Tq_K(T) - T\chi(K) = \begin{cases} (2T+1)((2T+1)^n - 1) & \text{if } n \text{ is even,} \\ (2T+1)^{n+1} - 1 & \text{if } n \text{ is odd.} \end{cases}$$

Recall that if K is a triangulated compact homology *n*-manifold, its face numbers satisfy the following Dehn-Sommerville relations ([5], see also for example [4]):

$$\forall 0 \le p \le n, \ f_p(K) = \sum_{i=p}^n (-1)^{i+n} \binom{i+1}{p+1} f_i(K).$$

The Dehn-Sommerville relations imply that  $R_K(T)$  satisfy the following striking symmetry property observed by I.G. Macdonald [6] which we recall here together with a proof for the reader's convenience.

**Theorem 8 (Theorem 2.1, [6])** Let K be a triangulated compact homology n-manifold. Then,  $R_K(-1-T) = (-1)^{n+1}R_K(T)$ . **Proof.** Observe that

$$R_{K}(-1-T) = \sum_{p=0}^{n} f_{p}(K)(-1-T)^{p+1} + \chi(K)(1+T)$$
  
$$= \sum_{p=0}^{n} f_{p}(K)(-1)^{p+1} \sum_{q=0}^{p+1} {p+1 \choose q} T^{q} + \chi(K)(1+T)$$
  
$$= \sum_{p=0}^{n} f_{p}(K)(-1)^{p+1} \sum_{q=0}^{p} {p+1 \choose q+1} T^{q+1} + \chi(K)T$$
  
$$= \sum_{q=0}^{n} T^{q+1} \sum_{p=q}^{n} {p+1 \choose q+1} f_{p}(K)(-1)^{p+1} + \chi(K)T.$$

Then, the Dehn-Sommerville relations imply

$$R_{K}(-1-T) = -\sum_{q=0}^{n} T^{q+1}(-1)^{n} f_{q}(K) + \chi(K)T$$
  
=  $(-1)^{n+1} R_{K}(T) + (1+(-1)^{n+1})\chi(K)T.$ 

Now, if n is even,  $1 + (-1)^{n+1} = 0$  while if n is odd,  $\chi(K) = 0$  by Poincaré duality with  $\mathbb{Z}/2\mathbb{Z}$  coefficients, see [7]. In both cases, we get  $R_K(-1-T) = (-1)^{n+1}R_K(T)$ .

**Corollary 9** Let K be a triangulated compact homology n-manifold.

- 1. If n is even, then  $q_K(-\frac{1}{2}) = \chi(K)$ .
- 2. If n is odd, the polynomial  $Tq_K(T)$  is preserved by the involution  $T \to -1 T$ .
- 3. If  $\chi(K) \leq 0$ , the real roots of  $R_K(T) = Tq_K(T) \chi(K)T$  lie on the interval [-1, 0].

**Proof.** When n is even,  $R_K$  has an odd number of real roots, invariant under the involution  $T \to -1 - T$  whose unique fixed point is  $-\frac{1}{2}$ . Theorem 8 thus implies that  $R_K(-\frac{1}{2}) = 0$ . Hence the first part. When n is odd,  $\chi(K) = 0$  by Poincaré duality so that  $R_K(T) = Tq_K(T)$ and the second part. Finally, if  $\chi(K) \leq 0$ , the coefficients of the polynomial  $R_K(T)$  are all positive, so that its real roots are all negative. It thus follows from Theorem 8 that they lie on the interval [-1,0].

**Remark 10** The first part of Corollary 9 was independently (not as a corollary of Theorem 8) observed by T. Akita [1]. In [8], we provide a probabilistic proof of it.

The third part of Corollary 9 always holds true when n is odd, since then  $\chi(K) = 0$ .

The first part of Corollary 9 raise the following question: given some dimension n, what are the universal parameters t such that  $q_K(t) = \chi(K)$  for every compact triangulated homology n-manifolds? We checked that t = -1 in odd dimensions and t = -1 with  $t = -\frac{1}{2}$  in even dimensions are the only ones, see Theorem 12.

#### 2.2 The asymptotic face polynomial

Let  $f(K) = (f_0(K), f_1(K), \ldots, f_n(K))$  be the face vector of K, that is the vector formed by the face numbers of the finite simplicial complex K. Now, for every d > 0, we set  $f_p^d(K) = f_p(\mathrm{Sd}^d(K))$ , where  $\mathrm{Sd}^d(K)$  denotes the  $d^{th}$  barycentric subdivision of K. How does the face vector change under barycentric subdivisions and what is the asymptotic behavior of  $f^d(K) = (f_0^d(K), f_1^d(K) \dots, f_n^d(K))$ ? These questions have been treated in [2], [3], leading to the following.

**Theorem 11 ([2], [3])** For every  $0 \le p \le n$ , there exist  $q_{p,n} > 0$  such that for every ndimensional finite simplicial complex K,  $\lim_{d\to+\infty} \frac{f_p^d(K)}{(n+1)!^d f_n(K)} = q_{p,n}$ . Moreover, the n+1 roots of the polynomial  $Tq_n^{\infty}(T)$  are simple, belong to the interval

Moreover, the n + 1 roots of the polynomial  $Tq_n^{\infty}(T)$  are simple, belong to the interval [-1,0] and are symmetric with respect to the involution  $T \in \mathbb{R} \mapsto -T - 1 \in \mathbb{R}$  whenever n > 0, where  $q_n^{\infty}(T) = \sum_{p=0}^n q_{p,n}T^p$ .

The symmetry property of  $Tq_n^{\infty}(T)$  follows from Theorem 8 and the first part of Theorem 11, since the Euler characteristic remains unchanged under subdivisions. This symmetry has been observed in [3] (with a different proof). It implies that  $q_n^{\infty}(-1) = 0$  and that  $q_n^{\infty}(-\frac{1}{2}) = 0$  whenever *n* is even, as the number of roots of  $Tq_n^{\infty}(T)$  is then odd and  $-\frac{1}{2}$  is the unique fixed point of the involution.

**Theorem 12** The reals t = -1 if n is odd and t = -1 together with  $t = -\frac{1}{2}$  if n is even are the only complex values on which the face polynomial  $q_K(T) = \sum_{p=0}^{\dim K} f_p(K)T^p$  equals  $\chi(K)$  for every compact triangulated homology n-manifold K.

**Proof.** Let us equip the *n*-dimensional sphere with the triangulation given by the boundary of the (n + 1)-simplex  $\Delta_{n+1}$ . Then, for every  $0 \leq p \leq n$ ,  $f_p(S^n) = \binom{n+2}{p+1}$  and  $q_{S^n}(T) = \frac{1}{T}((1+T)^{n+2} - 1 - T^{n+2})$ . Now, the polynomial  $q_{S^n}(T) - \chi(S^n)$  has only one real root if *n* is odd and two real roots if *n* is even. Indeed, differentiating the polynomial  $Tq_{S^n}(T) - \chi(S^n)T$  once if *n* is odd and twice if *n* is even, we get, up to a factor,  $(1+T)^{n+1} - T^{n+1}$  or respectively  $(1+T)^n - T^n$  which vanishes only for  $t = -\frac{1}{2}$  on the real line. From Rolle's theorem we deduce that 0 and -1 (respectively  $0, -\frac{1}{2}, -1$ ) are the only real roots of  $Tq_{S^n}(T) - \chi(S^n)T$  when *n* is odd (respectively, when *n* is even).

Finally, if  $t_0 \in \mathbb{C}$  is such that  $q_K(t_0) = \chi(K)$  for all triangulated manifolds of a given dimension *n*, then in particular,  $R_{\mathrm{Sd}^d(K)}(t_0) = 0$  for every d > 0. Dividing by  $f_n(K)(n+1)!^d$ and passing to the limit, we deduce that  $q_n^{\infty}(t_0) = 0$ . But from Theorem 11 we know that the roots of  $Tq^{\infty}(T)$  are all real, hence the result.

Let now  $\Lambda = (\lambda_{i,j})_{i,j\geq 1}$  be the infinite lower triangular matrix whose entries  $\lambda_{i,j}$  are the numbers of interior (j-1)-faces on the subdivided standard simplex  $\mathrm{Sd}(\Delta_{i-1})$  and let  $\Lambda_n = (\lambda_{i,j})_{1\leq i,j\leq n+1}$ , see Figure 1. The diagonal entries of  $\Lambda$  are given by Lemma 13. We set as a convention  $\lambda_{0,0} = 1$  and  $\lambda_{l,0} = 0$  whenever l > 0.

**Lemma 13** For every  $1 \leq j \leq i$ ,  $\lambda_{i,j} = \sum_{p=j-1}^{i-1} {i \choose p} \lambda_{p,j-1}$  where  ${i \choose j}$  denotes the binomial coefficient. In particular,  $\lambda_{i,i} = i!$ .



Figure 1: The matrix  $\Lambda_n$  for n = 1, 2, 3.

**Proof.** The interior (j-1)-faces of  $\mathrm{Sd}(\Delta_{i-1})$  are cones over the (j-2)-faces of the boundary of  $\mathrm{Sd}(\Delta_{i-1})$ . The latter are interior to some (p-1)-simplex of  $\partial \Delta_{i-1}$ ,  $j-1 \leq p \leq i-1$ . The result follows from the fact that for every  $1 \leq p \leq i-1$ ,  $\partial \Delta_{i-1}$  has  $\binom{i}{p}$  many (p-1)-dimensional faces while each such face contains  $\lambda_{p,j-1}$  many (j-2)-dimensional faces of  $\mathrm{Sd}(\Delta_{i-1})$  in its interior.

The first part of Theorem 11 is basically deduced in [2], [3] from the following observation: for every *n*-dimensional finite simplicial complex K, the face vector  $f(\mathrm{Sd}(K))$  is deduced from the face vector f(K) by multiplication on the right by  $\Lambda_n$ , that is  $f(\mathrm{Sd}(K)) = f(K)\Lambda_n$ , while the matrix  $\Lambda_n$  is diagonalizable with eigenvalues given by Lemma 13.

We deduce from [2], [3] that the vector  $(q_{p,n})_{0 \le p \le n}$  is the eigenvector of  $\Lambda_n^t$  associated to the eigenvalue  $\lambda_{n+1,n+1} = (n+1)!$  normalized by the relation  $q_{n,n} = 1$ . A geometric proof of this fact will be given in Section 4, see Corollary 27. This observation makes it possible to compute  $q_{p,n}$  in terms of the coefficients  $\lambda_{i,j}$ .

**Proposition 14** Let  $0 \le p < n$  and let  $\mathcal{P}_{p,n} = \{(p_1, \ldots, p_j) \in \mathbb{N}^j | j \ge 1 \text{ and } p + 1 = p_1 < \ldots < p_j < n + 1\}$ . Then

$$q_{p,n} = \sum_{(p_1,...,p_j)\in\mathcal{P}_{p,n}} \frac{\lambda_{n+1,p_j}\lambda_{p_j,p_{j-1}}\dots\lambda_{p_2,p_1}}{(\lambda_{n+1,n+1}-\lambda_{p_j,p_j})\dots(\lambda_{n+1,n+1}-\lambda_{p_1,p_1})}.$$

**Proof.** Having in mind that  $\Lambda_n$  is a lower triangular matrix and by Lemma 13,  $(n+1)! = \lambda_{n+1,n+1}$ . The equation  $\Lambda_n^t(q_{p,n}) = (n+1)!(q_{p,n})$  results in the following system.

For all  $0 \le p < n$ ,

$$q_{p,n} = \sum_{k=0}^{n-p-1} \frac{\lambda_{n+1-k,p+1}q_{n-k,n}}{(\lambda_{n+1,n+1} - \lambda_{p+1,p+1})}.$$

The solution of this system is obtained by induction on r = n - p by setting  $q_{n,n} = 1$ . The result follows from the fact that the partitions  $(p_1, \ldots, p_j)$  of integers between p+1 and n+1 such that  $p+1 = p_1 < \ldots < p_j < n+1$  are obtained (except the one with single term  $p_1 = p+1$ ) from those  $p+1+s = p'_1 < \ldots < p'_j < n+1$  for all  $1 \le s \le r$  by setting  $p_1 = p+1$  and  $p_{i+1} = p'_i$  for  $i \in \{1, \ldots, j\}$ .

Note that the coefficients  $\lambda_{i,j}$  of  $\Lambda$  can be computed. We recall their values obtained in [3] in the following proposition and suggest an alternative proof.

**Proposition 15 (Lemma 6.1, [3])** For every  $1 \le j \le i$ ,

$$\lambda_{i,j} = \sum_{p=0}^{j} {j \choose p} (-1)^{j-p} p^i.$$

(The left hand side in Lemma 6.1 of [3] should read  $\lambda_{i-1,j-1}$  and our  $\lambda_{i,j}$  corresponds to  $\lambda_{i-1,j-1}$  in [3].)

Let  $C = (c_{i,j})_{i,j\geq 1}$  be the infinite strictly lower triangular matrix such that  $c_{i,j} = {i \choose j}$  for  $i > j \geq 1$ . Also, for every  $r \geq 1$ , set  $(I + C)^r = (a_{i,j}^r)_{i,j\geq 1}$ .

**Lemma 16** For every  $i \ge j$ ,  $a_{i,j}^r = {i \choose j} r^{i-j}$ .

**Proof.** We proceed by induction on r. The statement holds true for r = 1. In the case r = 2, for every  $i \ge j$ ,

$$\begin{aligned} a_{i,j}^2 &= \sum_{j \le p \le i} {i \choose p} {p \choose j} \\ &= \frac{i!}{j!(i-j)!} \sum_{p=j}^i \frac{(i-j)!}{(p-j)!(i-p)!} \\ \overset{l=p-j}{=} {i \choose j} \sum_{l=0}^{i-j} {i-j \choose l} \\ &= {i \choose j} 2^{i-j}. \end{aligned}$$

The last line follows from the Newton binomial theorem. Now, let us suppose that the formula holds true for r - 1. Then, likewise,

**Proof of Proposition 15.** We deduce from Lemma 13 that the  $p^{th}$  column of the matrix  $\Lambda$  is obtained from the  $(p-1)^{th}$  one by multiplication on the left by C, so that it is equal to  $C^{p-1}v$  where v denotes the first column of  $\Lambda$  with 1 on every entry. Let  $C^r = (c_{i,j}^r)_{i\geq 1,j\geq 1}$ , then from the relation  $C^r = (I + C - I)^r = \sum_{p=0}^r {r \choose p} (I + C)^p (-1)^{r-p}$ , we deduce thanks to Lemma 16 that for all r > 0 and all  $i \geq j$ ,

$$c_{i,j}^{r} = \sum_{p=0}^{r} {r \choose p} {i \choose j} p^{i-j} (-1)^{r-p}$$
$$= {i \choose j} \sum_{p=0}^{r} {r \choose p} p^{i-j} (-1)^{r-p}$$

while  $c_{i,j}^r = 0$  whenever  $i \leq j$ . From the previous observation we now deduce that for all  $i \geq r+1$ ,

$$\begin{aligned} \lambda_{i,r+1} &= \sum_{j=1}^{i-1} {i \choose j} \sum_{p=0}^{r} {r \choose p} p^{i-j} (-1)^{r-p} \\ &= \sum_{p=0}^{r} {r \choose p} (-1)^{r-p} p^{i} \sum_{j=1}^{i-1} {i \choose j} p^{-j} \\ &= \sum_{p=0}^{r} {r \choose p} (-1)^{r-p} p^{i} \left( (1+\frac{1}{p})^{i} - 1 - p^{-i} \right) \\ &= \sum_{p=0}^{r} {r \choose p} (-1)^{r-p} (p+1)^{i} - \sum_{p=0}^{r} {r \choose p} (-1)^{r-p} p^{i} \end{aligned}$$

Now, we set l = p + 1 and get

$$\lambda_{i,r+1} = \sum_{l=1}^{r+1} {r \choose l-1} (-1)^{r-l+1} l^i - \sum_{p=0}^r {r \choose p} (-1)^{r-p} p^i$$
  
=  $(r+1)^i - \sum_{p=1}^r \left( {r \choose p-1} + {r \choose p} \right) (-1)^{r-p} p^i$   
=  $(r+1)^i - \sum_{p=1}^r {r+1 \choose p} (-1)^{r-p} p^i$   
=  $\sum_{p=1}^{r+1} {r+1 \choose p} (-1)^{r-p+1} p^i.$ 

Hence the result.

Finally, for every  $j \ge 1$ , let  $L_j(T) = \frac{1}{j!} \prod_{i=0}^{j-1} (T-i) \in \mathbb{R}[T]$  be the  $j^{th}$  Lagrange polynomial, so that  $L_j(p) = 0$  if  $0 \le p < j$  and  $L_j(p) = {p \choose j}$  if  $p \ge j$ . We deduce the following interpretation of the transpose matrix  $\Lambda^t$ .

**Corollary 17** For every  $j \ge 1$ ,  $T^j = \sum_{i=1}^j \lambda_{j,i} L_i(T)$ .

Corollary 17 means that  $\Lambda^t$  is the matrix of the vectors  $(T^j)_{j\geq 0}$  in the basis  $(L_i)_{i\geq 0}$  of  $\mathbb{R}[T]$ , setting  $T^0 = L_0 = 1$ .

**Proof.** Let  $i \ge 1$ . Then, for every  $l \ge i$ ,

$$\sum_{p=0}^{l} {l \choose p} (-1)^{l-p} L_i(p) = \sum_{p=i}^{l} {l \choose p} (-1)^{l-p} {p \choose i}$$
$$= {l \choose i} \sum_{p=i}^{l} {l-i \choose l-p} (-1)^{l-p}$$
$$= (-1)^{l-i} {l \choose i} \sum_{q=0}^{l-i} {l-i \choose q} (-1)^q$$
$$= \delta_{li},$$

where  $\delta_{li} = 0$  if  $l \neq i$  and  $\delta_{li} = 1$  otherwise. This result also holds true for  $l \in \{0, \ldots, i-1\}$ . We deduce that for  $0 \leq l \leq j$ ,

$$\sum_{p=0}^{l} \binom{l}{p} (-1)^{l-p} \left( \sum_{i=0}^{j} \lambda_{j,i} L_i(p) \right) = \lambda_{j,l}.$$

The result now follows from Proposition 15 and the fact that a degree j polynomial is uniquely determined by its values on the j + 1 integers  $\{0, \ldots, j\}$ , since the above linear combinations for  $l \in \{0, \ldots, j\}$  define an invertible triangular matrix.

### 3 Canonical measures on a simplicial complex

Let us equip the standard *n*-dimensional simplex  $\Delta_n$  with the Lebesgue measure  $d\operatorname{vol}_{\Delta_n}$ inherited by some affine embedding of  $\Delta_n$  in an Euclidian *n*-dimensional space  $\mathbb{E}$  in such a way that the total measure of  $\Delta_n$  is 1. This measure does not depend on the embedding  $\Delta_n \hookrightarrow \mathbb{E}$  for two such embeddings differ by an affine isomorphism which has constant Jacobian 1.

**Definition 18** For every n-dimensional locally finite simplicial complex K, we denote by  $dvol_K$  the measure  $\sum_{\sigma \in K^{[n]}} (f_{\sigma})_* (dvol_{\Delta_n})$  of |K| where  $K^{[n]}$  denotes the set of n-dimensional simplices of K and  $f_{\sigma} : \Delta_n \to \sigma$  a simplicial isomorphism.

If K is a finite n-dimensional simplicial complex, the total measure of |K| is thus  $f_n(K)$ and its (n-1)-skeleton has vanishing measure. This canonical measure  $d \operatorname{vol}_K$  is Radon with respect to the topology of |K|.

Now, for every  $p \in \{0, \ldots, n\}$ , we set  $\gamma_{p,K} = \sum_{\sigma \in K^{[p]}} \delta_{\hat{\sigma}}$ , where  $\delta_{\hat{\sigma}}$  denotes the Dirac measure on the barycenter  $\hat{\sigma}$  of  $\sigma$ . If K is finite, the total measure  $\int_{\sigma \in K^{[p]}} 1d\gamma_{p,K}(\sigma)$  equals  $f_p(K)$ . More generally, for every  $d \geq 0$ , we set  $\gamma_{p,K}^d = \frac{1}{(n+1)!^d} \sum_{\sigma \in \mathrm{Sd}^d(K)^{[p]}} \delta_{\hat{\sigma}}$ .

**Theorem 19** For every n-dimensional locally finite simplicial complex K and every  $p \in \{0, ..., n\}$ , the measure  $\gamma_{p,K}^d$  weakly converges to  $q_{p,n} dvol_K$  as d grows to  $+\infty$ .

By weak convergence, we mean that for every continuous function  $\varphi$  with compact support in |K|,  $\int_K \varphi d\gamma_{p,K}^d \xrightarrow[d \to +\infty]{} q_{p,n} \int_K \varphi d \operatorname{vol}_K$ . In order to prove Theorem 19, we need first the following lemma.

**Lemma 20** Let  $p \in \{0, \ldots, n\}$ . Then for every  $l, d \ge 0$ ,

$$\gamma_{p,\Delta_n}^{l+d} = \frac{1}{(n+1)!^l} \sum_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}} (f_\sigma)_* (\gamma_{p,\Delta_n}^d) - \theta_p^l(d),$$

where  $f_{\sigma}: \Delta_n \to \sigma$  denotes a simplicial isomorphism and the total measure of  $\theta_p^l(d)$  converges to zero as d grows to  $+\infty$ .

**Proof.** In a subdivided *n*-simplex  $\mathrm{Sd}^{l}(\Delta_{n})$ , every *p*-simplex  $\tau$  is a face of an *n*-simplex and the number of such *n*-simplices is by definition  $f_{n-p-1}(\mathrm{Lk}(\tau, \mathrm{Sd}^{l}(\Delta_{n})))$ . Since  $\mathrm{Sd}^{l+d}(\Delta_{n}) = \mathrm{Sd}^{d}(\mathrm{Sd}^{l}(\Delta_{n}))$ , we deduce that for every  $d \geq 0$ ,

$$\gamma_{p,\Delta_n}^{l+d} = \frac{1}{(n+1)!^l} \sum_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}} (f_\sigma)_* (\gamma_{p,\Delta_n}^d) - \frac{1}{(n+1)!^{l+d}} \sum_{\tau \in \mathrm{Sd}^l(\Delta_n)^{(n-1)}} (f_{n-\dim \tau-1}(\mathrm{Lk}(\tau, \mathrm{Sd}^l(\Delta_n))) - 1) \sum_{\alpha \in \mathrm{Sd}^d(\overset{\circ}{\tau})^{[p]}} \delta_{\hat{\alpha}},$$

where  $\operatorname{Sd}^{l}(\Delta_{n})^{(n-1)}$  denotes the (n-1)-skeleton of  $\operatorname{Sd}^{l}(\Delta_{n})$ .

We thus set  $\theta_p^l(d) = \frac{1}{(n+1)!^{l+d}} \sum_{\tau \in \mathrm{Sd}^l(\Delta_n)^{(n-1)}} \left( f_{n-\dim \tau-1}(\mathrm{Lk}(\tau, \mathrm{Sd}^l(\Delta_n))) - 1 \right) \sum_{\alpha \in \mathrm{Sd}^d(\overset{\circ}{\tau})^{[p]}} \delta_{\hat{\alpha}}.$ The total mass of this measure  $\theta_p^l(d)$  satisfies

$$\int_{\Delta_n} 1d\theta_p^l(d) \le \left(\frac{1}{(n+1)!^l} \sup_{\tau} \left(f_{n-\dim \tau-1}(\operatorname{Lk}(\tau, \operatorname{Sd}^l(\Delta_n))) - 1\right) \times \#\operatorname{Sd}^l(\Delta_n)^{(n-1)}\right) \frac{\sup_{\tau} f_p^d(\tilde{\tau})}{(n+1)!^d}.$$
  
Since dim  $\tau < n$ , we know from Theorem 11 that  $\frac{\sup_{\tau} f_p^d(\tilde{\tau})}{(n+1)!^d} \xrightarrow[d \to +\infty]{} 0$ . Hence the result.  $\Box$ 

**Proof of Theorem 19.** Let us first assume that  $K = \Delta_n$  and let  $\varphi \in C^0(\Delta_n)$ . We set, for every  $l, d \geq 0$ ,  $R_{l,d} = \int_{\Delta_n} \varphi d\gamma_{p,\Delta_n}^{l+d} - q_{p,n} \int_{\Delta_n} \varphi d\operatorname{vol}_{\Delta_n}$  and deduce from Lemma 20

$$R_{l,d} = \frac{1}{(n+1)!^l} \sum_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}} \left( \int_{\Delta_n} f_{\sigma}^* \varphi d\gamma_{p,\Delta_n}^d - q_{p,n} \int_{\Delta_n} f_{\sigma}^* \varphi d\mathrm{vol}_{\Delta_n} \right) - \int_{\Delta_n} \varphi d\theta_p^l(d),$$

since by definition  $(f_{\sigma})_* d \operatorname{vol}_{\Delta_n} = (n+1)!^l d \operatorname{vol}_{\Delta_n}|_{\sigma}$ . Thus,

$$R_{l,d} = \frac{1}{(n+1)!^l} \sum_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}} \left( \int_{\Delta_n} \left( f_{\sigma}^* \varphi - \varphi(\hat{\sigma}) \right) d\gamma_{p,\Delta_n}^d - q_{p,n} \int_{\Delta_n} \left( f_{\sigma}^* \varphi - \varphi(\hat{\sigma}) \right) d\mathrm{vol}_{\Delta_n} \right) \\ + \left( \frac{f_p(\mathrm{Sd}^d(\Delta_n))}{(n+1)!^d} - q_{p,n} \right) \frac{1}{(n+1)!^l} \sum_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}} \varphi(\hat{\sigma}) - \int_{\Delta_n} \varphi d\theta_p^l(d).$$

Now, since  $\varphi$  is continuous,  $\sup_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}}(\sup_{\sigma} |\varphi - \varphi(\hat{\sigma})|)$  converges to 0 as l grows to  $+\infty$ , while  $\frac{1}{(n+1)!^l} |\sum_{\sigma \in \mathrm{Sd}^l(\Delta_n)^{[n]}} \varphi(\hat{\sigma})|$  remains bounded by  $\sup_{\Delta_n} |\varphi|$ . Likewise by Theorem 11,  $\frac{f_p^d(\Delta_n)}{(n+1)!^d}$  converges to  $q_{p,n}$  as d grows to  $+\infty$ , while by Lemma 20,  $\int_{\Delta_n} 1d\theta_p^l(d)$  converges to 0. By letting d grow to  $+\infty$  and then l grow to  $+\infty$ , we deduce that  $R_{l,d}$  can be as small as we want for l, d large enough. This proves the result for  $K = \Delta_n$ .

Now, if K is a locally finite n-dimensional simplicial complex, we deduce the result by summing over all n-dimensional simplices of K, since from Theorem 11, the measure of the (n-1)-skeleton of K with respect to  $\gamma_{p,K}^d$  converges to 0 as d grows to  $+\infty$ .

Note that by integration of the constant function 1, Theorem 19 implies that for a finite simplicial complex K,  $\frac{f_p^d(K)}{(n+1)!^d} \xrightarrow[d \to +\infty]{} q_{p,n}$ , recovering the first part of Theorem 11. Also, since  $q_{n,n} = 1$ , it implies that  $\gamma_{n,K}^d \xrightarrow[d \to +\infty]{} d\text{vol}_K$ . This actually quickly follows from Riemann integration, since for every  $\varphi \in C^0(\Delta_n)$ ,

$$\begin{split} \int_{\Delta_n} \varphi d \mathrm{vol}_{\Delta_n} &= \lim_{d \to +\infty} \frac{1}{(n+1)!^d} \sum_{\sigma \in \mathrm{Sd}^d(\Delta_n)^{[n]}} \varphi(\hat{\sigma}) \\ &= \lim_{d \to +\infty} \int_{\Delta_n} \varphi d \gamma^d_{n,\Delta_n}. \end{split}$$

Let us give another point of view of this fact. For every  $\sigma \in \mathrm{Sd}(\Delta_n)^{[n]}$ , let us choose once for all a simplicial isomorphism  $f_{\sigma} : \Delta_n \to \sigma$ . Let us then consider the product space  $\Omega = Map(\mathbb{N}^*, \mathrm{Sd}(\Delta_n)^{[n]}) = (\mathrm{Sd}(\Delta_n)^{[n]})^{\mathbb{N}^*}$  of countably many copies of  $\mathrm{Sd}(\Delta_n)^{[n]}$  and equip it with the product measure  $\omega$ , where each copy of  $\mathrm{Sd}(\Delta_n)^{[n]}$  is equipped with the counting measure  $\frac{1}{(n+1)!} \sum_{\sigma \in \mathrm{Sd}(\Delta_n)^{[n]}} \delta_{\sigma}$ . It is a Radon measure with respect to the product topology on  $\Omega$ . We then set

$$\Phi: \quad \Omega \times \Delta_n \quad \to \quad \Delta_n \\
((\sigma_i)_{i \in \mathbb{N}^*}, x) \quad \mapsto \quad \lim_{d \to +\infty} f_{\sigma_1} \circ \ldots \circ f_{\sigma_d}(x).$$

**Theorem 21** The map  $\Phi$  is well defined, continuous, surjective and contracts the second factor  $\Delta_n$ . Moreover,  $dvol_{\Delta_n} = \Phi_*(\omega \times dvol_{\Delta_n}) = \Phi_*(\omega \times \delta_{\hat{\Delta}_n}) = \lim_{d \to +\infty} \gamma^d_{n,\Delta_n}$ .

(This result may be compared to the general Borel isomorphism theorem.)

For every  $d \ge 1$ , let us set

$$\Phi_d: \quad \Omega \times \Delta_n \quad \to \quad \Delta_n \\
((\sigma_i)_{i \in \mathbb{N}^*}, x) \quad \mapsto \quad f_{\sigma_1} \circ \ldots \circ f_{\sigma_d}(x).$$

**Proof.** For every  $(\sigma_i)_{i\in\mathbb{N}^*} \in \Omega$ , the sequence of compact subsets  $\operatorname{Im}(f_{\sigma_1} \circ \ldots \circ f_{\sigma_d})$  decreases as d grows to  $+\infty$ . These subsets are n-simplices of the barycentric subdivision  $\operatorname{Sd}^d(\Delta_n)$  so that their diameters converge to zero. We deduce the first part of Theorem 21. Since  $\Phi$  contracts the second factor and is measurable, the push forward  $\Phi_*(\omega \times \mu)$  does not depend on the probability measure  $\mu$  on  $\Delta_n$ . In particular,  $\Phi_*(\omega \times d\operatorname{vol}_{\Delta_n}) = \Phi_*(\omega \times \delta_{\hat{\Delta}_n})$ . Now, we have by definition  $(\Phi_d)_*(\omega \times d\operatorname{vol}_{\Delta_n}) = \frac{1}{(n+1)!^d} \sum_{\tau \in \operatorname{Sd}^d(\Delta_n)} (f_{\tau})_*(d\operatorname{vol}_{\Delta_n})$ , where  $f_{\tau}$  is the corresponding simplicial isomorphism  $f_{\sigma_1} \circ \ldots \circ f_{\sigma_d}$  between  $\Delta_n$  and  $\tau$ , so that  $(\Phi_d)_*(\omega \times d\operatorname{vol}_{\Delta_n}) = d\operatorname{vol}_{\Delta_n}$  for every d since  $(f_{\tau})_*(d\operatorname{vol}_{\Delta_n}) = (n+1)!^d \operatorname{dvol}_{\Delta_n}|_{\tau}$ . Likewise,  $(\Phi_d)_*(\omega \times \delta_{\hat{\Delta}_n}) = \frac{1}{(n+1)!^d} \sum_{\tau \in \operatorname{Sd}^d(\Delta_n)} (f_{\tau})_*(\delta_{\hat{\Delta}_n}) = \gamma_{n,\Delta_n}^d$  by definition. Since the sequence  $(\Phi_d)_{d\in\mathbb{N}^*}$  of continuous maps converge pointwise to  $\Phi$ , we deduce from Lebesgue's dominated convergence theorem that for every probability measure  $\mu$  on  $\Delta_n$ , the sequence  $(\Phi_d)_*(\omega \times \mu)$  weakly converges to  $\Phi_*(\omega \times \mu)$ .

Recall that by definition, the Dirac measure  $\delta_{\hat{\Delta}_n}$  in Theorem 21 coincides with the measure  $\gamma_{n,\Delta_n}$ . For p < n, we get

**Theorem 22** For every  $p \in \{0, ..., n\}$ ,

$$f_p(\Delta_n)d\mathrm{vol}_{\Delta_n} = \Phi_*(\omega \times \gamma_{p,\Delta_n}) = \lim_{d \to +\infty} f_{n-p-1}(\mathrm{Lk}(\sigma, \mathrm{Sd}^d(\Delta_n))) d\gamma_{p,\Delta_n}^d(\sigma).$$

Recall that  $f_p(\Delta_n) = {n+1 \choose p+1}$  and that by definition  $f_{-1}(\operatorname{Lk}(\sigma, \operatorname{Sd}^d(\Delta_n))) = 1$ . **Proof.** From Theorem 21,  $\Phi$  contracts the second factor. Since the mass of  $\gamma_{p,\Delta_n}$  equals

**Proof.** From Theorem 21,  $\Phi$  contracts the second factor. Since the mass of  $\gamma_{p,\Delta_n}$  equals  $f_p(\Delta_n)$  by definition, we deduce the first equality. Now, as in the proof of Theorem 21, we deduce from Lebesgue's dominated convergence theorem that the sequence  $(\Phi_d)_*(\omega \times \gamma_{p,\Delta_n})$  weakly converges to  $\Phi_*(\omega \times \gamma_{p,\Delta_n})$ . It remains thus to compute  $(\Phi_d)_*(\omega \times \gamma_{p,\Delta_n})$ . By definition  $(\Phi_d)_*(\omega \times \gamma_{p,\Delta_n}) = \frac{1}{(n+1)!^d} \sum_{\tau \in \mathrm{Sd}^d(\Delta_n)^{[n]}} (f_{\tau})_*(\gamma_{p,\Delta_n})$ , where  $f_{\tau}$  is the corresponding simplicial isomorphism  $f_{\sigma_1} \circ \ldots \circ f_{\sigma_d}$  between  $\Delta_n$  and  $\tau$ . In this sum, we see that each *p*-simplex of  $\mathrm{Sd}^d(\Delta_n)$  receives as many Dirac measures as the number of *n*-simplices adjacent to it. The number of *n*-simplices adjacent to  $\sigma \in \mathrm{Sd}^d(\Delta_n)^{[p]}$  is by definition  $f_{n-p-1}(\mathrm{Lk}(\sigma, \mathrm{Sd}^d(\Delta_n)))$ . We deduce

$$\begin{aligned} (\Phi_d)_*(\omega \times \gamma_{p,\Delta_n}) &= \frac{1}{(n+1)!^d} \sum_{\sigma \in \mathrm{Sd}^d(\Delta_n)^{[p]}} f_{n-p-1} \big( \mathrm{Lk}(\sigma, \mathrm{Sd}^d(\Delta_n)) \big) \delta_{\hat{\sigma}} \\ &= f_{n-p-1} \big( \mathrm{Lk}(\sigma, \mathrm{Sd}^d(\Delta_n)) \big) d\gamma_{p,\Delta_n}^d(\sigma). \end{aligned}$$

**Corollary 23** For every n-dimensional locally finite simplicial complex K and every  $p \in \{0, ..., n\}$ , the measure  $f_{n-p-1}(\operatorname{Lk}(\sigma, \operatorname{Sd}^d(K)))d\gamma_{p,K}^d(\sigma)$  weakly converges to  $f_p(\Delta_n)d\operatorname{vol}_K$  as d grows to  $+\infty$ .

**Proof.** By definition

$$\gamma_{p,K}^{d} = \sum_{\sigma \in K^{[n]}} \gamma_{p,\sigma}^{d} - \sum_{\tau \in K^{(n-1)}} \left( f_{n-\dim \tau - 1}(\operatorname{Lk}(\tau, K)) - 1 \right) \left( \frac{(\dim \tau + 1)!}{(n+1)!} \right)^{d} \gamma_{p,\tau}^{d}$$

since for every  $\tau \in K^{(n-1)}$  and every  $\sigma \in K^{[n]}$  such that  $\tau < \sigma$ ,  $\gamma_{p,\sigma}^d|_{\tau} = \left(\frac{(\dim \tau+1)!}{(n+1)!}\right)^d \gamma_{p,\tau}^d$ by definition and  $\tau$  is a face of exactly  $f_{n-\dim \tau-1}(\operatorname{Lk}(\tau, K))$  such  $\sigma$ 's. The result thus follows from Theorem 19 and Theorem 22.

#### 4 Limit density of links in a simplicial complex

Corollary 23 computes the limit density as d grows to  $+\infty$  of the top face numbers of the links of p-dimensional simplices in  $\mathrm{Sd}^d(K), p \in \{0, \ldots, n\}$ . We are going now to extend this result to all the face numbers of these links.

**Theorem 24** For every n-dimensional locally finite simplicial complex K and every  $0 \le p < n$ , the measure  $q_{\text{Lk}(\sigma, \text{Sd}^d(K))}(T) d\gamma_{p,K}^d(\sigma)$  (with value in  $\mathbb{R}_{n-p-1}[T]$ ) weakly converges to  $\left(\sum_{l=0}^{n-p-1} q_{p+l+1,n} f_p(\Delta_{p+l+1})T^l\right) d\text{vol}_K$  as d grows to  $+\infty$ .

**Proof.** Let  $\varphi \in C_c^0(|K|)$  be a continuous function with compact support on |K|. For every  $0 \le l \le n - p - 1$ , let us introduce the set

$$\mathcal{I}_{l} = \{ (\sigma, \tau) \in \mathrm{Sd}^{d}(K)^{[p]} \times \mathrm{Sd}^{d}(K)^{[p+l+1]} | \sigma < \tau \}.$$

$$\tag{1}$$

It is equipped with the projection  $p_1 : (\sigma, \tau) \in \mathcal{I}_l \mapsto \sigma \in \mathrm{Sd}^d(K)^{[p]}$  and  $p_2 : (\sigma, \tau) \in \mathcal{I}_l \mapsto \tau \in \mathrm{Sd}^d(K)^{[p+l+1]}$ . We observe that for every  $\sigma \in \mathrm{Sd}^d(K)^{[p]}$ ,  $\#p_1^{-1}(\sigma) = f_l(\mathrm{Lk}(\sigma, \mathrm{Sd}^d(K)))$  while for every  $\tau \in \mathrm{Sd}^d(K)^{[p+l+1]}$ ,  $p_2^{-1}(\tau)$  is in bijection with  $\tau^{[p]}$  (given by  $p_1$ ). Let us set

$$\varphi_1: (\sigma, \tau) \in \mathcal{I}_l \mapsto \varphi(\hat{\sigma}) \in \mathbb{R}; \ \varphi_2: (\sigma, \tau) \in \mathcal{I}_l \mapsto \varphi(\hat{\tau}) \in \mathbb{R}; \ \gamma_l = \frac{1}{(n+1)!^d} \sum_{(\sigma, \tau) \in \mathcal{I}_l} \delta_{(\sigma, \tau)}.$$
(2)

Then, we deduce

$$\begin{split} \int_{K} \varphi f_{l} \big( \mathrm{Lk}(\sigma, \mathrm{Sd}^{d}(K)) \big) d\gamma_{p,K}^{d}(\sigma) &= \int_{\mathcal{I}_{l}} \varphi_{1} d\gamma_{l} \\ &= \int_{\mathcal{I}_{l}} \varphi_{2} d\gamma_{l} + \int_{\mathcal{I}_{l}} (\varphi_{1} - \varphi_{2}) d\gamma_{l} \\ &= \int_{\mathrm{Sd}^{d}(K)^{[p+l+1]}} (p_{2})_{*} (\varphi_{2} d\gamma_{l}) + \int_{\mathcal{I}_{l}} (\varphi_{1} - \varphi_{2}) d\gamma_{l} \\ &= \int_{K} \varphi f_{p}(\tau) d\gamma_{p+l+1,K}^{d}(\tau) + \int_{\mathcal{I}_{l}} (\varphi_{1} - \varphi_{2}) d\gamma_{l} \end{split}$$

From Theorem 19, the first term  $\int_{K} \varphi f_p(\tau) d\gamma_{p+l+1,K}^d(\tau)$  in the right hand side converges to  $q_{p+l+1,n} f_p(\Delta_{p+l+1}) \int_{K} \varphi d\operatorname{vol}_K$  as d grows to  $+\infty$  while the second term  $\int_{\mathcal{I}_l} (\varphi_1 - \varphi_2) d\gamma_l$ converges to zero. Indeed,  $\varphi$  is continuous with compact support and the diameter of  $\tau \in$   $\mathrm{Sd}^d(K)^{[p+l+1]}$  uniformly converges to zero on this compact subset as d grows to  $+\infty$ . Thus, the suppremum of  $(\varphi_1 - \varphi_2)$  converges to zero as d grows  $+\infty$ . On the other hand, the total mass of  $\gamma_l$  remains bounded, since

$$\int_{\mathcal{I}_l} 1\gamma_l = \int_{\mathrm{Sd}^d(K)^{[p+l+1]}} (p_2)_*(d\gamma_l) = f_p(\Delta_{p+l+1}) \int_K \gamma_{p+l+1,K}^d$$

and the latter is bounded from Theorem 19. The result follows by definition of  $q_{\text{Lk}(\sigma, \text{Sd}^d(K))}(T)$ . 

Note that the (n-1)-skeleton of K has vanishing measure with respect to  $dvol_K$  while for every  $\sigma \in \mathrm{Sd}^d(K)^{[p]}$  interior to an *n*-simplex, its link is a homology (n-p-1)-sphere (Theorem 63.2) of [7]). After evaluation at T = -1 and integration of the constant function 1, Theorem 24 thus provides the following asymptotic Dehn-Sommerville relations:

$$\sum_{l=p}^{n} q_{l,n} \binom{l+1}{p+1} (-1)^{n+l} = q_{p,n}.$$

Now, recall that the dual block  $D(\sigma)$  of a simplex  $\sigma \in K$  is the union of all open simplices  $[\hat{\sigma}_0 \dots, \hat{\sigma}_p]$  of Sd(K) such that  $\sigma_0 = \sigma$ , see [7]. The closure  $D(\sigma)$  of  $D(\sigma)$  is called *closed block* dual to  $\sigma$  and following [7] we set  $\dot{D}(\sigma) = \overline{D}(\sigma) \setminus D(\sigma)$ . Then, we get the following.

**Theorem 25** For every n-dimensional locally finite simplicial complex K and every  $0 \le p \le 1$ n, the measure  $q_{D(\sigma)}(T)d\gamma_{p,K}^d(\sigma)$  weakly converges to  $\sum_{l=0}^{n-p} \left( \sum_{h=l}^{n-p} q_{p+h,n} f_p(\Delta_{p+h}) \lambda_{h,l} \right) T^l d\mathrm{vol}_K$ as d grows to  $+\infty$ .

**Proof.** By definition, the dual block  $D(\sigma)$  has only one face in dimension 0, namely  $\hat{\sigma}$ , so that for the coefficient l = 0, the result follows from Theorem 19. Let us now assume that  $0 < l \leq n - p$  and choose  $\varphi \in C_c^0(|K|)$ . We set

$$\mathcal{J}_{l} = \{(\sigma, \theta) \in \mathrm{Sd}^{d}(K)^{[p]} \times \mathrm{Sd}^{d+1}(K)^{[l-1]} | \theta \in \dot{D}(\sigma)\}.$$

Let  $p_1: (\sigma, \theta) \in \mathcal{J}_l \mapsto \sigma \in \mathrm{Sd}^d(K)^{[p]}$ . Then, for every  $\sigma \in \mathrm{Sd}^d(K)^{[p]}, \, \#p_1^{-1}(\sigma) = f_l(D(\sigma)),$ since  $p_1^{-1}(\sigma)$  is in bijection with  $\dot{D}(\sigma)$  and by taking the cone over  $\hat{\sigma}$  we get an isomorphism  $\tau \in \dot{D}(\sigma) \mapsto \hat{\sigma} * \tau \in D(\sigma) \setminus \hat{\sigma}$  where \* denotes the join operation. (Recall that if  $\tau = [e_0, \ldots, e_k]$ the join  $\hat{\sigma} * \tau$  is  $[\hat{\sigma}, e_0, \ldots, e_k]$ .)

Likewise by definition, every simplex  $\theta \in \dot{D}(\sigma)^{[l-1]}$  reads  $\theta = [\hat{\tau}_0, \dots, \hat{\tau}_{l-1}]$  where  $\sigma < \tau_0 < \tau_0$  $\ldots < \tau_{l-1}$  are simplices of  $\mathrm{Sd}^d(K)$  (see Theorem 64.1 of [7]). We deduce a map

$$\pi: \qquad \mathcal{J}_l \qquad \to \qquad \bigsqcup_{\substack{h=l-1\\ h=l-1}}^{n-p-1} \mathcal{I}_h$$
$$(\sigma, [\hat{\tau}_0, \dots, \hat{\tau}_{l-1}]) \qquad \mapsto \qquad (\sigma, \tau_{l-1})$$

where  $\mathcal{I}_h$  is the set defined in (1). We then set  $p_2: (\sigma, \tau) \in \bigsqcup_{h=l-1}^{n-p-1} \mathcal{I}_h \mapsto \tau \in \mathrm{Sd}^d(K) \setminus \mathrm{Sd}^d(K)^{(p+l-1)}$ . As in the proof of Theorem 24, for every  $\tau \in \mathrm{Sd}^d(K) \setminus \mathrm{Sd}^d(K)^{(p+l-1)}$ ,  $p_2^{-1}(\tau)$  is in bijection with  $\tau^{[p]}$  and  $\pi^{-1}(\sigma,\tau)$ with the set of interior (l-1)-dimensional simplices of Sd(Lk $(\sigma, \tau)$ ), so that  $\#\pi^{-1}((\sigma, \tau)) =$  $\lambda_{h+1,l}$  if dim  $\tau = p+h+1$ . Let us set  $\tilde{\varphi}_1 : (\sigma,\tau) \in \mathcal{J}_l \mapsto \varphi(\hat{\sigma}) \in \mathbb{R}$  and  $\tilde{\gamma}_l = \frac{1}{(n+1)!^d} \sum_{(\sigma,\theta) \in \mathcal{J}_l} \delta_{(\sigma,\theta)}$ . Then, we deduce

$$\int_{K} \varphi f_{l}(D(\sigma)) d\gamma_{p,K}^{d}(\sigma) = \int_{\mathcal{J}_{l}} \tilde{\varphi}_{l} d\tilde{\gamma}_{l}$$

$$= \sum_{h=l-1}^{n-p-1} \lambda_{h+1,l} \int_{\mathcal{I}_{h}} \varphi_{l} d\gamma_{h}$$

by pushing forward  $\tilde{\varphi}_1 d\tilde{\gamma}_l$  onto  $\bigsqcup_{h=l-1}^{n-p-1} \mathcal{I}_h$  with  $\pi$ , where  $\varphi_1$  and  $\gamma_h$  are defined by (2). Now, we have established in the proof of Theorem 24 that as d grows to  $+\infty$ ,  $\int_{\mathcal{I}_h} \varphi_1 d\gamma_h$ converges to  $f_p(\Delta_{p+h+1})q_{p+h+1}\int_K \varphi d\operatorname{vol}_K$ . We deduce that  $f_l(D(\sigma))d\gamma_{p,K}^d(\sigma)$  weakly converges to  $\left(\sum_{h=l}^{n-p}\lambda_{h,l}f_p(\Delta_{p+h})q_{p+h,n}\right)d\operatorname{vol}_K$ . Hence the result.

**Remark 26** In [8], we study the expected topology of a random subcomplex in a finite simplicial complex K and its barycentric subdivisions. The Betti numbers of such a subcomplex turn out to be asymptotically controlled by the measure given by Theorem 25.

Let us now finally observe that Theorem 25 provides a geometric proof of the following (compare Theorem A of [3]).

**Corollary 27** The vector  $(q_{p,n})_{0 \le p \le n}$  is the eigenvector of  $\Lambda_n^t$  associated to the eigenvalue (n+1)!, normalized by the relation  $q_{n,n} = 1$ .

**Proof.** By Theorem 64.1 of [7], we know that the dual blocks of a complex K are disjoint and that their union is |K|. We deduce that for every  $d \in \mathbb{N}^*$ ,

$$\frac{1}{(n+1)!^d} q_{\mathrm{Sd}^{d+1}(\Delta_n)}(T) = \sum_{p=0}^n \int_{\Delta_n} q_{D(\sigma)}(T) d\gamma_{p,\Delta_n}^d(\sigma).$$

By letting d grow to  $+\infty$ , we now deduce from Theorem 25, applied to  $K = \Delta_n$  and after integration of 1, that

$$(n+1)! \sum_{p=0}^{n} q_{p,n} T^{p} = \sum_{p=0}^{n} \left( \sum_{l=0}^{n-p} T^{l} \sum_{h=p+l}^{n} q_{h,n} f_{p}(\Delta_{h}) \lambda_{h-p,l} \right)$$
$$= \sum_{l=0}^{n} T^{l} \left( \sum_{h=l}^{n} q_{h,n} \sum_{p=0}^{h-l} f_{p}(\Delta_{h}) \lambda_{h-p,l} \right)$$

Now,  $\sum_{p=0}^{h-l} f_p(\Delta_h) \lambda_{h-p,l} = \sum_{p=l}^{h} {h+1 \choose p} \lambda_{p,l} = \lambda_{h+1,l+1}$  from Lemma 13. Hence, for every  $p \in \{0, \dots, n\}, (n+1)!q_{p,n} = \sum_{h=l}^{n} q_{h,n} \lambda_{h+1,l+1}.$ 

### References

- [1] T. Akita. A formula for the Euler characteristics of even dimensional triangulated manifolds. Proc. Amer. Math. Soc., 136(7):2571–2573, 2008.
- [2] F. Brenti and V. Welker. f-vectors of barycentric subdivisions. Math. Z., 259(4):849–865, 2008.

- [3] E. Delucchi, A. Pixton, and L. Sabalka. Face vectors of subdivided simplicial complexes. Discrete Math., 312(2):248–257, 2012.
- [4] S. Klain. Dehn-sommerville relations for triangulated manifolds. unpublished manuscript available at http://faculty.uml.edu/dklain/ds.pdf.
- [5] V. Klee. A combinatorial analogue of Poincaré's duality theorem. Canad. J. Math., 16:517–531, 1964.
- [6] I. G. Macdonald. Polynomials associated with finite cell-complexes. J. London Math. Soc. (2), 4:181–192, 1971.
- [7] J. R. Munkres. *Elements of algebraic topology*. Addison-Wesley Publishing Company, Menlo Park, CA, 1984.
- [8] N. Salepci and J.-Y. Welschinger. Asymptotic topology of random subcomplexes in a finite simplicial complex. *In preparation*, 2017.

Univ Lyon, Université Claude Bernard Lyon 1, CNRS UMR 5208, Institut Camille Jordan, 43 blvd. du 11 novembre 1918, F-69622 Villeurbanne cedex, France

salepci@math.univ-lyon1.fr, welschinger@math.univ-lyon1.fr.