

LARGE TIME BEHAVIOR OF SOLUTIONS OF LOCAL AND NONLOCAL NONDEGENERATE HAMILTON-JACOBI EQUATIONS WITH ORNSTEIN-UHLENBECK OPERATOR

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ABSTRACT. We study the well-posedness of second order Hamilton-Jacobi equations with an Ornstein-Uhlenbeck operator in \mathbb{R}^N and $\mathbb{R}^N \times [0, +\infty)$. As applications, we solve the associated ergodic problem associated to the stationary equation and obtain the large time behavior of the solutions of the evolution equation when it is nondegenerate. These results are some generalizations of the ones obtained by Fujita, Ishii & Loreti 2006 [19] by considering more general diffusion matrices or nonlocal operators of integro-differential type and general sublinear Hamiltonians. Our work uses as a key ingredient the *a-priori* Lipschitz estimates obtained in Chasseigne, Ley & Nguyen 2017 [10].

1. Introduction. The aims of this work are to study the existence and uniqueness of solutions of the equations

$$\lambda u^\lambda - \mathcal{F}(x, [u^\lambda]) + \langle b(x), Du^\lambda \rangle + H(x, Du^\lambda) = f(x), \quad x \in \mathbb{R}^N, \quad \lambda > 0, \quad (1)$$

$$\begin{cases} \frac{\partial u}{\partial t} - \mathcal{F}(x, [u]) + \langle b(x), Du \rangle + H(x, Du) = f(x), & (x, t) \in \mathbb{R}^N \times (0, \infty) \\ u(\cdot, 0) = u_0(\cdot) & \text{in } \mathbb{R}^N \end{cases} \quad (2)$$

and the large time behavior of solution $u(x, t)$ of (2), that is to prove that

$$u(\cdot, t) + ct \rightarrow v(\cdot) \quad \text{locally uniformly in } \mathbb{R}^N \text{ as } t \rightarrow \infty, \quad (3)$$

where $(c, v) \in \mathbb{R} \times C(\mathbb{R}^N)$ is a solution of the associated ergodic problem

$$c - \mathcal{F}(x, [v]) + \langle b(x), Dv \rangle + H(x, Dv) = f(x) \quad \text{in } \mathbb{R}^N. \quad (4)$$

Let us describe the main features of (1)-(2). The term $\langle b, D \rangle$ is an *Ornstein-Uhlenbeck drift*, i.e., there exists $\alpha > 0$ (the strength of the Ornstein-Uhlenbeck term) such that

$$\langle b(x) - b(y), x - y \rangle \geq \alpha |x - y|^2, \quad x, y \in \mathbb{R}^N, \quad (5)$$

the Hamiltonian H is continuous and *sublinear*, i.e., there exists $C_H > 0$ such that

$$|H(x, p)| \leq C_H(1 + |p|), \quad x, p \in \mathbb{R}^N, \quad (6)$$

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and the operator \mathcal{F} can be either *local*

$$\mathcal{F}(x, [u]) = \text{tr}(A(x)D^2u) \quad (\text{classical diffusion}) \quad (7)$$

where A is a nonnegative symmetric matrix, or *nonlocal*

$$\mathcal{F}(x, [u]) = \int_{\mathbb{R}^N} \{u(x+z) - u(x) - \langle Du(x), z \rangle \mathbb{I}_B(z)\} \nu(dz). \quad (8)$$

Since we work on an unbounded domain and deal with unbounded solutions, we need to restrict them in some class

$$\mathcal{E}_\mu = \left\{ g : \mathbb{R}^N \rightarrow \mathbb{R} : \lim_{|x| \rightarrow +\infty} \frac{g(x)}{\phi_\mu(x)} = 0 \right\}, \quad (9)$$

where we choose

$$\phi_\mu(x) = e^{\mu\sqrt{1+|x|^2}}, \quad \mu > 0. \quad (10)$$

Henceforth, we work on the datas f, u_0 which satisfy

$$|g(x) - g(y)| \leq C_g(\phi_\mu(x) + \phi_\mu(y))|x - y|, \quad g = f \text{ or } g = u_0, \quad x, y \in \mathbb{R}^N. \quad (11)$$

In the local case, the diffusion A is anisotropic and we assume that $A = \sigma\sigma^T$ where $\sigma \in W^{1,\infty}(\mathbb{R}^N; \mathcal{M}_N)$, i.e.,

$$|\sigma(x)| \leq C_\sigma, \quad |\sigma(x) - \sigma(y)| \leq L_\sigma|x - y| \quad x, y \in \mathbb{R}^N. \quad (12)$$

In the nonlocal case, \mathcal{F} has the form (8), where ν is a Lévy type measure, which is possibly singular and nonnegative. In order that (8) is well-defined for our solutions in \mathcal{E}_μ ,

$$\mathcal{I}(x, \psi, D\psi) := \int_{\mathbb{R}^N} \{\psi(x+z) - \psi(x) - \langle D\psi(x), z \rangle \mathbb{I}_B(z)\} \nu(dz) \quad (13)$$

has to be well-defined for any continuous $\psi \in \mathcal{E}_\mu$ which is C^2 in a neighborhood of x , which leads to assume that

$$\left\{ \begin{array}{l} \text{There exists a constant } C_\nu^1 > 0 \text{ such that} \\ \int_B |z|^2 \nu(dz), \int_{B^c} \phi_\mu(z) \nu(dz) \leq C_\nu^1. \end{array} \right. \quad (14)$$

An important example of ν is the *tempered β -stable law*

$$\nu(dz) = \frac{e^{-\mu|z|}}{|z|^{N+\beta}} dz, \quad (15)$$

where $\beta \in (0, 2)$ is the *order* of the integro-differential operator. Notice that, in the bounded framework when μ can be taken equal to 0, up to a normalizing constant, $-\mathcal{I} = (-\Delta)^{\beta/2}$ is the fractional Laplacian of order β , see [16] and [28] and references therein for further explanations about the integro-differential operator with Ornstein-Uhlenbeck drift.

Most of the results in this work are based on the Lipschitz estimates on the solutions of (1) and (2) obtained in [10], i.e.,

$$|u^\lambda(x) - u^\lambda(y)|, |u(x, t) - u(y, t)| \leq C(\phi_\mu(x) + \phi_\mu(y))|x - y|, \quad x, y \in \mathbb{R}^N, \quad (16)$$

where C is independent of $\lambda > 0$, $t \in [0, T)$, $T > 0$. The uniformity of these estimates with respect to λ, t is a crucial point for the applications, i.e., to be able to solve the ergodic problem (4) and to prove the large time behavior (3). They

are established for both degenerate and nondegenerate equations. Let us recall that the equations (1), (2) are called nondegenerate in [10] when

$$A(x) \geq \rho Id, \quad \text{for some } \rho > 0, \quad (17)$$

in the local case, which is the classical assumption of ellipticity. In the nonlocal one, we work with Lévy measures ν satisfying (14) and

$$\left\{ \begin{array}{l} \text{There exists } \beta \in (0, 2) \text{ such that for every } a \in \mathbb{R}^N \text{ there exist} \\ 0 < \eta < 1 \text{ and } C_\nu^2 > 0 \text{ such that, for all } \gamma > 0, \\ \int_{\mathcal{C}_{\eta, \gamma}(a)} |z|^2 \nu(dz) \geq C_\nu^2 \eta^{\frac{N-1}{2}} \gamma^{2-\beta}, \end{array} \right. \quad (18)$$

where $\mathcal{C}_{\eta, \gamma}(a) := \{z \in B_\gamma : (1 - \eta)|z||a| \leq |\langle a, z \rangle|\}$. We say that the nonlocal equation is nondegenerate when the order β belongs to the interval $(1, 2)$, since in this case, (18) gives a kind of ellipticity. This assumption, which holds true for the typical example (15), was introduced in [6] and allows to adapt Ishii-Lions' method to nonlocal integro-differential equation. We refer to [10] for details and comments.

As far as the long time behavior is concerned, there have been many results obtained for second order equations. But most of them are investigated in periodic settings. We refer to [5, 19, 18, 20, 6, 7, 9, 25, 26] and the references therein. There are few results in the unbounded settings, essentially the works of Fujita, Ishii & Loreti 2006 [19] and Ichihara & Sheu 2013 [21]. In both of these works, the authors are concerned with the local equation with a pure Laplacian diffusion. In particular, in [21], they deal with quadratic nonlinearity in gradients and use both PDE and probabilistic approach. Since our work is quite close to the one of [19], let us explain briefly the main differences. In [19], they consider

$$\frac{\partial u}{\partial t} - \Delta u + \alpha \langle x, Du \rangle + H(Du) = f(x), \quad (x, t) \in \mathbb{R}^N \times (0, \infty), \quad (19)$$

with the datas f, u_0 and the solutions belonging to the class (9) where

$$\phi_\mu(x) = e^{\mu|x|^2},$$

and $0 < \mu < \alpha$, which seems to be the optimal growth condition related to the density of the invariant measure associated with the Ornstein-Uhlenbeck process. The restriction on the growth in our case comes from the anisotropy of the diffusion (local case) or the nonlocal term. We do not know if the growth (10) is optimal. Moreover, in [19], H is Lipschitz continuous and independent of x , the authors can prove well-posedness of the equations in this growth class and avoid some technical difficulties. On the other hand, when considering the uniformly parabolic PDE (19), they can work with classical solutions thanks to Schauder theory for uniformly parabolic equations (see Ladyzhenskaya, Solonnikov & Uralseva [24]). The proofs are then less technical.

One of our main issues is to prove the existence of unbounded continuous viscosity solutions for the equations (1) and (2), see Theorems 2.1 and 2.3 for *nondegenerate* equations and Theorems 2.5 and 2.6 for degenerate equations. The key ingredient is the a priori Lipschitz estimate (16), which is a natural idea already used in [5, 25]. Let us underline that, in our case, (16) provides only *locally* Lipschitz estimates and the solutions are unbounded in the whole space \mathbb{R}^N . These lead to additional difficulties comparing to the classical uniformly continuous or globally Lipschitz case. Moreover, we are able to deal with Hamiltonians $H(x, p)$ which are merely sublinear. In this situation, we cannot use directly the classical viscosity solution

machinery since comparison principle between discontinuous viscosity solutions does not necessarily hold without the classical structure assumptions, see [22, 13, 3] for instance. In general this latter issue is overcome by using some global Lipschitz properties. More precisely, when doubling the variables using viscosity techniques, one has to prove that some quantities $H(x, p) - H(y, p)$ are small when x close to y . This is possible when p is bounded (due to the Lipschitz continuity of solutions) and when x, y lie in some bounded subset. In our case, both x, y and p are unbounded. The first idea is to recover some compactness by taking profit of the Ornstein-Uhlenbeck operator. From a PDE point of view, the property of the Ornstein-Uhlenbeck operator translates into a supersolution property for the growth function ϕ_μ (see [10, Lemma 2.1]), that is, there exist $C, K > 0$ such that

$$\mathcal{L}[\phi_\mu](x) := -\mathcal{F}(x, [\phi_\mu]) + \langle b(x), D\phi_\mu(x) \rangle - C|D\phi_\mu(x)| \geq \phi_\mu(x) - K, \quad x \in \mathbb{R}^N. \quad (20)$$

The second idea, which was already used in [5, 25] for instance, is to use a uniformly continuous truncation both for the Hamiltonian and the data f, u_0 in such a way that the Lipschitz estimate (16) for the approximate solutions still hold independently of the truncations. It is therefore possible to pass to the limit. The uniqueness of solutions of (1) and (2) is followed by comparison principle which holds when we suppose in addition

$$|H(x, p) - H(x, q)| \leq L_H |p - q|, \quad \text{for all } x, p, q \in \mathbb{R}^N. \quad (21)$$

For possibly degenerate equations, the results are true under stronger assumptions on the Hamiltonian, there is a function $\omega : [0, \infty) \rightarrow [0, \infty)$ satisfying $\omega(0) = 0$ such that

$$\begin{cases} |H(x, p) - H(y, p)| \leq \omega(|x - y|(1 + |p|)), & x, y, p, q \in \mathbb{R}^N, \\ |H(x, p) - H(x, q)| \leq L_H |p - q|, \\ |H(x, 0)| \leq L_H, \end{cases} \quad (22)$$

or

$$\begin{cases} |H(x, p) - H(y, p)| \leq \omega(|x - y|(1 + |p|)), & x, y, p, q \in \mathbb{R}^N, \\ |H(x, p) - H(x, q)| \leq L_H |p - q|(1 + |x|), \\ |H(x, 0)| \leq L_H(1 + |x|). \end{cases} \quad (23)$$

The proofs are done based on [19] using (20) as a crucial point. When dealing with (23), we need an additional condition on the strength α of the Ornstein-Uhlenbeck operator, see [10, Lemma 2.1]. We will only quote the results (Theorems 2.5 and 2.6) without proofs since they are closer classical ones in viscosity solutions.

As a by-product of the Lipschitz estimate in space (16), we obtain $\frac{1}{2}$ -Hölder estimates in time for the solutions of the evolution problem (2). This result is well known for local equations (Barles, Biton & Ley [4, Lemma 2.3]) but does not seem to be written for nonlocal ones. The result is interesting by itself so we provide a complete proof.

The other main result in our work is to obtain the convergence (3). We first study the ergodic problem (4) as an application of (16). The idea is classical, see the seminal work of Lions, Papanicolaou & Varadhan [27]. But in the unbounded setting, the solution of (4) does not belong to class (9) anymore but to a larger one. This brings an additional difficulty in the nonlocal case and we have to modify the proof. The proof of the convergence theorem is more classical and follows the arguments of [19]. But some adaptations are needed in presence of a nonlocal

operator and due to the fact that we work with nonsmooth solutions instead of C^2 -smooth ones.

The paper is organized as follows. In Section 2, we first study the well-posedness of the equations (1) and (2). At the end of this section we give a precise proof for the regularity of solution with respect to time in the nonlocal case. Section 3 is devoted to the ergodic problem (4) and to the proof of the convergence (3). Some classical and technical results are collected in Section 4.

Notation. In the whole paper, \mathcal{S}_N denotes the set of symmetric matrices of size N equipped with the norm $|A| = (\sum_{1 \leq i, j \leq N} a_{ij}^2)^{1/2}$, $B(x, \delta)$ is the open ball of center x and radius $\delta > 0$ (written B_δ if $x = 0$) and $B^c(x, \delta) = \mathbb{R}^N \setminus B(x, \delta)$.

Let $T \in (0, \infty)$, we write $Q_T = \mathbb{R}^N \times (0, T)$ and $Q = Q_\infty$, we introduce

$$\begin{aligned} \mathcal{E}_\mu^+(\mathbb{R}^N) &= \{v : \mathbb{R}^N \rightarrow \mathbb{R} : \limsup_{|x| \rightarrow +\infty} \frac{v(x)}{\phi_\mu(x)} \leq 0\}, \\ \mathcal{E}_\mu^+(\overline{Q}_T) &= \{v : \overline{Q}_T \rightarrow \mathbb{R} : \limsup_{|x| \rightarrow +\infty} \sup_{0 \leq t < T} \frac{v(x, t)}{\phi_\mu(x)} \leq 0\}, \end{aligned}$$

$\mathcal{E}_\mu^- := -\mathcal{E}_\mu^+$ and $\mathcal{E}_\mu := \mathcal{E}_\mu^+ \cap \mathcal{E}_\mu^-$, where ϕ_μ is defined by (10). Notice that $v \in \mathcal{E}_\mu(\mathbb{R}^N)$ if and only if for all $\epsilon > 0$, there exists $M(\epsilon) > 0$ such that

$$|v(x)| \leq \epsilon \phi(x) + M(\epsilon) \quad \text{for all } x \in \mathbb{R}^N. \quad (24)$$

In the whole article, we deal with viscosity solutions of (1), (2). Classical references in the local case are [13, 23, 17] and for the nonlocal integro-differential equations, we refer the reader to [8, 1, 10].

2. Well-posedness and regularity of the stationary and evolution problems. In two first parts of this Section, we build continuous solutions for (1)-(2) when supposing that the Hamiltonian is sublinear, i.e., (6) holds without further assumption, and that the equation is nondegenerate in the sense explained in the introduction part. The proofs in this case are strongly based on the a priori Lipschitz estimates obtained in [10], which hold thanks to the nondegeneracy of the equation together with the effect of the Ornstein-Uhlenbeck term. The last part is devoted to build solutions using the classical theory of viscosity solutions for possibly degenerate equations. Some additional assumptions on H and on the strength of the Ornstein-Uhlenbeck term are then needed (but we do not use the Lipschitz estimates (16)).

Throughout this Section, we write ϕ for ϕ_μ defined by (10).

2.1. Well-posedness of the stationary problem. We start with a comparison principle for solutions of (1) satisfying (16).

Proposition 1. Suppose that (5), (21), $f \in C(\mathbb{R}^N)$ and either (12) or (14) hold. Let $u \in USC(\mathbb{R}^N) \cap \mathcal{E}_\mu^+(\mathbb{R}^N)$ and $v \in LSC(\mathbb{R}^N) \cap \mathcal{E}_\mu^-(\mathbb{R}^N)$ be a viscosity sub and supersolution of (1), respectively. Assume that either u or v satisfies (16). Then $u \leq v$ in \mathbb{R}^N .

Proof of Proposition 1. We argue by contradiction assuming that $u(z) - v(z) \geq 2\eta > 0$ for some $z \in \mathbb{R}^N$. We consider

$$\Psi(x, y) = u(x) - v(y) - \frac{|x - y|^2}{2\epsilon^2} - \beta(\phi(x) + \phi(y)),$$

where ϵ, β are positive parameters. For small β we have $\Psi(z, z) \geq \eta$. Since $u \in \mathcal{E}_\mu^+(\mathbb{R}^N)$, $v \in \mathcal{E}_\mu^-(\mathbb{R}^N)$, Ψ attains a maximum at $(\bar{x}, \bar{y}) \in B(0, R_\beta) \times B(0, R_\beta)$, where R_β does not depend on ϵ . It follows that $u(x) - v(y) - \beta(\phi(x) + \phi(y))$ is bounded in $B(0, R_\beta) \times B(0, R_\beta)$, so the following classical properties (see [3]) hold up to some subsequence,

$$\frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} \rightarrow 0, \quad \bar{x}, \bar{y} \rightarrow \hat{x} \in B(0, R_\beta) \text{ as } \epsilon \rightarrow 0, \beta \text{ is fixed.} \quad (25)$$

Assuming that v for instance satisfies (16), since $\Psi(\bar{x}, \bar{x}) \leq \Psi(\bar{x}, \bar{y})$, we have

$$\begin{aligned} \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} &\leq v(\bar{x}) - v(\bar{y}) + \beta(\phi(\bar{x}) - \phi(\bar{y})) \\ &\leq C|\bar{x} - \bar{y}|(\phi(\bar{x}) + \phi(\bar{y})) + \beta\mu|\bar{x} - \bar{y}|(\phi(\bar{x}) + \phi(\bar{y})), \end{aligned}$$

using that

$$|\phi(\bar{x}) - \phi(\bar{y})| \leq \mu(\phi(\bar{x}) + \phi(\bar{y}))|\bar{x} - \bar{y}|. \quad (26)$$

This implies that $p_\epsilon := \frac{\bar{x} - \bar{y}}{\epsilon^2}$ remains bounded when $\epsilon \rightarrow 0$ and, up to some subsequence, $p_\epsilon \rightarrow \hat{p}$, for some $\hat{p} \in \mathbb{R}^N$.

We write the viscosity inequalities at (\bar{x}, \bar{y}) using [13, Theorem 3.2] in the local case and [8, Corollary 1] in the nonlocal one. In the local case, for every $\varrho > 0$, there exist $(p_\epsilon + \beta D\phi(\bar{x}), X) \in \bar{J}^{2,+}u(\bar{x})$, $(p_\epsilon - \beta D\phi(\bar{y}), Y) \in \bar{J}^{2,-}v(\bar{y})$ such that

$$\begin{pmatrix} X & O \\ O & -Y \end{pmatrix} \leq A + \varrho A^2, \text{ where } A = \frac{2}{\epsilon^2} \begin{pmatrix} I & -I \\ -I & I \end{pmatrix} + \beta \begin{pmatrix} D^2\phi(\bar{x}) & 0 \\ 0 & D^2\phi(\bar{y}) \end{pmatrix}$$

and $\varrho A^2 = O(\varrho)$ (ϱ will be sent to 0 first). It follows

$$\begin{aligned} &\lambda(u(\bar{x}) - v(\bar{y})) - (\mathcal{F}(\bar{x}, [u]) - \mathcal{F}(\bar{y}, [v])) + \langle b(\bar{x}) - b(\bar{y}), p_\epsilon \rangle + \beta \langle b(\bar{x}), D\phi(\bar{x}) \rangle \\ &\quad + \beta \langle b(\bar{y}), D\phi(\bar{y}) \rangle + H(\bar{x}, p_\epsilon + \beta D\phi(\bar{x})) - H(\bar{y}, p_\epsilon - \beta D\phi(\bar{y})) \\ &\leq f(\bar{x}) - f(\bar{y}), \end{aligned} \quad (27)$$

where $\mathcal{F}(\bar{x}, [u]) = \text{tr}(A(\bar{x})X)$ and $\mathcal{F}(\bar{y}, [v]) = \text{tr}(A(\bar{y})Y)$ in the local case and $\mathcal{F}(\bar{x}, [u]) = \mathcal{I}(\bar{x}, u, p_\epsilon + \beta D\phi(\bar{x}))$ and $\mathcal{F}(\bar{y}, [v]) = \mathcal{I}(\bar{y}, v, p_\epsilon - \beta D\phi(\bar{y}))$ in the nonlocal one.

We estimate the \mathcal{F} -terms by using the results of [10] for the test function

$$\frac{|x - y|^2}{2\epsilon^2} + \beta(\phi(x) + \phi(y)).$$

When \mathcal{F} is the local operator defined by (7), applying [10, Lemma 2.2], we obtain

$$\text{tr}(A(\bar{x})X - A(\bar{y})Y) \leq L_\sigma^2 \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} + \beta \text{tr}(A(\bar{x})D^2\phi(\bar{x})) + \beta \text{tr}(A(\bar{y})D^2\phi(\bar{y})) + O(\varrho).$$

When \mathcal{F} is the nonlocal operator defined by (8), applying [10, Proposition 2.1], we get

$$\mathcal{I}(\bar{x}, u, p_\epsilon + \beta D\phi(\bar{x})) - \mathcal{I}(\bar{y}, v, p_\epsilon - \beta D\phi(\bar{y})) \leq \beta \mathcal{I}(\bar{x}, \phi, D\phi) + \beta \mathcal{I}(\bar{y}, \phi, D\phi).$$

Therefore, in any case we have

$$\mathcal{F} \leq \beta \mathcal{F}(\bar{x}, \phi) + \beta \mathcal{F}(\bar{y}, \phi) + L_\sigma^2 \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} + O(\varrho). \quad (28)$$

Since $\Psi(\bar{x}, \bar{y}) \geq \Psi(z, z) \geq \eta$, we have $u(\bar{x}) - v(\bar{y}) \geq \eta$. Using (5), taking into account (28) and sending $\varrho \rightarrow 0$, inequality (27) leads to

$$\begin{aligned} & \lambda\eta - \beta\mathcal{F}(\bar{x}, [\phi]) - \beta\mathcal{F}(\bar{y}, [\phi]) - L_\sigma^2 \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} + \alpha \frac{|\bar{x} - \bar{y}|^2}{2\epsilon^2} + \beta\langle b(\bar{x}), D\phi(\bar{x}) \rangle \\ & + \beta\langle b(\bar{y}), D\phi(\bar{y}) \rangle + H(\bar{x}, p_\epsilon + \beta D\phi(\bar{x})) - H(\bar{y}, p_\epsilon - \beta D\phi(\bar{y})) \leq f(\bar{x}) - f(\bar{y}). \end{aligned}$$

Now sending ϵ to 0, using (25) and since $f \in C(\mathbb{R}^N)$ we obtain

$$\lambda\eta - 2\beta\mathcal{F}(\hat{x}, [\phi]) + 2\beta\langle b(\hat{x}), D\phi(\hat{x}) \rangle + H(\hat{x}, \hat{p} + \beta D\phi(\hat{x})) - H(\hat{x}, \hat{p} - \beta D\phi(\hat{x})) \leq 0.$$

Since $H(x, p)$ is lipschitz in p uniformly in x , i.e., (21) holds, we get

$$\lambda\eta - 2\beta\mathcal{F}(\hat{x}, [\phi]) + 2\beta\langle b(\hat{x}), D\phi(\hat{x}) \rangle - 2\beta L_H |D\phi(\hat{x})| \leq 0.$$

From (20), there exists a constant $K(L_H, \mathcal{F}) > 0$ such that

$$-\mathcal{F}(x, [\phi]) + \langle b(x), D\phi(x) \rangle - L_H |D\phi(x)| \geq \phi(x) - K \quad \forall x \in \mathbb{R}^N.$$

Therefore, we have

$$\lambda\eta + 2\beta\phi(\hat{x}) - 2\beta K \leq 0.$$

Since $\phi > 0$, sending β to 0, we get a contradiction. \square

Theorem 2.1. Suppose that (5), (6) and that $f \in C(\mathbb{R}^N) \cap \mathcal{E}_\mu(\mathbb{R}^N)$ satisfying (11). Assume either (12)-(17) or (14)-(18) with $\beta \in (1, 2)$ holds. For all $\lambda \in (0, 1)$, there exists a continuous viscosity solution u^λ of (1) such that

$$u^\lambda \in \mathcal{E}_\mu(\mathbb{R}^N), \quad (29)$$

$$|u^\lambda(x) - u^\lambda(y)| \leq C(\phi_\mu(x) + \phi_\mu(y))|x - y|, \quad x, y \in \mathbb{R}^N, \quad (30)$$

where $C > 0$ is a constant independent of λ . In addition, if (21) holds then the solution is unique in $C(\mathbb{R}^N) \cap \mathcal{E}_\mu(\mathbb{R}^N)$.

Proof of Theorem 2.1.

1. *Construction of a continuous viscosity solution to a truncated equation.* In order to recover the classical framework of viscosity solutions, we first truncate the datas on the equations.

Recall that $\phi(x) = e^{\mu\sqrt{|x|^2+1}}$ and $f \in \mathcal{E}_\mu(\mathbb{R}^N)$. By (24), for every $m \geq 1$, there exists $C(m) > 0$ such that

$$f(x) \geq -\frac{1}{2m}\phi(x) - C(m).$$

Therefore, there exists $R_m > 0$ such that

$$f(x) + \frac{1}{m}\phi(x) \geq m, \quad \text{for } |x| \geq R_m.$$

We then define

$$f_m(x) = \min\{f(x) + \frac{1}{m}\phi(x), m\}. \quad (31)$$

The function f_m is bounded by some constant C_m , still satisfies (11) with the constant $C_f + \frac{\mu}{m}$ and $f_m \rightarrow f$ locally uniformly in \mathbb{R}^N . Moreover, f_m is Lipschitz continuous in \mathbb{R}^N with

$$|f_m(x) - f_m(y)| \leq \left(\left(C_f + \frac{\mu}{m} \right) \sup_{\bar{B}(0, R_m)} 2\phi \right) |x - y| =: L_m |x - y|. \quad (32)$$

Indeed, from (31), if $x, y \notin \overline{B}(0, R_m)$, then $f_m(x) = f_m(y) = m$ and the property is true. If $x, y \in \overline{B}(0, R_m)$, it is trivial when $f_m(x) = m = f_m(y)$. When $f_m(x) = f(x) + \frac{1}{m}\phi(x)$ and $f_m(y) = f(y) + \frac{1}{m}\phi(y)$, then from (11) and (26), we have

$$\begin{aligned} |f_m(x) - f_m(y)| &\leq |f(x) - f(y)| + \frac{1}{m}|\phi(x) - \phi(y)| \\ &\leq C_f|x - y|(\phi(x) + \phi(y)) + \frac{\mu}{m}|x - y|(\phi(x) + \phi(y)) \\ &\leq L_m|x - y|. \end{aligned}$$

When $f_m(x) = f(x) + \frac{1}{m}\phi(x)$ whereas $f_m(y) = m$, then by (31), we have

$$|f_m(y) - f_m(x)| = |m - f(x) - \frac{1}{m}\phi(x)| \leq |f(y) + \frac{1}{m}\phi(y) - f(x) - \frac{1}{m}\phi(x)|,$$

thus, we conclude with the same argument as above.

Let $n \geq 1$, we now truncate the Hamiltonian by defining an Hamiltonian H_{mn} such that

$$H_{mn}(x, p) = \begin{cases} H_m(x, p) & \text{if } |p| \leq n \\ H_m(x, n\frac{p}{|p|}) & \text{if } |p| \geq n, \end{cases} \quad (33)$$

with

$$H_m(x, p) = \begin{cases} H(x, p) & \text{if } |x| \leq m \\ H(m\frac{x}{|x|}, p) & \text{if } |x| \geq m. \end{cases}$$

It is easy to verify that $H_{mn} \in BUC(\mathbb{R}^N \times \mathbb{R}^N)$ with a modulus of continuity depending on m, n and satisfies (6) with the same constant C_H . Indeed,

- for $|p| \geq n$,

$$\begin{aligned} |H_{mn}(x, p)| = |H_m(x, n\frac{p}{|p|})| &= \begin{cases} H(x, n\frac{p}{|p|}), |x| \leq m \\ H(m\frac{x}{|x|}, n\frac{p}{|p|}), |x| \geq m \end{cases} \\ &\leq C_H(1 + n) \leq C_H(1 + |p|), \end{aligned}$$

- for $|p| \leq n$,

$$|H_{mn}(x, p)| = |H_m(x, p)| = \begin{cases} H(x, p), |x| \leq m \\ H(m\frac{x}{|x|}, p), |x| \geq m \end{cases} \leq C_H(1 + |p|).$$

Obviously, H_{mn} converges locally uniformly in $\mathbb{R}^N \times \mathbb{R}^N$ to H_m when $n \rightarrow +\infty$ and H_m converges locally uniformly in $\mathbb{R}^N \times \mathbb{R}^N$ to H when $m \rightarrow +\infty$.

We then consider the new equation

$$\lambda u - \mathcal{F}(x, [u]) + \langle b(x), Du \rangle + H_{mn}(x, Du) = f_m(x) \quad \text{in } \mathbb{R}^N. \quad (34)$$

Classical strong comparison principle holds for bounded discontinuous viscosity sub and supersolutions (see Theorem 4.1 in the Appendix). Noticing that $u_{\lambda, mn}^{\pm}(x) = \pm\lambda^{-1}(C_m + C_H)$ are respectively a super and a subsolution of (34), we obtain by means of Perron's method, the existence and uniqueness of a continuous viscosity solution $u_{\lambda, mn}$ of (34) such that $|\lambda u_{\lambda, mn}| \leq \tilde{C}_m := C_m + C_H$ independent of n . We refer to classical references [13] for the details.

2. *Convergence of the solution of the approximate equation to a continuous solution of (1).* Recall that H_{mn} satisfies (6) with constants C_H independent of m, n . Moreover, from (32), we have f_m is L_m -lipschitz. Since either (12)-(17) or (14)-(18) with $\beta \in (1, 2)$ holds, then applying the a priori Lipschitz estimates [10, Theorem

2.1] for bounded solutions $u_{\lambda, mn}$ we obtain that $u_{\lambda, mn}$ is K_m -lipschitz continuous, i.e.,

$$|u_{\lambda, mn}(x) - u_{\lambda, mn}(y)| \leq K_m|x - y| \quad \text{for all } x, y \in \mathbb{R}^N. \quad (35)$$

Therefore, the family $(u_{\lambda, mn})_{n \geq 1}$ is uniformly equicontinuous in \mathbb{R}^N . By Ascoli Theorem, it follows that, up to some subsequence,

$$u_{\lambda, mn} \rightarrow u_{\lambda, m} \quad \text{as } n \rightarrow +\infty \text{ locally uniformly in } \mathbb{R}^N.$$

By stability ([1, 8, 13]), $u_{\lambda, m}$ is a continuous viscosity solution of (34) with H_m in place of H_{mn} and still satisfies (35) and $|\lambda u_{\lambda, m}| \leq \tilde{C}_m$.

Similarly H_m (respectively f_m) satisfies (6) (respectively (11)) with constants C_H and $C_f + \mu$ independent of $m \geq 1$. Applying [10, Theorem 2.1] again, we obtain that $u_{\lambda, m}$ satisfies (30) with C independent of λ, m . To apply Ascoli Theorem when sending $m \rightarrow \infty$, we need some local L^∞ bound for $u_{\lambda, m}$ independent of m . It is the purpose of the following Lemma.

Lemma 2.2. For every $\epsilon \in (0, 1)$, there exists $C(\epsilon) > 0$ independent of m and λ such that

$$|\lambda u_{\lambda, m}(x)| \leq \epsilon \phi(x) + C(\epsilon). \quad (36)$$

In particular, for all $R > 0$, there exists a constant $C_R > 0$ independent of m and $\lambda \in (0, 1)$ such that

$$|\lambda u_{\lambda, m}(x)| \leq C_R, \quad \text{for all } x \in B(0, R).$$

Proof of Lemma 2.2. Let $\epsilon \in (0, 1)$, $y \in \mathbb{R}^N$ such that

$$u_{\lambda, m}(y) - \epsilon \phi(y) = \max_{\mathbb{R}^N} \{u_{\lambda, m}(x) - \epsilon \phi(x)\}.$$

Since $u_{\lambda, m}$ is a viscosity solution of (34), at the maximum point, we have

$$\lambda u_{\lambda, m}(y) - \mathcal{F}(y, [\epsilon \phi]) + \langle b(y), \epsilon D\phi(y) \rangle + H_m(y, \epsilon D\phi(y)) \leq f_m(y).$$

Recall that H_m satisfies (6) with C_H independent of m . Hence using (20), we get

$$\lambda u_{\lambda, m}(y) \leq f_m(y) - \epsilon \phi(y) + \epsilon K + C_H. \quad (37)$$

Let $m \geq \frac{2}{\epsilon}$. Since $f \in \mathcal{E}_\mu(\mathbb{R}^N)$, by (24), there exists $M(\frac{\epsilon}{2}) > 0$ such that

$$f(y) \leq \frac{\epsilon}{2} \phi(y) + M(\frac{\epsilon}{2}).$$

Hence, from (37) and by the definition of f_m we obtain

$$\lambda u_{\lambda, m}(y) \leq f(y) + \frac{1}{m} \phi(y) - \epsilon \phi(y) + \epsilon K + C_H \leq M(\frac{\epsilon}{2}) + \epsilon K + C_H.$$

Moreover, since y is a maximum point of $u_{\lambda, m} - \epsilon \phi$, we have, for all $\lambda \in (0, 1)$, and $x \in B(0, R)$, $R > 0$,

$$\lambda u_{\lambda, m}(x) \leq \lambda \epsilon \phi(x) + \lambda u_{\lambda, m}(y) - \lambda \epsilon \phi(y) \leq \epsilon \phi(x) + M(\frac{\epsilon}{2}) + \epsilon K + C_H \leq C_R,$$

where $C_R = \max_{B(0, R)} \{\epsilon \phi(x) + M(\frac{\epsilon}{2}) + \epsilon K + C_H\}$ independent of m and λ .

The proof for the opposite inequality is the same by considering $\min_{\mathbb{R}^N} \{u_{\lambda, m}(x) + \epsilon \phi(x)\}$. \square

Now we can apply Ascoli Theorem to get, up to some subsequence, $u_{\lambda,m} \rightarrow u_\lambda$ as $m \rightarrow \infty$ locally uniformly in \mathbb{R}^N and u_λ is a continuous viscosity solution of (1) satisfying (30).

It remains to prove that $u_\lambda \in \mathcal{E}_\mu(\mathbb{R}^N)$. By (36), since $u_{\lambda,m} \rightarrow u_\lambda$ as $m \rightarrow \infty$, we get

$$|\lambda u_\lambda(x)| \leq \epsilon \phi(x) + C(\epsilon), \quad \text{for all } x \in \mathbb{R}^N.$$

This holds for any $\epsilon > 0$, it means that $u_\lambda \in \mathcal{E}_\mu(\mathbb{R}^N)$.

We conclude to the existence of a continuous solution u^λ of (1) belonging to the class (29)-(30).

3. *Uniqueness of the solution of (1) in $C(\mathbb{R}^N) \cap \mathcal{E}_\mu(\mathbb{R}^N)$.* Under the additional assumption (21), it is a direct consequence of comparison principle (see Proposition 1). \square

2.2. **Well-posedness of the evolution problem.** We recall that $Q_T = \mathbb{R}^N \times (0, T)$ and $Q = Q_\infty$.

Proposition 2. Suppose that (5), (21) and either (12) or (14) hold. Let $u \in USC(\bar{Q}_T) \cap \mathcal{E}_\mu^+(\bar{Q}_T)$ and $v \in LSC(\bar{Q}_T) \cap \mathcal{E}_\mu^-(\bar{Q}_T)$ be a viscosity sub and supersolution of (2) with $u(\cdot, 0) = u_0(\cdot)$, $f = f_1 \in C(\mathbb{R}^N)$ and $v(\cdot, 0) = v_0(\cdot)$, $f = f_2 \in C(\mathbb{R}^N)$, respectively. Assume either $u(\cdot, t)$ or $v(\cdot, t)$ satisfies (16) and $\sup_{\mathbb{R}^N} \{u_0(x) - v_0(x)\}, |(f_1 - f_2)^+|_\infty < +\infty$. Then, for all $(x, t) \in \bar{Q}_T$,

$$u(x, t) - v(x, t) \leq \sup_{\mathbb{R}^N} \{u_0(y) - v_0(y)\} + t|(f_1 - f_2)^+|_\infty.$$

The proof of this Proposition is a direct adaptation of the one of Proposition 1 which is extended in the parabolic case.

Theorem 2.3. Suppose (5), (6) and that $f, u_0 \in \mathcal{E}_\mu(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ satisfy (11) with constant C_f, C_0 . Assume either (12)-(17) or (14)-(18) with $\beta \in (1, 2)$ hold. Then, there exists a continuous viscosity solution u of (2) such that

$$u \in \mathcal{E}_\mu(\bar{Q}), \tag{38}$$

$$|u(x, t) - u(y, t)| \leq C|x - y|(\phi(x) + \phi(y)), \quad x, y \in \mathbb{R}^N, t \in [0, T], \tag{39}$$

where $C > 0$ is a constant independent of T . In addition, if (21) holds then the solution is unique in $C(\bar{Q}) \cap \mathcal{E}_\mu(\bar{Q})$.

Proof of Theorem 2.3. We only give a sketch of proof since it is similar with the proof of Theorem 2.1.

1. *Construction of a continuous viscosity solution to a truncated equation.* Let $m \geq 1$, we first truncate the initial data as we did for f_m in the proof of Theorem 2.1 by considering

$$u_{0m}(x) = \min\{u_0(x) + \frac{1}{m}\phi(x), m\}. \tag{40}$$

Since $u_0 \in \mathcal{E}_\mu(\mathbb{R}^N)$, we get

$$|u_{0m}(x)| \leq C_m, \tag{41}$$

$$|u_{0m}(x) - u_{0m}(y)| \leq L_m|x - y|. \tag{42}$$

Moreover, u_{0m} still satisfies (11) with the constant $C_0 + \mu$ and $u_{0m} \rightarrow u_0$ locally uniformly in \mathbb{R}^N .

We then introduce the truncated evolution problem (2) with H_{mn} (respectively f_m) defined by (33) (respectively (31)) for $m, n \geq 1$ and with the initial data defined

by (40). The classical comparison principle (see Theorem 4.2) holds for bounded discontinuous viscosity sub and supersolutions of

$$u_t - \mathcal{F}(x, [u]) + \langle b(x), Du \rangle + H_{mn}(x, Du) = f_m(x) \quad \text{in } Q_T, \quad (43)$$

with the initial data $u_{mn}(x, 0) = u_{0m}(x)$.

Notice that $u_{mn}^\pm(x, t) = \pm(C_m + (C_m + C_H)t)$ are respectively a super and a subsolution of (43) satisfying the initial conditions

$$u_{mn}^-(x, 0) = -C_m \leq u_{0m}(x) \leq C_m = u_{mn}^+(x, 0).$$

Then by means of Perron's method, we obtain the existence and uniqueness of a bounded continuous viscosity solution u_{mn} of (43) such that $|u_{mn}| \leq \tilde{C}_{mT}$ independent of n . We refer to classical references [13] for the details.

2. *Convergence of the solution of the truncated equation to a continuous solution of (2).* Recall that H_{mn} satisfies (6) with constant C_H independent of m, n . Moreover, from (32) and (42) we have f_m and u_{0m} are L_m -lipschitz. Since either (12)-(17) or (14)-(18) with $\beta \in (1, 2)$ hold, then applying [10, Theorem 3.1] for bounded solution u_{mn} , we obtain that u_{mn} is K_m -lipschitz continuous, i.e.,

$$|u_{mn}(x, t) - u_{mn}(y, t)| \leq K_m |x - y| \quad \text{for all } x, y \in \mathbb{R}^N, t \in [0, T],$$

where K_m is independent of T . Therefore, the family $(u_{mn})_{n \geq 1}$ is uniformly equicontinuous and bounded in \bar{Q} . It follows that, up to some subsequence,

$$u_{mn} \rightarrow u_m \quad \text{as } n \rightarrow +\infty \text{ locally uniformly in } \bar{Q}.$$

By stability [1, 8, 13], u_m is a viscosity solution of (34) with H_m in place of H_{mn} .

Similarly H_m (respectively f_m) satisfies (6) (respectively (11)) with constants C_H and $C_f + \mu$, u_{0m} satisfies (11) with constant $C_0 + \mu$ independent of m . By applying [10, Theorem 3.1] again, we obtain that u_m satisfies (39) with C independent of m and T .

To apply Ascoli Theorem sending $m \rightarrow \infty$, we need some local bound for u_m . Therefore we need to use following Lemma whose proof is omitted here since it is an adaptation of the one of [19, Theorem 2.2], which is extended in the case of general local diffusion or nonlocal operator and sublinear Hamiltonian by some routine calculations.

Lemma 2.4. Let $T > 0$. For all $\epsilon \in (0, 1)$, there exists $M(\epsilon) > 0$ such that

$$|u_m(x, t)| \leq \epsilon \phi(x) + M(\epsilon)(1 + |x| + t) \quad \text{for all } (x, t) \in \bar{Q}_T, \quad (44)$$

where $M(\epsilon)$ is independent of m and T . In particular, for all $R > 0$, there exists a constant $C_{RT} > 0$ independent of m such that

$$|u_m(x, t)| \leq C_{RT}, \quad \text{for all } x \in B(0, R), t \in [0, T],$$

and $u_m \in \mathcal{E}_\mu(\bar{Q}_T)$.

From Lemma 2.4, the family $(u_m)_{m \geq 1}$ is uniformly equicontinuous and bounded on compact subsets of \bar{Q}_T . By Ascoli Theorem, it follows that, up to some subsequence, $u_m \rightarrow u_T$ as $m \rightarrow +\infty$ locally uniformly in \bar{Q}_T . By stability, u_T is a continuous viscosity solution of (2) in \bar{Q}_T . Notice that u_T still satisfies (39) with C independent of T and (44). It is now easy to use a diagonal process to build a solution u of (2) in Q which also satisfies (39) and (44). In particular u is in $\mathcal{E}_\mu(\bar{Q}_T)$ for all $T > 0$ so is in $\mathcal{E}_\mu(\bar{Q})$. It ends the proof of existence.

3. *Uniqueness of the solution of (2) in the class $C(\overline{Q}) \cap \mathcal{E}_\mu(\overline{Q})$.* It is a direct consequence of the comparison principle (see Proposition 2) if we assume in addition (21) holds. \square

2.3. Well-posedness of the stationary and evolution equation by using classical techniques. The following results hold for possibly degenerate stationary and evolution equation in both local and nonlocal case.

2.3.1. *Results for the stationary problem.*

Theorem 2.5. Let $u \in USC(\mathbb{R}^N) \cap \mathcal{E}_\mu^+(\mathbb{R}^N)$ and $v \in LSC(\mathbb{R}^N) \cap \mathcal{E}_\mu^-(\mathbb{R}^N)$ be a viscosity sub and supersolution of (1), respectively. Suppose that (5) (11), (22) and either (12) or (14) hold. Then there is a unique solution $u^\lambda \in C(\mathbb{R}^N) \cap \mathcal{E}_\mu(\mathbb{R}^N)$ of (1).

Corollary 1. Under the assumptions of Theorem 2.5 with (22) is replaced by (23). Then for any $\alpha > 2C_H$, there is a unique solution $u^\lambda \in C(\mathbb{R}^N) \cap \mathcal{E}_\mu(\mathbb{R}^N)$ of (1).

In the above results, we do not assume anymore the equation is nondegenerate but we need to use stronger assumptions on H , which are the classical assumptions required in viscosity solutions (see [13, 3] and references therein). The key point of proof is to apply [10, Lemma 2.1] in order to build a sub and supersolution for (1), and the ideas are then based on [19], so we omit here. The restrictive on the strength α of the Ornstein-Uhlenbeck operator in the Corollary is to guarantee that [10, Lemma 2.1] holds when dealing with (23).

The same results hold true for the evolution equation.

2.3.2. *Results for the evolution problem.*

Theorem 2.6. Let $u \in USC(\overline{Q}_T) \cap \mathcal{E}_\mu^+(\overline{Q}_T)$ and $v \in LSC(\overline{Q}_T) \cap \mathcal{E}_\mu^-(\overline{Q}_T)$ be a viscosity sub and supersolution of (2), respectively. Suppose that (5), (11), (22) and either (12) or (14) hold. Assume that $u(x, 0) \leq v(x, 0)$ for all $x \in \mathbb{R}^N$, then there is a unique solution $u \in C(\overline{Q}_T) \cap \mathcal{E}_\mu(\overline{Q}_T)$ of (2).

Corollary 2. Under the assumptions of Theorem 2.6 with (23) in place of (22). Then for any $\alpha > 2C_H$, there is a unique solution $u \in C(\overline{Q}_T) \cap \mathcal{E}_\mu(\overline{Q}_T)$ of (2).

2.4. Regularity results with respect to time for the evolution problem.

The next lemma gives some time regularity estimates of a solution for which the space regularity is known. This is well known for local equations but does not seem to be written for nonlocal ones. We provide a general statement and a proof for the nonlocal case by adaptating the arguments of [4, Lemma 9.1].

Lemma 2.7. Let $R > 0$, $0 \leq t_0 < T$, $x_0 \in \mathbb{R}^N$, set $\Omega_{x_0, t_0, R, T} := B(x_0, R) \times (t_0, T)$ and consider a viscosity solution $u \in C(\overline{\Omega}_{x_0, t_0, R+1, T}) \cap \mathcal{E}_\mu(\overline{Q})$ of

$$u_t - \mathcal{F}(x, [u]) + \langle b(x), Du \rangle + H(x, Du) = f(x), \quad (x, t) \in \Omega_{x_0, t_0, R+1, T}, \quad (45)$$

where b, H are continuous and \mathcal{F} satisfies either (12) (local case) or (14) (nonlocal case). If

$$|u(y, t_0) - u(x, t_0)| \leq m(|y - x|) \quad \text{for } x, y \in \overline{B}(x_0, R+1), \quad (46)$$

for some modulus of continuity m , then there exists a modulus of continuity \tilde{m} depending only on $m, |u|_{L^\infty(\Omega_{x_0, t_0, R+1, T})}, b, H, \mu$ and σ or ν such that

$$|u(x, t) - u(x, t_0)| \leq \tilde{m}(|t - t_0|) \quad \text{for } x \in B(x_0, \frac{R}{2}), t \in [t_0, T]. \quad (47)$$

If $m(r) = Lr$, then $\tilde{m}(r) = \tilde{L}\sqrt{r}$, where \tilde{L} depends on $L, |u|_{L^\infty(\Omega_{x_0, t_0, R+1, T})}, b, H, \mu$ and σ or ν .

Remark 1. Notice that in our framework, (46) holds true for $m(r) = Lr$, see (16).

Proof of Lemma 2.7. We fix $\eta > 0$ and we want to find some constants $C, K > 0$ depending only on $m, |u|_{L^\infty(\Omega_{x_0, t_0, R+1, T})}, b, H, \sigma$ or ν and μ such that, for any $x \in B(x_0, R/2)$ and every $(y, t) \in \bar{\Omega}_{x_0, t_0, R+1, T}$, we have

$$-\eta - C|y - x|^2 - K(t - t_0) \leq u(y, t) - u(x, t_0) \leq \eta + C|y - x|^2 + K(t - t_0). \quad (48)$$

We prove only the second inequality, the first one being proved in a similar way. Let us fix $x \in B(x_0, R/2)$ and consider (y, t) as the running variable in the following.

At first, if we choose

$$C > \frac{8|u|_{L^\infty(\Omega_{x_0, t_0, R+1, T})}}{R^2}, \quad (49)$$

the desired inequality is fulfilled on $(\bar{B}(x_0, R+1) \setminus B(x_0, R)) \times [t_0, T]$ for every $\eta, K > 0$. Indeed, $|y - x| > R/2$ in this region. Notice that C is chosen independent of $x \in B(x_0, R/2)$.

Next, we want to ensure that the inequality holds on $\bar{B}(x_0, R+1) \times \{t_0\}$. We argue by contradiction assuming that there exists $\eta > 0$ such that, for every $C > 0$, there exists $y_C \in \bar{B}(x_0, R+1)$ such that

$$u(y_C, t_0) - u(x, t_0) > \eta + C|y_C - x|^2. \quad (50)$$

It follows that

$$|y_C - x| \leq \sqrt{\frac{2|u|_{L^\infty(\Omega_{x_0, t_0, R+1, T})}}{C}}.$$

Thus $|y_C - x| \rightarrow 0$ as $C \rightarrow +\infty$. Coming back to (50) and using (46), we infer

$$m(|y_C - x|) \geq u(y_C, t_0) - u(x, t_0) \geq \eta.$$

We obtain a contradiction if C is large enough since the left-hand side tends to 0 as $C \rightarrow +\infty$. Notice that the choice of C to obtain the inequality on $\bar{B}(x_0, R+1) \times \{t_0\}$ depends only on $\eta, |u|_{L^\infty(\Omega_{x_0, t_0, R+1, T})}$ and m .

Therefore, by choosing C large enough, the desired inequality holds on

$$((\bar{B}(x_0, R+1) \setminus B(x_0, R)) \times [t_0, T]) \cup (\bar{B}(x_0, R+1) \times \{t_0\}). \quad (51)$$

We then consider

$$\max_{\bar{\Omega}_{x_0, t_0, R+1, T}} \{u - \chi\} \quad \text{where } \chi(y, t) := u(x, t_0) + \eta + C|y - x|^2 + K(t - t_0). \quad (52)$$

If the maximum is nonpositive, then the desired inequality holds. Otherwise, the maximum is positive and, from (51), is achieved at an interior point (\bar{y}, \bar{t}) in $\Omega_{x_0, t_0, R, T}$. We can write the viscosity inequality for the subsolution u at this point using the smooth test-function χ . Since (\bar{y}, \bar{t}) is a maximum point of $u - \chi$ in $B(x_0, R) \times [t_0, T]$, we obtain (see [8, Definition 2])

$$\begin{aligned} & K - \int_B (\chi(\bar{y} + z, \bar{t}) - \chi(\bar{y}, \bar{t}) - \langle D\chi(\bar{y}, \bar{t}), z \rangle) \nu(dz) \\ & - \int_{B^c} (u(\bar{y} + z, \bar{t}) - u(\bar{y}, \bar{t})) \nu(dz) + \langle b(\bar{y}), D\chi(\bar{y}, \bar{t}) \rangle + H(\bar{y}, D\chi(\bar{y}, \bar{t})) \leq f(\bar{y}). \end{aligned} \quad (53)$$

We estimate the terms in the inequality using that $D\chi(y, t) = 2C(y-x)$, $D^2\chi(y, t) = 2CI$ and $|\bar{y} - x| \leq 2R$. We have

$$\begin{aligned} & |\langle b(\bar{y}), D\chi(\bar{y}, \bar{t}) \rangle + H(\bar{y}, D\chi(\bar{y}, \bar{t})) - f(\bar{y})| \\ & \leq \max_{y \in \bar{B}(x_0, R)} \{ |b(y)| |D\chi|_{L^\infty(\Omega_{x_0, t_0, R, T})} + |f(y)| + \max_{|\xi| \leq |D\chi|_{L^\infty(\Omega_{x_0, t_0, R, T})}} |H(y, \xi)| \} \\ & \leq \max_{y \in \bar{B}(x_0, R)} \{ 4CR|b(y)| + |f(y)| + \max_{|\xi| \leq 4CR} |H(y, \xi)| \}, \end{aligned}$$

and, using (14),

$$\begin{aligned} & \left| \int_B (\chi(\bar{y} + z, \bar{t}) - \chi(\bar{y}, \bar{t}) - \langle D\chi(\bar{y}, \bar{t}), z \rangle) \nu(dz) \right| \\ & = \left| \frac{1}{2} \int_B \int_0^1 \langle D^2\chi(\bar{y} + \theta z, \bar{t}) z, z \rangle d\theta \nu(dz) \right| \\ & \leq \frac{1}{2} |D^2\chi|_{L^\infty(\Omega_{x_0, t_0, R, T})} \int_B |z|^2 \nu(dz) \leq CC_\nu^1. \end{aligned}$$

Since $u \in \mathcal{E}_\mu(\bar{Q}) \subset \mathcal{E}_\mu(\bar{Q}_T)$, by (24) for $\epsilon = 1$, we have

$$|u(y, t)| \leq \phi_\mu(y) + M(1) = \phi_\mu(y) + M_T \quad \text{for all } y \in B(x_0, R), t \in [t_0, T]$$

for some constant M_T depending on T . It follows, using (14) again,

$$\begin{aligned} \left| \int_{B^c} (u(\bar{y} + z, \bar{t}) - u(\bar{y}, \bar{t})) \nu(dz) \right| & \leq \int_{B^c} (\phi_\mu(\bar{y} + z) + \phi_\mu(\bar{y}) + 2M_T) \nu(dz) \\ & \leq 2 \left(\max_{\bar{B}(x_0, R)} \phi_\mu + M_T \right) C_\nu^1. \end{aligned}$$

It follows that, if $K > 0$ is chosen such that

$$K > \max_{y \in \bar{B}(x_0, R)} \left\{ 4CR|b(y)| + |f(y)| + \max_{|\xi| \leq 4CR} |H(y, \xi)| + (C + 2M_T + 2\phi_\mu(y))C_\nu^1 \right\}, \quad (54)$$

then χ is a strict supersolution of (45) in $\Omega_{x_0, t_0, R, T}$ and (53) does not hold. Therefore, (52) is nonpositive and the desired inequality holds. Notice that K depends on x_0, t_0, R, T , the datas and $\eta, m, |u|_{L^\infty(\Omega_{x_0, t_0, R+1, T})}$ through the constant C .

By (48), we obtain that for every $\eta > 0$,

$$|u(x, t) - u(x, t_0)| \leq \eta + K(\eta)(t - t_0) \quad \text{for every } x \in B(x_0, \frac{R}{2}), t \in [t_0, T], \quad (55)$$

where we emphasize the dependence of K with respect to η . It is standard that by optimizing this estimate with respect to η we obtain a modulus of continuity, but let us do it for the sake of clarity. In order to solve $\eta = K(\eta)|t - t_0|$, we define $g : (0, \infty) \rightarrow (0, \infty)$ as the inverse function of $s \mapsto s/K(s)$. Notice that since $\eta \mapsto K(\eta)$ can be chosen as continuous, decreasing and such that $K(\eta) \rightarrow \infty$ as $\eta \rightarrow 0$, the function g is continuous on $(0, +\infty)$, increasing and such that $g(0^+) = 0$ (in other words, g is a modulus of continuity).

Now, choosing the specific value of $\eta := g(|t - t_0|)$ yields

$$|u(x, t) - u(x, t_0)| \leq 2g(|t - t_0|) \quad \text{for every } x \in B(x_0, \frac{R}{2}), t \in [t_0, T],$$

and this yields (47) with $\tilde{m} := 2g$ which is also modulus of continuity.

Now, assume that $m(r) = Lr$. Looking at the above proof, we notice on the one side that, since

$$|u(x, t_0) - u(y, t_0)| \leq L|x - y| \leq \eta + \frac{L^2}{4\eta}|x - y|^2,$$

the desired inequality (48) holds on $\overline{B}(x_0, R + 1) \times \{t_0\}$ if $C \geq \frac{L^2}{4\eta}$. Therefore (48) holds providing C satisfies the latter inequality and (49). On the other side, we see that

$$|D\chi(\bar{y}, \bar{t})| \leq |Du|_{L^\infty(\Omega_{x_0, t_0, R+1, T})} \leq L$$

coming back to (54), we see that it is enough to choose K such that

$$K > A_1 C + A_2 + 2M_T C_\nu^1,$$

where A_1, A_2 depends only on the datas, x_0, R and L . Choosing C and K as above, (55) then reads, for every $\eta > 0$ and $x \in B(x_0, \frac{R}{2}), t \in [t_0, T]$,

$$|u(x, t) - u(x, t_0)| \leq \eta + \left(A_1 \left(\frac{8|u|_{L^\infty(\Omega_{x_0, t_0, R+1, T})}}{R^2} + \frac{L^2}{4\eta} \right) + A_2 + 2M_T C_\nu^1 \right) |t - t_0|.$$

Minimizing the right-hand side with respect to $\eta > 0$, we get the conclusion. \square

3. Application to ergodic problem and long time behavior of solutions.

In this Section we will use some uniform estimates (16) obtained by [10] to solve (4) and then study the convergence (3). The idea comes back to the seminal work of Lions-Papanicolau-Varadhan [27]. Let u^λ be a solution of (1) satisfying (16) with constant independent of λ , we consider $w^\lambda(x) = u^\lambda(x) - u^\lambda(0)$ and aim at sending λ to 0. The family $(w^\lambda)_{\lambda \in (0, 1)}$ still satisfies (16). It is locally bounded since, by (16), we have $|w^\lambda(x)| \leq C(\phi_\mu(x) + \phi_\mu(0))|x|$ so, in this unbounded case, w^λ does not belong anymore to $\mathcal{E}_\mu(\mathbb{R}^N)$ but to a slightly bigger class. We therefore need to take a safety margin for the growth condition in the nonlocal case.

This create an additional difficulty in the nonlocal case. More precisely, from now on, we fix $\bar{\mu} > \mu > 0$ and we assume that

$$\text{the measure } \nu \text{ in (13) satisfies (14) with } \bar{\mu}. \quad (56)$$

Notice that the nonlocal operator \mathcal{I} given by (13) is well-defined for all function in $\mathcal{E}_\gamma, \gamma \leq \bar{\mu}$.

3.1. Application to ergodic problem.

Theorem 3.1. Under the assumptions of Theorem 2.1 (assuming in addition (56) in the nonlocal case), there exists a solution $(c, v) \in \mathbb{R} \times C(\mathbb{R}^N)$ of (4) such that

$$v \in \bigcap_{\mu < \gamma < \bar{\mu}} \mathcal{E}_\gamma(\mathbb{R}^N). \quad (57)$$

Proof of Theorem 3.1. Let $u^\lambda \in C(\mathbb{R}^N) \cap \mathcal{E}_\mu(\mathbb{R}^N)$, $\lambda \in (0, 1)$, be a solution of (1) given by Theorem 2.1. Define $w^\lambda, z^\lambda \in C(\mathbb{R}^N)$ by $w^\lambda(x) := u^\lambda(x) - u^\lambda(0)$ and $z^\lambda(x) := \lambda u^\lambda(x)$, respectively. Then in view of (30) and Lemma 2.2, there are

constant $C, C(1) > 0$ independent of λ such that, for all $x, y \in \mathbb{R}^N$,

$$\begin{aligned} |z^\lambda(0)| &\leq \phi_\mu(0) + C(1), \\ |z^\lambda(x) - z^\lambda(0)| &= |\lambda u^\lambda(x) - \lambda u^\lambda(0)| \leq C|x|(\phi_\mu(x) + \phi_\mu(0)), \\ |w^\lambda(x)| &\leq C|x|(\phi_\mu(x) + \phi_\mu(0)), \\ |w^\lambda(x) - w^\lambda(y)| &\leq C|x - y|(\phi_\mu(x) + \phi_\mu(y)). \end{aligned} \tag{58}$$

Therefore, $\{w^\lambda\}_{\lambda \in (0,1)}$ is a uniformly bounded and equi-continuous family on any balls of \mathbb{R}^N . By Ascoli's theorem, up to subsequences, we obtain

$$z^\lambda \rightarrow c, \quad w^\lambda \rightarrow v, \quad \text{locally uniformly in } \mathbb{R}^N \text{ as } \lambda \rightarrow 0,$$

for some $c \in \mathbb{R}$ and $v \in C(\mathbb{R}^N)$. By the stability of viscosity solutions (see [1, 8, 13]), we find that v satisfies (4) in the viscosity sense. Let $\mu < \gamma < \bar{\mu}$. Since

$$\lim_{|x| \rightarrow \infty} \frac{|x|\phi_\mu(x)}{\phi_\gamma(x)} = 0,$$

we see from (58) that $v \in \mathcal{E}_\gamma(\mathbb{R}^N)$. \square

To prove the uniqueness of the ergodic constant and the solution up to additive constants in (4), we need to linearize the equation in order to apply the strong maximum principle. To do that, we need to assume that (21) and (22) hold.

Theorem 3.2. Under the assumptions of Theorem 2.1 (assuming in addition (56) in the nonlocal case), let $(c, v_1), (d, v_2) \in \mathbb{R} \times (C(\mathbb{R}^N) \cap \mathcal{E}_\gamma(\mathbb{R}^N))$ with $\mu < \gamma < \bar{\mu}$ be respectively a subsolution and a supersolution of (4).

- (i) If (21) holds, then $c \leq d$;
- (ii) If (22) holds and $c = d$, then there is a constant $C \in \mathbb{R}$ such that $v_1 - v_2 = C$ in \mathbb{R}^N .

Proof of Theorem 3.2.

- (i) We argue by contradiction assuming that $c > d$ and choose $\epsilon > 0$ small enough so that

$$2\epsilon K_\gamma < c - d,$$

where $K = K_\gamma$ appearing in (20). Since $(c, v_1), (d, v_2)$ are sub- and supersolutions of (4), we can easily verify that $\tilde{v}_1(x, t) = v_1(x) - \epsilon\phi_\gamma(x) + ct$ is viscosity subsolution of

$$v_t - \mathcal{F}(x, [v]) + \langle b(x), Dv(x, t) \rangle + H(x, Dv(x, t)) = f_1(x) \quad \text{in } Q_T = \mathbb{R}^N \times (0, T)$$

and $\tilde{v}_2(x, t) = v_2(x) + \epsilon\phi_\gamma(x) + dt$ is viscosity supersolution of

$$v_t - \mathcal{F}(x, [v]) + \langle b(x), Dv(x, t) \rangle + H(x, Dv(x, t)) = f_2(x) \quad \text{in } Q_T,$$

where $f_1(x) = f(x) - \epsilon\phi_\gamma(x) + \epsilon K_\gamma$, $f_2(x) = f(x) + \epsilon\phi_\gamma(x) - \epsilon K_\gamma$. Since (21) holds, we can apply Proposition 2 for \tilde{v}_1 and \tilde{v}_2 to obtain that, for all $(x, t) \in Q_T$,

$$\begin{aligned} v_1(x) - v_2(x) - 2\epsilon\phi_\gamma(x) + (c - d)t \\ \leq \sup_{\mathbb{R}^N} \{v_1(y) - v_2(y) - 2\epsilon\phi_\gamma(y)\} + t|(2\epsilon K_\gamma - 2\epsilon\phi_\gamma)^+|_\infty. \end{aligned}$$

Taking x as close as we want to where the sup is achieved, this implies that

$$(c - d)t \leq 2\epsilon K_\gamma t, \quad \text{for all } t > 0,$$

which is a contradiction. Thus $c \leq d$.

- (ii) For the proof of the second statement, we use the following Lemma, the proof of which is classical and given in the Appendix:

Lemma 3.3. Under the assumptions of Theorem 3.2 (ii), the function $\omega = v_1 - v_2$ is a continuous viscosity subsolution of

$$-\mathcal{F}(x, [\omega]) + \langle b(x), D\omega(x) \rangle - L_H |D\omega| = 0. \quad (59)$$

Now let $\epsilon > 0$. Since $\omega = v_1 - v_2 \in \mathcal{E}_\gamma(\mathbb{R}^N)$, $\omega - \epsilon\phi_\gamma$ attains a maximum at some $x_\epsilon \in \mathbb{R}^N$. From Lemma 3.3, using $\epsilon\phi_\gamma$ as a test function for w we have

$$-\mathcal{F}(x_\epsilon, [\epsilon\phi_\gamma]) + \langle b(x_\epsilon), D\epsilon\phi_\gamma(x_\epsilon) \rangle - L_H |D\epsilon\phi_\gamma(x_\epsilon)| \leq 0. \quad (60)$$

Recall from (20) that there is a constant $K_\gamma > 0$ such that

$$-\mathcal{F}(x, [\phi_\gamma]) + \langle b(x), D\phi_\gamma(x) \rangle - L_H |D\phi_\gamma(x)| \geq \phi_\gamma(x) - K_\gamma \quad \text{for } x \in \mathbb{R}^N.$$

Therefore, there is $R_\gamma > 0$ independent of ϵ such that, for $x \in \mathbb{R}^N \setminus B(0, R_\gamma)$,

$$-\mathcal{F}(x, [\epsilon\phi_\gamma]) + \langle b(x), D\epsilon\phi_\gamma(x) \rangle - L_H |D\epsilon\phi_\gamma(x)| \geq \epsilon(\phi_\gamma(x) - K_\gamma) > 0. \quad (61)$$

From (60) and (61) we deduce that $\omega - \epsilon\phi_\gamma$ can only attain a maximum at $x_\epsilon \in B(0, R_\gamma)$. Then we argue as [19, Theorem 4.5] based on the strong maximum principle (see [2, 14] in the local case and [12, 11] in the nonlocal one) to get that ω is constant in \mathbb{R}^N . \square

3.2. Application to long time behavior of solutions. We study the long time behavior of solutions of (2) in the non-degenerate case.

Theorem 3.4. Let $\mu > 0$. Suppose (5), (6), (22) and that $f, u_0 \in \mathcal{E}_\mu(\mathbb{R}^N) \cap C(\mathbb{R}^N)$ satisfying (11). Assume either (12)-(17) (local case) or (14)-(18)-(56) with $\beta \in (1, 2)$ and $\bar{\mu} > \mu$ (nonlocal case). Let $u \in \mathcal{E}_\mu(\bar{Q}) \cap C(\bar{Q})$ be the unique solution of (2) and $(c, v) \in \mathbb{R} \times (C(\mathbb{R}^N) \cap \mathcal{E}_\gamma(\mathbb{R}^N))$ a solution of (4) for some $\mu < \gamma \leq \bar{\mu}$. Then there is a constant $a \in \mathbb{R}$ such that

$$\lim_{t \rightarrow \infty} \max_{B(0, R)} |u(x, t) - (ct + v(x) + a)| = 0 \quad \text{for all } R > 0. \quad (62)$$

Notice that, under our assumptions, Theorems 2.1, 2.3, 3.1 and 3.2 hold.

Before giving the proof, let us state some preliminaries. The key ingredient is the Lipschitz estimates (16) obtained in [10]. Then, the proof of Theorem 3.4 is quite close to the one of [19, Theorem 5.1]. We follow its lines but there are changes, first because the equation may be nonlocal, and second because we do not work with C^2 -smooth solutions.

At first, up to replace $f(x)$ by $f(x) - c$ (which still satisfies (11)) and the solution $u(x, t)$ by $u(x, t) - ct$, we may assume, without loss of generality, that $c = 0$.

In what follows we introduce the function

$$w(x, t) = u(x, t) - v(x) \quad \text{on } \bar{Q}. \quad (63)$$

Since $c = 0$, v is a viscosity solution of

$$-\mathcal{F}(x, [v]) + \langle b(x), Dv(x) \rangle + H(x, Dv(x)) = f(x) \quad \text{in } \mathbb{R}^N \quad (64)$$

and u is the viscosity solution of

$$u_t - \mathcal{F}(x, [u]) + \langle b(x), Du(x, t) \rangle + H(x, Du(x, t)) = f(x) \quad \text{in } Q,$$

then, by Lemma 3.3 (actually we use the parabolic version of this Lemma, which is obtained by straightforward adaptations in its proof), w is a viscosity subsolution of

$$\mathcal{P}[w](x, t) := w_t - \mathcal{F}(x, [w]) + \langle b(x), Dw \rangle - L_H |Dw| = 0 \quad \text{in } Q. \quad (65)$$

Thanks to (20) (with γ instead of μ), there exists $K = K(\gamma, L_H)$ such that

$$-\mathcal{F}(x, [\phi_\gamma]) + \langle b(x), D\phi_\gamma(x) \rangle - L_H |D\phi_\gamma(x)| \geq \phi_\gamma(x) - K \quad \text{in } \mathbb{R}^N.$$

Therefore

$$\varphi(x, t) := (\phi_\gamma(x) - K)e^{-t} \quad (66)$$

is a smooth supersolution of

$$\mathcal{P}[\varphi](x, t) \geq 0 \quad \text{in } Q. \quad (67)$$

We divide the proof of Theorem 3.4 into several lemmas. The following lemma gives some boundedness of w with respect to t (recall that $c = 0$).

Lemma 3.5. Under the assumptions of Theorem 3.4, for every $0 < \epsilon < 1$, there exists $C(\epsilon) > 0$ such that

$$|w(x, t)| \leq \epsilon \phi_\gamma(x) + C(\epsilon), \quad (x, t) \in \overline{Q}. \quad (68)$$

We refer to [19, Lemma 5.3] for the proof of this lemma.

Lemma 3.6. Under the assumptions of Theorem 3.4, for every $R > 0$, there exists $L_R > 0$ (independent of t) such that

$$|u(x, t) - u(x, s)| \leq L_R \sqrt{|t - s|} \quad \text{for all } x \in B(0, R), t, s \in [0, +\infty). \quad (69)$$

Proof of Lemma 3.6. It is a direct consequence of Lemma 2.7. Indeed, take $x_0 = 0$, $t_0 = 0$ and $\Omega_{0,0,2R+1,T} = B(0, 2R+1) \times (0, T)$ in Lemma 2.7. By Lemma 3.5, $|u|_{L^\infty(\Omega_{0,0,2R+1,T})}$ depends only on R (but not on T). Notice also that M_T which appears in the proof of Lemma 2.7 can be chosen independent of T thanks to Lemma 3.5. The conclusion follows. Indeed, Lemma 3.5 implies the following time-independent bound $|u(x, t)| \leq |v(x)| + \epsilon \phi_\gamma(x) + C(\epsilon)$. \square

Lemma 3.7. Under the assumptions of Theorem 3.4, the sets $\{u(\cdot, t) : t \geq 0\}$ and $\{u(\cdot, \cdot + t) : t \geq 0\}$ are precompact in $C(\mathbb{R}^N)$ and $C(\overline{Q})$, respectively.

Proof of Lemma 3.7. The proof is a straightforward consequence of the boundedness and equicontinuity of both families on bounded subsets of \mathbb{R}^N . These properties follow from Theorem 2.3 and Lemmas 3.5 and 3.6. \square

We introduce the half-relaxed limits (see [13, 8])

$$\bar{u}(x) = \limsup_{t \rightarrow \infty}^* u(x, t), \quad \underline{u}(x) = \liminf_{t \rightarrow \infty}^* u(x, t).$$

Lemma 3.8. Under the assumptions of Theorem 3.4, there exist a solution $v \in C(\mathbb{R}^N) \cap \mathcal{E}_\gamma(\mathbb{R}^N)$ of (64) satisfying (16) and $\bar{C}, \underline{C} \in \mathbb{R}$ such that

$$\bar{u} + \bar{C} = \underline{u} + \underline{C} = v. \quad (70)$$

Proof of Lemma 3.8. By Theorem 2.3 and Lemma 3.5, we obtain easily that \bar{u} and \underline{u} are well-defined, belong to $\mathcal{E}_\gamma(\mathbb{R}^N)$, satisfy the Lipschitz estimates (16). By classical stability results ([13, 8]), \bar{u} is a viscosity subsolution and \underline{u} a viscosity supersolution of (64). By Theorem 3.1 (under our assumptions which leads to $c = 0$), there exists a solution $(0, v)$ of (4). The existence of \bar{C}, \underline{C} such that (70) holds follows directly from Theorem 3.2 (ii). \square

Now, we are ready to prove Theorem 3.4.

Proof of Theorem 3.4. To prove the convergence (62), in view of Lemma 3.7, it is sufficient to prove that $\overline{C} = \underline{C}$ in Lemma 3.8. Since $\overline{u} \geq \underline{u}$, we have $\overline{C} \leq \underline{C}$ and it remains to establish $\overline{C} \geq \underline{C}$.

We claim that there exists $u_\infty \geq \underline{u}$ in the ω -limit set

$$\Omega(u) = \{\omega \in C(\overline{Q}) : \text{there exists } t_j \rightarrow +\infty \text{ such that } u(\cdot, \cdot + t_j) \rightarrow \omega \text{ in } C(\overline{Q})\}$$

such that

$$u_\infty(0, 1) = \underline{u}(0). \quad (71)$$

Indeed, by (16) for u , we have

$$\underline{u}(x) = \liminf_{t \rightarrow +\infty} u(x, t), \quad (72)$$

hence, there exists $t_j \rightarrow +\infty$ such that $u(0, t_j) \rightarrow \underline{u}(0)$. Therefore, using Lemma 3.7 again, there exists a subsequence (still denoted (t_j)) and $u_\infty \in C(\overline{Q})$ such that $u(\cdot, \cdot + t_j - 1) \rightarrow u_\infty$ in $C(\overline{Q})$. It is clear that $\underline{u} \leq u_\infty \in \Omega(u)$ and, since $u(0, 1 + t_j - 1) = u(0, t_j) \rightarrow \underline{u}(0)$, we get (71) and the claim is proved.

Now, we prove that there is a sequence $s_j \rightarrow +\infty$ such that

$$u(\cdot, s_j) \rightarrow \underline{u} \quad \text{in } C(\mathbb{R}^N) \quad \text{as } j \rightarrow \infty. \quad (73)$$

From the previous claim, the function $\zeta \in C(\overline{Q})$ defined by $\zeta(x, t) = \underline{u}(x) - u_\infty(x, t)$ attains a maximum over \overline{Q} at the point $(0, 1)$. Moreover, \underline{u} is a viscosity solution (so subsolution) of (64) and, by stability, u_∞ is a viscosity solution (so supersolution) of (2). Thanks to Lemma 3.3, we get ζ is a viscosity subsolution of (65). By applying the strong maximum principle to ζ (adaptating the proof of Theorem 3.2 to the case of parabolic equations), we find that ζ is constant in \overline{Q} . Since $\zeta(0, 1) = 0$, we obtain $\underline{u}(x) = u_\infty(x, t)$ for all $(x, t) \in \overline{Q}$. But, by the definition of $\Omega(u)$, there is a sequence $s_j \rightarrow +\infty$ such that $u(\cdot, \cdot + s_j) \rightarrow u_\infty$ in $C(\overline{Q})$. This shows (73).

For $j \in \mathbb{N}$ and $\epsilon > 0$, define

$$w_j(x, t) := u(x, t + s_j) - v(x) + \underline{C} = w(x, t + s_j) + \underline{C} = u(x, t + s_j) - \underline{u}(x),$$

where s_j is defined in (73) and we used (63) and (70) for the last two equalities. By Lemma 3.5,

$$w_j(x, 0) = w(x, s_j) + \underline{C} \leq \frac{\epsilon}{2} \phi_\gamma(x) + C\left(\frac{\epsilon}{2}\right) + \underline{C},$$

hence there exists $R = R_\epsilon > 0$ large enough such that

$$w_j(x, 0) \leq \epsilon(\phi_\gamma(x) - K) = \epsilon\varphi(x, 0) \quad \text{for } x \in \mathbb{R}^N \setminus B(0, R_\epsilon),$$

where φ is defined in (66). For x in the compact subset $\overline{B}(0, R_\epsilon)$, up to fix j big enough, by (73), we infer

$$w_j(x, 0) = u(x, s_j) - \underline{u}(x) \leq \epsilon \leq \epsilon\varphi(x, 0) + (K + 1)\epsilon.$$

Therefore

$$w_j(x, 0) \leq \epsilon\varphi(x, 0) + (K + 1)\epsilon \quad \text{for } x \in \mathbb{R}^N.$$

Since $w_j \in \mathcal{E}_\gamma(\overline{Q})$ is a subsolution and $\epsilon\varphi \in \mathcal{E}_\gamma(\overline{Q})$ is a supersolution of (65) in \overline{Q}_T for any $T > 0$, by the comparison principle of Proposition 2 in $\mathcal{E}_\gamma(\overline{Q}_T)$, we obtain

$$w_j(x, t) - \epsilon\varphi(x, t) \leq \sup_{\mathbb{R}^N} \{w_j(\cdot, 0) - \epsilon\varphi(\cdot, 0)\} \leq (K + 1)\epsilon \quad \text{for } (x, t) \in \overline{Q}_T.$$

Since this bound does not depend on $T > 0$, the previous inequality holds in \overline{Q} and it follows that

$$u(x, t + s_j) \leq v(x) - \underline{C} + \epsilon(\phi_\gamma(x) - K)e^{-t} + (K + 1)\epsilon$$

and therefore, using Lemma 3.8,

$$\limsup_{t \rightarrow \infty}^* u(x, t + s_j) = \overline{u}(x) = v(x) - \overline{C} \leq v(x) - \underline{C} + (K + 1)\epsilon.$$

Sending ϵ to 0, we get the desired inequality $\overline{C} \geq \underline{C}$. It ends the proof. \square

4. Appendix.

Theorem 4.1. Let $u \in USC(\mathbb{R}^N)$ and $v \in LSC(\mathbb{R}^N)$ be bounded viscosity sub and supersolution of (34), respectively. Assume that $f \in BUC(\mathbb{R}^N)$, $H \in BUC(\mathbb{R}^N \times \mathbb{R}^N)$, (5) and either (12) or (14) hold. Then $u \leq v$ in \mathbb{R}^N .

Theorem 4.2. Let $u \in USC(\overline{Q}_T)$ and $v \in LSC(\overline{Q}_T)$ be bounded viscosity sub and supersolution of (43), respectively. Assume that $f \in BUC(\mathbb{R}^N)$, $H \in BUC(\mathbb{R}^N \times \mathbb{R}^N)$, (5), either (12) or (14) hold and $u(x, 0) \leq v(x, 0)$, $x \in \mathbb{R}^N$. Then $u \leq v$ in \overline{Q}_T .

The proofs of these two above Theorems are classical and easily adapted by [19] in the bounded case using $H \in BUC(\mathbb{R}^N \times \mathbb{R}^N)$ and $f \in BUC(\mathbb{R}^N)$.

Proof of Lemma 3.3. Since the proof is classical, we only give it in the nonlocal case. The local one is the same, for this case, one can see for instance [15, Lemma 2.2].

We divide the proof in several steps.

Step 1. Viscosity inequalities for v_1 and v_2 . Let $\varphi \in C^2(\mathbb{R}^N)$ and $\bar{x} \in \mathbb{R}^N$ be a local maximum point of $\omega - \varphi$. We can assume that this maximum is strict in the same ball $\overline{B}(\bar{x}, R)$ for some $R > 0$. Let $\Theta(x, y) = \varphi(x) + \frac{|x-y|^2}{\epsilon^2}$ and consider

$$M_\epsilon := \max_{x, y \in \overline{B}(\bar{x}, R)} \{v_1(x) - v_2(y) - \Theta(x, y)\}.$$

This maximum is achieved at a point (x_ϵ, y_ϵ) and, since the maximum is strict, we know [3] that

$$x_\epsilon, y_\epsilon \rightarrow \bar{x}, \quad \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2} \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0 \quad (74)$$

$$M_\epsilon = v_1(x_\epsilon) - v_2(y_\epsilon) - \Theta(x_\epsilon, y_\epsilon) \rightarrow v_1(\bar{x}) - v_2(\bar{x}) - \varphi(\bar{x}) = \omega(\bar{x}) - \varphi(\bar{x}).$$

Setting $p_\epsilon = 2\frac{x_\epsilon - y_\epsilon}{\epsilon^2}$, we have

$$D_x \Theta(x_\epsilon, y_\epsilon) = p_\epsilon + D\varphi(x_\epsilon), \quad D_y \Theta(x_\epsilon, y_\epsilon) = -p_\epsilon. \quad (75)$$

Applying [8, Corollary 1], we write the viscosity inequalities for v_1 and v_2 at (x_ϵ, y_ϵ)

$$\begin{aligned} & -(\mathcal{I}(x_\epsilon, v_1, D_x \Theta) - \mathcal{I}(y_\epsilon, v_2, -D_y \Theta)) + \langle b(x_\epsilon), D_x \Theta \rangle - \langle b(y_\epsilon), -D_y \Theta \rangle \\ & + H(x_\epsilon, D_x \Theta) - H(y_\epsilon, -D_y \Theta) \leq f(x_\epsilon) - f(y_\epsilon). \end{aligned} \quad (76)$$

Step 2. Estimate of $\mathcal{T} := \mathcal{I}(x_\epsilon, v_1, D_x \Theta) - \mathcal{I}(y_\epsilon, v_2, -D_y \Theta)$. For each $\delta > 0$, we have

$$\begin{aligned} \mathcal{T} &= \mathcal{I}[B_\delta](x_\epsilon, \Theta, D_x \Theta) + \mathcal{I}[B^\delta](x_\epsilon, v_1, D_x \Theta) \\ &\quad - \mathcal{I}[B_\delta](y_\epsilon, \Theta, -D_y \Theta) - \mathcal{I}[B^\delta](y_\epsilon, v_2, -D_y \Theta). \end{aligned}$$

From (75), we first estimate

$$\begin{aligned} \mathcal{T}_1 &:= \mathcal{I}[B_\delta](x_\epsilon, \Theta, D_x \Theta) - \mathcal{I}[B_\delta](y_\epsilon, \Theta, -D_y \Theta) \\ &= \int_{B_\delta} \left\{ \varphi(x_\epsilon + z) - \varphi(x_\epsilon) + \frac{|x - y + z|^2 - |x - y - z|^2}{\epsilon^2} - \langle D\varphi(x_\epsilon), z \rangle \right\} \nu(dz) \\ &= \mathcal{I}[B_\delta](x_\epsilon, \varphi, D\varphi) + \frac{1}{\epsilon^2} o_\delta(1). \end{aligned} \quad (77)$$

On the other hand, at the maximum point (x_ϵ, y_ϵ) we have

$$v_1(x_\epsilon + z) - v_2(y_\epsilon + z) - (v_1(x_\epsilon) - v_2(y_\epsilon)) \leq \varphi(x_\epsilon + z) - \varphi(x_\epsilon),$$

for each $z \in B$. Hence, for each $0 < \delta < \kappa < 1$, using this inequality we obtain

$$\mathcal{T}_2 := \mathcal{I}[B^\delta](x_\epsilon, v_1, D_x \Theta) - \mathcal{I}[B^\delta](y_\epsilon, v_2, -D_y \Theta) \leq J^\kappa + \mathcal{I}[B_\kappa \setminus B_\delta](x_\epsilon, \varphi, D\varphi), \quad (78)$$

where

$$J^\kappa = \int_{B_\kappa^c} \left\{ v_1(x_\epsilon + z) - v_2(y_\epsilon + z) - (v_1(x_\epsilon) - v_2(y_\epsilon)) - \langle D\varphi(x_\epsilon), z \rangle \mathbb{1}_B(z) \right\} \nu(dz).$$

Therefore from (77) and (78), we conclude that for all $0 < \delta < \kappa < 1$

$$\mathcal{T} = \mathcal{T}_1 + \mathcal{T}_2 \leq J^\kappa + \mathcal{I}[B_\kappa](x_\epsilon, \varphi, D\varphi) + \frac{1}{\epsilon^2} o_\delta(1). \quad (79)$$

Since $v_1, v_2 \in \mathcal{E}_\gamma(\mathbb{R}^N)$, there exists $C > 0$ such that $|v_i(x)| \leq C\phi_\gamma(x)$, $\forall i = 1, 2$, $x \in \mathbb{R}^N$. Let $\gamma < \bar{\mu}$, thanks to (56) we have $\int_{B_\kappa^c} \phi_\gamma(z) \nu(dz) < +\infty$. Hence, applying Dominated convergence Theorem and using (74), we get, for each $\kappa > 0$ fixed,

$$\limsup_{\epsilon \rightarrow 0} J^\kappa \leq \mathcal{I}[B_\kappa^c](\bar{x}, \omega, D\varphi).$$

Therefore, letting $\delta \rightarrow$ and then $\epsilon \rightarrow 0$ in (79), using (74) we obtain

$$\limsup_{\epsilon \rightarrow 0} \mathcal{T} \leq \mathcal{I}(\bar{x}, \omega, D\varphi). \quad (80)$$

Step 3. Estimate of $\mathcal{B} := \langle b(x_\epsilon), D_x \Theta \rangle - \langle b(y_\epsilon), -D_y \Theta \rangle$. From (5) and (75) we have

$$\mathcal{B} = \langle b(x_\epsilon), p_\epsilon + D\varphi(x_\epsilon) \rangle - \langle b(y_\epsilon), p_\epsilon \rangle \geq 2\alpha \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2} + \langle b(x_\epsilon), D\varphi(x_\epsilon) \rangle. \quad (81)$$

Step 4. Estimate of $\mathcal{H} := H(x_\epsilon, D_x \Theta) - H(y_\epsilon, -D_y \Theta)$. From (22) and (75) we have

$$\mathcal{H} \geq -L_H |D\varphi(x_\epsilon)| - L_H |x_\epsilon - y_\epsilon| - 2L_H \frac{|x_\epsilon - y_\epsilon|^2}{\epsilon^2}. \quad (82)$$

Step 5. Estimate of $\mathcal{F} := f(y_\epsilon) - f(x_\epsilon)$. Since $f \in C(\mathbb{R}^N)$, hence we have

$$\mathcal{F} \leq o_\epsilon(1). \quad (83)$$

Step 6. Conclusion. Combining (80), (81), (82), (83) to (76) and sending $\epsilon \rightarrow 0$, we obtain

$$-\mathcal{F}(\bar{x}, [\omega]) + \langle b(\bar{x}), D\varphi(\bar{x}) \rangle - L_H |D\varphi(\bar{x})| \leq 0,$$

which means exactly that ω is a subsolution of (59). \square

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