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ON THE EIGENVALUES OF WEIGHTED DIRECTED GRAPHS

MARWA BALTI

ABSTRACT. We consider a non self-adjoint Laplacian on a directed graph with non symmetric edge weights. We analyse spectral properties of this Laplacian under a Kirchhoff's assumption. Moreover we establish isoperimetric inequalities in terms of the numerical range to show the lack of essential spectrum of Laplacian on *heavy ends* directed graphs. We introduce a special self-adjoint operator and compare its essential spectrum with that of the non self-adjoint Laplacian considered.

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INTRODUCTION

This article follows up on the ideas in [4] and [12] on monotonicity eigenvalues which are relative to continuous domains. The purpose of this work is to explore in the case of weighted directed graphs, some familiar facts of monotonicity proved on domains of \mathbb{R}^n and on compact Riemannian manifolds. Specially, our main question is: "can one study the behavior of the eigenvalues of a special operator under perturbations on finite graphs?" First, we consider a finite, directed and connected graph G with non symmetric edge weights. Then, we introduce the associated non symmetric Laplacian Δ_G . We introduce the self-adjoint operator $S_G = \Delta_G + \Delta_G^*$, such that it is easier to examine its spectrum

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thanks to selfadjointness. We give some spectral properties of Δ_G and we show that the real part of its eigenvalues can coincide with the eigenvalues of $\frac{1}{2}S_G$. Secondly, we study the monotonicity of eigenvalues relative to vertices or edges of G . We prove that the k^{th} eigenvalue λ_k of S_G is decreasing only in a class of graphs called here *flower-like-graphs*, and it is monotone increasing in the set of edges. These results are inspired by the classical results going back from M. Fiedler [6] and P. Kurasov, G. Malenova, S. Naboko [11] on the first nonzero eigenvalue of a simple graph. We extend these results for the higher eigenvalues λ_k of our special operator. In the second part of our work, we try to establish and improve the Proposition 2.1 of [3] for a Riemannian manifold M which gives upper bounds on the higher eigenvalues $\lambda_k(M)$ in terms of Dirichlet eigenvalues on components of a partition of M .

Let us briefly outline the contents of this article. We shall start with a short section of preliminaries consisting on some basic properties of the non symmetric Laplacian Δ_G on G and the associated Green formula. In section 3, we establish a generalization of some monotonicity eigenvalue results. Furthermore, we study the example of flower-like graphs and insist on the interest of the subgraph concept. Hence, we can remark that these considerations of graph help to give an upper bound of the eigenvalues of a simple tree. This section includes also similar Weyl and Cauchy theorems for the matrices [10]. In section 4, we are interested on the study of the eigenvalues of the Dirichlet Laplacian. We involve a comparison on the eigenvalues to use the decomposition of G into two components A and B and we give an upper bound on the eigenvalues of G in terms of Dirichlet eigenvalues of A and B .

1. PRELIMINARIES

We will review in this section some basic definitions and introduce the notation used in the article. They are introduced in [2] for the infinite graph.

1.1. Notion of Graphs. We call oriented or directed graph, the couple $G = (V, \vec{E})$, where V is a set of vertices, and $\vec{E} \subset V \times V$ is a set of directed edges. For two vertices x, y of V , we denote by (x, y) the edge that connects x to y , we also say that x and y are *neighbors*.

For all $x \in V$, we set:

- $E = \left\{ \{x, y\}, (x, y) \in \vec{E} \text{ or } (y, x) \in \vec{E} \right\}$
- $V_x^+ = \left\{ y \in V, (x, y) \in \vec{E} \right\}$
- $V_x^- = \left\{ y \in V, (y, x) \in \vec{E} \right\}$
- $V_x = V_x^+ \cup V_x^-$.

The valency on G is given by:

$$v(x) = \#V_x \quad \text{for all } x \in V.$$

We introduce some definitions given in [1], [2], [5], [14] for the case of symmetric graphs.

Definitions 1.1. • A path between two vertices x and y in V is a finite set of directed edges $(x_1, y_1); (x_2, y_2); \dots; (x_n, y_n)$, $n \geq 1$ such that

$$x_1 = x, y_n = y \text{ and } x_i = y_{i-1} \quad \forall 2 \leq i \leq n.$$

- $G = (V, \vec{E})$ is called *connected* if two vertices are always related by a path.
- $G = (V, \vec{E})$ is called *strongly connected* if there is for all vertices x, y a path from x to y and one from y to x .

Example 1.1. The cycle graph $C_n = \{0, 1, \dots, n-1\}$, with

$$(0, 1) \in \vec{E}, (1, 2) \in \vec{E}, \dots, (n-2, n-1) \in \vec{E}, (n-1, 0) \in \vec{E}$$

is strongly connected.

- Define for a finite subset Ω of V , the interior, the vertex boundary and the edge boundary of Ω respectively by:

$$\overset{\circ}{\Omega} = \{y \in \Omega, V_y \subset \Omega\}$$

$$\partial\Omega = \{y \in \Omega^c, y \in V_x \text{ for some } x \in \Omega\}$$

$$\partial_E\Omega = \{(x, y) \in \vec{E} : (x \in \Omega, y \in \Omega^c) \text{ or } (x \in \Omega^c, y \in \Omega)\}.$$

We remark that the strong connectedness of G assures that:

$$\forall x \in V, \#V_x^+ \neq 0 \text{ and } \#V_x^- \neq 0. \quad (1)$$

In this work we suppose that G is **finite, connected** and satisfies the **Hypothesis (1)**. In this work, we take the following definition.

Definition 1.1. Directed weighted Graph: A weighted graph (G, b) is the data of a graph (V, \vec{E}) and a weight $b : V \times V \rightarrow \mathbb{R}_+$ satisfying the following conditions:

- $b(x, x) = 0$ for all $x \in V$, (no loops in \vec{E})
- $b(x, y) > 0$ iff $(x, y) \in \vec{E}$
- Assumption (β) .

Assumption (β) : for all $x \in V$, $\beta^+(x) = \beta^-(x)$
where

$$\beta^+(x) = \sum_{y \in V_x^+} b(x, y) \text{ and } \beta^-(x) = \sum_{y \in V_x^-} b(y, x).$$

The weight β_G on a vertex $x \in V$ is given by:

$$\beta_G(x) = \beta^+(x) + \beta^-(x) = 2\beta^+(x).$$

Remark 1.1. The Assumption (β) is natural, it looks like the Kirchhoff's law in the electrical networks.

The weighted graph is *symmetric* if for all $x, y \in V$, $b(x, y) = b(y, x)$, as a consequence $(x, y) \in \vec{E} \Rightarrow (y, x) \in \vec{E}$ (the graph is symmetric).

In addition, we consider a weight m on V :

$$m : V \rightarrow \mathbf{R}_+^*.$$

1.2. Functional spaces. Let us introduce the following function spaces associated to the graph G :

$$\mathcal{C}_m(V) = \{f : V \rightarrow \mathbb{C}\}$$

endowed with the following inner product:

$$(f, g)_m = \sum_{x \in V} m(x) f(x) \overline{g(x)}.$$

We define its associated norm by:

$$\|f\|_m = \sqrt{(f, f)_m}.$$

A particular case called normalized is for $m = \beta^+$.
For a subset U of V , Let

$$\mathcal{C}_m(U) = \{f \in \mathcal{C}_m(V), f \text{ with support in } U\}.$$

The weights m and b are called simple if they are constant equal to 1 on V and E respectively. We denote by G^s the simple graph (with simple weights).

2. LAPLACIAN ON DIRECTED GRAPHS

For a weighted connected directed graph (G, b) , we introduce the combinatorial Laplacians:

Definitions 2.1. • We define the Laplacian Δ_G on $\mathcal{C}_m(V)$ by:

$$\Delta_G f(x) = \frac{1}{m(x)} \sum_{y \in V_x^+} b(x, y) (f(x) - f(y)).$$

- In particular, if for all $x \in V$, $\beta^+(x) = m(x)$, the Laplacian is said to be the normalized Laplacian and is defined on $\mathcal{C}_{\beta^+}(V)$ by:

$$\tilde{\Delta}_G f(x) = \frac{1}{\beta^+(x)} \sum_{y \in V_x^+} b(x, y) (f(x) - f(y)).$$

- For any operator A on $\mathcal{C}_m(V)$, the Dirichlet operator A_U^D , where U is a subset of V , is defined by:

$$f \text{ is with support in } U, \quad A_U^D(f) = A(f)|_U.$$

Thanks to Hypothesis (β) the adjoint of Δ has a simple expression.

Definition 2.1. Adjoint of an operator: The adjoint operator Δ^* of Δ is defined by:

$$\forall \phi, \psi \in \mathcal{C}_m(V), \quad (\Delta \psi, \phi) = (\psi, \Delta^* \phi).$$

Proposition 2.1. Let f be a function of $\mathcal{C}_m(V)$, we have

$$\Delta_G^* f(x) = \frac{1}{m(x)} \sum_{y \in V_x^-} b(y, x) (f(x) - f(y)).$$

Proof:

The following calculation for all $f, g \in \mathcal{C}_m(V)$ gives:

$$\begin{aligned}
(\Delta_G f, g)_m &= \sum_{(x,y) \in \vec{E}} b(x,y) (f(x) - f(y)) \overline{g(x)} \\
&= \sum_{x \in V} \overline{g(x)} f(x) \sum_{y \in V_x^+} b(x,y) - \sum_{(y,x) \in \vec{E}} b(y,x) \overline{g(y)} f(x) \\
&= \sum_{x \in V} \overline{g(x)} f(x) \sum_{y \in V_x^-} b(y,x) - \sum_{(y,x) \in \vec{E}} b(y,x) \overline{g(y)} f(x) \\
&= \sum_{x \in V} f(x) \sum_{y \in V_x^-} b(y,x) \overline{(g(x) - g(y))} \\
&= (f, \Delta_G^* g)_m.
\end{aligned}$$

□

The Green's formula is one of the main tools when we are working with the symmetric Laplace operator. In the following we establish it for the non symmetric Laplacian.

Lemma 2.1. Green's Formula. *Let f and g be two functions of $\mathcal{C}_m(V)$. Then*

$$(\Delta_G f, g)_m + (\Delta_G^* f, g)_m = \sum_{(x,y) \in \vec{E}} b(x,y) (f(x) - f(y)) \overline{(g(x) - g(y))}.$$

Proof:

The proof is a simple calculation:

$$\begin{aligned}
(\Delta_G f, g)_m + (\Delta_G^* f, g)_m &= \sum_{(x,y) \in \vec{E}} b(x,y) (f(x) - f(y)) \overline{g(x)} \\
&\quad + \sum_{(y,x) \in \vec{E}} b(y,x) (f(x) - f(y)) \overline{g(x)} \\
&= \sum_{(x,y) \in \vec{E}} b(x,y) \left(f(x) \overline{g(x)} + f(x) \overline{g(x)} - f(y) \overline{g(x)} - f(x) \overline{g(y)} \right) \\
&= \sum_{(x,y) \in \vec{E}} b(x,y) (f(x) - f(y)) \overline{(g(x) - g(y))}.
\end{aligned}$$

□

Definition 2.2. Special Laplacian. *We define a special Laplacian S_G as the sum of the two non self-adjoint Laplacians Δ_G and Δ_G^* , given by:*

$$\begin{aligned}
S_G f(x) &= (\Delta_G + \Delta_G^*) f(x) \\
&= \frac{1}{m(x)} \sum_{y \in V_x^+ \cup V_x^-} (b(x,y) + b(y,x)) (f(x) - f(y)) \\
&= \frac{1}{m(x)} \sum_{y \in V_x} a(x,y) (f(x) - f(y))
\end{aligned}$$

where $a(x,y) = b(x,y) + b(y,x)$ for any $x, y \in V$.

Remark 2.1. (1) S_G is a symmetric operator on $C_m(V)$, because $(\Delta_G^*)^* = \Delta_G$.

(2) S_G is a positive operator: for all $f \in C_m(V)$,

$$\begin{aligned} (S_G f, f) &= (\Delta_G f, f) + (\Delta_G^* f, f) \\ &= \sum_{(x,y) \in \vec{E}} b(x,y) |f(x) - f(y)|^2 \\ &\geq 0. \end{aligned}$$

In our discussion on the study of eigenvalues of a self-adjoint operator, it is natural to introduce the different characterization by variational principles [9].

2.1. Variational principles and Properties. Let A be a bounded from below self-adjoint operator. The eigenvalues of A can be characterized by three fundamental variational principles: the Rayleigh's principle, the Poincaré-Ritz max-min principle and the Courant-Fischer-Weyl principle applied to the Rayleigh quotients $\mathcal{R}(f) = \frac{(Af, f)}{(f, f)}$, $f \neq 0$.

Let us arrange the eigenvalues of A as

$$\lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_n$$

counted according to their multiplicities.

In this case we have :

(1) The *Rayleigh's principle* states:

$$\lambda_k = \min_{\substack{f \neq 0, (f, f_i) = 0 \\ i=1, \dots, k-1}} \mathcal{R}(f) \quad (2)$$

where f_i are eigenvectors corresponding to the eigenvalues λ_i and the minimum is reached at the eigenvector f_k .

(2) The *Poincaré-Ritz principle* establishes:

$$\lambda_k = \min_{\dim \Omega = k} \max_{f \in \Omega, f \neq 0} \mathcal{R}(f). \quad (3)$$

(3) The *Courant-Fischer-Weyl principle* is given in the form:

$$\lambda_k = \max_{\dim \Omega = k-1} \min_{\substack{f \perp \Omega \\ f \neq 0}} \mathcal{R}(f). \quad (4)$$

The result below establishes a link between the eigenvalues of S_G and Δ_G . We assume that the eigenvalues of Δ_G are ordered as follows respectively:

$$\mathcal{R}e(\lambda_1(\Delta_G)) \leq \mathcal{R}e(\lambda_2(\Delta_G)) \dots \leq \mathcal{R}e(\lambda_n(\Delta_G)).$$

Lemma 2.2.

$$2\mathcal{R}e(\lambda_n(\Delta_G)) \leq \lambda_n(S_G)$$

Proof:

Let f be an eigenfunction associated to $\lambda_n(\Delta_G)$, we have

$$\begin{aligned} \lambda_n(S_G) &\geq \frac{(S_G f, f)_m}{(f, f)_m} \\ &\geq \frac{(\Delta_G f, f)_m + \overline{(\Delta_G f, f)_m}}{(f, f)_m} \\ &\geq 2\mathcal{R}e(\lambda_n(\Delta_G)). \end{aligned}$$

□

Remark 2.2. In a particular case, the previous inequalities are strict. Let us consider the following example, where $m \equiv 1$, we have $\sigma(S_G) = \{0, 3, 7\}$ and

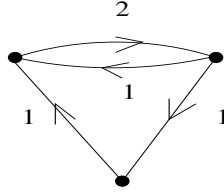


FIGURE 1. Graph with real spectrum

$$\sigma(\Delta_G) = \{0, 2, 3\}.$$

We introduce in the following a particular case of graphs whose $\sigma(S_G) = 2\mathcal{R}e(\sigma(\Delta_G))$.

Example 2.1. Let us consider the simple cycle graph C_3 , see the Figure 2, we have ,

$$\sigma(\Delta_{C_3}) = \left\{ 0, \frac{3}{2} + i\frac{\sqrt{3}}{2}, \frac{3}{2} - i\frac{\sqrt{3}}{2} \right\} \text{ and } \sigma(S_{C_3}) = \{0, 3, 3\}.$$

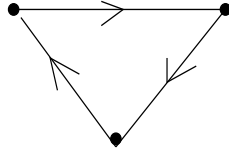


FIGURE 2. Cycle graph

In the following proposition we determine the spectrum of the non symmetric Laplacian Δ_{C_n} . We follow the same approach as Grigoryan for the symmetric Laplacian [7] page 49.

Proposition 2.2. The eigenvalues of $\Delta_{C_n} = I - P$ are as follows:

- (1) If n is odd then the eigenvalues are $\lambda = 0$ (simple) and $\lambda = 1 - e^{\pm i\frac{2l\pi}{n}}$ for all $l = 1, \dots, \frac{n-1}{2}$ (simple).

(2) If n is even then the eigenvalues are $\lambda = 0$, 2 (simple) and $\lambda = 1 - e^{\pm i \frac{2l\pi}{n}}$ for all $l = 1, \dots, \frac{n}{2} - 1$ (simple).

Proof:

To compute the eigenvalues of $\Delta_{C_n} = I - P$, it is sufficient to determine the spectrum of P . Let α be an eigenvalue of the operator $Pf(k+1) = f(k)$, for $k = 0, \dots, n-1$, which leads to $f(k) = \alpha^k f(0)$ but $f(n) = f(0)$ thus $\alpha^n = 1$. As f is an eigenfunction ($f(0) \neq 0$), then $\alpha = e^{\pm i\theta}$. As f is n -periodic provided $n\theta$ is a multiple of 2π , hence,

$$\theta = \frac{2l\pi}{n},$$

where l is an integer of $(0, \frac{n}{2})$. □

Observe that an interesting corollary concerning the spectra of S_{C_n} and Δ_{C_n} .

Corollary 2.1.

$$\sigma(S_{C_n}) = 2\mathcal{R}e(\sigma(\Delta_{C_n})).$$

Proof:

We refer the Lemma 2.7 of [7], we remark that the eigenvalues of the operator $\frac{1}{2}(P+P^*)f(k) = \frac{1}{2}(f(k+1)+f(k-1))$ coincide with the real part of the eigenvalues of P . □

Using the Green's formula, we establish some properties of the spectrum on any graph G .

Proposition 2.3. (1) 0 is a simple eigenvalue of \tilde{S}_G and S_G .

(2) All the eigenvalues of \tilde{S}_G are contained in $[0, 4]$.

(3) The real part of the eigenvalues of $\tilde{\Delta}_G$ are also contained in $[0, 2]$.

Proof:

(1) As in the case of undirected graph [7], we have for all $f \in \mathcal{C}_{\beta^+}(V)$,

$$(\tilde{S}_G f, f)_{\beta^+} = \sum_{(x,y) \in \tilde{E}} b(x,y) |f(x) - f(y)|^2.$$

Clearly, the constant function is an eigenfunction of 0. Assume now that f is an eigenfunction of the eigenvalue 0. By the connectedness of G , f is constant, which will imply that 0 is a simple eigenvalue. It is similar for \tilde{S}_G .

(2) It is sufficient to prove that \tilde{S}_G is bounded by 4 because \tilde{S}_G is non negative by the Green's formula. In fact, for all

$f \in \mathcal{C}_{\beta^+}(V)$ and thanks to Assumption (β) , we obtain

$$\begin{aligned}
 (\tilde{S}_G f, f)_{\beta^+} &= \sum_{(x,y) \in \vec{E}} b(x,y) |f(x) - f(y)|^2 \\
 &\leq 2 \sum_{(x,y) \in \vec{E}} b(x,y) (|f(x)|^2 + |f(y)|^2) \\
 &\leq 2 \sum_{(x,y) \in \vec{E}} b(x,y) |f(x)|^2 + 2 \sum_{(x,y) \in \vec{E}} b(x,y) |f(y)|^2 \\
 &\leq 2 \sum_{x \in V} \sum_{y \in V_x^+} b(x,y) |f(x)|^2 + 2 \sum_{y \in V} \sum_{x \in V_x^-} b(x,y) |f(y)|^2 \\
 &\leq 2 \sum_{x \in V} |f(x)|^2 \beta^+(x) + 2 \sum_{y \in V} |f(y)|^2 \beta^-(y) \\
 &\leq 4(f, f)_{\beta^+}.
 \end{aligned}$$

(3) We deduce directly our inclusion thanks to the Lemma 2.2. \square

3. DOMAIN MONOTONICITY OF EIGENVALUES

The purpose of this part is to give an overview of some results concerning the monotonicity with regard to the domain, of eigenvalues of S_G , the special self-adjoint Laplacian associated to directed graphs with non symmetric edge weights. We could be concerned with the related question:

Does a given eigenvalue increases or decreases under a given perturbation of G ?

3.1. Definitions on G . Before discussing the study of variation of eigenvalues, let us recall some basic definitions: let $G = (V, \vec{E})$ be a graph,

- The graph $G_1 = (V, \vec{E}_1)$ is called a partial graph of G , if \vec{E}_1 is included in \vec{E} .
- A graph $H = (V_H, \vec{E}_H)$ is called a subgraph of $G = (V_G, \vec{E}_G)$ if $V_H \subset V_G$ and $\vec{E}_H = \{(x, y); x, y \in V_H\} \cap \vec{E}_G$.
- A graph (V_U, \vec{E}_U) is called a part of a graph $G = (V_G, \vec{E}_G)$ if $V_U \subset V_G$ and $\vec{E}_U = \{(x, y), x, y \in V_U\} \subset \vec{E}_G$.

Remark 3.1. A subgraph is a part of G but the converse is not true, for example let us give the following undirected graphs, see the Figure 3.



FIGURE 3. H part of G

H is a part of G , but not a subgraph.

Remark 3.2. *A confusion between a subgraph and a part of a graph can create a false interpretation on the monotonicity of eigenvalue.*

3.2. Monotonicity relative to vertices. We study the monotonicity of eigenvalues under the variation of the set of vertices.

By the Courant-Fischer-Weyl principle we establish the following statement.

Theorem 3.1. *Let $H = (V_H, \vec{E}_H)$ be a connected subgraph of a graph $G = (V_G, \vec{E}_G)$, $\#V_G = n$, then for any $1 \leq k \leq \#V_H = r$:*

$$\lambda_k(S_H) \leq \lambda_{n-r+k}(S_G).$$

Proof:

Let f_1, f_2, \dots, f_k be k eigenfunctions associated to $\lambda_1(S_H), \lambda_2(S_H), \dots, \lambda_k(S_H)$; and $F = \{f_1, f_2, \dots, f_k, \delta_1, \dots, \delta_{n-r}\}$, $\delta_1, \dots, \delta_{n-r}$ are the Dirac measures on G relative to the vertices in $V_G \setminus V_H$. It is clear that $\dim F = k + n - r$. Then using (4), we obtain ;

$$\lambda_{n-r+k+1}(S_G) \geq \min_{\varphi \in F^\perp \setminus \{0\}} \frac{(S_G \varphi, \varphi)_m}{(\varphi, \varphi)_m}$$

hence $\exists \varphi_k \in F^\perp$ with support in H such that

$$\begin{aligned} \lambda_{n-r+k+1}(S_G)(\varphi_k, \varphi_k)_m &\geq (S_G \varphi_k, \varphi_k)_m \\ &= (S_H \varphi_k, \varphi_k)_m \\ &\geq \lambda_{k+1}(S_H)(\varphi_k, \varphi_k)_m. \end{aligned}$$

□

For studying the behavior of eigenvalues relative to perturbations, we propose a special construction of graphs.

Definition 3.1. *Let $G = (V_G, \vec{E}_G)$ be a weighted graph. G is called H -flower-like with respect to the subgraph $H = (V_H, \vec{E}_H)$ of G if there exists $(H_i)_{i \in I}$ a family of subgraphs of G such that:*

- (1) $V_G = V_H \cup (\uplus_i V_{H_i}^\circ)$
- (2) $\forall i, j \in I, \quad x \in V_{H_i}^\circ, \quad y \in V_{H_j}^\circ, \quad i \neq j \Rightarrow x \notin V_y.$
- (3) $\forall i \in I, \quad \exists x_i \in V_G, \quad V_H \cap V_{H_i} = \{x_i\}.$

These special graphs are used to create a rule of monotonicity of λ_k under a given graph perturbation.

Theorem 3.2. *Let $G = (V_G, \vec{E}_G)$ be a H -flower-like graph, we have then for any $1 \leq k \leq \#V_H = r$,*

$$\lambda_k(S_H) \geq \lambda_k(S_G). \tag{5}$$

Proof:

We use the variational principle (2). Let be f_1, f_2, \dots, f_{k-1} eigenfunctions of G in $\mathcal{C}_m(V)$ associated to $\lambda_1(S_G), \lambda_2(S_G), \dots, \lambda_{k-1}(S_G)$,

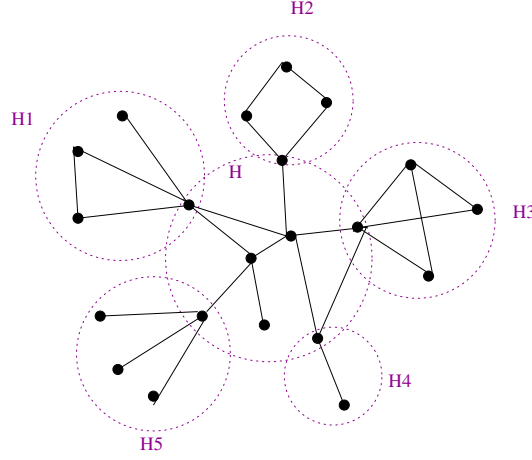


FIGURE 4. H-flower-like graph

and g_j be the eigenfunction associated to $\lambda_j(S_H)$, for $j = 1, \dots, k$. We define a function ϕ_j on V_G by:

$$\phi_j(x) = \begin{cases} g_j(x) & \text{if } x \in V_H \\ g_j(x_1) & \text{if } x \in V_{H_1} \setminus \{x_1\} \\ \cdot & \\ \cdot & \\ g_j(x_r) & \text{if } x \in V_{H_r} \setminus \{x_r\} \end{cases}$$

where $\{x_s\} = V_H \cap V_{H_s}$, $\forall s \in I = \{1, \dots, r\}$

Let $F = \text{Span}\{\phi_j\}_{1 \leq j \leq k}$, $\dim F = k$ so, there exist k reals $\alpha_1, \alpha_2, \dots, \alpha_k$ not all equal to zero, satisfying:

$$\left(\sum_{j=1}^k \alpha_j \phi_j, f_l \right)_m = 0 \quad \forall 1 \leq l \leq k-1.$$

Therefore the function $\psi(x) = \sum_{j=1}^k \alpha_j \phi_j(x) \neq 0$ is orthogonal to

f_l for all $1 \leq l \leq k-1$, we define for $x \in H$, $\psi'(x) = \sum_{j=1}^k \alpha_j g_j(x)$.

Then we get :

$$\begin{aligned} \lambda_k(S_G)(\psi, \psi)_m &\leq (S_G \psi, \psi)_m \\ &= (S_H \psi', \psi')_m \quad \text{because } \psi \text{ is constant on each } H_i \\ &\leq \lambda_k(S_H)(\psi', \psi')_m \end{aligned}$$

as

$$\begin{aligned} (\psi, \psi)_m &= \sum_{x \in V_G} m(x) |\psi(x)|^2 \\ &\geq \sum_{x \in V_H} m(x) |\psi'(x)|^2 \\ &= (\psi', \psi')_m \end{aligned}$$

then

$$\lambda_k(S_G)(\psi', \psi')_m \leq \lambda_k(S_H)(\psi', \psi')_m.$$

□

The above results have several important consequences, for instance on a tree seen as a flower-like graph.

Corollary 3.1. *Let G^s be a simple symmetric tree and $2q = \max_{x \in V} v(x)$ then the eigenvalues of G^s satisfy for all $k = 1, \dots, q$:*

$$\lambda_k(S_{G^s}) \leq 2$$

and

$$\lambda_{q+1}(S_{G^s}) \leq 2(q+1).$$

Proof:

Clearly there exists a symmetric star graph S_q with $q+1$ vertices seen as a subgraph of G^s . It can be considered as a S_q -flower-like graph. Therefore G^s satisfies the assumptions of the Proposition 3.2. Hence we get the result because the spectrum of S_q is : $0, 2, \dots, 2, 2(q+1)$, see [13].

□

In the following, we will show more general results : instead of adding only one vertex and one edge, we would also add a graph.

Corollary 3.2. *Let G be a graph with n vertices, let G_1 connected to G by a single edge. Then for $k = 1, \dots, n$:*

$$\lambda_k(S_G) \geq \lambda_k(S_{G_1}).$$

Remark 3.3. *The previous corollary is an immediate consequence of the Proposition 3.2. In addition this is an interesting generalization of Proposition 2 in [11] which shows that $\lambda_2(S_G) \geq \lambda_2(S_{G_1})$ where G_1 is obtained from G by adding one edge between one vertex of G and one new vertex.*

3.3. Monotonicity relative to edges. We apply the Weyl and interlacing Theorems for matrices to study how the spectrum of the special operator S_G of a directed graph changes under adding an edge or a set of edges.

Now we have several opportunities to refer to the following basic observation about subspace intersections (see [10] page 235).

Lemma 3.1. *Let W be a finite dimensional vector space and let S_1, S_2, \dots, S_k be subspaces of W , if*

$$\delta = \dim S_1 + \dots + \dim S_k - (k-1) \dim W \geq 1$$

then $\dim(S_1 \cap \dots \cap S_k) \geq \delta$ and hence $S_1 \cap \dots \cap S_k$ contains δ linearly independent vectors, in particular, it contains a nonzero vector.

Observation 3.1. *Let S_G be the special self-adjoint operator with eigenvalues $\lambda_1(S_G) \leq \lambda_2(S_G) \leq \dots \leq \lambda_n(S_G)$. Then the ordered eigenvalues of $-S$ are $\lambda_1(-S_G) \leq \lambda_2(-S_G) \leq \dots \leq \lambda_n(-S_G)$, that is, $\lambda_k(-S_G) = -\lambda_{n-k+1}(S_G)$, $k = 1, \dots, n$.*

We show that the k^{th} eigenvalue $\lambda_k(S_G)$ is monotonously increasing functions of the set of edges. The following results are the generalizations of the Proposition 1 and the Proposition 2 in [11].

Proposition 3.1. *Let $G = (V, \vec{E})$ be a connected finite weighted graph ($\#V = n$), consider the partial graphs $G_1 = (V, \vec{E}_1)$ and $G_2 = (V, \vec{E}_2)$ where $E = E_1 \sqcup E_2$ (disjoint union). Then for all $k = 1, \dots, n$ and $r, s = 1, 2$, $r \neq s$:*

$$\lambda_k(S_G) \leq \lambda_{k+j}(S_{G_s}) + \lambda_{n-j}(S_{G_r}) \quad j = 0, \dots, n-k \quad (6)$$

and

$$\lambda_k(S_G) \geq \lambda_{k-j+1}(S_{G_s}) + \lambda_j(S_{G_r}) \quad j = 1, \dots, k.$$

Proof:

Let g_k , h_k , and f_k be the eigenfunctions associated to $\lambda_k(S_{G_1})$, $\lambda_k(S_{G_2})$ and $\lambda_k(S_G)$ respectively for $k = 1, \dots, n$.

For $j = 0, \dots, n-k$, we define $S_1 = \text{Span}\{g_1, \dots, g_{k+j}\}$, $S_2 = \text{Span}\{h_1, \dots, h_{n-j}\}$ and $S_3 = \text{Span}\{f_k, \dots, f_n\}$ by the Lemma (3.1) there exists a non zero function ψ in $S_1 \cap S_2 \cap S_3$ and so we will have

$$\begin{aligned} \lambda_k(S_G)(\psi, \psi)_m &\leq (S_G \psi, \psi)_m & (7) \\ &= \sum_{(x,y) \in \vec{E}} b(x,y) |\psi(x) - \psi(y)|^2 \\ &= (S_{G_s} \psi, \psi)_m + (S_{G_r} \psi, \psi)_m \\ &\leq \lambda_{k+j}(S_{G_s})(\psi, \psi)_m + \lambda_{n-j}(S_{G_r})(\psi, \psi)_m \\ &\leq (\lambda_{k+j}(S_{G_s}) + \lambda_{n-j}(S_{G_r}))(\psi, \psi)_m. \end{aligned}$$

In the following we apply the equality (6) to the operator $-S_G$ because the inequalities (7) does not depend on positivity of b , and by:

$$\lambda_k(-S_G) = -\lambda_{n-k+1}(S_G).$$

We obtain by re-indexing:

$$\lambda_k(S_G) \geq \lambda_{k-j+1}(S_{G_s}) + \lambda_j(S_{G_r}) \quad j = 1, \dots, k.$$

□

We can easily deduce,

Corollary 3.3. *Let $G = (V, \vec{E})$ be a connected finite graph, $G_1 = (V, \vec{E}_1)$ and $G_2 = (V, \vec{E}_2)$ two partial graphs of G where $\sharp V = n$ and $E = E_1 \sqcup E_2$, then for all $k = 1, \dots, n$ and $r, s = 1, 2, r \neq s$:*

$$\lambda_k(S_{G_s}) \leq \lambda_k(S_G) \leq \lambda_k(S_{G_s}) + \lambda_n(S_{G_r}).$$

Proof:

By applying the proposition 3.1 to $j = 0$ and $j = 1$ respectively, we obtain the result because $\lambda_1(S_{G_r}) = 0$.

□

In other words, adding a subset of edges to \vec{E} while keeping the same set of vertices always induces an increasing of the k^{th} eigenvalue or keeps it unchanged.

Corollary 3.4. *Let $G = (V, \vec{E})$ be a connected finite graph with n vertices, and G_1 a graph obtained by adding a set of edges to G then for all $k = 1, \dots, n$:*

$$\lambda_k(S_G) \leq \lambda_k(S_{G_1}).$$

4. COMPARISON EIGENVALUES OF DIRICHLET LAPLACIAN ON GRAPHS

In this section, we present some results about the spectrum comparison between the Laplacian and the Dirichlet Laplacian. The purpose of this part is to find the relation between the usual vertex weight on a subgraph H of G and its boundary weight to compare eigenvalues.

This is done by establishing a clear and explicit link between the eigenvalues and the Dirichlet eigenvalues on G .

In the following proposition, we treat the Dirichlet Laplacian case :

$$\forall x \in U, \quad S_U^D f(x) = \frac{1}{m(x)} \sum_{\substack{y \in V_x \\ y \in V_G}} a(x, y)(f(x) - f(y)).$$

By the same techniques used in the Lemma 2.2, we can show the following Lemma.

Lemma 4.1.

$$\lambda_1(S_U^D) \leq 2\mathcal{R}e(\lambda_1(\Delta_U^D))$$

and

$$\lambda_n(S_U^D) \geq 2\mathcal{R}e(\lambda_n(\Delta_U^D)).$$

Proof:

Let f and g be eigenfunctions associated to $\lambda_1(\Delta_U^D)$ and $\lambda_n(\Delta_U^D)$ respectively. By the variational principle of $\lambda_1(S_U^D)$ and $\lambda_n(S_U^D)$, we have

$$\lambda_1(S_U^D) \leq \frac{(S_U^D f, f)_m}{(f, f)_m} = \frac{(\Delta_U^D f, f)_m}{(f, f)_m} + \frac{\overline{(\Delta_U^D f, f)_m}}{(f, f)_m} = \lambda_1(\Delta_U^D) + \overline{\lambda_1(\Delta_U^D)}$$

and

$$\lambda_n(S_U^D) \geq \frac{(S_U^D g, g)_m}{(g, g)_m} = \frac{(\Delta_U^D g, g)_m}{(g, g)_m} + \frac{\overline{(\Delta_U^D g, g)_m}}{(g, g)_m} = \lambda_n(\Delta_U^D) + \overline{\lambda_n(\Delta_U^D)}$$

□

In the same spirit as the Cauchy interlacing theorem concerning hermitian bordered matrices (see [9] theorem 4.3.28 for a generalized statement) one can prove the following.

Proposition 4.1. *Consider a connected subgraph $H = (V_H, \vec{E}_H)$ of $G = (V_G, \vec{E}_G)$, ($\#V_G = n, \#V_H = r$), then the eigenvalues on G of the normalized Laplacian S_G satisfies:*

$$\lambda_k(S_H^D) \leq \lambda_{k+n-r}(S_G).$$

Proof:

Let h_1, \dots, h_r and f_1, \dots, f_n be the eigenfunctions associated to S_H^D and S_G respectively, define the function g_k for $k = 1, \dots, r$ by:

$$g_k = \begin{cases} h_k & \text{on } V_H \\ 0 & \text{otherwise} \end{cases}$$

Let $1 \leq k \leq r$ and fix $S_1 = \text{Span}\{g_k, \dots, g_r\}$ and $S_2 = \text{Span}\{f_1, \dots, f_{k+n-r}\}$ by the Lemma 3.1 there exists a function ψ in $S_1 \cap S_2$. Since $\psi \in S_1$, it has the form

$$\psi = \begin{cases} g & \text{on } V_H \\ 0 & \text{otherwise} \end{cases}$$

for some $g \in \text{Span}\{g_k, \dots, g_r\}$. Observe that:

$$\begin{aligned} \lambda_k(S_H^D)(g, g)_m &\leq (S_H^D g, g)_m \\ &\leq (S_G \psi, \psi)_m \end{aligned}$$

since $(g, g)_m = (\psi, \psi)_m$, we get

$$\lambda_k(S_H^D) \leq \frac{(S_G \psi, \psi)_m}{(\psi, \psi)_m}$$

then,

$$\lambda_k(S_H^D) \leq \lambda_{k+n-r}(S_G).$$

□

We deduce easily from the Proposition 4.1 an estimation of the eigenvalues of S_G and Δ_G , thanks to Theorem 3.1, as follows:

Corollary 4.1. *Consider a connected subgraph $H = (V_H, \vec{E}_H)$ of $G = (V_G, \vec{E}_G)$, ($\#V_G = n, \#V_H = r$), then the eigenvalues on G satisfies:*

$$\lambda_{k+n-r}(S_G) \geq \max(\lambda_k(S_H), \lambda_k(S_H^D)).$$

Corollary 4.2. *Consider a connected subgraph $H = (V_H, \vec{E}_H)$ of $G = (V_G, \vec{E}_G)$, ($\#V_G = n, \#V_H = r$), then:*

$$\lambda_n(S_G) \geq \max\left(2\mathcal{R}e((\lambda_r(\Delta_H))), 2\mathcal{R}e(\lambda_r(\Delta_H^D))\right).$$

Corollary 4.3. Consider a connected subgraph $H = (V_H, \vec{E}_H)$ of the cycle graph C_n , ($\#V_H = r$), then:

$$\mathcal{Re}((\lambda_n(\Delta_{C_n}) \geq \max(\mathcal{Re}((\lambda_r(\Delta_H))), \mathcal{Re}(\lambda_r(\Delta_H^D))))).$$

In the following Proposition we prove how to give an upper bound for $\lambda_{k+l}(S_G)$ in terms of a Rayleigh quotient. We give a discrete version of the Proposition 2.1 [3] by applying the Poincaré min-max principle. The methods we use follow closely the arguments given in B. Benson [3] in the case of the Laplacian of Riemannian manifolds. Next we provide an upper bound of the eigenvalues of G according to Dirichlet eigenvalues on such repartition of G as in the Figure 5.

Proposition 4.2. Let $G = (V, \vec{E})$ a finite connected graph, $U = (V_U, \vec{E}_U)$ a part of G and $A = (V_A, \vec{E}_A)$, $B = (V_B, \vec{E}_B)$ two subgraphs satisfying the following conditions:

- (1) $V = V_A \sqcup V_B = V_A^\circ \sqcup V_B^\circ \sqcup V_U$ (disjoint union)
- (2) $\vec{E} = \vec{E}_A \sqcup \vec{E}_B \sqcup \vec{E}_U$
- (3) $\partial_{\vec{E}} A = \partial_{\vec{E}} B = \vec{E}_U$.

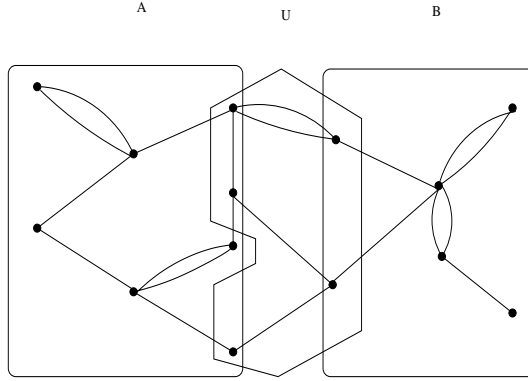


FIGURE 5. Repartition of G

Then we get for $1 \leq k, l \leq \min(\#V_A^\circ, \#V_B^\circ)$:

$$\lambda_{k+l}(S_G) \leq \max(\lambda_k(S_A^D), \lambda_l(S_B^D)).$$

Proof:

Let h_k and g_l be the eigenvectors associated to $\lambda_k(S_A^D)$ and $\lambda_l(S_B^D)$ respectively, for $1 \leq k \leq \#V_A^\circ$ and $1 \leq l \leq \#V_B^\circ$.

We define f on $\mathcal{C}_m(\overset{\circ}{A} \oplus \overset{\circ}{B})$ by:

$$f(x) = \begin{cases} h_k(x) & \text{if } x \in V_A^\circ \\ g_l(x) & \text{if } x \in V_B^\circ \\ 0 & \text{if } x \in V_U. \end{cases}$$

We have

$$(Sf, f)_m = (S_A^D h_k, h_k)_m + (S_B^D g_l, g_l)_m.$$

Using the Poincaré min-max principle (3), we obtain:

$$\begin{aligned} \lambda_{k+l}(S_G) &\leq \max_{f \in F_A^{\circ} + F_B^{\circ}} \frac{(S_G f, f)_m}{(f, f)_m} = \max_{h \in F_A^{\circ}, g \in F_B^{\circ}} \frac{(S_A^D h, h)_m + (S_B^D g, g)_m}{(h, h)_m + (g, g)_m} \\ &\leq \frac{\lambda_k(S_A^D)(h, h)_m + \lambda_l(S_B^D)(g, g)_m}{(h, h)_m + (g, g)_m} \\ &\leq \max(\lambda_k(S_A^D), \lambda_l(S_B^D)). \end{aligned}$$

Hence

$$\lambda_{k+l}(S_G) \leq \max(\lambda_k(S_A^D), \lambda_l(S_B^D)).$$

□

Remark 4.1. *The previous Proposition remains true in the particular case of the Laplacian \tilde{S}_G .*

An estimate of $\lambda_2(S_G)$ can also be obtained with respect to $\lambda_1(\Delta_B^D)$ and $\lambda_1(\Delta_A^D)$.

Corollary 4.4. *Under the same hypothesis of the previous Proposition we have*

$$\lambda_2(S_G) \leq \max\left(2\mathcal{R}e(\lambda_1(\Delta_A^D)), 2\mathcal{R}e(\lambda_1(\Delta_B^D))\right).$$

Corollary 4.5. *Under the same hypothesis of the previous Proposition, from the cycle graph C_n , we have*

$$\mathcal{R}e(\lambda_2(\Delta_{C_n})) \leq \max\left(\mathcal{R}e(\lambda_1(\Delta_A^D)), \mathcal{R}e(\lambda_1(\Delta_B^D))\right).$$

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