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I. Introduction

The well-known Kalman filter is an optimal unbiased estimator for linear dynamic systems with Gaussian noise and non-Gaussian noise [1, pp. 480]. In a number of practical situations, the state of the system is subject to some constraints. Examples of such systems include camera tracking [2], target tracking [3, 4], vision-based systems [5], attitude determination [6–8], and orbit determination [9, 10]. If the state constraints are either ignored or dealt with heuristically, the use of the Kalman filter may result in a nonoptimal estimation [11]. Therefore, it is necessary to incorporate the state constraints into the Kalman filter.

Many researchers have studied Kalman filters for linear and nonlinear state constraints, and for equality and inequality constraints [12]. Methods of dealing with linear constraints include model reduction, perfect measurements, estimate projection, gain projection, probability density function truncation, and system projection. If the state constraint is nonlinear, a typical approach is to linearize the constraint around the current state estimate [9]. However, the linearized method may degrade the performance of the filter or may even cause the filter to diverge.

In [13], a Kalman filter with a single nonlinear equality constraint was proposed that allows for the exact use of second-order nonlinear state constraints. Using the Lagrangian multiplier technique, a first-order necessary condition for a constrained least-square optimization problem is derived to form the estimation and Lagrangian multiplier polynomial equations. The Lagrangian multiplier is

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solved iteratively and then provides the constrained estimate. To implement the filter in [13], the second-order coefficient matrix of the constraint should be positive-definite. The method proposed in [13] is extended to multiple nonlinear-linear mixing state equality constraints in [14, 15], and the condition of the positive definite second-order coefficient matrix is relaxed to include any symmetric matrices. In addition, a H_∞ filter is introduced to deal with the situation of noise with unknown statistics in [15]. However, in [13–15], the use of a Newton's iteration method to solve the Lagrangian multiplier is limited by the initial guess. Moreover, because the polynomial equation has multiple roots, the root found may not correspond to a minimal index. In [7], a norm-constrained Kalman filter (NCKF) is developed for the norm state equality constraint. For this situation, the Lagrangian multiplier forms solutions of a quadratic polynomial equation and the two solutions are given in a closed form. The second-order sufficient condition for a constrained optimization problem is then derived to choose the solution that minimizes the performance index. However, the approach in [7] is specific to the attitude estimation problem.

In this note, the NCKF is generalized to the case of a quadratic form that has more general applications in aerospace engineering. The coefficient matrix for these quadratic form constraints can either be positive-semidefinite or positive-indefinite. For a positive-semidefinite matrix, the NCKF can be applied by transforming the constraint to a norm constraint. Methods to incorporate constraints with a positive-indefinite matrix into filters may require specific derivations for different applications. A typical constraint with a positive-indefinite matrix is that the position vector of a spacecraft is orthogonal to the velocity at the perigee/apogee. In Ref. [10], the linearized method is used by projecting the unconstrained estimate into the constrained space. However, the application of the constraint at the perigee and apogee positions causes a discontinuity in the covariance because this type of constraints is isolated. Two scalar weights can be introduced to smoothen the state and covariance estimation. Another typical quadratic form constraint with a positive-indefinite matrix is in Markley variables [16]. The elements of Markley variables are the angular momentum components in an inertial frame, in a body frame and a rotation angle. Markley variables are used for spinning spacecraft attitude estimation, because they have fewer rapidly-varying elements compared with quaternion-based representation. The constraint that the magnitude of the angular momentum vector is the same in the inertial and body frames allows the filters to employ a six-component error state instead of an error vector with the full seven-component state. Such a procedure is similar to that commonly used to estimate the constrained four component-quaternion representation [17]. The filter employing Markley variables is referred to as SpinKF and several different varieties of SpinKF can be derived depending on the choice of error state: SpinKF-1 [18], SpinKF-I, or SpinKF-B [6].

The main of focus of this note is to extend the NCKF to a situation for a general quadratic form state constraint using the standard Lagrangian multiplier technique and to obtain a mathematically robust method for finding solutions of Lagrangian multiplier, using an eigenvalue decomposition of a companion matrix of a polynomial function in place of a less robust Newton-Raphson iteration. An analytical criterion derived from the second-order optimal sufficient condition is used to select the Lagrangian multiplier corresponding to the minimum point of the performance index. The proposed filter is mathematically equivalent to the norm-constrained filter when the quadratic-form matrix is positive-semidefinite.

II. Problem Statement

Let $\hat{\boldsymbol{x}}^-$ be a priori state estimation of state $\boldsymbol{x}$ ($\boldsymbol{x} \in \mathcal{R}^n$) before employing the measurement $\boldsymbol{y}$ ($\boldsymbol{y} \in \mathcal{R}^m$) and let $\boldsymbol{P}^-$ be the corresponding error covariance matrix. Then

$$\boldsymbol{P}^- = \mathrm{E}\left\{(\hat{\boldsymbol{x}}^- - \boldsymbol{x})(\hat{\boldsymbol{x}}^- - \boldsymbol{x})^T\right\} \tag{1}$$

It is assumed that the measurement is a linear combination of state with the form

$$\boldsymbol{y} = \boldsymbol{H}\boldsymbol{x} + \boldsymbol{\eta} \tag{2}$$

where $\boldsymbol{\eta}$ is measurement noise with the covariance $\boldsymbol{R} = \mathrm{E}\left\{\boldsymbol{\eta}\boldsymbol{\eta}^T\right\}$. Let $\hat{\boldsymbol{x}}^+$ be a posteriori estimation of $\boldsymbol{x}$ given by

$$\hat{\boldsymbol{x}}^+ = \boldsymbol{K}\boldsymbol{y} + \boldsymbol{N}\hat{\boldsymbol{x}}^- + \boldsymbol{n} \tag{3}$$

where $\boldsymbol{K} \in \mathcal{R}^{n\times m}, \boldsymbol{N} \in \mathcal{R}^{n\times n}$, and $\boldsymbol{n} \in \mathcal{R}^n$ are matrices to be determined. Note that a perfect priori state estimation ($\hat{\boldsymbol{x}}^- = \boldsymbol{x}$) and a perfect measurement ($\boldsymbol{\eta} = \boldsymbol{0}$) would result in a perfect posteriori estimation ($\hat{\boldsymbol{x}}^+ = \boldsymbol{x}$), which means that

$$\boldsymbol{x} = (\boldsymbol{K}\boldsymbol{H} + \boldsymbol{N})\boldsymbol{x} + \boldsymbol{n} \tag{4}$$

Therefore, $\boldsymbol{K}, \boldsymbol{N}$ and $\boldsymbol{n}$ must satisfy the constraints

$$\boldsymbol{K}\boldsymbol{H} + \boldsymbol{N} \;=\; \boldsymbol{I} \tag{5a}$$

$$\boldsymbol{n} \;=\; \boldsymbol{0} \tag{5b}$$

, and Eq. (3) can be rewritten as,

$$\hat{\boldsymbol{x}}^+ = \hat{\boldsymbol{x}}^- + \boldsymbol{K}\boldsymbol{\epsilon} \tag{6}$$

where, $\boldsymbol{\epsilon} = \boldsymbol{y} - \boldsymbol{H}\hat{\boldsymbol{x}}^-$.

For an unconstrained linear optimal estimation problem, the optimal $\boldsymbol{K}$ can be determined by minimizing the loss function

$$J = \frac{1}{2}\mathrm{Tr}\boldsymbol{P}^+ = \frac{1}{2}\mathrm{Tr}\left[\mathrm{E}\left\{(\hat{\boldsymbol{x}}^+ - \boldsymbol{x})(\hat{\boldsymbol{x}}^+ - \boldsymbol{x})^T\right\}\right] \tag{7}$$

where, $\boldsymbol{P}^+$ is the posteriori error covariance.

In this note, a quadratic form state constraint is considered, and any quadratic form state constraint can be written in the form of

$$\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} = l \tag{8}$$

with $\boldsymbol{A} = \boldsymbol{A}^T \in \mathcal{R}^{n\times n}$. The performance index Eq. (7) can be redefined as

$$J = \frac{1}{2}\mathrm{Tr}\left[\mathrm{E}\left\{(\hat{\boldsymbol{x}}^+ - \boldsymbol{x})(\hat{\boldsymbol{x}}^+ - \boldsymbol{x})^T\right\}\right] + \frac{1}{2}\lambda((\hat{\boldsymbol{x}}^+)^T\boldsymbol{A}\hat{\boldsymbol{x}}^+ - l) \tag{9}$$

where, λ is a Lagrangian multiplier associated with the constraint in Eq. (8).

Assuming that the measurement noise $\boldsymbol{\eta}$ is not correlated with $\hat{\boldsymbol{x}}^-$ and substituting Eq. (6) and Eq. (2) into Eq. (9) yields

$$J \;=\; \frac{1}{2}\mathrm{Tr}\left[\boldsymbol{K}\boldsymbol{R}\boldsymbol{K}^T + (\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H})\boldsymbol{P}^-(\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H})^T\right] + \frac{1}{2}\lambda[(\hat{\boldsymbol{x}}^- + \boldsymbol{K}\boldsymbol{\epsilon})^T\boldsymbol{A}(\hat{\boldsymbol{x}}^- + \boldsymbol{K}\boldsymbol{\epsilon}) - l] \tag{10}$$

The goal is to determine $\boldsymbol{K}$ and λ while minimizing the loss function Eq. (10).

III. Analytical Solution of $\boldsymbol{K}$

According to Appendix A, the necessary conditions for minimizing J in Eq. (10) are

$$\nabla_{\boldsymbol{K}} J = \boldsymbol{K}\boldsymbol{R} - (\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H})\boldsymbol{P}^{-}\boldsymbol{H}^{T} + \lambda\boldsymbol{A}(\hat{\boldsymbol{x}}^{-} + \boldsymbol{K}\boldsymbol{\epsilon})\boldsymbol{\epsilon}^{T} = \boldsymbol{0} \tag{11a}$$

$$\nabla_{\boldsymbol{\lambda}} J = (\hat{\boldsymbol{x}}^{+})^{T}\boldsymbol{A}\hat{\boldsymbol{x}}^{+} - l = 0 \tag{11b}$$

where, the identities of $\frac{\partial}{\partial \boldsymbol{x}}(\boldsymbol{x}^{\boldsymbol{T}}\boldsymbol{C}\boldsymbol{x}) = (\boldsymbol{C} + \boldsymbol{C}^{\boldsymbol{T}})\boldsymbol{x}$ and $\frac{\partial}{\partial \boldsymbol{A}}(\mathrm{Tr}[\boldsymbol{A}\boldsymbol{B}\boldsymbol{A}^{\boldsymbol{T}}]) = (\boldsymbol{A}\boldsymbol{B} + \boldsymbol{A}\boldsymbol{B}^{\boldsymbol{T}})$ [19, pp. 679] are used. Eq. (11) can be rearranged as,

$$\boldsymbol{K} + \lambda\boldsymbol{A}\boldsymbol{K}\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{T}\boldsymbol{W}^{-1} = (\boldsymbol{P}^{-}\boldsymbol{H}^{T} - \lambda\boldsymbol{A}\hat{\boldsymbol{x}}^{-}\boldsymbol{\epsilon}^{T})\boldsymbol{W}^{-1} \tag{12a}$$

$$\boldsymbol{\epsilon}^{T}\boldsymbol{K}^{T}\boldsymbol{A}\boldsymbol{K}\boldsymbol{\epsilon} + 2\boldsymbol{\epsilon}^{T}\boldsymbol{K}^{T}\boldsymbol{A}\hat{\boldsymbol{x}}^{-} + (\hat{\boldsymbol{x}}^{-})^{T}\boldsymbol{A}\hat{\boldsymbol{x}}^{-} - l = 0 \tag{12b}$$

where $\boldsymbol{W} = \boldsymbol{H}\boldsymbol{P}^{-}\boldsymbol{H}^{T} + \boldsymbol{R}$.

Eq. (12) has two unknown parameters to be solved: $\boldsymbol{K}$ and λ. A possible way is to solve K in terms of λ and other variables from Eq. (12a) and then solve λ from Eq. (12b) by substituting the solution of $\boldsymbol{K}$ into Eq. (12b). $\boldsymbol{K}$ clearly depends on the prior estimate of the state and measurement of residuals, which indicates that $\boldsymbol{K}$ and λ fluctuate with the noise.

Eq. (12a) is known as the discrete-time Sylvester equation for $\boldsymbol{K}$. If $\boldsymbol{A} = \boldsymbol{I}$, $\boldsymbol{K}$ can be solved directly from Eq. (12a), as presented in [7]. However, for a general $\boldsymbol{A}$, the solution of $\boldsymbol{K}$ cannot be obtained intuitively. For a m dimensional space, there exist $m-1$ vectors $\boldsymbol{\beta}_i (i = 1, \cdots, m-1)$ such that $\boldsymbol{\beta}_i \perp \boldsymbol{\beta}_j (i \neq j)$ and $\boldsymbol{\epsilon} \perp \boldsymbol{\beta}_i$. Therefore, the m eigenvalues of $\lambda\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{T}\boldsymbol{W}^{-1}$ are $0, \cdots, 0, \lambda\varepsilon$ with $\varepsilon = \boldsymbol{\epsilon}^{T}\boldsymbol{W}^{-1}\boldsymbol{\epsilon}$, and the eigenvectors corresponding to these m eigenvalues are given by $\boldsymbol{W}\boldsymbol{\beta}_1, \boldsymbol{W}\boldsymbol{\beta}_2, \cdots, \boldsymbol{W}\boldsymbol{\beta}_{m-1}, \boldsymbol{\epsilon}$. Suppose there exist m $\alpha_i \in \mathcal{R}$ $(i = 1, 2, \cdots, m)$ such that

$$\alpha_1\boldsymbol{\epsilon} + \alpha_2\boldsymbol{W}\boldsymbol{\beta}_1 + \cdots + \alpha_m\boldsymbol{W}\boldsymbol{\beta}_{m-1} = \boldsymbol{0} \tag{13}$$

Because $\boldsymbol{W} > 0$, a left-multiplication of Eq. (13) by $\boldsymbol{\epsilon}^{T}\boldsymbol{W}^{-1}$ gives $\alpha_1 = 0$ and then a left-multiplication of Eq. (13) by $\boldsymbol{\beta}_j^{T}\boldsymbol{W}^{-1}(j = 1, \cdots, m-1)$ gives $\alpha_i = 0 (i = 2, \cdots, m)$, which indicates that the m eigenvectors of $\lambda\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{T}\boldsymbol{W}^{-1}$ are linearly independent. In other words, $\lambda\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{T}\boldsymbol{W}^{-1}$ can be diagonalized as

$$\lambda\boldsymbol{\epsilon}\boldsymbol{\epsilon}^{T}\boldsymbol{W}^{-1} = \boldsymbol{V}\boldsymbol{\Gamma}\boldsymbol{V}^{-1} \tag{14}$$

where, $\boldsymbol{\Gamma} := \mathrm{diag}([\gamma_1, \cdots, \gamma_m]) = \mathrm{diag}([0, \cdots, 0, \lambda\varepsilon])$ and $\boldsymbol{V} = [\boldsymbol{W\beta}_1, \boldsymbol{W\beta}_2, \cdots, \boldsymbol{W\beta}_{m-1}, \boldsymbol{\epsilon}]$. Noting that a symmetrical matrix can be diagonalized by an orthogonal matrix, $\boldsymbol{A}$ can be written as,

$$\boldsymbol{A} \;=\; \boldsymbol{U\Xi U}^T \tag{15}$$

where $\boldsymbol{U}$ is an orthogonal matrix, and $\boldsymbol{\Xi} = \mathrm{diag}([\xi_1, \cdots, \xi_n])$. Substituting Eq. (15) into Eq. (12a) and multiplying it on the left by $\boldsymbol{U}^T$ and on the right by $\boldsymbol{V}$, we obtain

$$\tilde{\boldsymbol{K}} + \boldsymbol{\Xi}\tilde{\boldsymbol{K}}\boldsymbol{\Gamma} = \tilde{\boldsymbol{C}} \tag{16}$$

where, $\tilde{\boldsymbol{K}} = \boldsymbol{U}^T\boldsymbol{KV}$, $\tilde{\boldsymbol{C}} = \boldsymbol{U}^T\boldsymbol{CV}$ and $\boldsymbol{C} = (\boldsymbol{P}^-\boldsymbol{H}^T - \lambda\boldsymbol{A}\hat{\boldsymbol{x}}^-\boldsymbol{\epsilon}^T)\boldsymbol{W}^{-1}$

Denoting the (i,j)th elements of $\tilde{\boldsymbol{K}}$ and $\tilde{\boldsymbol{C}}$ as $\tilde{k}_{ij}$ and $\tilde{c}_{ij}$, Eq. (16) reads as

$$\tilde{k}_{ij} + \xi_i\tilde{k}_{ij}\gamma_j = \tilde{c}_{ij} \tag{17}$$

which means that

$$\tilde{k}_{ij} = \frac{\tilde{c}_{ij}}{1+\xi_i\gamma_j} \tag{18}$$

Once $\tilde{\boldsymbol{K}}$ is calculated, $\boldsymbol{K}$ is given by $\boldsymbol{U}\tilde{\boldsymbol{K}}\boldsymbol{V}^{-1}$. Following the extended Kalman filter (EKF) derivation assumption, the posteriori error covariance is given by

$$\boldsymbol{P}^+ = (\boldsymbol{I} - \boldsymbol{KH})\boldsymbol{P}^-(\boldsymbol{I} - \boldsymbol{KH})^T + \boldsymbol{KRK}^T \tag{19}$$

IV. Analytical solution of λ

Noting that $\tilde{c}_{ij}$ and γ_j depend on λ, λ should be calculated before computing $\boldsymbol{K}$. In this section, an analytical solution of λ will be derived.

A right-multiplication of Eq. (12a) by $\boldsymbol{\epsilon}$ gives

$$\boldsymbol{K\epsilon} + \lambda\varepsilon\boldsymbol{AK\epsilon} = \boldsymbol{d} - \lambda\varepsilon\boldsymbol{A}\hat{\boldsymbol{x}}^- \tag{20}$$

where $\boldsymbol{d} = \boldsymbol{P}^-\boldsymbol{H}^T\boldsymbol{W}^{-1}\boldsymbol{\epsilon}$. Then

$$\boldsymbol{K\epsilon} = [\boldsymbol{I} + \lambda\varepsilon\boldsymbol{A}]^{-1}[\boldsymbol{d} - \lambda\varepsilon\boldsymbol{A}\hat{\boldsymbol{x}}^-] \tag{21}$$

Substituting Eq. (21) into Eq. (12b) yields,

$$[\boldsymbol{d} - \lambda\varepsilon \boldsymbol{A}\hat{\boldsymbol{x}}^-]^T[I + \lambda\varepsilon\boldsymbol{A}]^{-1}\boldsymbol{A}[\boldsymbol{I} + \lambda\varepsilon\boldsymbol{A}]^{-1}[\boldsymbol{d} - \lambda\varepsilon\boldsymbol{A}\hat{\boldsymbol{x}}^-] \tag{22a}$$

$$+2[\boldsymbol{d} - \lambda\varepsilon\boldsymbol{A}\hat{\boldsymbol{x}}^-]^T[I + \lambda\varepsilon\boldsymbol{A}]^{-1}\boldsymbol{A}\hat{\boldsymbol{x}}^- \tag{22b}$$

$$= l - (\hat{\boldsymbol{x}}^-)^T\boldsymbol{A}\hat{\boldsymbol{x}}^- \tag{22c}$$

A further simplification of Eq. (22), obtained by substituting Eq. (15) into Eq. (22), gives,

$$\boldsymbol{h}^T(\boldsymbol{I} + \lambda\varepsilon\boldsymbol{\Xi})^{-2}\boldsymbol{\Xi}\boldsymbol{h} = l \tag{23}$$

where, $\boldsymbol{h} = \tilde{\boldsymbol{d}} + \tilde{\boldsymbol{x}}$, $\tilde{\boldsymbol{d}} = \boldsymbol{U}^T\boldsymbol{d}$, and $\tilde{\boldsymbol{x}}^- = \boldsymbol{U}^T\hat{\boldsymbol{x}}^-$. Eq. (23) can be written in a scalar form as

$$s_1(\tilde{\lambda}) = \sum_{j=1}^{n} \frac{h_j^2\xi_j}{(1 + \tilde{\lambda}\xi_j)^2} - l = 0 \tag{24}$$

where $\tilde{\lambda} = \lambda\varepsilon$.

Eq. (24) is similar to the constraint equation expressed in terms of a Lagrangian multiplier in [13] for a constrained least-square problem and a Newton's iteration method is proposed for solving λ by starting with $\lambda_0 = 0$ in [13]. However, this method can only find one root of the constraint equation and this root may not be the one corresponding to the minimum value of the loss function because the number of the roots of Eq. (24) is $2p$, where $p = q$ for $l \neq 0$ or $p = q - 1$ for $l = 0$, and q is the number of distinct nonzero eigenvalues of $\boldsymbol{A}$. Therefore, a more effective method should be developed to find $2p$ roots of Eq. (24) and determine the root that minimizes the loss function.

Let $\tilde{\xi}_1, \tilde{\xi}_1, \cdots, \tilde{\xi}_q$ be q distinct nonzero eigenvalues of $\boldsymbol{A}$ and define a monic polynomial function as

$$s_2(\tilde{\lambda}) = \begin{cases} \frac{1}{\prod_{j=1}^{q}\tilde{\xi}_j^2}\prod_{j=1}^{q}(1 + \tilde{\lambda}\tilde{\xi}_j)^2(1 - \frac{1}{l}\sum_{i=1}^{n}\frac{h_i^2\xi_i}{(1+\tilde{\lambda}\xi_i)^2}) & \text{if } l \neq 0 \\ \frac{1}{\prod_{j=1}^{q}\tilde{\xi}_j^2}\prod_{j=1}^{q}(1 + \tilde{\lambda}\tilde{\xi}_j)^2\sum_{i=1}^{n}\frac{h_i^2\xi_i}{(1+\tilde{\lambda}\xi_i)^2} & \text{if } l = 0 \end{cases} \tag{25}$$

Eq. (25) can be rewritten as,

$$s_2(\tilde{\lambda}) = \beta_0 + \beta_1\tilde{\lambda} + \beta_2\tilde{\lambda}^2 + \cdots + \tilde{\lambda}^{2p} \tag{26}$$

where, $\beta_i = \frac{1}{i!}\frac{d^i}{d\tilde{\lambda}^i}s_2(0)$ can be calculated by expanding Eq. (25) or by using difference methods. It is noticed that the roots of function (26) are the same as the solutions of Eq. (24) and the Frobenius companion matrix corresponding to Eq. (26) is defined by

$$\boldsymbol{G} = \begin{bmatrix} 0 & 0 & \cdots & 0 & -\beta_0 \\ 1 & 0 & \cdots & 0 & -\beta_1 \\ 0 & 1 & \cdots & 0 & -\beta_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -\beta_{2p-1} \end{bmatrix} \tag{27}$$

Because the eigenvalues of the matrix (27) are the roots of function (25), the $2p$ roots of (25) can be calculated by performing an eigenvalue decomposition of $\boldsymbol{G}$. Compared with the Newton method, the proposed method does not require an iteration process and can find all roots of (25).

Because $\boldsymbol{G}$ has $2p$ eigenvalues, one has to determine the value that minimizes the performance index. The following lemma, derived from the second-order sufficient condition for a constrained optimization problem, can be used to select the Lagrangian multiplier corresponding to the local minimum point.

Lemma .1 *If $\tilde{\lambda}^*$ is an eigenvalue of (27), and $\prod_{j=1}^{r}(1+\tilde{\lambda}^*\xi_j) > 0 \quad j \neq i$ for $\forall i \in [1, r]$ with $r = 2, \cdots n$, then $\tilde{\lambda}^*$ corresponds to a minimal performance index.*

Proof .2 *See Appendix C.*

Remark .3 *If $\boldsymbol{A} = \boldsymbol{I}$, the constraint reduces to a norm constraint. Because $\tilde{\xi} = 1$ and $q = 1$, Eq. (25) can be simplified as*

$$s_2(\lambda) = \frac{1}{\varepsilon^2}[\ (1+\lambda\epsilon)^2 - \frac{\boldsymbol{h}^T\boldsymbol{h}}{l}\ ] = 0 \tag{28}$$

and its solution is given by

$$\lambda = -\frac{1}{\varepsilon} \pm \frac{|\boldsymbol{h}|}{\varepsilon\sqrt{l}} \tag{29}$$

It is easy to verify from Lemma .1 that the plus sign in Eq. (29) corresponds a minimum performance. The Kalman filter gain can be directly computed from Eq. (12a), which is the same as the gain derived in [7]. Moreover, the proposed constrained Kalman filter (CKF) is mathematically equivalent to the NCKF for a positive- semidefinite $\boldsymbol{A}$.

Remark .4 *A transformation* $\boldsymbol{z} = \boldsymbol{\Upsilon}\tilde{\boldsymbol{x}}$ *with the* (i,j) *element of* $\boldsymbol{\Upsilon}$ *given by*

$$\boldsymbol{\Upsilon}(i,j) = \begin{cases} 1, & \text{if } \xi_i = 0 \text{ and } i = j \\ \sqrt{|\xi_i|}, & \text{if } \xi_i \neq 0 \text{ and } i = j \\ 0 & \text{if } i \neq j \end{cases} \tag{30}$$

can be introduced to simplify Eq. (8) *as*

$$\boldsymbol{z}^T \tilde{\boldsymbol{\Xi}} \boldsymbol{z} = l \tag{31}$$

where $\tilde{\boldsymbol{\Xi}}$ *is a diagonal matrix whose elements include only* $0, 1, -1$*.*

For variable $\boldsymbol{z}$*, there are at most 2 distinct nonzero eigenvalues for its quadratic-form matrix, which indicates that there exists at most four roots for the Lagrangian multiplier.*

V. Simulation Results

In this section, two numerical examples are presented to demonstrate the CKF developed in the preceding sections. One is the tracking of a moving target along a hyperbolic road segment and the other is the constrained estimation for the attitude of a spinning spacecraft using Markley variables.

A. Tracking a target along a hyperbolic road segment

In this subsection, a simple example of tracking a moving target is considered to evaluate the effectiveness of the proposed CKF and to compare its performance with the traditional EKF and the linearized constrained Kalman filter (LCKF) [9]. For the LCKF, the constrained estimation is constructed by projecting the unconstrained estimation obtained from the EKF onto the linearized constraint surface. The primary purpose of this example is to illustrate the implementation of the CKF. A 2D situation is considered and the hyperbolic road constraint can be written as,

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{32}$$

where $a = b = 1$, and the center of the hyperbolic road is chosen as the origin of the $x-y$ coordinates.

Eq. (32) is in the form of Eq. (8) with

$$\boldsymbol{A} = \mathrm{diag}([\frac{1}{a^2}, -\frac{1}{b^2}, 0, 0]) \tag{33}$$

$l = 1$, and $\boldsymbol{x} = [x, y, \dot{x}, \dot{y}]^T$. The kinematic model of the target can be written as

$$\dot{x} = a\omega \sec\theta \tan\theta \quad (34a)$$

$$\dot{y} = b\omega \sec^2\theta \quad (34b)$$

$$\ddot{x} = a\omega^2(\sec\theta \tan^2\theta + \sec^3\theta) \quad (34c)$$

$$\ddot{y} = 2b\omega^2 \sec^3\theta \sin\theta \quad (34d)$$

where $\theta = \omega t$, and $\omega = 0.015$ rad/s is the angular velocity. The initial conditions of the target are chosen as $\boldsymbol{x}_0 = [1, -0.0075, -0.0001, 0.015]^T$. The target is equipped to measure its range relative to two reference points, r_a and r_b, where $\boldsymbol{r}_a = [-1; -1]^T$ and $\boldsymbol{r}_b = [5; 9]^T$. Then, the measurement equation is given by

$$\boldsymbol{y} = \begin{bmatrix} \sqrt{(\boldsymbol{r} - \boldsymbol{r}_a)^T(\boldsymbol{r} - \boldsymbol{r}_a)} \\ \sqrt{(\boldsymbol{r} - \boldsymbol{r}_b)^T(\boldsymbol{r} - \boldsymbol{r}_b)} \end{bmatrix} + \boldsymbol{w} \quad (35)$$

where $\boldsymbol{r} = [x, y]^T$, and the standard deviation of the measurement noise $\boldsymbol{w}$ is assumed to be 0.1.

The measurement sensitivity matrix corresponding to Eq. (35) is given by

$$\boldsymbol{H} = \begin{bmatrix} (\boldsymbol{r} - \boldsymbol{r}_a)^T / \sqrt{(\boldsymbol{r} - \boldsymbol{r}_a)^T(\boldsymbol{r} - \boldsymbol{r}_a)} & \mathbf{0}_{1\times 2} \\ (\boldsymbol{r} - \boldsymbol{r}_b)^T / \sqrt{(\boldsymbol{r} - \boldsymbol{r}_b)^T(\boldsymbol{r} - \boldsymbol{r}_b)} & \mathbf{0}_{1\times 2} \end{bmatrix} \quad (36)$$

The initial estimation position and velocity error are chosen as 0.05 and 2×10^{-3} for each axis, respectively, and the initial estimation error covariance is selected to be $\boldsymbol{P}_0 = \text{diag}[2.5 \times 10^{-3}, 2.5 \times 10^{-3}, 4 \times 10^{-6}, 4 \times 10^{-6}]$.

A simulation flowchart of the CKF is shown in Fig. 1. For the CKF, the essential difference between a hyperbolic constraint and the circular constraint presented in [13] or the norm constraint presented in [7] lies in the fact that, for a hyperbolic constraint, $\boldsymbol{A}$ is a positive-indefinite matrix, and $p = q = 2$, which indicates that Eq. (25) is a polynomial equation with degree four and that analytical solutions of the polynomial equation can not be obtained intuitively. Because $\boldsymbol{A}$ is a diagonal matrix, $\boldsymbol{\Xi} = \boldsymbol{A}$ and $\boldsymbol{U} = \boldsymbol{I}_2$. After some algebraic manipulations, the coefficients of

Eq. (25) are obtained as

$$\beta_0 = a^4b^4 - (a^2b^4h_1^2 - b^2a^4h_2^2) \tag{37a}$$

$$\beta_1 = -2a^4b^2 + 2a^2b^4 + 2a^2b^2(h_1^2 + h_2^2) \tag{37b}$$

$$\beta_2 = a^4 - 4a^2b^2 + b^4 - (a^2h_1^2 - b^2h_2^2) \tag{37c}$$

$$\beta_3 = -2b^2 + 2a^2 \tag{37d}$$

Four solutions of $\tilde{\lambda}$ are given by the eigenvalues of $\boldsymbol{G}$ in Eq. (27), and to determine the solution that minimizes the loss function, the second-order sufficient condition is used. According to Lemma 1, the $\tilde{\lambda}$ satisfying $-a^2 < \tilde{\lambda} < b^2$ corresponds an minimal index. In practice, an alternative simple approach to determine the optimal $\tilde{\lambda}$ is to evaluate the loss function (52) directly, noting that only two real eigenvalues exist for the companion matrix.

The Forbenius norms of the Kalman gains for the three filters and Lagrangian multiplier for the CKF are shown in Fig. 2 and Fig. 3, respectively. The norms of the gains for the LCKF and the EKF are convergent. However, the norms of the gain and Lagrangian multiplier for the CKF fluctuate with the noise, which coincides with the analysis in Sec. III. The effects of this phenomenon on the stability and convergence of the filter are unknown, and require further study in the future. As a direct consequence, the $\boldsymbol{P}^+$ in Eq. (19) is only a covariance-like matrix and is not strictly the error covariance, similar to the case of the unit-norm constraint [7]. Hence, the roots of the trace of $\boldsymbol{P}^+$ as shown in Fig. 4 may not accurately represent the performance of the CKF.

The root-mean-square-error (rmse) [10] values of the EKF, the LCKF, and the CKF are computed to compare the performance of these three filters. The values are shown in Fig 5. The average rmses of the EKF, the LCKF, and the CKF are 0.0186, 0.0068, and 0.0058, respectively. The three filters are convergent. Fig. 5 shows that the LCKF outperforms the EKF. The error of the constrained estimation of the LCKF is smaller than the unconstrained estimation error as proved in [9]. The overall performance of the CKF is better than that of the LCKF because the LCKF is subject to approximation errors depending on nonlinear terms and the point around which linearization occurs, whereas the proposed CKF can enforce a nonlinear constraint.

B. Constrained Attitude Estimation for Spinning Spacecraft Using Markley Variables

Attitude estimation is often more difficult for a spinning spacecraft than for a three-axis stabilized spacecraft because of the need to follow rapidly-varying state vector elements and the lack of three-axis rate measurements form gyros. Hence, Markley variables [6, 16, 18] may be employed instead of a quaternion-based representation for a spinning spacecraft.

The state for the Markley variables is defined as $\boldsymbol{x}^T = [\boldsymbol{L}_B^T, \boldsymbol{L}_I^T, \alpha]$, where $\boldsymbol{L}_I$ and $\boldsymbol{L}_B$ are the angular momentum components in an inertial frame ($\mathcal{F}_I$) and spacecraft's body frame ($\mathcal{F}_B$), respectively, and α will be defined later. Then the equations of motion of $\boldsymbol{L}_I$ and $\boldsymbol{L}_B$ can be written as [20, Chaps. 12 and 16],

$$\dot{\boldsymbol{L}}_I = \boldsymbol{R}_{BI}^T \boldsymbol{N}_B \tag{38}$$

$$\dot{\boldsymbol{L}}_B = \boldsymbol{N}_B - \boldsymbol{\omega}_B \times \boldsymbol{L}_B \tag{39}$$

where $\boldsymbol{N}_B$ is the external torque expressed in $\mathcal{F}_B$, $\boldsymbol{R}_{BI}$ is the attitude matrix from $\mathcal{F}_I$ to $\mathcal{F}_B$, and $\boldsymbol{\omega}_B = \boldsymbol{J}^{-1}(\boldsymbol{L}_B - \boldsymbol{L}_{\text{int}})$. Here, $\boldsymbol{J}$ is the spacecraft moment of inertia tensor and $\boldsymbol{L}_{\text{int}}$ is the angular momentum relative to the body frame of any internal moving components.

Denoting $\boldsymbol{l}_B$ and $\boldsymbol{l}_I$ as unit vectors of $\boldsymbol{L}_B$ and $\boldsymbol{L}_I$, respectively, consider a transformation from $\mathcal{F}_I$ to an intermediate frame ($\mathcal{F}_E$) that rotates about $\boldsymbol{l}_B \times \boldsymbol{l}_I$ by an angle ϕ ($\phi = \cos(\boldsymbol{l}_B \cdot \boldsymbol{l}_L)$). Then, define another transformation from $\mathcal{F}_E$ to $\mathcal{F}_B$ that rotates about $\boldsymbol{L}_B$ by an angle α. Therefore,

$$\boldsymbol{R}_{BI} = \boldsymbol{R}_{BE} \boldsymbol{R}_{EI} \tag{40}$$

where $\boldsymbol{R}_{EI}$ and $\boldsymbol{R}_{BE}$ are the attitude matrices from $\mathcal{F}_I$ to $\mathcal{F}_E$ and from $\mathcal{F}_E$ to $\mathcal{F}_B$, respectively, which can be written as [6, 21],

$$\boldsymbol{R}_{EI} = (\boldsymbol{l}_B \cdot \boldsymbol{l}_I)\boldsymbol{I}_3 - \boldsymbol{l}_I \boldsymbol{l}_B^T + \boldsymbol{l}_B \boldsymbol{l}_I^T + (1 + \boldsymbol{l}_B \cdot \boldsymbol{l}_I)^{-1}(\boldsymbol{l}_B \times \boldsymbol{l}_I)(\boldsymbol{l}_B \times \boldsymbol{l}_I)^T \tag{41}$$

$$\boldsymbol{R}_{BE} = \cos\alpha \boldsymbol{I}_3 + \frac{1 - \cos\alpha}{L^2} \boldsymbol{L}_B \boldsymbol{L}_B^T - \frac{\sin\alpha}{L}[\boldsymbol{L}_B \times] \tag{42}$$

Here, $\boldsymbol{I}_3$ is the 3×3 identity matrix, $[\boldsymbol{v}\times]$

$$[\boldsymbol{v}\times] = \begin{bmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{bmatrix} \tag{43}$$

denotes the cross-product matrix for any vector $\boldsymbol{v}$, and

$$L \equiv \|\boldsymbol{L}_B\| = \|\boldsymbol{L}_I\| \tag{44}$$

The equations of motion of α can be written as [6]

$$\dot{\alpha} = (1 + \boldsymbol{l}_B \cdot \boldsymbol{l}_I)^{-1}[(\boldsymbol{l}_B + \boldsymbol{l}_I) \cdot \boldsymbol{\omega}_{BI} + L^{-1}(\boldsymbol{l}_B \times \boldsymbol{l}_I)(\boldsymbol{N}_B + \boldsymbol{R}_{BI}^T \boldsymbol{N}_B)] \tag{45}$$

The constraint in Eq. (44) is in the form of Eq. (8) with

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{I}_3 & \mathbf{0}_{3\times3} & \mathbf{0}_{3\times1} \\ \mathbf{0}_{3\times3} & -\boldsymbol{I}_3 & \mathbf{0}_{3\times1} \\ \mathbf{0}_{1\times3} & \mathbf{0}_{1\times3} & 0 \end{bmatrix} \tag{46}$$

and $l = 0$. Therefore, the CKF developed in this note can be implemented directly. It should be noted that the analytical solution of $\tilde{\lambda}$ for this example can be explicitly derived from Eq. (24), which can be written as,

$$\tilde{\lambda}_1 = \frac{g_1 - g_2}{g_1 + g_2}\ , \tilde{\lambda}_2 = \frac{g_1 + g_2}{g_1 - g_2} \tag{47}$$

where, $g_1 = (h_1^2 + h_2^2 + h_3^2)^{0.5}$, $g_2 = (h_4^2 + h_5^2 + h_6^2)^{0.5}$ and h_i $(i = 1, 2, \cdots, 6)$ is the ith element of $\boldsymbol{h}$. According to Lemma 1, the $\tilde{\lambda}$ satisfying $|\tilde{\lambda}| < 1$ corresponds to a minimum index.

The initial orbital elements are the following: semimajor axis $=$ 43052.445 km, eccentricity $=$ 0.7778, inclination of the orbit $=$ 0.1658 rad, right ascension of the ascending node $=$ 6.2817 rad, and argument of perigee $=$ 0.00276 rad and true latitude $= 0$. The initial quaternion and angular velocity are $\boldsymbol{q}_0^T = [-0.7435, 0.6042, 0.1943, 0.2107]$ and $\boldsymbol{\omega}_0^T = [0.3747, 0, 2.0944]$ rad/s, respectively. The inertia tensor is given by $\boldsymbol{J} = \text{diag}[200, 200, 384]$ kg-m^2 and $\boldsymbol{L}_{\text{int}}$ is assumed to be zero. Then, the initial values of the attitude matrix $\boldsymbol{R}_{BI}$ and α can be computed from $\boldsymbol{q}_0$ [19, pp.612] and Eq. (40), respectively. The initial values of $\boldsymbol{L}_B$ and $\boldsymbol{L}_I$ are given by $\boldsymbol{J}\boldsymbol{\omega}_0$ and $\boldsymbol{R}_{BI}^T(t_0)\boldsymbol{L}_B$, respectively.

The spacecraft is assumed to carry three-axis magnetometers (TAM) to measure the direction of the Earth's magnetic fields and a sun sensor to measure the direction vector to the Sun in $\mathcal{F}_B$. The $1 - \sigma$ of the measurement noise for both sensors is 10^{-2} in each axis.

Similar with the scenarios in [6], three scenarios are considered here. The nominal scenario consists of torque-free motion, a 2-deg nutation angle, no sensor misalignments, and uses 30 observations per spin period. The second scenario is the same with the nominal scenario, but with misalignments of 0.5 deg on the sensors. The third scenario is also similar with the nominal scenario, but with a torque applied in the body frame to precess the angular momentum vector by 45-deg during the 1400 s time span, and the torque values are not passed to the filter. For these three scenarios, the initial attitude error and rate error is 10 deg and 5 deg/s in each axis, respectively.

The performance of the CKF is compared with two of other Markley variables based filters, known as SpinKF: SpinKF-I, and SpinKF-B [6]. The error states of SpinKF-I and SpinKF-B are the vector of infinitesimal attitude error angles and the angular momentum vector expressed in the inertial frame and the vector of infinitesimal attitude error angles and the angular momentum vector expressed in the spacecraft body frame, respectively. State update preserves the constraint in the CKF, but may violate it in the SpinKF filters. Therefore, both SpinKF-I and SpinKF-B require restoration of the constraint after the update by trusting the magnitude of the angular momentum in the inertial frame more than that in the spacecraft body frame or vice versa.

Monte Carlo simulation is used to generate the mean errors in the estimate of the angular momentum direction of these three scenarios after the filters have converged. The results are based on a total of 100 runs and summarized in Table 1. Analysis revealed that the three filters filters are performing well and that the mean error in angular momentum direction using the CKF is smaller than that those using other filters for the nominal scenario and the scenario with misaligned sensors. However, SpinKF-B performs better than the other filters with the attitude slew. This may due to the fact that an estimate of L is required to propagate $\boldsymbol{L}_I$ and $\boldsymbol{\alpha}$. The CKF can preserve the constraint, but cannot ensure that the estimate of L is more accurate than that in the SpinKF filters. It should be noted that the performance of the CKF is also affected by other factors as well, such as number of observations per spin period, propagation step size, and error in the inertia tensor model. In future work, these factors will be considered to evaluate the CKF's robustness.

Table 1 Mean error in angular momentum direction (deg)

Scenario	SpinKF-B	SpinKF-I	CKF
Nominal	0.157	0.145	0.141
Misaligned sensors	0.314	0.291	0.287
Attitude slew (no torque data to filters)	1.307	1.343	1.374

VI. Conclusions

A constrained Kalman filter (CKF) with a quadratic state equality constraint that generalizes a norm equality constraint was developed by Lagrangian multiplier technique. Robust solutions for the Lagrangian multiplier were determined through eigenvalue decomposition of a companion matrix of a polynomial function instead of Newton-Raphson iteration, and second-order optimality conditions determine the minimizing value of the Lagrangian multipliers.

The performance of the CKF was evaluated by tracking a target on a hyperbolic road segment and attitude estimation for a spinning spacecraft using a seven-parameter angular-momentum-based representation, referred to as Markley variables. The main advantage of the CKF is that it can help correct unrealistic estimates and can enforce the constraint exactly, thereby avoiding approximation errors caused by linearization of the linearized Constrained Kalman filter (LCKF). However, the Kalman gain and Lagrangian multiplier for the CKF fluctuate with noise, leading to a suboptimal estimation in the sense that only a covariance-like matrix, an approximation of error covariance, is utilized for covariance update.

For spinning spacecraft attitude estimation, the CKF was tested against two versions of other filters using the same representation but accounting for the constraint by employing a reduced six-component error state, known as the SpinKF filters. Numerical results showed that the CKF outperforms the SpinKF filters except in the situation where a torque is applied to precess the spin axis and the torque values are not passed to the filters. The difference in performance is attributed to the fact that an estimate of the magnitude of the angular momentum is required to perform state propagation in the filters. The CKF can enforce the constraint on the state vector, but cannot ensure that the estimate of the magnitude of the angular momentum is more accurate than that in the SpinKF filters.

In the future, we plan to establish a more accurate approximation of error covariance for constrained estimation and to evaluate the stability and convergence of the CKF. We also plan to test the robustness of the CKF with application in attitude estimation using Markley variables.

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Appendix

A. Necessary and sufficient conditions of a constrained optimization problem

Consider the problem

$$\min f(\boldsymbol{x}) \text{ subject to } g(\boldsymbol{x}) = 0 \tag{48}$$

Define the Lagrangian as

$$J(\boldsymbol{x}, \lambda) = f(\boldsymbol{x}) + \lambda g(\boldsymbol{x}) \tag{49}$$

Then, $\boldsymbol{x}^*$ is a local minimum if and only if there exist a unique λ^* s.t.

1. $\nabla J_x(\boldsymbol{x}^*, \lambda^*) = \mathbf{0}$ and $\nabla J_\lambda(\boldsymbol{x}^*, \lambda^*) = 0$ (necessary condition)

2. $\boldsymbol{z}^T \nabla^2_{xx} J(\boldsymbol{x}^*, \lambda^*) \boldsymbol{z} \geq 0 \quad \forall \boldsymbol{z}$ s.t. $\nabla_{\boldsymbol{x}} g(\boldsymbol{x}^*)^T \boldsymbol{z} = 0$ (sufficient condition)

B. Condition for a quadratic form to be positive (negative) subject to linear constraints

Let $\boldsymbol{A}$ be a symmetric $n \times n$ matrix and $\boldsymbol{B}$ an $m \times n$ matrix with full row rank m. Let $\boldsymbol{A}_{rr}$ denote the $r \times r$ matrix in the top left corner of $\boldsymbol{A}$ and $\boldsymbol{B}_r$ the $m \times r$ matrix whose columns are the first r columns of $\boldsymbol{B}$. Assume that $|\boldsymbol{B}_m| \neq 0$. Define the $(m + r) \times (m + r)$ matrices

$$\boldsymbol{\Delta}_r = \begin{bmatrix} 0 & \boldsymbol{B}_r \\ \boldsymbol{B}_r^T & \boldsymbol{A}_{rr} \end{bmatrix} \quad (r = 1, 2, \cdots, n) \tag{50}$$

and let $\Omega = (\boldsymbol{x} \in \mathcal{R}^n : \boldsymbol{x} \neq \boldsymbol{0}, \boldsymbol{B}\boldsymbol{x} = \boldsymbol{0})$ Then, $\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{x} > 0$ for all $\boldsymbol{x} \in \Omega$ if and only if [22, pp.61-62]

$$(-1)^m|\boldsymbol{\Delta}_r| > 0 \tag{51}$$

C. Derivation of Lemma 1

The loss function (10) is rewritten in terms of $\boldsymbol{K}$ as

$$J = f(\boldsymbol{K}) + \lambda g(\boldsymbol{K}) \tag{52}$$

where,

$$f(\boldsymbol{K}) = \frac{1}{2}\mathrm{Tr}\left[\boldsymbol{K}\boldsymbol{R}\boldsymbol{K}^T + (\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H})\boldsymbol{P}^-(\boldsymbol{I} - \boldsymbol{K}\boldsymbol{H})^T\right] \tag{53}$$

$$g(\boldsymbol{K}) = \frac{1}{2}\left[\boldsymbol{\epsilon}^T\boldsymbol{K}^T\boldsymbol{A}\boldsymbol{K}\boldsymbol{\epsilon} + 2\boldsymbol{\epsilon}^T\boldsymbol{K}^T\boldsymbol{A}\hat{\boldsymbol{x}}^- + (\hat{\boldsymbol{x}}^-)^T\boldsymbol{A}\hat{\boldsymbol{x}}^- - l\right] \tag{54}$$

Denote $L(\boldsymbol{K}^*, \lambda^*) = \nabla^2_{KK}J(\boldsymbol{K}^*, \lambda^*) = \boldsymbol{W} \otimes \boldsymbol{I}_n + \lambda\boldsymbol{\epsilon}\boldsymbol{\epsilon}^T \otimes \boldsymbol{A}$, where $\otimes$ is the Kronecker product. Suppose $\boldsymbol{K}^*$ is a solution of Eq. (21) and λ^* is given by Eq. (24). Then, according to Appendix A, $\boldsymbol{K}^*$ is a local minimum point of $f(\boldsymbol{K})$ subject to the constraint $g(\boldsymbol{K}) = 0$ if

$$Q(\boldsymbol{z}) = \mathrm{vec}(\boldsymbol{z})^T L(\boldsymbol{K}^*, \lambda^*)\mathrm{vec}(\boldsymbol{z}) > 0 \tag{55}$$

for all $\boldsymbol{z}$ satisfying $\boldsymbol{z} \cdot \nabla_K g(K^*) = (\boldsymbol{I}_m \otimes \nabla g^T)\mathrm{vec}(\boldsymbol{z}) = 0$, where

$$\nabla_K g(\boldsymbol{K}^*) = \boldsymbol{A}(\boldsymbol{I} + \lambda\varepsilon\boldsymbol{A})^{-1}(\boldsymbol{d} + \hat{\boldsymbol{x}}^-)\boldsymbol{\epsilon}^T \tag{56}$$

According to Appendix B, to determine the definiteness of $\boldsymbol{Q}(\boldsymbol{z})$ in Eq. (55) in the tangent space of $\nabla g(\boldsymbol{K^*})$, the following matrices are constructed:

$$\boldsymbol{S}_r = \begin{bmatrix} \boldsymbol{0} & (\boldsymbol{I}_m \otimes \nabla g^T)_r \\ (\boldsymbol{I}_m \otimes \nabla g)_r & \boldsymbol{L}_{rr}(\boldsymbol{K^*}, \boldsymbol{\lambda^*}) \end{bmatrix} \tag{57}$$

where $\boldsymbol{L}_{rr}(\boldsymbol{K^*}, \boldsymbol{\lambda^*})$ is the $r \times r$ matrix in the top left corner of $\boldsymbol{L}(\boldsymbol{K^*}, \boldsymbol{\lambda^*})$, and $(\boldsymbol{I}_m \otimes \nabla g^T)_r$ is the matrix whose columns are the first r columns of $\boldsymbol{I}_m \otimes \nabla g^T$. It can be seen that $\boldsymbol{S}_r$ is in the form of Eq. (50), and $\boldsymbol{Q} > 0$ for all $\boldsymbol{z} \cdot \nabla g(K^*) = 0$ if and only if

$$(-1)^m|\boldsymbol{S}_r| > 0 \quad (r = m(m+1), \cdots, mn) \tag{58}$$

Only a scalar measurement situation ($m = 1$) is considered in this note for analyzing $|\boldsymbol{S}_r|$, as there is no essential difference between a scalar measurement and vector measurements. Eq. (57) can be written as

$$\boldsymbol{S}_r = \begin{bmatrix} 0 & \boldsymbol{h}_r^T \epsilon (\boldsymbol{I} + \lambda \varepsilon \boldsymbol{A}_r)^{-1} \boldsymbol{A}_r \\ \boldsymbol{A}_r (\boldsymbol{I}_r + \lambda \varepsilon \boldsymbol{A}_r)^{-1} \boldsymbol{h}_r \epsilon & W(\boldsymbol{I}_r + \lambda \varepsilon \boldsymbol{A}_r) \end{bmatrix} \tag{59a}$$

$\boldsymbol{S}_r$ is a symmetrical partitioned matrix and its determinant is given by

$$|\boldsymbol{S}_r| = -\varepsilon |W| |\boldsymbol{I}_r + \lambda \epsilon \boldsymbol{A}_r| \boldsymbol{h}_r^T (\boldsymbol{I} + \lambda \varepsilon \boldsymbol{A}_r)^{-1} \boldsymbol{A}_r (\boldsymbol{I}_r + \lambda \varepsilon \boldsymbol{A}_r)^{-1} \boldsymbol{A}_r (\boldsymbol{I}_r + \lambda \varepsilon \boldsymbol{A}_r)^{-1} \boldsymbol{h}_r \tag{60a}$$

$$= -\varepsilon W |\boldsymbol{I}_r + \lambda \varepsilon \boldsymbol{\Xi}_r| \boldsymbol{h}_r^T \boldsymbol{U}_r (\boldsymbol{I}_r + \lambda \varepsilon \boldsymbol{\Xi}_r)^{-3} \boldsymbol{\Xi}_r^2 \boldsymbol{U}_r^T \boldsymbol{h}_r \tag{60b}$$

Generally, the sign of $|\boldsymbol{S}_r|$ should be verified using Eq. (60), but it can be seen that a sufficient condition of $|\boldsymbol{S}_r| < 0$ is $|(\boldsymbol{I} + \lambda \varepsilon \boldsymbol{\Xi}_r)|(\boldsymbol{I}_r + \lambda \varepsilon \boldsymbol{\Xi}_r)^{-1} > 0$. Because the ith eigenvalue of $(\boldsymbol{I}_r + \lambda \varepsilon \boldsymbol{\Xi}_r)|(\boldsymbol{I}_r + \lambda \varepsilon \boldsymbol{\Xi}_r)^{-1}$ is given by $\prod_{j=1}^{r} (1 + \varepsilon \lambda \xi_j) \quad j \neq i$. Therefore, Lemma 1 is derived.

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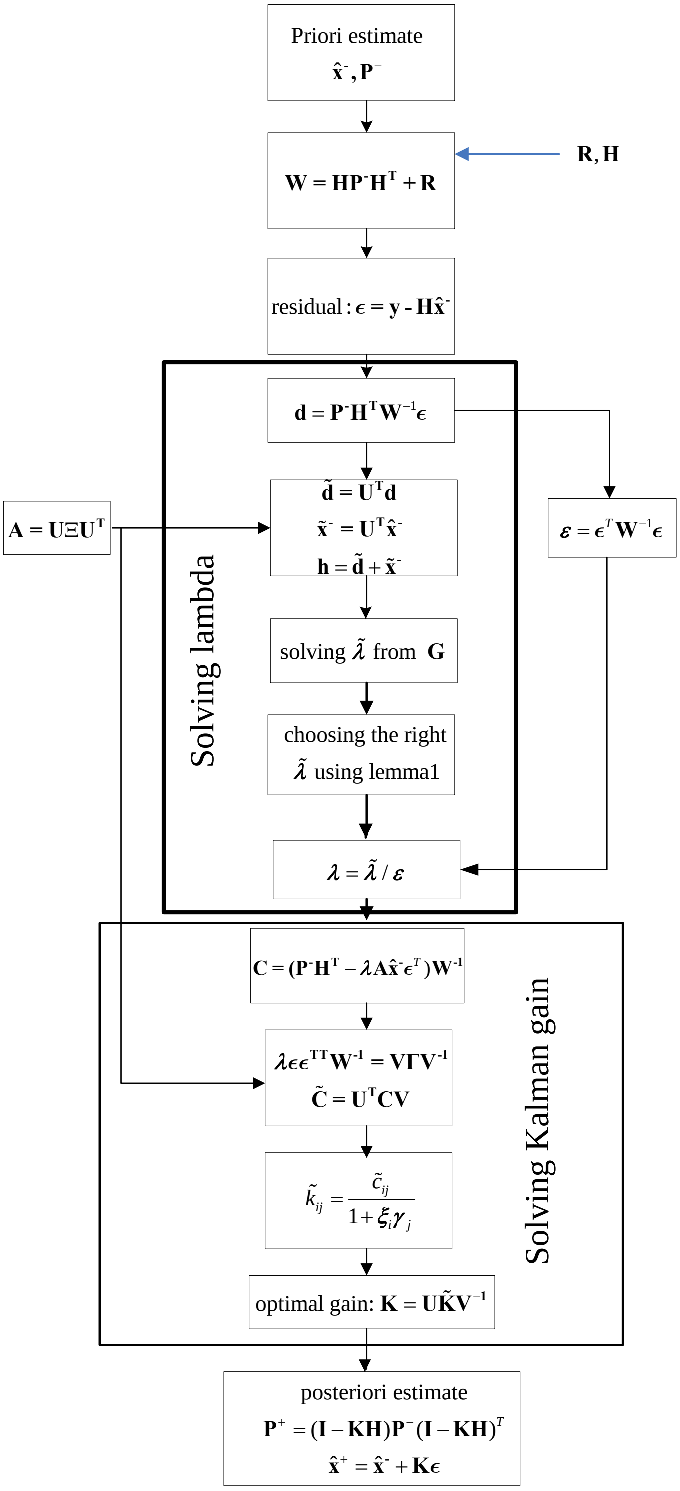

Fig. 1 Simulation flowchart of the CKF

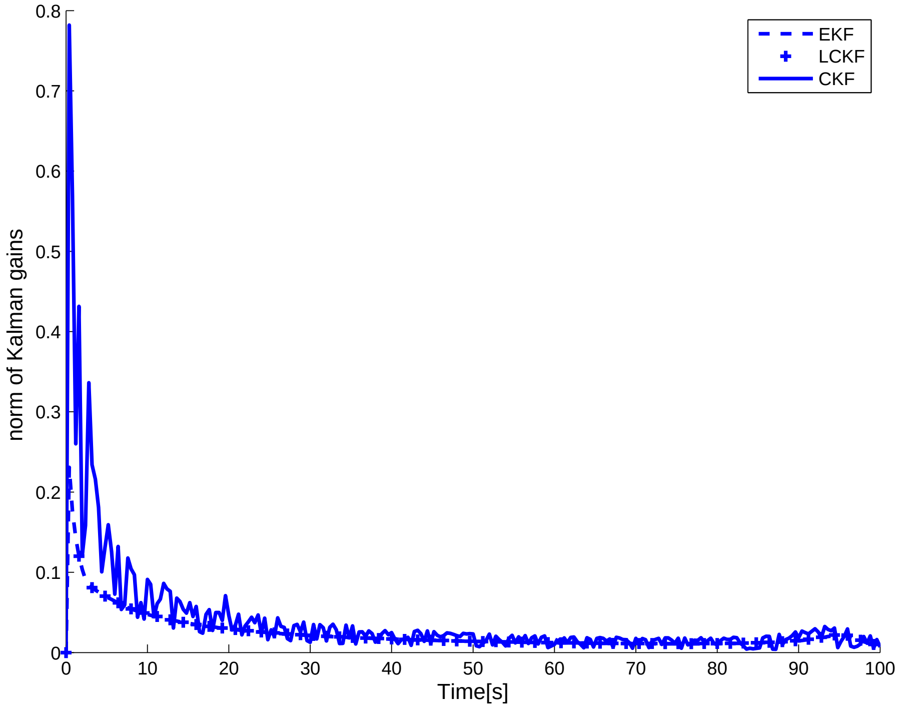

Fig. 2 norm of Kalman gains

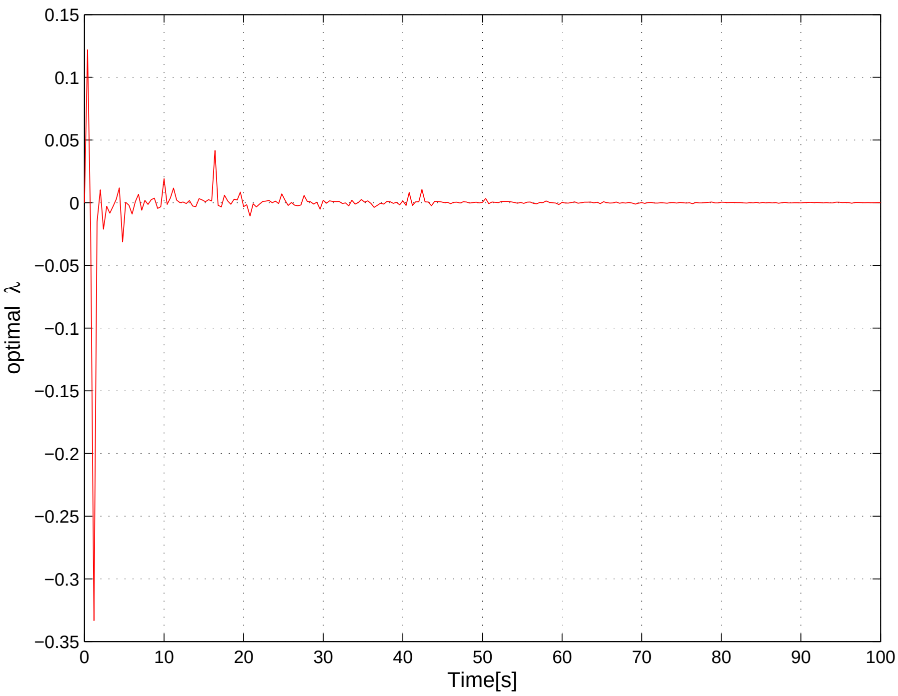

Fig. 3 Optimal Lagrangian Multiplier

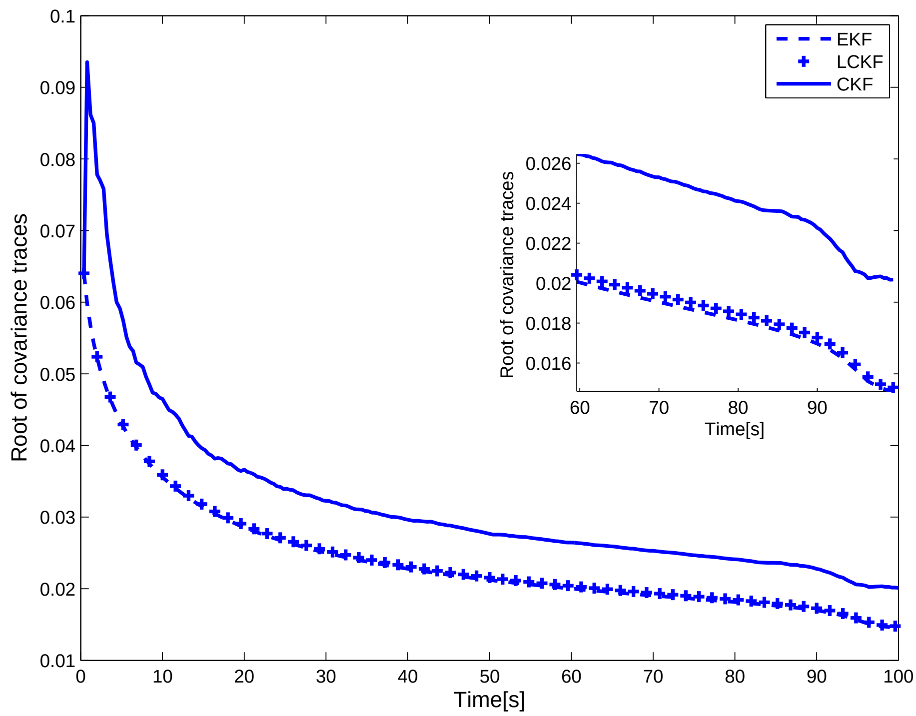

Fig. 4 Root of trace of the error covariance

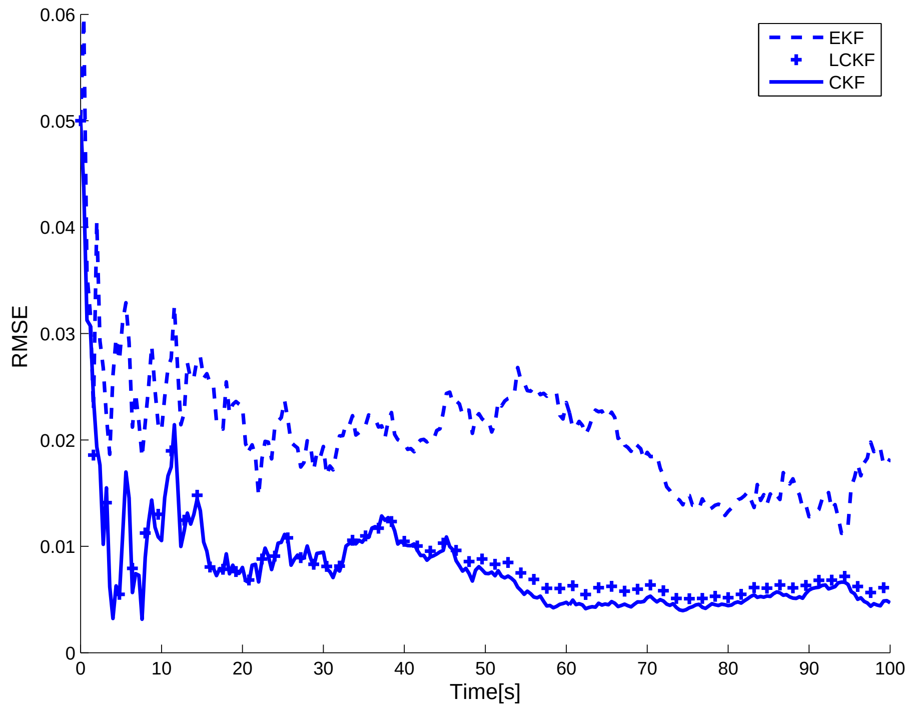

Fig. 5 Rmse of the EKF, LCKF, and CKF