Asymptotic behavior of critical points of an energy involving a " circular-well " potential

Petru Mironescu, Itai Shafrir

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Abstract

We describe the asymptotic behavior of critical points of $\int_\Omega [(1/2)|\nabla u|^2 + W(u)/\varepsilon^2]$ when $\varepsilon \to 0$. Here, W is a Ginzburg-Landau type potential, vanishing on a simple closed curve Γ. Unlike the case of the standard Ginzburg-Landau potential $W(u) = (1-|u|^2)^2/4$, studied by Bethuel, Brezis and Hélein, we do not assume any symmetry on W or Γ. In order to overcome the difficulties due to the lack of symmetry, we develop new tools which might be of independent interest.

1 Statement of the problem

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded star-shaped domain. Let $\Gamma \subset \mathbb{R}^2$ be a smooth simple curve and let $g : \partial\Omega \to \Gamma$ be a smooth boundary datum of degree d. Consider, for every $\varepsilon > 0$, a critical point $u_\varepsilon \in H^1_g(\Omega;\mathbb{R}^2)$ of the energy

$$E_\varepsilon(u) = \int_\Omega \left[\frac{1}{2}|\nabla u|^2 + \frac{W(u)}{\varepsilon^2}\right]. \tag{1.1}$$

Here, $W : \mathbb{R}^2 \to [0,\infty)$ is a smooth potential vanishing precisely on Γ; for the exact assumptions on W, see (1.5)–(1.10) below.

In the Ginzburg-Landau (GL) case, i.e., when $W(u) = (1-|u|^2)^2/4$, the asymptotic behavior of $\{u_\varepsilon\}$ when $\varepsilon \to 0$ was studied by Bethuel, Brezis and Hélein, first for minimizers when the boundary condition has zero degree in [4], and later for minimizers and, more generally, for critical points for arbitrary boundary datum in the seminal work [5].

The analysis in [5] for *minimizers* of the GL energy can be adapted with no significant difficulty to the case of general W, at least when W is *non-degenerate*, see (1.9). Using more involved arguments, it is even possible to describe the asymptotic behavior of minimizers in the case of a general boundary condition g that does not necessarily take values into Γ; see André and Shafrir [3].

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We address here the question of the asymptotic behavior of critical points of the energy (1.1), i.e., of solutions of

$$\begin{cases} \Delta u_\varepsilon = \dfrac{1}{\varepsilon^2}\nabla W(u_\varepsilon) & \text{in } \Omega \\ u_\varepsilon = g & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

that need not be energy minimizing with respect to their own boundary condition. As we will see below, the answer to this question requires new ideas and ingredients.

The method of proof in [5, Chapter X] for critical points of the GL energy is based on a clever decomposition of the gradient ∇u_ε. Its starting point is the identity

$$\frac{\partial}{\partial x_1}\left(u_\varepsilon \times \frac{\partial}{\partial x_1}u_\varepsilon\right) + \frac{\partial}{\partial x_2}\left(u_\varepsilon \times \frac{\partial}{\partial x_2}u_\varepsilon\right) = 0, \tag{1.3}$$

which is a direct consequence of the fact that $W(u) = W(|u|)$ in the GL case. We could not find an analogous identity to (1.3) for general W. Our method is different and relies on two main tools:

1. Selection of "good rays" (see Subsection 5.2).

2. A maximum principle for the phase (see Proposition 2.1).

Combined, they allow us to prove a crucial estimate, namely

$$E_\varepsilon(u_\varepsilon) \le C(|\log\varepsilon| + 1). \tag{1.4}$$

The first ingredient is new even for the GL energy (and leads to a simplification of the original arguments in [5, Chapter X]), and the second one is much more subtle in the case of a general potential W than in the GL case.

For the analysis of solutions to (1.2) we will need, in the spirit of [5], the additional assumption that Ω is strictly star-shaped. This assumption enables us to prove that the second term in the energy (1.1) remains bounded when $\varepsilon \to 0$, and then to perform the "bad discs" construction à la Bethuel-Brezis-Hélein [5], which is the starting point of the study of the location of the vortices.

The remaining part of the analysis is similar to the one in [5] (with some technical complications), and leads to our main result, Theorem 1.1 below. In order to state it, we first present all the assumptions on Γ and W.

$$W : \mathbb{R}^2 \to [0,\infty) \text{ is a smooth function satisfying } W^{-1}(\{0\}) = \Gamma \tag{1.5}$$

and

$$\Gamma \text{ is a simple closed smooth curve in } \mathbb{R}^2. \tag{1.6}$$

We assume without loss of generality that

$$|\Gamma| = 2\pi \tag{1.7}$$

and consider

$$\tau : \mathbb{S}^1 \to \Gamma \text{ an arc length parametrization of } \Gamma. \tag{1.8}$$

We also suppose that W is *non-degenerate* in the following sense:

$$W(\zeta) \geq \mu \operatorname{dist}^2(\zeta, \Gamma) \text{ if } \operatorname{dist}(\zeta, \Gamma) < \delta, \tag{1.9}$$

for some $\mu, \delta > 0$ (and then it follows from (1.5) that (1.9) holds on any compact subset of $\mathbb{R}^2$).

In addition, we impose the following coercivity assumption on the behavior of W at infinity:

$$\frac{\partial W}{\partial r}(z) \geq 0 \text{ for } |z| = r > R_0, \tag{1.10}$$

for some $R_0 > \max\{|z|; z \in \Gamma\}$.

1.1 Theorem. *Let Ω be a smooth, bounded, strictly star-shaped domain in $\mathbb{R}^2$. Let W, Γ and τ satisfy (1.5)–(1.10). Let $g : \partial\Omega \to \Gamma$ be a smooth boundary condition of degree d. For each $\varepsilon > 0$, let u_ε denote a solution of (1.2). Then up to a subsequence we have*

$$u_{\varepsilon_n} \to u_* = \tau\left(e^{\imath\eta(z)}\left(\frac{z-a_1}{|z-a_1|}\right)^{D_1} \cdots \left(\frac{z-a_N}{|z-a_N|}\right)^{D_N}\right) \text{ in } C^{1,\alpha}\left(\overline{\Omega} \setminus \{a_1, \ldots, a_N\}\right), \tag{1.11}$$

where

1. *$a_1, \ldots, a_N \in \Omega$ are mutually distinct points.*
2. *$D_1, \ldots, D_N \in \mathbb{Z} \setminus \{0\}$ satisfy the compatibility condition $\sum_{j=1}^N D_j = d$.*
3. *η is a harmonic function in $\overline{\Omega}$.*
4. *$\alpha \in (0,1)$.*

In the spirit of [5], we may also prove that the configuration $(a_1, \ldots, a_N)$ is a critical point of a suitable renormalized energy associated with the degrees $(D_j)_{j=1}^N$ and the boundary condition; see Remark 5.17 in Section 5.

Let us mention that non minimizing solutions do exist. For the GL energy, their existence was established in different situations. In the special case where Ω is the unit disc and $g(z) = z^d$, with $|d| \geq 2$, the GL energy has critical points of the form $u_\varepsilon(re^{\imath\theta}) = f_\varepsilon(r)e^{\imath d\theta}$, and these solutions are not minimizing for sufficiently small ε [5]. Non minimizing critical points also exist when $d = 0$: F.H. Lin [10] constructed examples of "mixed vortex-antivortex solutions". More specifically, for all $N \geq 1$ there exists $g_N : \partial\Omega \to \mathbb{S}^1$ of degree 0 and non minimizing corresponding critical points u_{ε_n} such that

$$u_{\varepsilon_n} \to u_* = e^{\imath\eta_N(z)} \prod_{j=1}^{2N} \left(\frac{z - a_{j,N}}{|z - a_{j,N}|}\right)^{(-1)^{j-1}}.$$

Other existence results concerning non minimizing solutions for the the GL energy were proved by Almeida and Bethuel [1] and by F. Zhou and Q. Zhou [13], using variational and topological methods. We believe that at least some of these methods lead to the existence of non minimizing critical points of (1.2) for a general W, but we did not investigate this issue.

Except for the upper bound (1.4), we did not establish a more precise estimate for the energy $E_\varepsilon(u_\varepsilon)$. In the case of the GL-energy, Comte and Mironescu [6] proved that the following is true:

$$E_\varepsilon(u_\varepsilon) = \pi \left(\sum_{j=1}^{N} D_j^2 \right) |\log \varepsilon| + O(1). \tag{1.12}$$

It would be interesting to generalize the validity of (1.12) to our setting.

The paper is organized as follows. In Section 2 we introduce some notation and prove a maximum principle for the phase, that plays an important role in the remaining part of the paper. In Section 3 we study the case of boundary data of zero degree ($d = 0$) under the additional assumption that the solutions stay close to Γ, i.e., no vortices appear. The techniques of this section are used in Section 4 to treat the more general case of a boundary data depending on ε (again, for vortex-less solutions). This latter case is very useful in the proof of convergence away from the vortices in Theorem 1.1. Section 5 is devoted to the proof of the main result, Theorem 1.1.

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Contents

2 Preliminaries

2.1 Coordinates and Euler-Lagrange equations

Consider $\Gamma_\delta := \{z \in \mathbb{R}^2; \operatorname{dist}(z, \Gamma) < \delta\}$. For sufficiently small δ_Γ (depending on Γ) the Euclidean nearest point projection Π on Γ is well-defined and smooth in Γ_{δ_Γ} (see e.g. [8, Sec. 14.6]).

Assume in what follows that $u : \omega \to \mathbb{R}^2$ is a smooth map such that

$$u(x) \in \Gamma_{\delta_\Gamma}, \ \forall\, x \in \omega. \tag{2.1}$$

(Here, $\omega \subset \mathbb{R}^2$ is some open set.) Locally in ω, we can associate to u two smooth coordinates, t and φ, such that $\Pi \circ u = \tau(e^{\imath\varphi})$ and t is the signed distance of u to Γ (taken with the plus sign inside Γ). Analytically, this means that the functions t and φ satisfy $(t(x), \varphi(x)) \in (-\delta_\Gamma, \delta_\Gamma) \times \mathbb{R}$ and

$$u(x) = \tau\left(e^{\imath\varphi(x)}\right) + t(x)\,\vec{n}\left(\tau\left(e^{\imath\varphi(x)}\right)\right). \tag{2.2}$$

Here, $\vec{n}(z)$ denotes the inward unit normal to Γ at the point $z \in \Gamma$.

Equivalently, we have

$$\Pi(u(x)) = \tau\left(e^{\imath\varphi(x)}\right) \text{ and } t(x) = (u(x) - \Pi(u(x))) \cdot \vec{n}\,(\Pi(u(x))). \tag{2.3}$$

Note that t is globally defined, but φ is only *locally* defined in ω, and that φ is (locally) unique mod 2π. It is useful to note that φ is *globally* defined when ω is simply connected.

A simple calculation (see [2, Lemma 4.1]) shows that for u satisfying (2.1) we have (denoting by $\kappa(z)$ the curvature of Γ at the point $z \in \Gamma$)

$$|\nabla u|^2 = \left(1 - t\kappa\left(\tau(e^{\imath\varphi})\right)\right)^2 |\nabla\varphi|^2 + |\nabla t|^2 = (1 - t\kappa(\Pi \circ u))^2\,|\nabla\varphi|^2 + |\nabla t|^2. \tag{2.4}$$

Moreover, for such u we have (using (1.9)) that

$$W(u) = \alpha(\varphi, t)\,t^2 \tag{2.5}$$

where $\alpha(\varphi, t)$ is a smooth positive function, 2π-periodic in the φ-variable.

Assume next that $u = u_\varepsilon$ is a solution of (1.2) in Ω and that $\omega \subset \Omega$ is such that (2.1) holds. Then locally in ω we may use (2.5) to write the Euler-Lagrange equations (1.2) for the function u in the new coordinates t and φ as follows.

$$-\operatorname{div}(a\nabla\varphi) = b|\nabla\varphi|^2 - \frac{\alpha_\varphi t^2}{\varepsilon^2}, \tag{2.6a}$$

$$-\Delta t + (2\alpha + \alpha_t t)\frac{t}{\varepsilon^2} = c|\nabla\varphi|^2. \tag{2.6b}$$

In (2.6), the coefficients $a = a(\varphi, t)$, $b = b(\varphi, t)$ and $c = c(\varphi, t)$ are given by

$$\begin{cases} a = \left(1 - t\kappa\left(\tau(e^{\imath\varphi})\right)\right)^2 = 1 + O(t) \\ b = -\dfrac{1}{2}a_\varphi = O(t) \\ c = -\dfrac{1}{2}a_t = O(1) \end{cases}. \tag{2.7}$$

2.2 A maximum principle for the phase

By (2.5)–(2.7), for sufficiently small $\delta_0 \in (0, \delta_\Gamma)$ there exist positive constants $c_0, \ldots, c_5$ such that for $|t| \le \delta_0$ there holds:

$$|1 - a| \le c_0|t|, \tag{2.8a}$$

$$2\alpha - |\alpha_t t| \ge c_1, \tag{2.8b}$$

$$|c| \le c_2, \tag{2.8c}$$

$$\left|\frac{b}{a}\right| \le c_3|t|, \tag{2.8d}$$

$$\left|\frac{a_t}{a}\right| \le 2c_4, \tag{2.8e}$$

$$\left|\frac{\alpha_\varphi}{a}\right| \le c_5. \tag{2.8f}$$

Note that δ_0 depends only on Γ.

Next we prove a maximum principle for the phase φ, that will be useful throughout the paper. For this purpose, we introduce two numbers, $0<\delta_1<\delta_0$ and $m>0$, satisfying

$$\frac{c_5}{c_1}\le m \tag{2.9}$$

and

$$2c_4\delta_1+m(mc_2+c_3)\delta_1^3<1. \tag{2.10}$$

Note that δ_1 and m depend only on Γ and W.

2.1 Proposition. *Let $u=u_\varepsilon$ be a critical point of E_ε in a bounded simply connected domain ω, continuous on $\overline{\omega}$ and satisfying* $\operatorname{dist}(u(x),\Gamma)\le\delta_1$, $\forall\, x\in\overline{\omega}$. *Consider $t=t_\varepsilon,\varphi=\varphi_\varepsilon$ associated to u via (2.2). Then*

$$\min_{\overline{\omega}}\left(\varphi-\frac{mt^2}{2}\right)=\min_{\partial\omega}\left(\varphi-\frac{mt^2}{2}\right), \tag{2.11a}$$

$$\max_{\overline{\omega}}\left(\varphi+\frac{mt^2}{2}\right)=\max_{\partial\omega}\left(\varphi+\frac{mt^2}{2}\right). \tag{2.11b}$$

2.2 Corollary. *If, in addition to the hypotheses of Proposition 2.1, we have $u_\varepsilon(\partial\omega)\subseteq\Gamma$, then*

$$\min_{\partial\omega}\varphi\le\varphi(x)-\frac{mt^2(x)}{2}\le\varphi(x)+\frac{mt^2(x)}{2}\le\max_{\partial\omega}\varphi,\quad\forall\, x\in\omega. \tag{2.12}$$

In particular,

$$\min_{\partial\omega}\varphi\le\varphi(x)\le\max_{\partial\omega}\varphi,\ \forall\, x\in\overline{\omega}. \tag{2.13}$$

Proof of Proposition 2.1. First, we may rewrite the equation (2.6a) as

$$-\Delta\varphi=\frac{1}{a}\nabla a\cdot\nabla\varphi+\frac{b}{a}|\nabla\varphi|^2-\frac{\alpha_\varphi t^2}{a\varepsilon^2}.$$

Using

$$\nabla a\cdot\nabla\varphi=a_\varphi|\nabla\varphi|^2+a_t\nabla\varphi\cdot\nabla t=-2b|\nabla\varphi|^2+a_t\nabla\varphi\cdot\nabla t$$

yields

$$-\Delta\varphi=-\frac{b}{a}|\nabla\varphi|^2+\frac{a_t}{a}\nabla\varphi\cdot\nabla t-\frac{\alpha_\varphi}{a}\frac{t^2}{\varepsilon^2}. \tag{2.14}$$

From (2.6b) we deduce

$$-\Delta\left(\frac{t^2}{2}\right)=-|\nabla t|^2-t\Delta t=-|\nabla t|^2+ct|\nabla\varphi|^2-(2\alpha+\alpha_t t)\frac{t^2}{\varepsilon^2}. \tag{2.15}$$

Combining (2.14)–(2.15) and invoking (2.8) gives

$$\begin{aligned}-\Delta\left(\frac{mt^2}{2}-\varphi\right)\leq&(mc_2+c_3)|t||\nabla\varphi|^2-m|\nabla t|^2+2c_4|\nabla\varphi||\nabla t|\\&+(c_5-mc_1)\frac{t^2}{\varepsilon^2}.\end{aligned}\tag{2.16}$$

We also have

$$\begin{aligned}\left|\nabla\left(\frac{mt^2}{2}-\varphi\right)\right|^2=&m^2t^2|\nabla t|^2-2mt\nabla\varphi\cdot\nabla t+|\nabla\varphi|^2\\\geq&m^2t^2|\nabla t|^2-2m|t||\nabla\varphi||\nabla t|+|\nabla\varphi|^2.\end{aligned}\tag{2.17}$$

By (2.16)–(2.17) we obtain, for any $k>0$,

$$\begin{aligned}-\Delta\left(\frac{mt^2}{2}-\varphi\right)-k\left|\nabla\left(\frac{mt^2}{2}-\varphi\right)\right|^2\leq&(mc_2|t|+c_3|t|-k)|\nabla\varphi|^2-\left(m+km^2t^2\right)|\nabla t|^2\\&+(2c_4+2km|t|)|\nabla\varphi||\nabla t|+(c_5-mc_1)\frac{t^2}{\varepsilon^2}.\end{aligned}\tag{2.18}$$

Next we are looking for conditions that will insure that the right-hand side of (2.18) is nonpositive. First, by our assumption (2.9) the last term is indeed nonpositive. The sum of the first three terms on the right-hand side of (2.18) is a quadratic form in the two variables $|\nabla\varphi|,|\nabla t|$ whose discriminant Δ is given by

$$\begin{aligned}\Delta/4=&(c_4+km|t|)^2-(k-mc_2|t|-c_3|t|)m\left(1+kmt^2\right)\\=&c_4^2+m(mc_2+c_3)|t|-km\left(1-2c_4|t|-m(mc_2+c_3)|t|^3\right).\end{aligned}\tag{2.19}$$

By (2.10) and (2.19) it follows that for sufficiently large k we have $\Delta\leq 0$, implying that the right-hand side of (2.18) is nonpositive. For such k it follows that the function $v:=mt^2/2-\varphi$ satisfies

$$\Delta(e^{kv})=ke^{kv}(\Delta v+k|\nabla v|^2)\geq 0 \text{ in } \omega.$$

By the maximum principle, $\max_{\overline{\omega}}v=\max_{\partial\omega}v$, which is equivalent to (2.11b).

By similar calculations, the function $w:=mt^2/2+\varphi$ satisfies $\Delta(e^{kw})\geq 0$, implying (2.11a). □

3 Asymptotic behavior of solutions without vortices

In this section we shall study the asymptotic behavior of solutions u_ε of (1.2) in a smooth bounded simply connected domain Ω in $\mathbb{R}^2$. We assume a priori that the solutions are vortex-less. Actually, we shall assume a stronger condition, namely that the solutions are "sufficiently close" to Γ, in a sense to be precised below (see (3.1)). We are given a smooth boundary condition $g:\partial\Omega\to\Gamma$ of degree zero and a family of solutions $\{u_\varepsilon\}$ of (1.2). Since g is of degree zero, we may globally write it as $g=\tau(e^{\imath\varphi_0})$ for some smooth $\varphi_0:\partial\Omega\to\mathbb{R}$.

We next assume that

$$\operatorname{dist}(u_\varepsilon(x),\Gamma)\leq\delta_1,\ \forall\, x\in\Omega,\tag{3.1}$$

where δ_1 is chosen to satisfy the hypotheses of Proposition 2.1.

Then we may write, globally in $\overline{\Omega}$ and with smooth t_ε and φ_ε,

$$u_\varepsilon(x) = \tau\left(e^{\imath\varphi_\varepsilon(x)}\right) + t_\varepsilon(x)\,\vec{n}\left(\tau\left(e^{\imath\varphi_\varepsilon(x)}\right)\right),\ \forall x \in \overline{\Omega}. \tag{3.2}$$

Let ζ denote the harmonic extension of φ_0 to Ω and define the Γ-valued map u_0 by

$$u_0 := \tau\left(e^{\imath\zeta}\right). \tag{3.3}$$

The main result of this section establishes, in the spirit of [4], a convergence result of u_ε to the limit u_0.

3.1 Theorem. *Let, for $0 < \varepsilon < \varepsilon_0$, u_ε denote a solution of (1.2) satisfying (3.1). Then we have*

$$u_\varepsilon \to u_0 \text{ in } C^{1,\alpha}(\overline{\Omega}) \text{ as } \varepsilon \to 0, \forall\, \alpha < 1, \tag{3.4}$$
$$\|\Delta u_\varepsilon\|_\infty \le C, \tag{3.5}$$
$$\|u_\varepsilon - u_0\|_\infty \le C\varepsilon^2, \tag{3.6}$$
$$\|\nabla(u_\varepsilon - u_0)\|_\infty \le C\varepsilon. \tag{3.7}$$

Theorem 3.1 is an immediate consequence of several intermediate estimates (Lemma 3.2 to Proposition 3.9) that we now state and prove.

We start with two simple estimates satisfied by the solutions. These estimates are valid in any bounded domain Ω provided $|u_\varepsilon| \le R_0$ on $\partial\Omega$.

3.2 Lemma. *We have*

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le R_0, \tag{3.8}$$

where R_0 is given by (1.10).

Proof. We claim that the set $E := \{x \in \Omega; |u_\varepsilon(x)| > R_0\}$ is empty. Indeed, this follows from the maximum principle for subharmonic functions since, on the one hand, we have $|u_\varepsilon| = R_0$ on ∂E and, on the other hand, u_ε satisfies in E

$$\Delta(|u_\varepsilon|^2) = 2\left(|\nabla u_\varepsilon|^2 + \Delta u_\varepsilon \cdot u_\varepsilon\right) \ge \frac{2}{\varepsilon^2}\nabla W(u_\varepsilon)\cdot u_\varepsilon \ge 0$$

(the latter inequality following from (1.10)). □

From Lemma 3.2 we deduce the following gradient bound.

3.3 Lemma. *We have for some constant C,*

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \le \frac{C}{\varepsilon}. \tag{3.9}$$

The proof of Lemma 3.3 uses the same rescaling argument as in [4] and is therefore omitted.

Next we prove:

3.4 Lemma. *We have* $\lim_{\varepsilon\to 0} t_\varepsilon = 0$ *uniformly on* $\overline{\Omega}$.

Proof. Arguing by contradiction, assume that for a subsequence $\varepsilon_n \to 0$ and a sequence of points $\{x_n\} \subset \Omega$ we have $\lim_{n\to\infty} t_{\varepsilon_n}(x_n) = T$ with $T \neq 0$. We distinguish two cases:

1. $\displaystyle\lim_{n\to\infty} \frac{\operatorname{dist}(x_n, \partial\Omega)}{\varepsilon_n} = \infty.$

2. $\displaystyle\liminf_{n\to\infty} \frac{\operatorname{dist}(x_n, \partial\Omega)}{\varepsilon_n} < \infty.$

In Case 1 we define a rescaled sequence on $B_{R_n}(0)$, with $R_n := \dfrac{\operatorname{dist}(x_n, \partial\Omega)}{\varepsilon_n}$, by

$$\widetilde{u}_{\varepsilon_n}(x) := u_{\varepsilon_n}(x_n + \varepsilon_n x). \tag{3.10}$$

By our assumptions, $R_n \to \infty$ and, by standard elliptic estimates, a further subsequence, still denoted by $\{\widetilde{u}_{\varepsilon_n}\}$, converges in $C^{1,\beta}_{\text{loc}}(\mathbb{R}^2)$ to a limit $\widetilde{u}$, solution of $\Delta\widetilde{u} = \nabla W(\widetilde{u})$ on all of $\mathbb{R}^2$ and such that $\operatorname{dist}(\widetilde{u}(x), \Gamma) \leq \delta_1$, $\forall\, x \in \mathbb{R}^2$. The associated $\widetilde{t}, \widetilde{\varphi}$ then solve the system (2.6), with $\varepsilon = 1$ on $\mathbb{R}^2$ and $\widetilde{t}(0) = T \neq 0$. But then the proof of Proposition 2.1 shows that the two functions $e^{k\widetilde{v}}$ and $e^{k\widetilde{w}}$, where

$$\widetilde{v} := \frac{m\widetilde{t}^2}{2} - \widetilde{\varphi} \text{ and } \widetilde{w} := \frac{m\widetilde{t}^2}{2} + \widetilde{\varphi},$$

are subharmonic and bounded on $\mathbb{R}^2$. It follows that both $\widetilde{v}$ and $\widetilde{w}$ are identically constant in $\mathbb{R}^2$, and therefore the same holds for $\widetilde{t}$ and $\widetilde{\varphi}$. In particular $\widetilde{t} \equiv T \neq 0$ and $\nabla\widetilde{\varphi} \equiv 0$. But then, in view of (2.8b), equation (2.6b) is violated. Contradiction.

Consider next Case 2. We may assume that $L = \displaystyle\lim_{n\to\infty} \frac{\operatorname{dist}(x_n, \partial\Omega)}{\varepsilon_n}$ exists. By Lemma 3.3, we have $L > 0$. Arguing similarly to Case 1 we define the rescaled sequence $\{\widetilde{u}_{\varepsilon_n}\}$ by (3.10). Again, a subsequence converges to a solution of $\Delta\widetilde{u} = \nabla W(\widetilde{u})$, this time on a half-plane H, with a constant boundary condition $\widetilde{u} = \gamma$ on ∂H, for some point $\gamma \in \Gamma$.

With no loss of generality, we may assume that $H = \mathbb{R} \times (0, \infty)$. We know that for some point $(x_0, y_0) \in H$ with $y_0 = L$ we have $\widetilde{t}(x_0, L) = T \neq 0$. In addition, the boundary condition $\widetilde{u} = \gamma$ implies that the corresponding coordinates $\widetilde{t}$ and $\widetilde{\varphi}$ satisfy $\widetilde{t} = 0$ on ∂H and $\widetilde{\varphi} = \Phi =$const. on ∂H.

As above, the functions $e^{k\widetilde{v}}$ and $e^{k\widetilde{w}}$ are subharmonic. Since they are also bounded, the maximum principle applies on H and we obtain that both functions attain their maximum on ∂H. We obtain that

$$\Phi \leq \widetilde{\varphi}(x) - \frac{m\widetilde{t}^2(x)}{2} \leq \widetilde{\varphi}(x) + \frac{m\widetilde{t}^2(x)}{2} \leq \Phi, \ \forall\, x \in H.$$

It follows that $\widetilde{t} \equiv 0$, contradicting $\widetilde{t}(x_0) = T \neq 0$. □

Next we prove strong convergence of $\{u_\varepsilon\}$ to u in H^1.

3.5 Proposition. *As* $\varepsilon \to 0$, *we have*

$$u_\varepsilon \to u_0 \text{ in } H^1(\Omega) \text{ and } E_\varepsilon(u_\varepsilon) \to \frac{1}{2}\int_\Omega |\nabla u_0|^2. \tag{3.11}$$

Proof. Write $\varphi_\varepsilon = \psi_\varepsilon + \zeta$ (see (3.3)). The phase φ_ε is determined up to an integer multiple of 2π. We fix φ_ε by imposing

$$\psi_\varepsilon = 0 \text{ on } \partial\Omega. \tag{3.12}$$

Note that by Corollary 2.2 we have

$$\|\psi_\varepsilon\|_\infty \le 2M := 2\|\varphi_0\|_\infty. \tag{3.13}$$

We rewrite (2.6a) (dropping the subscript ε) as

$$-\operatorname{div}(a\nabla\psi) = \operatorname{div}((a-1)\nabla\zeta) + b\left(|\nabla\psi|^2 + 2\nabla\psi\cdot\nabla\zeta + |\nabla\zeta|^2\right) - \frac{\alpha_\varphi t^2}{\varepsilon^2}. \tag{3.14}$$

Multiplying (3.14) by $\psi \in H_0^1(\Omega)$ and integrating yields

$$\int_\Omega a|\nabla\psi|^2 = \int_\Omega \left[(1-a)\nabla\zeta\cdot\nabla\psi + b\left(|\nabla\psi|^2 + 2\nabla\psi\cdot\nabla\zeta + |\nabla\zeta|^2\right)\psi - \frac{\alpha_\varphi t^2}{\varepsilon^2}\psi\right].$$

Using Cauchy-Schwarz inequality, (3.13), (2.8a), (2.8d), (2.8f), Lemma 3.4 and Poincaré inequality, it follows that for some constant $C = C(g)$ and for sufficiently small ε we have

$$\int_\Omega |\nabla\psi|^2 \le C\int_\Omega \frac{t^2}{\varepsilon^2}. \tag{3.15}$$

Similarly, we rewrite (2.6b) as

$$-\Delta t + (2\alpha + \alpha_t t)\frac{t}{\varepsilon^2} = c\left(|\nabla\psi|^2 + 2\nabla\psi\cdot\nabla\zeta + |\nabla\zeta|^2\right). \tag{3.16}$$

Multiplying (3.16) by $t \in H_0^1(\Omega)$, integrating and using (2.8b) leads to

$$\int_\Omega \left[|\nabla t|^2 + (2\alpha + \alpha_t t)\frac{t^2}{\varepsilon^2}\right] = \int_\Omega ct\left(|\nabla\psi|^2 + 2\nabla\psi\cdot\nabla\zeta + |\nabla\zeta|^2\right). \tag{3.17}$$

Using (2.8b) and (2.8c) in (3.17) gives

$$\int_\Omega \left[|\nabla t|^2 + c_1\frac{t^2}{\varepsilon^2}\right] \le C\|t\|_\infty\left(1 + \int_\Omega |\nabla\psi|^2\right). \tag{3.18}$$

Plugging (3.15) into (3.18) yields (using Lemma 3.4)

$$\int_\Omega \left[|\nabla t|^2 + \frac{t^2}{\varepsilon^2}\right] = o(1). \tag{3.19}$$

Combining (3.15) and (3.19), we find that

$$\int_\Omega |\nabla\psi|^2 = o(1). \tag{3.20}$$

The conclusion (3.11) clearly follows from (3.19)–(3.20). □

3.6 Remark. From the Euler-Lagrange equation (1.2), Lemma 3.2 and standard elliptic estimates, we get the following uniform bounds, when ε is bounded away from zero, i.e., for $\{u_\varepsilon\}_{\varepsilon>\varepsilon_0}$:

$$\|u_\varepsilon\|_{W^{2,p}} \le C(p,\varepsilon_0),\ \forall\, p<\infty,\ \text{ and }\ \|u_\varepsilon\|_{C^{1,\alpha}} \le C(\alpha,\varepsilon_0),\ \forall\, \alpha<1. \tag{3.21}$$

Combining (3.21) with Proposition 3.5 we obtain a uniform bound for $E_\varepsilon(u_\varepsilon)$ for all $\varepsilon>0$.

3.7 Lemma. *$\{\varphi_\varepsilon\}$ is bounded in $W^{1,4}(\Omega)$.*

Proof. We use the same notation as in the proof of Proposition 3.5 and write $\varphi_\varepsilon = \psi_\varepsilon + \zeta = \psi + \zeta$. We will actually show that

$$\int_\Omega |\nabla\psi_\varepsilon|^4 = o(1), \tag{3.22}$$

that clearly implies the result for small ε (and then the result for any $\varepsilon>0$ follows from Remark 3.6). Rewrite (2.6a) as

$$\begin{cases} -\Delta\psi = b|\nabla\psi|^2 + \underbrace{2b\nabla\psi\cdot\nabla\zeta + b|\nabla\zeta|^2 + \operatorname{div}\big((a-1)\nabla\zeta + (a-1)\nabla\psi\big)}_{R=R_\varepsilon} \\ \qquad \underbrace{-\dfrac{\alpha_\varphi t^2}{\varepsilon^2}}_{S=S_\varepsilon} = b|\nabla\psi|^2 + R + S \text{ in } \Omega, \\ \psi = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.23}$$

We split $\psi = \psi_1+\psi_2+\psi_3$ where

$$\begin{cases} -\Delta\psi_1 = b|\nabla\psi|^2,\ -\Delta\psi_2 = R,\ -\Delta\psi_3 = S \text{ in } \Omega, \\ \psi_1=\psi_2=\psi_3 = 0 \text{ on } \partial\Omega. \end{cases} \tag{3.24}$$

Fix any $p>2$. By standard elliptic estimates, using (2.8a) and (2.8d),

$$\begin{aligned} \|\nabla\psi_2\|_p \le& C_1\big\{\|2b\nabla\psi\cdot\nabla\zeta\|_p + \|b|\nabla\zeta|^2\|_p + \|(a-1)\nabla\zeta\|_p + \|(a-1)\nabla\psi\|_p\big\} \\ \le& C_2\|t\|_\infty\big(\|\nabla\psi\|_p + 1\big). \end{aligned} \tag{3.25}$$

Next we estimate ψ_1. Let $p>1$ and set $q := \dfrac{2p}{p+2}$. Then, by Sobolev embedding (in two dimensions), $W^{2,q}(\Omega)\hookrightarrow W^{1,p}(\Omega)$. Note also that $\dfrac{1}{2q} = \dfrac{1/2}{2} + \dfrac{1/p}{2}$, hence

$$\|f\|_{2q}^2 \le \|f\|_2\|f\|_p,\ \forall\, f\in L^p(\Omega). \tag{3.26}$$

By elliptic estimates, (2.8d) and (3.26) we obtain

$$\begin{aligned} \|\nabla\psi_1\|_p \le& C_1\|\psi_1\|_{W^{2,q}} \le C_2\|b|\nabla\psi|^2\|_q \le C_3\|t\|_\infty\|\nabla\psi\|_{2q}^2 \\ \le& C_4\|t\|_\infty\|\nabla\psi\|_2\|\nabla\psi\|_p \le o(1)\cdot\|t\|_\infty\|\nabla\psi\|_p, \end{aligned} \tag{3.27}$$

where we used (3.20) in the last inequality.

Finally, we turn to ψ_3. Multiplying (2.6b) by t, integrating and using (2.8b) and the Cauchy-Schwarz inequality yields

$$\frac{c_1}{\varepsilon^2}\|t\|_2^2 \le \int_\Omega ct|\nabla\varphi|^2 \le C\|t\|_2\|\nabla\varphi\|_4^2,$$

implying that (for small ε),

$$\int_\Omega t^4 \le \int_\Omega t^2 \le C\varepsilon^4\|\nabla\varphi\|_4^4. \tag{3.28}$$

Recall also that by (3.19),

$$\|t\|_2 = o(\varepsilon). \tag{3.29}$$

Again by elliptic estimates and (3.26) we get

$$\|\nabla\psi_3\|_p \le \frac{C}{\varepsilon^2}\|t^2\|_q = \frac{C}{\varepsilon^2}\|t\|_{2q}^2 \le \frac{C}{\varepsilon^2}\|t\|_2\|t\|_p. \tag{3.30}$$

Choose $p = 4$. Using (3.28)–(3.29) in (3.30) gives:

$$\|\nabla\psi_3\|_4 \le \frac{C}{\varepsilon^2}\cdot o(\varepsilon)\cdot\varepsilon\|\nabla\varphi\|_4 \le o(1)\cdot\left(\|\nabla\psi\|_4 + 1\right). \tag{3.31}$$

Combining (3.25),(3.27) and (3.31) and using Lemma 3.4 we obtain

$$\|\nabla\psi\|_4 \le o(1)\cdot\left(\|\nabla\psi\|_4 + 1\right),$$

and (3.22) follows. □

3.8 Lemma. *$\{u_\varepsilon\}$ is bounded in $H^2(\Omega)$.*

Proof. Again by Remark 3.6, it suffices to consider small ε. Using the L^4-bound of Lemma 3.7 for $\nabla\varphi_\varepsilon$ in (3.28) yields

$$\int_\Omega t^2 \le C\varepsilon^4. \tag{3.32}$$

Since $|\nabla W(u_\varepsilon)| = O(t_\varepsilon)$, we deduce from (3.32) that the right-hand side of the equation in (1.2) is bounded in $L^2(\Omega)$ and the conclusion follows from elliptic estimates. □

3.9 Proposition. *We have*

$$\|t_\varepsilon\|_\infty \le C\varepsilon^2, \tag{3.33}$$
$$\|\nabla t_\varepsilon\|_\infty \le C\varepsilon, \tag{3.34}$$
$$\|\psi_\varepsilon\|_\infty \le C\varepsilon^2, \tag{3.35}$$
$$\|\nabla\psi_\varepsilon\|_\infty \le C\varepsilon. \tag{3.36}$$

Proof. We use an argument from [4, Step B.4]. Fix $q > 2$. Multiplying (2.6b) by $|t|^{q-2}t/(\varepsilon^2)^{q-1}$ and integrating gives

$$\begin{aligned} c_1 \int_\Omega \left(\frac{|t|}{\varepsilon^2}\right)^q &\le \int_\Omega \frac{(q-1)}{\varepsilon^{2(q-1)}}|t|^{q-2}|\nabla t|^2 + (2\alpha + \alpha_t t)\left(\frac{|t|}{\varepsilon^2}\right)^q \\ &= \int_\Omega c|\nabla\varphi|^2 \left(\frac{|t|}{\varepsilon^2}\right)^{q-2} \frac{t}{\varepsilon^2}. \end{aligned} \tag{3.37}$$

We conclude, using Hölder inequality and (2.8c), that the function $f_\varepsilon = f := \dfrac{t}{\varepsilon^2}$ satisfies

$$c_1\|f\|_q^q \le \int_\Omega c|\nabla\varphi|^2|f|^{q-1} \le c_2\|\nabla\varphi\|_{2q}^2\|f\|_q^{q-1},$$

i.e.,

$$\|f\|_q \le \frac{c_2}{c_1}\|\nabla\varphi\|_{2q}^2. \tag{3.38}$$

By Lemma 3.8 and Sobolev embedding, $\{\nabla u_\varepsilon\}$ is uniformly bounded in $L^r(\Omega)$ for every $r \in [1,\infty)$, and we obtain from (3.38) that $\|f\|_q \le C_q$. It follows that for each $q > 2$ the right-hand side of the equation in (1.2) is bounded in $L^q(\Omega)$. Hence $\{\nabla u_\varepsilon\}$ is uniformly bounded in $L^\infty(\Omega)$, and therefore

$$\|\nabla\varphi\|_\infty \le \overline{C}, \tag{3.39}$$

for some constant $\overline{C}$. Going back to (3.38) we obtain that

$$\|f\|_q \le \left(\frac{c_2}{c_1}\right)\overline{C}^2|\Omega|^{1/q}. \tag{3.40}$$

Passing to the limit $q \to \infty$ in (3.40) yields

$$\|f\|_\infty \le \left(\frac{c_2}{c_1}\right)\overline{C}^2,$$

and (3.33) follows.

Next, using (3.39) and (3.33) in (2.6b) gives the $\|\Delta t\|_\infty \le C$. Combining this estimate with (3.33) and applying an interpolation inequality (see [4, Lemma A.2]) yields (3.34). To prove (3.35)–(3.36) for ψ, we use (3.39) and the estimates

$$a - 1 = O(t) = O(\varepsilon^2) \quad \text{and} \quad b = O(t) = O(\varepsilon^2),$$

which allow us to rewrite (3.23) in the form

$$\Delta\psi = F + \operatorname{div} G, \text{ with } \|F\|_\infty = O(\varepsilon^2) \text{ and } \|G\|_\infty = O(\varepsilon^2).$$

The estimate (3.35) follows by elliptic estimates and finally (3.36) is deduced via interpolation as above. □

The proof of the main result of this section is an easy consequence of our previous estimates.

Proof of Theorem 3.1. Since, by (2.5), $|\nabla W(u_\varepsilon)| = O(t_\varepsilon)$, (3.5) follows from (1.2) and (3.33). By standard elliptic estimates we obtain that $\{u_\varepsilon\}$ is uniformly bounded in $C^{1,\beta}(\overline{\Omega})$ for all $\beta < 1$, and (3.4) follows by the Arzelà-Ascoli theorem (the identification of the limit as u_0 follows from Proposition 3.5). Finally, (3.6) is a consequence of (3.33) and (3.35), while (3.7) follows from (3.34) and (3.36). □

4 Boundary condition depending on ε

In the next sections we shall also need a version of Theorem 3.1 in the case where the boundary condition depends on ε, and does not necessarily take values into Γ (analogously to [4, Theorem 2] which deals with minimizers for the GL energy). For Ω as in Section 3, assume that the family $\{g_\varepsilon\}$ of maps $g_\varepsilon : \partial\Omega \to \mathbb{R}^2$, $\varepsilon > 0$, satisfies:

$$\|g_\varepsilon\|_{H^1(\partial\Omega)} \le C, \tag{4.1}$$

$$\int_{\partial\Omega} W(g_\varepsilon) \le C\varepsilon^2. \tag{4.2}$$

From (4.1)–(4.2) it follows in particular that, possibly up to a subsequence,

$$g_\varepsilon \to g \text{ in } H^s(\partial\Omega),\ \forall\, 0 < s < 1, \text{ and thus in } C^\alpha(\partial\Omega),\ \forall\, \alpha \in (0, 1/2), \tag{4.3}$$

for some $g \in H^1(\partial\Omega;\Gamma)$.

For each $\varepsilon > 0$ (or $\varepsilon \in (0, \varepsilon_0)$), let u_ε denote a solution of

$$\begin{cases} \Delta u_\varepsilon = \dfrac{1}{\varepsilon^2}\nabla W(u_\varepsilon) & \text{in } \Omega \\ u_\varepsilon = g_\varepsilon & \text{on } \partial\Omega \end{cases}. \tag{4.4}$$

We now make the crucial assumption that u_ε satisfies (3.1) (at least for small ε). Then we have

$$\deg \Pi \circ g_\varepsilon = 0 \text{ and thus } \deg g = 0. \tag{4.5}$$

(Recall that Π is the Euclidean projection on Γ.)

As before, we write $g(x) = \tau(e^{\imath\varphi_0(x)})$, with $\varphi_0 \in H^1(\partial\Omega;\mathbb{R})$. Define, in Ω, the Γ-valued map u_0 by (3.3), i.e., $u_0 = \tau(e^{\imath\zeta})$, where ζ is the harmonic extension of φ_0 to Ω. Our main result establishes the convergence of $\{u_\varepsilon\}$ towards u_0 when ε goes to zero:

4.1 Theorem. *Under the assumptions (4.1)–(4.4) and (3.1) we have, as $\varepsilon \to 0$,*

$$u_\varepsilon \to u_0 \text{ strongly in } H^1(\Omega) \text{ and in } C^0(\overline{\Omega}), \tag{4.6}$$

$$\|\Delta u_\varepsilon\|_{L^\infty(K)} \le C_K, \tag{4.7}$$

$$u_\varepsilon \to u_0 \text{ strongly in } C^{1,\alpha}(K),\ \forall\, \alpha < 1, \tag{4.8}$$

$$\|u_\varepsilon - u_0\|_{L^\infty(K)} \le C_K\varepsilon^2 \text{ and } \|\nabla(u_\varepsilon - u_0)\|_{L^\infty(K)} \le C_K\varepsilon, \tag{4.9}$$

for every compact $K \subset\subset \Omega$.

The proof follows similar steps to those of Section 3 and part of the analysis carries over with slight modifications to the current situation. This is the case for the analogous results to Lemma 3.2 and Lemma 3.4 that we state in the next proposition.

4.2 Proposition. *We have $\|u_\varepsilon\|_\infty \le R_0$ and $\lim_{\varepsilon\to 0} t_\varepsilon = 0$ uniformly on $\overline{\Omega}$.*

Next we turn to an H^1-convergence result, generalizing Proposition 3.5.

4.3 Proposition. *We have*

$$u_\varepsilon \to u_0 \text{ in } H^1(\Omega) \text{ and } E_\varepsilon(u_\varepsilon) \to \frac{1}{2}\int_\Omega |\nabla u_0|^2. \tag{4.10}$$

Proof. We define the pair of functions t_ε and φ_ε associated with u_ε via (3.2). We let ζ_ε denote the harmonic extension of $\varphi_{\varepsilon|\partial\Omega}$ to Ω. Analogously to the proof of Proposition 3.5, we then write $\varphi_\varepsilon = \psi_\varepsilon + \zeta_\varepsilon$, with $\psi_\varepsilon = 0$ on $\partial\Omega$.

Clearly, (4.3) implies that, possibly after subtracting suitable integer multiples of 2π from the φ_ε's, we have $\varphi_{\varepsilon|\partial\Omega} \to \varphi_0$ in $H^{1/2}(\partial\Omega)$, and thus

$$\lim_{\varepsilon\to 0}\int_\Omega |\nabla\zeta_\varepsilon|^2 = \int_\Omega |\nabla\zeta|^2. \tag{4.11}$$

Repeating the calculations at the beginning of the proof of Proposition 3.5, with ζ_ε playing the role of ζ, yields, analogously to (3.15),

$$\int_\Omega |\nabla\psi_\varepsilon|^2 \le C\int_\Omega \frac{t_\varepsilon^2}{\varepsilon^2}. \tag{4.12}$$

Now, since in the current setting t_ε is not identically zero on $\partial\Omega$, multiplying (3.16) by t_ε, integrating and using (2.8b) yields

$$\begin{aligned}\int_\Omega \left[|\nabla t_\varepsilon|^2 + (2\alpha + \alpha_t t_\varepsilon)\frac{t_\varepsilon^2}{\varepsilon^2}\right] &= \int_{\partial\Omega} t_\varepsilon \frac{\partial t_\varepsilon}{\partial n} \\ &\quad + \int_\Omega c\, t_\varepsilon \left(|\nabla\psi_\varepsilon|^2 + 2\nabla\psi_\varepsilon\cdot\nabla\zeta_\varepsilon + |\nabla\zeta_\varepsilon|^2\right)\end{aligned}, \tag{4.13}$$

where n stands for the outward normal on $\partial\Omega$. In order to deal with the boundary term in (4.13), we use a Pohozaev identity type argument, as in [4, Proposition 3]. So let $V = (V_1, V_2)$ be a smooth vector field on Ω satisfying $V = n$ on $\partial\Omega$. We consider the vector field $V\cdot\nabla u_\varepsilon = (V\cdot\nabla(u_\varepsilon)_1, V\cdot\nabla(u_\varepsilon)_2)$. We take the scalar product of both sides of the equation in (4.4) and $V\cdot\nabla u_\varepsilon$ and integrate. A direct computation (see [4]) gives

$$\begin{aligned}\int_\Omega (\Delta u_\varepsilon)\cdot(V\cdot\nabla u_\varepsilon) &= \int_\Omega \left[\frac{1}{2}\operatorname{div} V\,|\nabla u_\varepsilon|^2 - (V_1)_{x_1}|(u_\varepsilon)_{x_1}|^2 - (V_2)_{x_2}|(u_\varepsilon)_{x_2}|^2\right] \\ &\quad - \int_\Omega \big((V_1)_{x_2} + (V_2)_{x_1}\big)(u_\varepsilon)_{x_1}\cdot(u_\varepsilon)_{x_2} \\ &\quad + \frac{1}{2}\int_{\partial\Omega}\left(\left|\frac{\partial u_\varepsilon}{\partial n}\right|^2 - \left|\frac{\partial g_\varepsilon}{\partial\sigma}\right|^2\right).\end{aligned} \tag{4.14}$$

(Here, $\partial\sigma$ stands for the tangential derivative on $\partial\Omega$.)

On the other hand, we have

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_\Omega \nabla W(u_\varepsilon)\cdot(V\cdot\nabla u_\varepsilon) &= \frac{1}{\varepsilon^2}\int_\Omega V\cdot\nabla\big(W(u_\varepsilon)\big) \\ &= \frac{1}{\varepsilon^2}\left(-\int_\Omega (\operatorname{div} V)\,W(u_\varepsilon) + \int_{\partial\Omega} W(u_\varepsilon)\right).\end{aligned} \tag{4.15}$$

Equating (4.14) and (4.15), using (4.1), (4.2), (4.11) and (2.5) yields

$$\begin{aligned}\int_{\partial\Omega}\left|\frac{\partial t_\varepsilon}{\partial n}\right|^2 &\le C_1\int_{\partial\Omega}\left|\frac{\partial u_\varepsilon}{\partial n}\right|^2 \le C_2\left(1 + \int_\Omega\left[|\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2}\right]\right) \\ &\le C_3\left(1 + \int_\Omega\left[|\nabla t_\varepsilon|^2 + |\nabla\psi_\varepsilon|^2 + \frac{t_\varepsilon^2}{\varepsilon^2}\right]\right).\end{aligned} \tag{4.16}$$

By (4.16), the Cauchy-Schwarz inequality and (4.2) we obtain

$$\int_{\partial\Omega} t_\varepsilon \frac{\partial t_\varepsilon}{\partial n} \le C\varepsilon \left(\int_{\partial\Omega} \left| \frac{\partial t_\varepsilon}{\partial n} \right|^2 \right)^{1/2} \le C'\varepsilon \left(1 + \int_\Omega \left[|\nabla t_\varepsilon|^2 + |\nabla \psi_\varepsilon|^2 + \frac{t_\varepsilon^2}{\varepsilon^2} \right] \right)^{1/2}. \tag{4.17}$$

Substituting (4.17) in (4.13) leads to

$$\begin{aligned} \int_\Omega \left[|\nabla t_\varepsilon|^2 + c_1 \frac{t_\varepsilon^2}{\varepsilon^2} \right] \le & C\varepsilon \left(1 + \int_\Omega \left[|\nabla t_\varepsilon|^2 + |\nabla \psi_\varepsilon|^2 + \frac{t_\varepsilon^2}{\varepsilon^2} \right] \right)^{1/2} \\ & + C\|t_\varepsilon\|_\infty \left(1 + \int_\Omega |\nabla \psi_\varepsilon|^2 \right). \end{aligned} \tag{4.18}$$

Combining (4.12), (4.18) and Proposition 4.2 we get

$$\int_\Omega \left[|\nabla t_\varepsilon|^2 + \frac{t_\varepsilon^2}{\varepsilon^2} \right] = o(1). \tag{4.19}$$

Using (4.19) in (4.12) finally gives

$$\int_\Omega |\nabla \psi_\varepsilon|^2 = o(1), \tag{4.20}$$

and (4.10) follows from (4.19)–(4.20) and (4.11). □

Analogously to Lemma 3.7, and in particular to (3.22), we have:

4.4 Lemma. *$\psi_\varepsilon \to 0$ in $W^{1,4}(\Omega)$.*

Proof. We first notice that since $\{\varphi_{\varepsilon|\partial\Omega}\}$ is bounded in $H^1(\partial\Omega)$ by (4.1), the family $\{\zeta_\varepsilon\}$ is bounded in $H^{3/2}(\Omega)$. Since $H^{3/2}(\Omega) \hookrightarrow W^{1,4}(\Omega)$, we get:

$$\{\zeta_\varepsilon\} \text{ is bounded in } W^{1,4}(\Omega). \tag{4.21}$$

Arguing as in the proof of Lemma 3.7 we use (3.23) to split

$$\psi_\varepsilon = \psi_{1,\varepsilon} + \psi_{2,\varepsilon} + \psi_{3,\varepsilon}.$$

The same arguments that led to (3.25) and (3.27) (with $p = 4$) yield

$$\|\nabla \psi_{2,\varepsilon}\|_4 \le C\|t_\varepsilon\|_\infty \left(\|\nabla \psi_\varepsilon\|_4 + 1 \right) \tag{4.22}$$

and

$$\|\nabla \psi_{1,\varepsilon}\|_4 \le C\|\psi_{1,\varepsilon}\|_{W^{2,4/3}} \le o(1) \cdot \|t_\varepsilon\|_\infty \|\nabla \psi_\varepsilon\|_4. \tag{4.23}$$

The only difference with respect to the case where $g_\varepsilon \equiv g$ stands in the estimate of $\psi_{3,\varepsilon}$. Multiplying (2.6b) by t_ε and integrating gives

$$\|\nabla t_\varepsilon\|_2^2 + \frac{c_1}{\varepsilon^2}\|t_\varepsilon\|_2^2 \le \int_\Omega c\, t_\varepsilon |\nabla \varphi_\varepsilon|^2 + \int_{\partial\Omega} t_\varepsilon \frac{\partial t_\varepsilon}{\partial n} \le C\|t_\varepsilon\|_2 \|\nabla \varphi_\varepsilon\|_4^2 + C\varepsilon, \tag{4.24}$$

where in the last inequality we used the Cauchy-Schwarz inequality and (4.17) combined with (4.19)–(4.20).

Next we claim that

$$\|t_\varepsilon\|_2 \|\nabla \varphi_\varepsilon\|_4^2 \le \varepsilon, \tag{4.25}$$

for sufficiently small ε. Indeed, arguing by contradiction, assume that (4.25) does not hold, i.e., for a sequence $\varepsilon_n \to 0$ we have

$$\|t_{\varepsilon_n}\|_2 \|\nabla \varphi_{\varepsilon_n}\|_4^2 > \varepsilon_n. \tag{4.26}$$

Then, from (4.24) we get that

$$\frac{c_1}{\varepsilon_n^2}\|t_{\varepsilon_n}\|_2^2 \le C\|t_{\varepsilon_n}\|_2 \|\nabla \varphi_{\varepsilon_n}\|_4^2,$$

and the argument of the proof of Lemma 3.7 applies, so thanks to (4.21) we get, as in (3.31), that

$$\|\nabla \psi_{3,\varepsilon_n}\|_4 \le \frac{C}{\varepsilon_n^2} \cdot o(\varepsilon_n) \cdot \varepsilon_n \|\nabla \varphi_{\varepsilon_n}\|_4 = o(1) \cdot \left(\|\nabla \psi_{\varepsilon_n}\|_4 + 1\right). \tag{4.27}$$

From (4.22),(4.23) and (4.27) we obtain that (3.22) holds, and therefore

$$\|\nabla \varphi_{\varepsilon_n}\|_4 \le A, \tag{4.28}$$

for some constant $A > 0$. It follows from (4.28) and (4.26) that

$$\left\| \frac{t_{\varepsilon_n}}{\varepsilon_n} \right\|_2 \ge \frac{1}{A^2},$$

which contradicts (4.19).

Using (4.25) in (4.24) gives

$$\|\nabla t_\varepsilon\|_2^2 + \frac{c_1}{\varepsilon^2}\|t_\varepsilon\|_2^2 \le C\varepsilon, \tag{4.29}$$

which implies, in particular, that

$$\int_\Omega |t_\varepsilon|^q = o(1) \int_\Omega t_\varepsilon^2 = o(\varepsilon^3), \tag{4.30}$$

for any $q > 2$. By (4.30), Sobolev embedding and elliptic estimates we obtain

$$\|\nabla \psi_{3,\varepsilon}\|_4 \le C\|\Delta \psi_{3,\varepsilon}\|_{4/3} \le C\left(\int_\Omega \frac{t_\varepsilon^{8/3}}{\varepsilon^{8/3}}\right)^{3/4} \le \frac{C}{\varepsilon^2} \cdot o(\varepsilon^{9/4}) = o(\varepsilon^{1/4}). \tag{4.31}$$

Combining (4.22)–(4.23) with (4.31) we are led to

$$\|\nabla \psi_\varepsilon\|_4 \le o(1) \cdot \left(\|\nabla \psi_\varepsilon\|_4 + 1\right),$$

implying that $\|\nabla \psi_\varepsilon\|_4 = o(1)$, as claimed. □

We next prove local estimates in Ω. It suffices to consider a sequence $\varepsilon_n \to 0$, but for simplicity we will drop the subscript n.

Fix some small $r_0 > 0$, depending on Ω, such that the nearest point projection onto $\partial\Omega$ is smooth in the set $\{x \in \Omega; \operatorname{dist}(x, \partial\Omega) < r_0\}$. Set, for $0 < r < r_0$, $\Omega_r := \{x \in \Omega; \operatorname{dist}(x, \partial\Omega) > r\}$, which is a smooth domain. Using (4.29) and the Fubini theorem we can find some $r = r_\varepsilon$ such that $r_0/2 < r < r_0$ and

$$\int_{\partial\Omega_r} \left[|\nabla t_\varepsilon|^2 + \frac{t_\varepsilon^2}{\varepsilon^2} \right] \le C\varepsilon. \tag{4.32}$$

For such r, we claim the following.

4.5 Lemma. *We have*

$$\int_{\Omega_r}\left[|\nabla t_\varepsilon|^2+\frac{t_\varepsilon^2}{\varepsilon^2}\right]\le C\varepsilon^2. \tag{4.33}$$

Proof. By (4.32) and the Cauchy-Schwarz inequality we have

$$\left|\int_{\partial\Omega_r} t_\varepsilon\frac{\partial t_\varepsilon}{\partial n}\right|\le C\varepsilon^{1/2}\cdot\varepsilon^{3/2}=C\varepsilon^2. \tag{4.34}$$

Applying (4.24) on Ω_r, using (4.34) and Lemma 4.4 yields

$$\int_{\Omega_r}\left[|\nabla t_\varepsilon|^2+c_1\frac{t_\varepsilon^2}{\varepsilon^2}\right]\le C\left(\int_{\Omega_r}t_\varepsilon^2\right)^{1/2}+C\varepsilon^2, \tag{4.35}$$

which clearly implies (4.33). □

4.6 Lemma. *We have*

$$\|\nabla u_\varepsilon\|_{L^\infty(\Omega_r)}\le C. \tag{4.36}$$

Proof. Choose $\widetilde{r}\in(r_0/6,r_0/5)$ satisfying (4.32) on $\partial\Omega_{\widetilde{r}}$. Then the above arguments apply for $\Omega_{\widetilde{r}}$. In particular, (4.33) holds on $\Omega_{\widetilde{r}}$, and using Fubini theorem we can find $s\in(\widetilde{r}_0/4,r_0/3)$ such that

$$\int_{\partial\Omega_s}\left[|\nabla t_\varepsilon|^2+\frac{t_\varepsilon^2}{\varepsilon^2}\right]\le C\varepsilon^2. \tag{4.37}$$

Since $|\nabla W(u_\varepsilon)|=O(t_\varepsilon)$, the estimate (4.33) on $\Omega_{\widetilde{r}}$ implies that $\|\Delta u_\varepsilon\|_{L^2(\Omega_{\widetilde{r}})}=O(1)$. By standard interior elliptic estimates, it follows that

$$\|u_\varepsilon\|_{H^2(\Omega_s)}\le C,$$

and then, by Sobolev embeddings,

$$\|\nabla u_\varepsilon\|_{L^p(\Omega_s)}\le C_p,\ \forall\, p\in[1,\infty). \tag{4.38}$$

Next we argue similarly to the proof of Proposition 3.9. For any $q>2$, multiplying (2.6b) by $|t_\varepsilon|^{q-2}t_\varepsilon/(\varepsilon^2)^{q-1}$ and integrating over Ω_s gives

$$\begin{aligned} c_1\int_{\Omega_s}\left(\frac{|t_\varepsilon|}{\varepsilon^2}\right)^q &\le \int_{\Omega_s}\left[\frac{(q-1)}{\varepsilon^{2(q-1)}}|t_\varepsilon|^{q-2}|\nabla t_\varepsilon|^2+(2\alpha+\alpha_t t_\varepsilon)\left(\frac{|t_\varepsilon|}{\varepsilon^2}\right)^q\right]\\ &=\int_{\Omega_s}c|\nabla\varphi_\varepsilon|^2\left(\frac{|t_\varepsilon|}{\varepsilon^2}\right)^{q-2}\frac{t_\varepsilon}{\varepsilon^2}+\int_{\partial\Omega_s}\left(\frac{|t_\varepsilon|}{\varepsilon^2}\right)^{q-2}\left(\frac{t_\varepsilon}{\varepsilon^2}\right)\left(\frac{\partial t_\varepsilon}{\partial n}\right). \end{aligned} \tag{4.39}$$

We apply the above with $q=5/2$. Using (4.37) and Cauchy-Schwarz inequality we estimate the boundary integral by

$$\begin{aligned} \left|\int_{\partial\Omega_s}\left(\frac{|t_\varepsilon|}{\varepsilon^2}\right)^{1/2}\left(\frac{t_\varepsilon}{\varepsilon^2}\right)\left(\frac{\partial t_\varepsilon}{\partial n}\right)\right| &\le\frac{1}{\varepsilon^3}\left(\int_{\partial\Omega_s}|t_\varepsilon|^3\right)^{1/2}\left(\int_{\partial\Omega_s}|\nabla t_\varepsilon|^2\right)^{1/2}\\ &=\frac{1}{\varepsilon^3}\cdot o(\varepsilon^2)\cdot O(\varepsilon)=o(1). \end{aligned} \tag{4.40}$$

From (4.39)–(4.40) and Hölder inequality we deduce that the function $f_\varepsilon := \dfrac{t_\varepsilon}{\varepsilon^2}$ satisfies

$$c_1\|f_\varepsilon\|^{5/2}_{L^{5/2}(\Omega_s)} \le \int_{\Omega_s} c|\nabla\varphi_\varepsilon|^2|f_\varepsilon|^{3/2} + o(1) \le c_2\|\nabla\varphi_\varepsilon\|^2_{L^5(\Omega_s)}\|f_\varepsilon\|^{3/2}_{L^{5/2}(\Omega_s)} + o(1).$$

Applying (4.38) to the above yields

$$\|f_\varepsilon\|^{5/2}_{L^{5/2}(\Omega_s)} \le C\|f_\varepsilon\|^{3/2}_{L^{5/2}(\Omega_s)} + o(1),$$

implying that $\|f_\varepsilon\|_{L^{5/2}(\Omega_s)} = O(1)$ and therefore $\|\Delta u_\varepsilon\|_{L^{5/2}(\Omega_s)} = O(1)$. By elliptic interior estimates we obtain that $\|u_\varepsilon\|_{W^{2,5/2}(\Omega_r)} = O(1)$, and (4.36) follows by Sobolev embedding. □

We are now ready to complete the proof of the main result of this section.

Proof of Theorem 4.1. The strong convergence $u_\varepsilon \to u_0$ in $H^1(\Omega)$ was established in Proposition 4.3. To complete the proof of (4.6) we need to prove the uniform convergence. This follows from the two uniform convergences on $\overline{\Omega}$: $t_\varepsilon \to 0$ (see Proposition 4.2) and $\psi_\varepsilon \to 0$ (which results, by Morrey theorem, from the $W^{1,4}$-convergence that was established in Lemma 4.4).

For the proof of (4.7) we only need to verify the following estimate:

$$\|t_\varepsilon\|_{L^\infty(K)} \le C_K\varepsilon^2, \tag{4.41}$$

for every compact $K \subset\subset \Omega$. We shall prove (4.41) using an argument from [4]. We first use Kato's inequality in (2.6b) to get

$$\Delta|t_\varepsilon| \ge \operatorname{sgn}(t_\varepsilon)\Delta t_\varepsilon = (2\alpha + \alpha_t t_\varepsilon)\frac{|t_\varepsilon|}{\varepsilon^2} - c|\nabla\varphi_\varepsilon|^2\operatorname{sgn}(t_\varepsilon).$$

Hence, by (2.8b) and (4.36),

$$-\Delta|t_\varepsilon| + c_1\frac{|t_\varepsilon|}{\varepsilon^2} \le C_r \text{ in } \Omega_r. \tag{4.42}$$

Now recall [4, Lemma 2] that states that the radial solution $\omega = \omega(r)$ of

$$\begin{cases} -\varepsilon^2\Delta\omega + \omega = 0 & \text{in } B_R(0) \\ \omega = 1 & \text{on } \partial B_R(0) \end{cases} \tag{4.43}$$

satisfies, for $\varepsilon < \dfrac{3}{4}R$,

$$\omega(r) \le e^{(r^2-R^2)/(4\varepsilon R)} \text{ in } B_R(0). \tag{4.44}$$

Let $d := \operatorname{dist}(K, \partial\Omega)$, so that (4.42) is satisfied with $r := d/2$. Let x_0 be an arbitrary point in K. With no loss of generality we may assume $x_0 = 0$. From (4.42)–(4.44) and the maximum principle we obtain that

$$|t_\varepsilon| \le C\varepsilon^2 + \exp\left[\sqrt{c_1}(|x|^2 - d^2/4)/(2\varepsilon d)\right] \text{ in } B_{d/2}(0).$$

In particular,

$$\frac{|t_\varepsilon(0)|}{\varepsilon^2} \le C + \frac{1}{\varepsilon^2}\exp\left[-d\sqrt{c_1}/(8\varepsilon)\right]. \tag{4.45}$$

Since the right-hand side of (4.45) remains bounded as $\varepsilon \to 0$, (4.41) follows, completing the proof of (4.7).

From (4.7) and elliptic estimates we obtain that u_ε is bounded in $W^{2,p}_{\mathrm{loc}}(\Omega)$ for every $p < \infty$, and (4.8) follows from Morrey theorem. Finally, (4.9) follows from the previous estimates by the same arguments as in the proof of Theorem 3.1. □

We will need in the next section also the following variant of Theorem 4.1 and Theorem 3.1. The proof is very similar to the proofs of these theorems, and is therefore omitted.

4.7 Theorem. *Let Ω be a smooth bounded and simply connected domain in $\mathbb{R}^2$. Let $x_0 \in \partial\Omega$ and suppose that $R > 0$ is sufficiently small such that $\partial B_R(x_0) \cap \partial\Omega$ consists of exactly two points.*

Suppose that $g : \partial(B_R(x_0)\cap\Omega) \to \Gamma$ is a continuous map of degree zero such that the restriction $g_{|\partial\Omega\cap\overline{B}_R(x_0)}$ is smooth. Let φ_0 be a continuous function such that $g = \tau(e^{\imath\varphi_0})$. Let ζ be the harmonic extension of φ_0 to $\Omega\cap B_R(x_0)$ and set $u_0 := \tau(e^{\imath\zeta})$.

For each $\varepsilon > 0$ let $g_\varepsilon : \partial(\Omega\cap B_R(x_0)) \to \mathbb{R}^2$ satisfy:

$$g_\varepsilon = g \text{ on } \partial\Omega\cap B_R(x_0) \tag{4.46}$$

$$\|g_\varepsilon\|_{H^1(\partial B_R(x_0)\cap\Omega)} \le C, \tag{4.47}$$

$$\int_{\partial B_R(x_0)\cap\Omega} W(g_\varepsilon) \le C\varepsilon^2, \tag{4.48}$$

$$g_\varepsilon \to g \text{ in } H^s(\partial B_R(x_0)\cap\Omega),\ 0 < s < 1. \tag{4.49}$$

Let u_ε be a solution of (4.4) on $\Omega\cap B_R(x_0)$ (instead of Ω) satisfying (3.1). Then for every $R_1 \in (0,R)$ we have:

$$\|\Delta u_\varepsilon\|_{L^\infty(\overline{\Omega}\cap B_{R_1}(x_0))} \le C_{R_1}, \tag{4.50}$$

$$u_\varepsilon \to u_0 \text{ in } C^{1,\alpha}(\overline{\Omega}\cap B_{R_1}(x_0)), \tag{4.51}$$

$$\|u_\varepsilon - u_0\|_{L^\infty(\overline{\Omega}\cap B_{R_1}(x_0))} \le C_{R_1}\varepsilon^2, \tag{4.52}$$

$$\|\nabla(u_\varepsilon - u_0)\|_{L^\infty(\overline{\Omega}\cap B_{R_1}(x_0))} \le C_{R_1}\varepsilon. \tag{4.53}$$

Note that (possibly after passing to a subsequence) the condition (4.49) actually follows from conditions (4.46)–(4.48) via the compact embedding $H^1(\partial B_R(x_0)\cap\Omega) \hookrightarrow H^s(\partial B_R(x_0)\cap\Omega)$, $0 < s < 1$.

5 General solutions

5.1 Preliminary estimates

Assume that Ω is a smooth bounded domain in $\mathbb{R}^2$, strictly star-shaped with respect to a point $z \in \Omega$. With no loss of generality, we may assume that $z = 0$, and thus

$$x\cdot n = x\cdot n(x) \ge c > 0,\ \forall\, x \in \partial\Omega \tag{5.1}$$

(with $n = n(x)$ the outward normal to $\partial\Omega$ at $x \in \partial\Omega$).

Let $g : \partial\Omega \to \Gamma$ be a smooth boundary datum of degree d. For each $\varepsilon > 0$, let u_ε denote a solution of (1.2). As in the previous sections, we denote by $t(x) = t_\varepsilon(x)$ the signed distance of $u_\varepsilon(x)$ to Γ. In contrast with the previous sections, we do not impose a condition like (3.1), and thus we allow solutions with vortices.

We start with some basic estimates satisfied by the solutions u_ε. We first notice that the results of Lemma 3.2 and Lemma 3.3 hold true since their proofs do not rely on the degree of g.

Next we prove a Pohozaev identity that does rely heavily on the star-shapeness assumption.

5.1 Lemma. *We have*

$$\int_\Omega \frac{W(u_\varepsilon)}{\varepsilon^2} + \int_{\partial\Omega} \left|\frac{\partial u_\varepsilon}{\partial n}\right|^2 \le C, \tag{5.2}$$

for some C independent of ε.

Proof. The proof is standard and requires only a simple adaptation of the proof in [5]. We argue as in the proof of Proposition 4.3 multiplying both side of the equation in (1.2) by $V \cdot \nabla u_\varepsilon$, but this time with $V = (x_1, x_2)$. For this choice of V, (4.14) reads

$$\int_\Omega \Delta u_\varepsilon \cdot (V \cdot \nabla u_\varepsilon) = \int_{\partial\Omega} \left[\frac{\partial u_\varepsilon}{\partial n} \cdot (x \cdot \nabla u_\varepsilon) - \frac{1}{2}(x \cdot n)|\nabla u_\varepsilon|^2 \right], \tag{5.3}$$

while (4.15) becomes

$$\frac{1}{\varepsilon^2}\int_\Omega \nabla W(u_\varepsilon) \cdot (V \cdot \nabla u_\varepsilon) = \frac{1}{\varepsilon^2}\int_\Omega V \cdot \nabla(W(u_\varepsilon)) = -\frac{2}{\varepsilon^2}\int_\Omega W(u_\varepsilon). \tag{5.4}$$

Combining (5.3) with (5.4) yields

$$\frac{2}{\varepsilon^2}\int_\Omega W(u_\varepsilon) + \frac{1}{2}\int_{\partial\Omega}(x \cdot n)\left|\frac{\partial u_\varepsilon}{\partial n}\right|^2 = \int_{\partial\Omega}\left[\frac{1}{2}(x \cdot n)\left|\frac{\partial g}{\partial \sigma}\right|^2 - (x \cdot \sigma)\frac{\partial u_\varepsilon}{\partial n} \cdot \frac{\partial g}{\partial \sigma}\right],$$

which, in view of (5.1), clearly implies (5.2). □

Since by (1.9) there exists $0 < \mu_0 < \mu$ such that $W(\zeta) \ge \mu_0 \operatorname{dist}^2(\zeta, \Gamma)$ for $\zeta \in B_{R_0}$, it follows from (5.2) that

$$\int_\Omega \frac{\operatorname{dist}^2(u_\varepsilon(x), \Gamma)}{\varepsilon^2} \le C. \tag{5.5}$$

Using the two estimates (5.2) and (3.9), we can show, using the argument of [5, Chapter 4], that for any small $\delta_2 > 0$ (we will always take $\delta_2 < \delta_1$, see Proposition 2.1) the set

$$S_{\varepsilon,\delta_2} := \{x \in \Omega;\, \operatorname{dist}(u_\varepsilon(x), \Gamma) > \delta_2\} \tag{5.6}$$

can be covered by a finite number of "bad discs" $\{B_{\lambda\varepsilon}(x_j^\varepsilon)\}_{j=1}^{k_\varepsilon}$ with

$$\{x_j^\varepsilon\}_{j=1}^{k_\varepsilon} \subset S_{\varepsilon,\delta_2}, \tag{5.7}$$

where k_ε is bounded uniformly in ε.

Indeed, we first use (3.9) to choose $\lambda > 0$ such that

$$\operatorname{dist}(u_\varepsilon(x),\Gamma) > \delta_2 \Longrightarrow \left\{B_{\lambda\varepsilon/4}(x) \subset \Omega \text{ and } \operatorname{dist}(u_\varepsilon(y),\Gamma) > \delta_2/2,\ \forall y \in B_{\lambda\varepsilon/4}(x)\right\}. \tag{5.8}$$

Then, we take a collection of mutually disjoint discs $\{B_{\lambda\varepsilon/4}(x_j^\varepsilon)\}_{j=1}^{k_\varepsilon}$ which is maximal with respect to the property that (5.7) holds true. Note that by (1.5) there exists $\eta = \eta(\delta_2)$ such that

$$W(z) > \eta,\ \forall z \in B_{R_0} \setminus \Gamma_{\delta_2/2}, \tag{5.9}$$

where $\Gamma_{\delta_2/2} = \{z \in \mathbb{R}^2; \operatorname{dist}(z,\Gamma) < \delta_2/2\}$. Taking into account (3.8) we get from (5.8)–(5.9) that

$$\frac{1}{\varepsilon^2}\int_{B_{\lambda\varepsilon/4}(x_j^\varepsilon)} W(u_\varepsilon) \geq \pi\lambda^2\eta/16,\ j = 1,\dots,k_\varepsilon. \tag{5.10}$$

The uniform bound for k_ε follows by combining (5.10) with (5.2). By construction $S_{\varepsilon,\delta_2} \subset \bigcup_{j=1}^{k_\varepsilon} B_{\lambda\varepsilon}(x_j^\varepsilon)$. Next, by increasing λ if necessary, we may also assume that the bad discs are well-separated, in the sense that $B_{4\lambda\varepsilon}(x_j^\varepsilon) \cap B_{4\lambda\varepsilon}(x_\ell^\varepsilon) = \emptyset$ if $j \neq \ell$ (this may results in decreasing the value of k_ε).

Passing to a subsequence $\varepsilon_n \to 0$, but continuing to denote ε_n by ε, for simplicity, we may assume $k_\varepsilon = k$ is independent of ε. Note that outside the bad discs the function $t(x)$ is well-defined and that we have

$$|t(x)| \leq \delta_2,\ \forall x \in \Omega_\varepsilon := \Omega \setminus \bigcup_{j=1}^{k} \overline{B_{\lambda\varepsilon}(x_j^\varepsilon)}. \tag{5.11}$$

The definitive value of δ_2 satisfying $\delta_2 \leq \delta_1$ will be chosen in Section 5.3; see the proof of Proposition 5.12.

We next prove that the x_j^ε's are relatively far away from $\partial\Omega$.

5.2 Lemma. *We have*

$$\lim_{\varepsilon\to 0} \frac{\operatorname{dist}(x_j^\varepsilon,\partial\Omega)}{\varepsilon} = \infty,\ j = 1,\dots,k. \tag{5.12}$$

Proof. We argue by contradiction and assume that (5.12) does not hold for some j along some sequence $\varepsilon_n \to 0$. For notational simplicity, we drop the subscript n. We will obtain a contradiction via a blow up analysis. Let, for small ε, y^ε denote the projection of x_j^ε onto $\partial\Omega$, and let $\mathscr{R}^\varepsilon$ denote the rotation of $\mathbb{R}^2$ such that $\mathscr{R}^\varepsilon(0,-1) = n(y^\varepsilon)$. Consider

$$v_\varepsilon(x) := u\left(y^\varepsilon + \varepsilon\mathscr{R}^\varepsilon x\right),\ x \in \frac{1}{\varepsilon}\left(\mathscr{R}^\varepsilon\right)^{-1}\left(\Omega - y^\varepsilon\right).$$

Using (3.8) and (5.2), together with the boundary condition in (1.2), we find that, up to a subsequence and uniformly on compacts of $H := \{x \in \mathbb{R}^2; x_2 > 0\}$, v_ε converges to a solution v of

$$\begin{cases} \Delta w = \nabla W(v) & \text{in } H \\ w = w_0 \in \Gamma & \text{on } \partial H \\ \dfrac{\partial w}{\partial x_2} = 0 & \text{on } \partial H \end{cases}; \tag{5.13}$$

here, w_0 is a constant. Let us note that w is not a constant. Indeed, we assumed by contradiction that (5.12) does not hold, and then the fact that w is not constant follows from (5.7).

Consider now the map

$$\widetilde{w} = \begin{cases} w, & \text{in } H \\ w_0, & \text{in } \mathbb{R}^2 \setminus H \end{cases}. \tag{5.14}$$

In view of (5.13), the map $\widetilde{w}$ satisfies $\Delta\widetilde{w} = \nabla W(\widetilde{w})$ in $\mathbb{R}^2$, first in the distributions sense, then, by elliptic regularity, in the classical sense. By unique continuation, we have $\widetilde{w} = w_0$. (The unique continuation property follows from [12]; there, $\widetilde{w}$ is a scalar function, but this is not relevant for the proof. For an explicit result relevant for vector-valued functions, see e.g. [11, Appendix].) This contradicts the fact that w is not a constant, and achieves the proof of the proposition. □

Now that we know that the "bad discs" $B_{\lambda\varepsilon}(x_j^\varepsilon)$ are well-inside Ω, we may define the integer d_j^ε as the degrees of u_ε on $\partial B_{\lambda\varepsilon}(x_j^\varepsilon)$. By (3.9), these integers are uniformly bounded, so we may assume that their values are independent of ε as well, and thus

$$\deg(u, \partial B_{\lambda\varepsilon}(x_j^\varepsilon)) = d_j, \ \forall\, \varepsilon, \ j = 1, \dots, k. \tag{5.15}$$

In the sequel, in case there is no risk of confusion, we shall often drop the subscript ε.

Our next estimate yields in particular a simple answer to Open Problem 19 in the book [5] (previously solved in [6] using a different method); see Corollary 5.5 below.

5.3 Proposition. *We have* $\int_{\Omega_\varepsilon} |\nabla t|^2 \le C$.

Proof. The proof uses the following pointwise inequality:

$$|\nabla W(z)|^2 \le M W(z), \ \forall z \in B_{R_0}, \tag{5.16}$$

for some $M > 0$. The validity of (5.16) for z in a neighborhood of Γ follows from (2.5); the extension to arbitrary $z \in B_{R_0}$ is clear (see also Remark 5.4 below for a simple alternative argument valid also for degenerate W). Arguing as in [5, Ch. V], we obtain using the Galgardo-Nirenberg inequality, (5.16) and (5.2) that

$$\begin{aligned} \|\nabla u\|_{L^4(\Omega)} \le & C_1 \|u\|_{H^2}^{1/2} \|u\|_\infty^{1/2} \le C_2 \|u\|_{H^2}^{1/2} \le C_3 \left\{ \frac{1}{\varepsilon^2} \|\nabla W(u)\|_2 + 1 \right\}^{1/2} \\ \le & C_4 \left\{ \frac{1}{\varepsilon^2} \|W(u)\|_1^{1/2} + 1 \right\}^{1/2} \le \frac{C_5}{\varepsilon^{1/2}}. \end{aligned} \tag{5.17}$$

Next we multiply (2.6b) by t and integrate over Ω_ε. Using (5.17), (2.8b) (recall that $\delta_2 \le \delta_1 \le \delta_0$) and (5.5) we get

$$\begin{aligned} \int_{\Omega_\varepsilon} \left[|\nabla t|^2 + \frac{c_1 t^2}{\varepsilon^2} \right] \le & \widetilde{C}_1 \int_{\Omega_\varepsilon} |\nabla \varphi|^2 |t| + \sum_{j=1}^k \int_{\partial B_{\lambda\varepsilon}(x_j^\varepsilon)} t \frac{\partial t}{\partial n} \\ \le & \widetilde{C}_1 \left(\int_{\Omega_\varepsilon} |\nabla u|^4 \right)^{1/2} \left(\int_{\Omega_\varepsilon} t^2 \right)^{1/2} + \widetilde{C}_2 \le \widetilde{C}_1 C_5^2 \left(\int_{\Omega_\varepsilon} \frac{t^2}{\varepsilon^2} \right)^{1/2} + \widetilde{C}_2 \le \widetilde{C}_3. \end{aligned} \tag{5.18}$$

For the bound of the boundary integrals we used the estimate $\left| \frac{\partial t}{\partial n} \right| \le \frac{C}{\varepsilon}$ on $\partial B_{\lambda\varepsilon}(x_j^\varepsilon)$ (by (3.9)). The conclusion of the proposition is a direct consequence of (5.18). □

5.4 Remark. The inequality (5.16) was proved by Dieudonné in [7], in connection to his simplfied proof to a result of Glaeser [9] about the square root of a nonnegative C^2-function. A variant of Dieudonné's argument, valid for any $W \in C^2(\mathbb{R}^2)$, goes as follows. Fix a function $\xi \in C_c^\infty(\mathbb{R}^2;[0,1])$ such that $\xi \equiv 1$ on B_{R_0} and set, in $\mathbb{R}^2$, $F(z) := \xi(z)W(z) + (1-\xi(z))|z|^2$. Note that F is a smooth nonnegative function on $\mathbb{R}^2$. Let

$$K := \frac{1}{2}\max_{z\in\mathbb{R}^2} \|D^2F(z)\|,$$

where $\|A\|$ stands for the spectral norm of the matrix A. By Taylor formula

$$0 \le F(z+h) \le F(z) + \nabla F(z)\cdot h + K|h|^2, \tag{5.19}$$

for every $z,h \in \mathbb{R}^2$. Applying (5.19) for $h := -\dfrac{\nabla F(z)}{2K}$ yields $|\nabla F(z)|^2 \le 4KF(z)$, whence (5.16).

5.5 Corollary. *Let u_ε satisfy (1.2). Then*

$$\int_\Omega |\nabla(\mathrm{dist}(u_\varepsilon,\Gamma))|^2 \le C, \ \forall\, \varepsilon > 0. \tag{5.20}$$

In particular, in the GL case, i.e., $W(u) = (1-|u|^2)^2/4$, we have

$$\int_\Omega \big|\nabla|u_\varepsilon|\big|^2 \le C, \forall\, \varepsilon > 0. \tag{5.21}$$

Proof. Since

$$\|\nabla(\mathrm{dist}(u_\varepsilon,\Gamma))\|_\infty \le \|\nabla u_\varepsilon\|_\infty \le \frac{C}{\varepsilon}$$

(by (3.9)), we have

$$\int_{\cup_{j=1}^k B_{\lambda\varepsilon}(x_j^\varepsilon)} |\nabla(\mathrm{dist}(u_\varepsilon,\Gamma))|^2 \le C. \tag{5.22}$$

The result of the corollary readily follows from Proposition 5.3 and (5.22). (Recall that, in Ω_ε, we have $\mathrm{dist}(u_\varepsilon,\Gamma) = |t_\varepsilon|$.)

In the GL case, it suffices to note that $\mathrm{dist}(u_\varepsilon,\Gamma) = 1-|u_\varepsilon|$. □

5.2 A O(|log ε|) bound for the energy

The main result of this section is the following.

5.6 Proposition. *We have $E_\varepsilon(u_\varepsilon) \le C(|\log\varepsilon|+1), \ \forall\, \varepsilon > 0$.*

In view of Proposition 5.3, of (3.9) and (5.2), it suffices to obtain the following bound for the energy of the phase φ:

$$\int_{\Omega_\varepsilon} |\nabla\varphi|^2 \le C(|\log\varepsilon|+1), \ \forall\, \varepsilon > 0. \tag{5.23}$$

Since φ is defined only locally in Ω_ε (only its gradient $\nabla\varphi$ is defined globally), it will be convenient to introduce a new function, which is globally defined in Ω_ε.

5.7 Definition. Let Π denote the nearest point projection on Γ in a δ_2-tubular neighborhood of Γ. The $\mathbb{S}^1$-valued map

$$\Omega_\varepsilon \ni z \mapsto \tau^{-1}(\Pi(u)) \cdot \prod_{j=1}^{k} \left(\frac{z - x_j}{|z - x_j|} \right)^{-d_j}$$

(with d_j as in (5.15)) has zero degree around each of the holes $B_{\lambda\varepsilon}(x_j)$, $j = 1, \dots, k$. Hence, there exists a unique (up to addition of an integer multiple of 2π) scalar function $\eta = \eta_\varepsilon$ such that

$$\tau^{-1}(\Pi(u)) = e^{\imath\eta} \prod_{j=1}^{k} \left(\frac{z - x_j}{|z - x_j|} \right)^{d_j} \text{ in } \Omega_\varepsilon. \tag{5.24}$$

By adding an appropriate multiple of 2π we may assume that

$$\min_{\partial\Omega} \eta_\varepsilon \in [0, 2\pi). \tag{5.25}$$

Since g is smooth, we deduce from (5.25) that

$$\|\eta_\varepsilon\|_{L^\infty(\partial\Omega)} \leq C(g). \tag{5.26}$$

Our first step consists of proving an L^∞ bound for η. In order to be able to apply the maximum principle of Proposition 2.1 we will remove from Ω_ε a collection of rays, connecting the boundaries of the holes $B_{\lambda\varepsilon}(x_j)$, $j = 1, \dots, k$, to the boundary of Ω. The choice of these "good rays" will depend on energy considerations. For any $j = 1, \dots, k$ and $\alpha \in [0, 2\pi)$, we let $D_j(\alpha)$ be the half-line

$$D_j(\alpha) := \{x_j^\varepsilon + r\, e^{\imath\alpha}; r \in [\lambda\varepsilon, \infty)\},$$

and then set

$$R_j(\alpha) := D_j \cap \Omega_\varepsilon.$$

5.8 Lemma. *For each $j = 1, \dots, k$ and $0 < \varepsilon < 1/2$, there exists $\alpha_j = \alpha_j(\varepsilon) \in [0, 2\pi)$ such that $R_j := R_j(\alpha_j)$ satisfies*

$$\int_{R_j} \left| \frac{\partial \eta}{\partial r} \right| \leq C |\log \varepsilon|^{1/2} \|\nabla \eta\|_{L^2(\Omega_\varepsilon)}. \tag{5.27}$$

Here, $\partial/\partial r$ stands for the tangential derivative along R_j.

Proof. Since

$$\int_{\Omega_\varepsilon} |\nabla \eta|^2 \geq \int_0^{2\pi} \left(\int_{R_j(\alpha)} |\nabla \eta|^2 \, r dr \right) d\alpha,$$

there exists $\alpha_j \in [0, 2\pi)$ such that

$$\int_{R_j(\alpha_j)} |\nabla \eta|^2 \, r dr \leq \frac{1}{2\pi} \|\nabla \eta\|^2_{L^2(\Omega_\varepsilon)}. \tag{5.28}$$

Therefore,

$$\int_{R_j(\alpha_j)} \left| \frac{\partial \eta}{\partial r} \right| \leq \left(\int_{\lambda\varepsilon}^{\operatorname{diam}\Omega} \frac{dr}{r} \right)^{1/2} \left(\int_{R_j(\alpha_j)} \left| \frac{\partial \eta}{\partial r} \right|^2 r dr \right)^{1/2} \leq C |\log \varepsilon|^{1/2} \|\nabla \eta\|_{L^2(\Omega_\varepsilon)}. \qquad \square$$

Next, we denote $\omega_\varepsilon := \Omega_\varepsilon \setminus \bigcup_{j=1}^{k} R_j$.

For each j, let θ_j denote the polar coordinate around the point x_j, taking values in $[\alpha_j, \alpha_j + 2\pi)$. Then the function

$$\Theta = \Theta_\varepsilon := \sum_{j=1}^{k} d_j \theta_j, \tag{5.29}$$

is *smooth* in ω_ε and satisfies

$$\|\Theta\|_{L^\infty(\omega_\varepsilon)} \le 4\pi \sum_{j=1}^{k} |d_j|. \tag{5.30}$$

We define $\varphi = \varphi_\varepsilon := \eta + \Theta$ in ω_ε. Note that

$$\tau^{-1}(\Pi(u)) = e^{\imath\eta} \prod_{j=1}^{k} \left(\frac{z - x_j}{|z - x_j|} \right)^{d_j} = e^{\imath(\Theta+\eta)} = e^{\imath\varphi} \text{ in } \omega_\varepsilon,$$

so that φ is a well-defined phase of u in ω_ε.

5.9 Lemma. *We have*

$$\|\eta\|_{L^\infty(\partial\omega_\varepsilon)} \le C \left(|\log\varepsilon|^{1/2} \|\nabla\eta\|_{L^2(\Omega_\varepsilon)} + 1 \right) \tag{5.31}$$

and

$$\limsup_{\delta\to 0} \sup\{|\varphi(x)|;\, x \in \omega_\varepsilon,\, \operatorname{dist}(x, \partial\omega_\varepsilon) \le \delta\} \le C \left(|\log\varepsilon|^{1/2} \|\nabla\eta\|_{L^2(\Omega_\varepsilon)} + 1 \right). \tag{5.32}$$

Proof. We may assume that $0 < \varepsilon < 1/2$. Let $r_j(\alpha)$ be the smallest $r > \lambda\varepsilon$ such that $x_j^\varepsilon + r e^{\imath\alpha} \in \partial\Omega$. By Lemma 5.8 and (3.9), for each $x \in [x_j^\varepsilon + \lambda e^{\imath\alpha}, x_j^\varepsilon + r_j(\alpha) e^{\imath\alpha}]$ we have

$$\left| \eta(x) - \eta\left(x_j^\varepsilon + r_j(\alpha_j) e^{\imath\alpha_j} \right) \right| \le C \left(|\log\varepsilon|^{1/2} \|\nabla\eta\|_{L^2(\Omega_\varepsilon)} + 1 \right). \tag{5.33}$$

Note that (3.9) is needed in case R_j intersects some of the other discs $\{B_{\lambda\varepsilon}(x_l^\varepsilon)\}_{l\neq j}$ before reaching $\partial\Omega$ for the first time, at $x_j^\varepsilon + r_j(\alpha_j)$. In particular, the following holds:

$$\left| \eta(x_j^\varepsilon + \lambda\varepsilon e^{\imath\alpha_j}) - \eta\left(x_j^\varepsilon + r_j(\alpha_j) e^{\imath\alpha_j} \right) \right| \le C \left(|\log\varepsilon|^{1/2} \|\nabla\eta\|_{L^2(\Omega_\varepsilon)} + 1 \right). \tag{5.34}$$

On the other hand, by (3.9) we have

$$|\eta(x) - \eta(y)| \le C,\ j = 1, \dots, k,\ \forall\, x, y \in \partial B_{\lambda\varepsilon}(x_j^\varepsilon). \tag{5.35}$$

We obtain (5.31) by combining (5.33)–(5.35) with (5.26).
Finally, (5.32) follows from (5.31) and (5.30). □

5.10 Lemma. *We have* $\|\eta\|_{L^\infty(\Omega_\varepsilon)} \le C \left(|\log\varepsilon|^{1/2} \|\nabla\eta\|_{L^2(\Omega_\varepsilon)} + 1 \right)$.

Proof. We apply the maximum principle in Proposition 2.1 to φ on each component of the open set $\{x \in \omega_\varepsilon; \operatorname{dist}(x, \partial\omega_\varepsilon) > \delta\}$, then we let $\delta \to 0$ (with fixed ε). Using (5.32), we find that

$$\sup_{\omega_\varepsilon} |\varphi| \le C\left(|\log\varepsilon|^{1/2}\|\nabla\eta\|_{L^2(\Omega_\varepsilon)} + 1\right). \tag{5.36}$$

The bound for η is a consequence of (5.30) and (5.36). □

Proof of Proposition 5.6. By (2.6), η satisfies in Ω_ε

$$\begin{aligned} -\operatorname{div}(a\nabla\eta) &= -\operatorname{div}(a\nabla\varphi) + \operatorname{div}(a\nabla\Theta) = b|\nabla\varphi|^2 - \frac{\alpha_\varphi t^2}{\varepsilon^2} + \operatorname{div}(a\nabla\Theta) \\ &= f + \operatorname{div}(a\nabla\Theta), \end{aligned} \tag{5.37}$$

with

$$f = f_\varepsilon := b\,|\nabla\varphi|^2 - \frac{\alpha_\varphi t^2}{\varepsilon^2}. \tag{5.38}$$

Above we denoted by $\nabla\Theta$ the vector field

$$\nabla\Theta = \sum_{j=1}^{k} d_j \nabla\theta_j = \sum_{j=1}^{k} d_j \frac{(x - x_j^\varepsilon)^\perp}{|x - x_j^\varepsilon|},$$

which is smooth in $\mathbb{R}^2 \setminus \{x_1^\varepsilon, \dots, x_k^\varepsilon\}$. We claim that

$$\|f_\varepsilon\|_{L^1(\Omega_\varepsilon)} \le C. \tag{5.39}$$

Indeed, the second term on the right-hand side of (5.38) is bounded in $L^1(\Omega_\varepsilon)$ by Proposition 5.3. The L^1 boundedness of the first term $b\,|\nabla\varphi|^2$ follows from the calculation (5.18) and the inequality (2.8d).

Multiplying (5.37) by η and integrating yields

$$\begin{aligned} \int_{\Omega_\varepsilon} a|\nabla\eta|^2 &= \int_{\Omega_\varepsilon} f\eta - \int_{\Omega_\varepsilon} a\nabla\Theta\cdot\nabla\eta + \int_{\partial\Omega_\varepsilon} a\,\frac{\partial\varphi}{\partial n}\eta \\ &\le \|f\|_1\|\eta\|_\infty + C|\log\varepsilon|^{1/2}\|\nabla\eta\|_{L^2(\Omega_\varepsilon)} + C\|\eta\|_\infty \\ &\le C\left(|\log\varepsilon|^{1/2}\|\nabla\eta\|_{L^2(\Omega_\varepsilon)} + 1\right). \end{aligned} \tag{5.40}$$

The second line in (5.40) uses (5.2) on $\partial\Omega$ and (3.9) on $\partial B_{\lambda\varepsilon}(x_j^\varepsilon)$. The third line follows from Lemma 5.10.

From (5.40) we get

$$\|\nabla\eta\|^2_{L^2(\Omega_\varepsilon)} \le C(|\log\varepsilon| + 1), \tag{5.41}$$

and therefore

$$\|\nabla\varphi\|^2_{L^2(\Omega_\varepsilon)} \le 2\left(\|\nabla\Theta\|^2_{L^2(\Omega_\varepsilon)} + \|\nabla\eta\|^2_{L^2(\Omega_\varepsilon)}\right) \le C(|\log\varepsilon| + 1). \tag{5.42}$$

As explained above, estimate (5.42) implies Proposition 5.6. □

Combining Lemma 5.10 with (5.41) we obtain the following.

5.11 Corollary. *We have* $\|\eta\|_{L^\infty(\Omega_\varepsilon)} \le C(|\log\varepsilon| + 1)$.

5.3 An L^p-bound for the gradient, $p \in [1,2)$

The main result of this section is

5.12 Proposition. *We have* $\|\nabla u_\varepsilon\|_{L^p(\Omega)} \le C_p$, $1 \le p < 2$.

Proof. Fix any $p \in (1,2)$. We will apply the bad discs construction from Subsection 5.1 with a $\delta_2 = \delta_2(p) \le \delta_1$, that we define below. By standard elliptic estimates, there exists a constant $A_p = A_p(\Omega)$ such that the solution w of the problem

$$\begin{cases} -\Delta w = \operatorname{div} \mathbf{g} & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{5.43}$$

with $\mathbf{g} \in (L^p(\Omega))^2$ satisfies

$$\|\nabla w\|_{L^p(\Omega)} \le A_p \|\mathbf{g}\|_{L^p(\Omega)}. \tag{5.44}$$

We require from $\delta_2(p)$ to satisfy

$$0 < \delta_2(p) \le \min\left(\delta_1, \frac{1}{2c_0 A_p}\right), \tag{5.45}$$

where c_0 is defined in (2.8a). We choose $\delta_2 = \delta_2(p)$ accordingly such that (5.11) holds. In the sequel, Ω_ε denotes the set given in (5.11) for this choice of δ_2. Note that the number of discs and the value of λ may change with δ_2, but we shall use the same notation as before.

Let $H = H_\varepsilon$ denote the harmonic function in Ω satisfying $H = \eta_\varepsilon$ on $\partial\Omega$. By (5.26) and the maximum principle,

$$\|H_\varepsilon\|_{L^\infty(\Omega)} = \|\eta_\varepsilon\|_{L^\infty(\partial\Omega)} \le C(g). \tag{5.46}$$

Note that

$$\|\eta_\varepsilon\|_{W^{1-1/p,p}(\partial\Omega)} \le C,$$

since

$$\left\| \prod_{j=1}^{k} \left(\frac{z - x_j}{|z - x_j|} \right)^{d_j} \right\|_{W^{1,p}(\Omega)} \le C,$$

see (5.24). Therefore, we also have

$$\|H\|_{W^{1,p}(\Omega)} \le C. \tag{5.47}$$

Consider a function $\xi = \xi_\varepsilon \in C^\infty(\overline{\Omega})$ satisfying

$$0 \le \xi \le 1,\ \xi \equiv 1 \text{ on } \Omega \setminus \bigcup_{j=1}^{k} B_{2\lambda\varepsilon}(x_j^\varepsilon),\ \xi \equiv 0 \text{ on } \bigcup_{j=1}^{k} B_{3\lambda\varepsilon/2}(x_j^\varepsilon),\ \|\nabla\xi\|_\infty \le \frac{4}{\lambda\varepsilon}. \tag{5.48}$$

Note that, by (5.12), for small ε the discs $\{B_{2\lambda\varepsilon}(x_j)\}$ do not intersect the boundary, and thus $\xi = 1$ on $\partial\Omega$. From the properties of ξ we obtain, in particular, that

$$\|\nabla\xi\|_p = \|\nabla\xi\|_{L^p(\bigcup_{j=1}^{k} B_{2\lambda\varepsilon}(x_j^\varepsilon))} \le C\,\varepsilon^{2/p-1} = o(1). \tag{5.49}$$

In Ω, we set $\widetilde{\eta} := \widetilde{\eta}_\varepsilon = \xi^2\eta$ and $\widetilde{H} = \widetilde{H}_\varepsilon := \xi^2 H$. From (5.46)–(5.49) we conclude that

$$\|\widetilde{H}\|_{W^{1,p}(\Omega)} \le C. \tag{5.50}$$

Note that, although η is defined only in Ω_ε, the function $\widetilde{\eta}$ is globally defined (and smooth), since $\eta = 0$ on a neighborhood of $\overline{B}_{\lambda\varepsilon}(x_j^\varepsilon)$.

The function $\widetilde{\eta}$ satisfies

$$\begin{aligned}
-\operatorname{div}(a\nabla\widetilde{\eta}) &= -\operatorname{div}(a\xi^2\nabla\eta) - \operatorname{div}(a\eta\nabla(\xi^2)) \\
&= \underbrace{-\xi^2\operatorname{div}(a\nabla\varphi)}_{F_1}\underbrace{-a\nabla(\xi^2)\cdot\nabla\varphi}_{F_2} + \operatorname{div}(\underbrace{a\xi^2\nabla\Theta}_{G_1}) + \operatorname{div}(\underbrace{-2a\eta\xi\nabla\xi}_{G_2}) \\
&:= F_1 + F_2 + \operatorname{div}G_1 + \operatorname{div}G_2.
\end{aligned}$$

Therefore,

$$\begin{cases} -\Delta(\widetilde{\eta} - \widetilde{H}) &= F_1 + F_2 + \operatorname{div}G_1 + \operatorname{div}G_2 + \operatorname{div}(a\nabla\widetilde{H}) \\ & \quad + \operatorname{div}((a-1)\nabla(\widetilde{\eta} - \widetilde{H})) \text{ in } \Omega \\ \widetilde{\eta} - \widetilde{H} &= 0 \text{ on } \partial\Omega \end{cases}. \tag{5.51}$$

By elliptic estimates, there exists $B_p = B_p(\Omega) > 0$ such that the solution w of the problem

$$\begin{cases} -\Delta w = v & \text{in } \Omega \\ w = 0 & \text{on } \partial\Omega \end{cases} \tag{5.52}$$

with $v \in L^1(\Omega)$, satisfies

$$\|\nabla w\|_p \le B_p \|v\|_1. \tag{5.53}$$

Note that $F_1 = \xi^2 f$ is bounded in $L^1(\Omega)$; here, f is defined in (5.38). The same holds for F_2 since, by (3.9),

$$\int_\Omega |a\nabla(\xi^2)\cdot\nabla\varphi| = \int_{\bigcup_{j=1}^k B_{2\lambda\varepsilon}(x_j^\varepsilon)\setminus B_{3\lambda\varepsilon/2}(x_j^\varepsilon)} |a\nabla(\xi^2)\cdot\nabla\varphi| \le C_1\varepsilon^2\|\nabla\xi\|_\infty\|\nabla u\|_\infty \le C_2.$$

Using the inequality

$$|\nabla\Theta(x)| \le \frac{C}{r}, \text{ with } r = r(x) := \operatorname{dist}(x, \{x_1^\varepsilon, \dots, x_k^\varepsilon\}),$$

we find that G_1 is bounded in $L^p(\Omega)$. Similarly, G_2 is bounded in $L^p(\Omega)$, since

$$\int_\Omega |G_2|^p \le C_3\|\eta\|_\infty^p \left(\frac{1}{\varepsilon}\right)^p \varepsilon^2 \le C_4\varepsilon^{2-p}|\log\varepsilon|^p = o(1),$$

by Corollary 5.11 and (5.48). Finally, $a\nabla\widetilde{H}$ is bounded in $L^p(\Omega)$ by (5.50).

We also note that

$$|t(x)| \le \delta_2(p) \text{ on } \operatorname{supp}(\nabla(\widetilde{\eta} - \widetilde{H})) \subset \Omega_\varepsilon.$$

Using the above in (5.51) we get by (5.44) and (5.53) that

$$\begin{aligned}\|\nabla(\widetilde{\eta}-\widetilde{H})\|_{L^p} &\le A_p\left(\|(a-1)\nabla(\widetilde{\eta}-\widetilde{H})\|_{L^p}+\|G_1\|_{L^p}+\|G_2\|_{L^p}+\|a\nabla\widetilde{H}\|_{L^p}\right)\\ &\quad + B_p\left(\|F_1\|_{L^1}+\|F_2\|_{L^1}\right)\\ &\le A_p c_0\delta_2(p)\|\nabla(\widetilde{\eta}-\widetilde{H})\|_{L^p}+C.\end{aligned} \tag{5.54}$$

Combining (5.45) and (5.54), we find that $\|\nabla(\widetilde{\eta}-\widetilde{H})\|_{L^p}\le C$, which in conjunction with (5.50) implies that $\|\nabla\widetilde{\eta}\|_{L^p}\le C$. Since $\|\nabla\Theta\|_{L^p(\Omega_\varepsilon)}\le C$, we obtain that

$$\|\nabla u_\varepsilon\|_{L^p(\Omega\setminus\cup_{j=1}^k B_{2\lambda\varepsilon}(x_j^\varepsilon))}\le C. \tag{5.55}$$

The conclusion of Proposition 5.12 follows from (5.55) and the fact that, by (3.9), $\{\nabla u_\varepsilon\}$ is bounded in $L^p(B_{2\lambda\varepsilon}(x_j^\varepsilon))$. □

5.4 A bound for the energy away from the singularities

We denote by $a_1,\dots,a_N\in\overline{\Omega}$ the different limits of the families $\{x_j^\varepsilon\}$, $j=1,\dots,k$ (possibly along a subsequence). Since two different families can converge to the same limit, we have $N\le k$. At this point we do not exclude the possibility that some of the a_i's belong to $\partial\Omega$. Consider some $r>0$ such that

$$r<\min\{|a_i-a_j|;\, i\ne j\}\text{ and }r<\operatorname{dist}(a_j,\partial\Omega),\ \forall\, j\text{ such that }a_j\in\Omega. \tag{5.56}$$

We denote

$$\widetilde{\Omega}_r:=\Omega\setminus\bigcup_{j=1}^N\overline{B_r(a_j)},$$

and by D_j the degree of u_ε on $\partial(B_\rho(a_j)\cap\Omega)$ for small ε and (small but fixed) ρ. The following equality is clear: if $J_j:=\{\ell;\, x_\ell^\varepsilon\to a_j\}$, then $D_j=\sum_{\ell\in J_j}d_\ell$.

5.13 Theorem. *For each r as in (5.56) we have*

$$E_\varepsilon(u_\varepsilon;\widetilde{\Omega}_r)\le C(r). \tag{5.57}$$

Proof. By the boundedness of $\{\nabla\eta\}$ in $L^1(\Omega_\varepsilon)$ (see Proposition 5.12), it follows that there exists $\widetilde{r}=\widetilde{r}(\varepsilon)\in(r/2,r)$ such that

$$\sum_{j=1}^N\int_{\partial B_{\widetilde{r}}(a_j)\cap\Omega}|\nabla\eta|\,d\sigma\le C_1(r). \tag{5.58}$$

Similarly, we can find for each $j\in\{1,\dots,N\}$ a number $\beta_j\in[0,2\pi)$ such that the set

$$\widetilde{R}_j=\widetilde{R}_j(\beta_j):=\{a_j+s\,e^{\imath\beta_j};\, s\ge\widetilde{r}\}\cap\widetilde{\Omega}_{\widetilde{r}}$$

satisfies

$$\int_{\widetilde{R}_j}\left|\frac{\partial\eta}{\partial s}\right|ds\le C_2(r). \tag{5.59}$$

Repeating the proof of Lemma 5.10 and using (5.58) and (5.59), we find that

$$\|\eta\|_{L^\infty(\widetilde{\Omega}_{\tilde{r}})} \le C_3(r). \tag{5.60}$$

For ε sufficiently small we have

$$|x_\ell^\varepsilon - a_j| < \tilde{r}/2, \ \forall \ell \in J_j, j = 1,\dots,N. \tag{5.61}$$

Next, we multiply the equation (5.37) satisfied by η and integrate over $\widetilde{\Omega}_{\tilde{r}}$. This yields as in (5.40)

$$\int_{\widetilde{\Omega}_{\tilde{r}}} a|\nabla\eta|^2 = \int_{\widetilde{\Omega}_{\tilde{r}}} f\eta - \int_{\widetilde{\Omega}_{\tilde{r}}} a\nabla\Theta\cdot\nabla\eta + \int_{\partial\widetilde{\Omega}_{\tilde{r}}} a\frac{\partial\varphi}{\partial n}\eta := I_1 + I_2 + I_3. \tag{5.62}$$

By (5.39) and (5.60) we have $|I_1| \le C_4(r)$. We claim that also $|I_3| \le C_5(r)$. Indeed, we use (5.2) and (5.60) for the integral on $\partial\widetilde{\Omega}_{\tilde{r}} \cap \partial\Omega$ and for the integral on $\partial B_{\tilde{r}}(a_j) \cap \Omega$ we use (5.58) and the fact that thanks to (5.61) we have

$$\left|\frac{\partial\Theta}{\partial n}\right| \le \frac{C}{\tilde{r}} \ \text{ on } \partial B_{\tilde{r}}(a_j).$$

Applying the Cauchy-Schwarz inequality to I_2 and the above estimates in (5.62) leads to

$$\int_{\widetilde{\Omega}_{\tilde{r}}} a|\nabla\eta|^2 \le C_6(r) + \int_{\widetilde{\Omega}_{\tilde{r}}} \frac{a}{2}|\nabla\eta|^2 + \int_{\widetilde{\Omega}_{\tilde{r}}} \frac{a}{2}|\nabla\Theta|^2. \tag{5.63}$$

Since $\int_{\widetilde{\Omega}_{\tilde{r}}} (a/2)|\nabla\Theta|^2 \le C_7(r)(|\log r| + 1)$, we get from (5.63) that $\int_{\widetilde{\Omega}_{\tilde{r}}} |\nabla\eta|^2 \le C_8(r)$. It follows that also $\int_{\widetilde{\Omega}_{\tilde{r}}} |\nabla\varphi|^2 \le C_9(r)$, which clearly implies (5.57). □

5.5 Convergence of $\{u_{\varepsilon_n}\}$

The bound of Proposition 5.12 implies that for a subsequence $\{u_{\varepsilon_n}\}$ we have

$$u_{\varepsilon_n} \rightharpoonup u_* \text{ weakly in } W^{1,p}(\Omega), \ \forall p \in [1,2), \tag{5.64}$$

for some $u_* \in \bigcap_{p\in[1,2)} W^{1,p}(\Omega;\Gamma)$. The fact that u_* is Γ-valued follows from (5.64) and the estimate (5.2) that implies the convergence $t_{\varepsilon_n} \to 0$ in $L^2(\Omega)$.

We can now further state

5.14 Proposition. *We have*

$$u_{\varepsilon_n} \to u_* \text{ in } C^{1,\alpha}(\overline{\Omega} \setminus \{a_1,\dots,a_N\}), \ \forall \alpha \in (0,1). \tag{5.65}$$

The limit u_* is a Γ-valued harmonic map in $\Omega \setminus \{a_1,\dots,a_N\}$.

Proof. We argue as in [5, Proof of Theorem VI.1]. For notational simplicity, we drop in what follows the subscript n. It suffices to show that for every $x_0 \in \overline{\Omega} \setminus \{a_1,\dots,a_N\}$ there exists $R > 0$ such that $u_\varepsilon \to u_*$ in $C^{1,\alpha}(\overline{\Omega} \cap B_R(x_0))$. Consider first the case $x_0 \in \Omega$. We choose $R > 0$ such that $B_{2R}(x_0) \Subset \Omega \setminus \{a_1,\dots,a_N\}$. Since, by (5.57),

$$E_\varepsilon(u_\varepsilon; B_{2R}(x_0)) \le C,$$

we can use Fubini's theorem to find $R' \in (R, 2R)$ such that (after passing to a further subsequence),

$$\int_{\partial B_{R'}(x_0)} \left[\frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2} \right] \leq C.$$

Then, applying Theorem 4.1 we obtain that, up to a further subsequence, $u_\varepsilon \to u_0$ in $C^{1,\alpha}(\overline{B}_R(x_0))$, and that u_0 is a harmonic map in $B_R(x_0)$ (since it can be written as $u_0 = \tau(e^{\imath\zeta})$ where ζ is a harmonic function in $B_R(x_0)$). Using the uniqueness of the limit, we find that $u_0 = u_*$, and that the original subsequence $\{u_{\varepsilon_n}\}$ converges to u_* in $C^{1,\alpha}(\overline{B}_R(x_0))$.

It remains to consider the case $x_0 \in \partial\Omega \setminus \{a_1, \dots, a_N\}$ (at this stage we do not exclude the possibility that some of the a_j's belong to $\partial\Omega$). We choose a small $R > 0$ such $\partial B_R(x_0) \cap \partial\Omega$ consists of exactly two points and

$$R < \min_{1 \leq j \leq N} |x_0 - a_j|.$$

Again by (5.57), we have

$$E_\varepsilon(u_\varepsilon; \Omega \cap B_{2R}(x_0)) \leq C,$$

and by Fubini's theorem there exists $R' \in (R, 2R)$ such that (after passing to a further subsequence),

$$\int_{\partial B_{R'}(x_0) \cap \Omega} \left[\frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{W(u_\varepsilon)}{\varepsilon^2} \right] \leq C.$$

Applying Theorem 4.7 we obtain that $u_\varepsilon \to u_*$ in $C^{1,\alpha}(B_R(x_0) \cap \overline{\Omega})$. □

Next we deduce further properties of the map u_* that will enable us to conclude the proof of Theorem 1.1.

5.15 Proposition. *We have* $\{a_1, \dots, a_N\} \subset \Omega$.

Proof. The proof is the same as that of [5, Theorem X.4], so we just mention the main idea. By Pohozaev identity (5.2) and Proposition 5.14 it follows that

$$\int_{\partial\Omega} \left| \frac{\partial u_*}{\partial n} \right|^2 < \infty. \tag{5.66}$$

The map $v_* := \tau^{-1} \circ u_*$ is an $\mathbb{S}^1$-valued smooth harmonic map on $\overline{\Omega} \setminus \{a_1, \dots, a_N\}$, and satisfies: $v_* \in W^{1,p}(\Omega; \mathbb{S}^1)$ for all $p \in [1, 2)$, $v_* = \tau^{-1} \circ g$ on $\partial\Omega$, and thanks to (5.66), also

$$\int_{\partial\Omega} \left| \frac{\partial v_*}{\partial n} \right|^2 < \infty.$$

Therefore, all the hypotheses of [5, Lemma X.14] are satisfied, and we can conclude that v_* is smooth in a neighborhood of $\partial\Omega$. Clearly, the same holds for u_*. □

To conclude the proof of Theorem 1.1 we need to show that the limit u_* has the form given in (1.11).

5.16 Proposition. *We have*

$$u_*(z) = \tau\left(e^{\imath\eta(z)}\left(\frac{z-a_1}{|z-a_1|}\right)^{D_1}\cdots\left(\frac{z-a_N}{|z-a_N|}\right)^{D_N}\right), \tag{5.67}$$

for some smooth harmonic function η in $\overline{\Omega}$ and $D_1,\dots,D_N \in \mathbb{Z}\setminus\{0\}$.

Equivalently, Proposition 5.16 asserts that the $\mathbb{S}^1$-valued harmonic map $\tau^{-1}\circ u_*$ is the canonical harmonic map associated with $\tau^{-1}\circ g$ and $\{(a_j,D_j)\}_{j=1}^N$, as defined in [5, Sec I.3].

Proof. We apply the same argument as in [5, Ch. VII], which uses the Hopf differential. Setting

$$\omega = \omega_\varepsilon := \left|(u_\varepsilon)_{x_1}\right|^2 - \left|(u_\varepsilon)_{x_2}\right|^2 - 2\imath\,(u_\varepsilon)_{x_1}\cdot(u_\varepsilon)_{x_2},$$

we find by a direct computation

$$\frac{\partial\omega}{\partial\overline{z}} = \frac{1}{2}\left(\omega_{x_1}+\imath\,\omega_{x_2}\right) = \Delta u_\varepsilon\cdot\left((u_\varepsilon)_{x_1}-\imath\,(u_\varepsilon)_{x_2}\right) = 2\Delta u_\varepsilon\cdot\frac{\partial u_\varepsilon}{\partial z}. \tag{5.68}$$

Moreover, by (1.2),

$$\frac{\partial}{\partial z}W(u_\varepsilon) = \nabla W(u_\varepsilon)\cdot\frac{\partial u_\varepsilon}{\partial z} = \varepsilon^2\Delta u_\varepsilon\cdot\frac{\partial u_\varepsilon}{\partial z}. \tag{5.69}$$

By (5.68)–(5.69),

$$\frac{\partial\omega}{\partial\overline{z}} = \frac{\partial}{\partial z}\left(\frac{2W(u_\varepsilon)}{\varepsilon^2}\right).$$

Note that up to a further subsequence we have

$$\frac{W(u_\varepsilon)}{\varepsilon^2} \overset{*}{\rightharpoonup} \sum_{j=1}^N m_j\delta_{a_j}, \tag{5.70}$$

for some positive m_j's. (Convergence is in the weak star topology of $C(\overline{\Omega})$.) Indeed, combining (4.9) and (4.52) we obtain, for any sufficiently small $R>0$,

$$W(u_\varepsilon) \le C_R\varepsilon^2 \text{ in } \overline{\Omega}\setminus\bigcup_{j=1}^N B_R(a_j),$$

which clearly implies (5.70) with $m_j\ge 0$. The fact that $m_j>0$ for all j follows from (5.10).

Defining the distribution

$$\alpha = \alpha_\varepsilon := \frac{\partial}{\partial z}\left[\left(\frac{1}{\pi z}\right)*\left(\chi_\Omega\frac{W(u_\varepsilon)}{\varepsilon^2}\right)\right],$$

we obtain by a direct calculation that $\beta = \beta_\varepsilon := \omega - 2\alpha$ is a holomorphic function in Ω (see also [5]).

Since, by (4.7)–(4.8),

$$\left\|\frac{1}{\varepsilon^2}\nabla W(u_\varepsilon)\frac{\partial u_\varepsilon}{\partial x_j}\right\|_{L^\infty(\overline{\Omega}\setminus\bigcup_{j=1}^N B_R(a_j))} \le C,\ j=1,2,$$

we obtain that $\left\{\dfrac{W(u_\varepsilon)}{\varepsilon^2}\right\}$ is bounded in $C^1(\overline{\Omega}\setminus\bigcup_{j=1}^N B_R(a_j))$, and we deduce by the argument of [5] that $\{\beta_\varepsilon\}$ is bounded in $C^0_{\rm loc}(\Omega)$. It follows that, up to a further subsequence, $\beta_\varepsilon\to\beta_*$ in $C^k_{\rm loc}(\Omega)$, $\forall\, k$, for some holomorphic function β_* in Ω. In addition, using (5.70) we find that

$$\alpha_\varepsilon\to\alpha_*:=\frac{\partial}{\partial z}\left(\frac{1}{\pi z}\right)*\sum_{j=1}^N m_j\delta_{a_j}=-\frac{1}{\pi}\sum_{j=1}^N m_j\,\text{p.v.}\,\frac{1}{(z-a_j)^2}\ \text{in}\ \mathscr{D}'(\mathbb{R}^2). \tag{5.71}$$

Therefore, $\omega_\varepsilon=\beta_\varepsilon+2\alpha_\varepsilon\to\beta_*+2\alpha_*$ in $\mathscr{D}'(\Omega)$.
Since Proposition 5.14 implies that

$$\omega_\varepsilon\to\omega_*:=\left|(u_*)_{x_1}\right|^2-\left|(u_*)_{x_2}\right|^2-2\imath\,(u_*)_{x_1}\cdot(u_*)_{x_2}\ \text{in}\ C^0_{\rm loc}(\overline{\Omega}\setminus\{a_1,\dots,a_N\}),$$

we obtain

$$\omega_*=\beta_*+2\alpha_*\ \text{in}\ \mathscr{D}'(\Omega\setminus\{a_1,\dots,a_N\}). \tag{5.72}$$

Fix any $j\in\{1,\dots,N\}$ and assume without loss of generality that $a_j=0$. Recall that $\tau^{-1}\circ u$ is a harmonic map in $\Omega\setminus\{a_1,\dots,a_N\}$ (Proposition 5.14) and belongs to $W^{1,p}(\Omega)$ when $1\le p<2$ (Proposition 5.12). In addition, we have $\deg(u_*,0)=D_j$. Arguing as in [5, Remark I.1] we may write, near 0,

$$u_*=\tau\left(\exp(\imath D_j\theta+\imath c_j\log r+\imath h)\right),$$

where h is a harmonic function.

It follows that if we write, locally near 0, $u_*=\tau(e^{\imath\varphi})$ with $\varphi:=D_j\theta+c_j\log r+h$, then we have

$$\omega_*=\left|(u_*)_{x_1}\right|^2-\left|(u_*)_{x_2}\right|^2-2\imath\,(u_*)_{x_1}\cdot(u_*)_{x_2}=\left(\varphi_{x_1}-\imath\,\varphi_{x_2}\right)^2=\left(\frac{c_j-\imath D_j}{z}+2\frac{\partial h}{\partial z}\right)^2. \tag{5.73}$$

From (5.71)–(5.73) we obtain

$$(c_j-\imath D_j)^2=-\frac{2m_j}{\pi},$$

implying that $c_j=0$ and also $m_j=\dfrac{\pi D_j^2}{2}$. The fact that $c_j=0$ for all j implies that u_* has the form (5.67). Since we know already that $m_j\neq 0$ for all j, it follows that also $D_j\neq 0$ for all j. □

5.17 Remark. Arguing as in [5, Ch. VII], we may conclude from (5.73) that $\partial h/\partial z=0$. This implies that the configuration $(a_1,\dots,a_N)$ is a critical point of the renormalized energy associated with the degrees $(D_j)_{j=1}^N$ and the $\mathbb{S}^1$-valued boundary condition $\tau^{-1}\circ g$, see [5, Corollary VIII.1].

References

[1] Almeida, L., Bethuel, F. (1998). Topological methods for the Ginzburg-Landau equations. *J. Math. Pures Appl. (9)* 77(1), 1–49.

[2] André, N., Shafrir, I. (2003). On a singular perturbation problem involving the distance to a curve. *J. Anal. Math. 90*, 337–396.

[3] André, N., Shafrir, I. (2007). On a singular perturbation problem involving a "circular-well" potential. *Trans. Amer. Math. Soc. 359*(10), 4729–4756 (electronic).

[4] Bethuel, F., Brezis, H., Hélein, F. (1993). Asymptotics for the minimization of a Ginzburg-Landau functional. *Calc. Var. and Partial Differential Equations 1*, 123–148.

[5] Bethuel, F., Brezis, H., Hélein, F. (1994). *Ginzburg-Landau Vortices*. Birkhäuser.

[6] Comte, M., Mironescu, P. (1996). Remarks on nonminimizing solutions of a Ginzburg-Landau type equation. *Asymptotic Anal. 13*(2), 199–215.

[7] Dieudonné, J. (1970). Sur un théorème de Glaeser. *J. Analyse Math. 23*, 85–88.

[8] Gilbarg, D., Trudinger, N.S. (2001). *Elliptic partial differential equations of second order*. Classics in Mathematics. Berlin: Springer-Verlag.

[9] Glaeser, G. (1963). Racine carrée d'une fonction différentiable. *Ann. Inst. Fourier (Grenoble) 13*(fasc. 2), 203–210.

[10] Lin, F.H. (1995). Mixed vortex-antivortex solutions of Ginzburg-Landau equations. *Arch. Rational Mech. Anal. 133*(2), 103–127.

[11] Lopes, O. (1996). Radial symmetry of minimizers for some translation and rotation invariant functionals. *J. Differential Equations 124*(2), 378–388.

[12] Müller, C. (1954). On the behavior of the solutions of the differential equation $\Delta U = F(x,U)$ in the neighborhood of a point. *Comm. Pure Appl. Math. 7*, 505–515.

[13] Zhou, F., Zhou, Q. (1999). A remark on multiplicity of solutions for the Ginzburg-Landau equation. *Ann. Inst. H. Poincaré Anal. Non Linéaire 16*(2), 255–267.