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On feedback stabilization of linear switched systems via switching signal control

Raphaël M. Jungers*, Paolo Mason†

Abstract

Motivated by recent applications in control theory, we study the feedback stabilizability of switched systems, where one is allowed to choose the switching signal as a function of $x(t)$ in order to stabilize the system. We propose new algorithms and analyze several mathematical features of the problem which were unnoticed up to now, to our knowledge. We prove complexity results, (in-)equivalence between various notions of stabilizability, existence of Lyapunov functions, and provide a case study for a paradigmatic example introduced by Stanford and Urbano in the nineties.

1 Introduction

Switched systems are a paradigmatic family of complex systems, which has sparked many interesting research efforts in the last decades. They appear naturally in many engineering situations, or as abstractions of more complicated systems. In this framework, the basic stability and stabilizability questions which one might ask turn out to be extremely challenging. This is true even in the particular case in which the dynamics switch within a given set of linear modes, and even if only a finite set of linear modes is available, which is the case considered in this paper.

In order to support this claim, let us mention one question: Is it possible to design a switching signal giving rise to an asymptotically stable behaviour, i.e. such that for any starting point the corresponding trajectory converges to zero? It is well known that the answer to this question for discrete time linear switched systems depends on the so-called *joint spectral subradius*, and this quantity is known to be Turing-uncomputable even if we restrain to the case of two available linear modes [3],[20, Section 2.2.4].

If the joint spectral subradius determines the possibility of stabilizing the system in open loop assuming the switching signal independent of the initial state, the quantity we focus on here is instead related to the feedback stabilization problem: suppose that one can observe at every time t the value of $x(t)$, and is allowed to control the system based on this sole information; is there a strategy allowing to globally stabilize the system?

This question appears naturally in several applications, and has motivated the interest of control theorists in recent years. Let us mention [24] for a study of this problem motivated by supervisory control and measurement scheduling and [17] concerning the problem of optimizing drug treatments of some viral pathologies, like HIV. Other important applications are in bisimulation of hybrid systems [15], where the switched system is an abstraction of a more complex one. See [5, 7, 18, 27] for more works and applications around the stabilization of switched systems.

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Concerning the stabilization issue for discrete-time switched systems, to our knowledge the first work mentioning the advantage of making use of switching laws depending on the initial condition, is [36]. The examples provided in [36] suggest in particular that the existence of a stabilizing switching sequence independent on the initial condition is not equivalent to the stabilizability by a feedback law. Subsequent papers have provided algorithmic ways of constructing a stabilizing feedback switching rule: In [41, 42], a first sufficient LMI (Linear Matrix Inequalities) condition is given. This condition ensures the existence of an ellipsoid which can be left invariant under a proper state feedback switching rule. Characterizations of the existence of such an invariant ellipsoid (often termed as *quadratic stability*), are given in [34, 40]. In [14], a more general condition is given, which allows for more complex invariant sets, described by piecewise quadratic functions. This condition (under its most general form) is a BMI (Bilinear Matrix Inequality), but relaxations of it are given, which can be more easily checked, and still are more general than the above mentioned condition. Other conditions can be found in [11, 14, 29]. Interesting improvements on the BMI approach have been obtained recently [10].

A further improvement has been obtained in [43], where it is shown how to iterate such conditions in order to decrease their conservativeness, at the cost of increasing the computation time. Asymptotically, these conditions become necessary; that is, if the system is stabilizable via control of the switching signal, there is a condition in the hierarchy that will be satisfied. See [24] for similar results with a slightly different notion of stabilizability. As pointed out by the authors, in practice, for reasonably large systems, the conditions could become too computationally expensive before to reach one that is satisfied.

Let us notice that the stabilization problem for discrete-time switched systems has been largely studied even beyond the framework considered in this paper, namely in the case in which additional linear control terms are taken into account. See, for instance, [2, 25, 26, 28] for the feedback stabilization problem under arbitrary switching, and [5, 23, 43] for the stabilization problem through the joint action of a linear controller and a controller-ruled switching signal.

Concerning continuous-time systems, the feedback stabilization problem has attracted a considerable attention in the non-linear control community since many decades. In particular let us mention that many noteworthy results have been obtained starting from the 80s, dealing with necessary and sufficient conditions for asymptotic controllability and stabilizability as well as existence and properties of control-Lyapunov functions, see for instance [1, 6, 8, 35]. In many cases these results directly apply to the switched systems setting. However, there exist only few works specializing on the stabilizability problem in the case of continuous-time switched systems. In this context let us mention that for two dimensional linear continuous-time switched systems the stabilizability property can be characterized in a quite explicit way (see e.g. [9]), namely a switched system is stabilizable if and only if it satisfies a few simple algebraic inequalities.

In this paper we deal with both the analytic and the algorithmic side of the stabilization problem, mainly in the discrete-time case. In the first part we perform a qualitative analysis of the stabilizability issue for discrete-time linear switched systems, showing in particular, in Proposition 1 and Corollary 1, that such systems admit a stabilizable switching law depending on the initial point if and only if they are stabilizable by means of an homogeneous feedback. One of the main ingredients in order to prove this result consists in the construction of a suitable control-Lyapunov function. The possibility of stabilizing the dynamics by a switching law depending on the initial point is measured by the *pointwise stabilization radius* $\tilde{\rho}$ which describes the lowest possible exponential growth rate of the system. This object is analogous to the joint spectral subradius, which measures the possibility of stabilizing the system by means of a switching rule *independent* of the initial point. These two quantities turn out to be different in general, as shown in Example 1 below, inspired by the results in [36]. On the other hand they are equal in the particular case in which all the available linear modes share a pair of nested proper invariant cones, as shown in Theorem 1. Concerning the pointwise stabilization radius, we show in Proposition 2 the rather surprising fact that there does not always exist an homogeneous feedback law that exactly achieves $\tilde{\rho}$.

In the second part of the paper we analyze the algorithmic side of the stabilizability problem for discrete-time linear switched system. Unsurprisingly, many negative results can be derived: we

show in Proposition 6 that the stabilizability problem is NP-hard, even for nonnegative matrices, and in Proposition 7 that it is undecidable. We provide two algorithms that respectively deliver a lower and an upper bound on the stabilization radius (to our knowledge this is the first systematic lower bound in the literature). However, the problem of approximating the stabilization radius appears very hard to tackle in general, and even for low dimensional systems. In order to support this claim, we analyze a specific example in Section 5. For this system, based on the theoretical results obtained in the preceding sections, we propose a few approaches in order to approximate the best convergence rate that one can ensure. Then we present a technical result providing an obstruction to the computation of the optimal achievable rate of decay, namely we show that there exists a dense set of initial conditions for which the achievable rate of decay is constant and strictly lower than the upper bound obtained algorithmically. As a consequence, a nontrivial lower bound for the stabilization radius appear to be hard to obtain without resorting to *ad hoc* techniques. Note that many of the obtained results may be easily transposed in the framework of continuous-time linear switched system as we observe in Section 6.

2 Setting and elementary properties

Given a compact set of matrices $\mathcal{M} = \{A_\sigma \mid \sigma \in \Sigma\} \subset \mathbb{R}^{d \times d}$, we consider the discrete-time switched system defined by the difference equations of the form

$$x(s+1) = A_{\sigma(s)}x(s), \quad x(0) = x_0,$$

associated with a switching sequence $\sigma(\cdot)$. We denote by $x_{\sigma, x_0}(t)$ the corresponding solution, that is $x_{\sigma, x_0}(t) = A_{\sigma(t)} \dots A_{\sigma(1)}x_0$ for $t \in \mathbb{N}$.

In this section we formalize some basic notions of stabilizability we will deal with and some related elementary properties. In the definition below we introduce two natural notions of stabilizability related to the use of open loop and closed loop switching laws, respectively.

Definition 1. *We say that \mathcal{M} is pointwise stabilizable if for any $x_0 \in \mathbb{R}^d$ there exists a switching law $\sigma_{x_0}(\cdot)$ such that $\lim_{t \rightarrow \infty} x_{\sigma_{x_0}, x_0}(t) = 0$.*

We say that \mathcal{M} is feedback stabilizable if there exists a 0-homogeneous¹ function $\sigma : \mathbb{R}^d \rightarrow \mathcal{M}$ such that, for any $x_0 \in \mathbb{R}^d$, $\lim_{t \rightarrow \infty} x(t) = 0$, where $x(\cdot)$ is a solution of the corresponding closed loop system starting at x_0 . In addition we say that \mathcal{M} is uniformly feedback stabilizable if there exists $\sigma(\cdot)$ as above such that for any positive ϵ_1, ϵ_2 there exists $T \in \mathbb{N}$ such that if $|x_0| < \epsilon_1$ the corresponding trajectory $x(\cdot)$ satisfies $|x(t)| < \epsilon_2$ if $t \geq T$.

The difference between the definitions of pointwise and feedback stabilizability is subtle. In the former pointwise stabilizability, one has to decide the switching signal once and for all at time zero. In the latter feedback stabilizability, we assume that the controller has the knowledge of the state $x(t)$ at every instant, and the controller has to decide its actuation based on this sole knowledge.

In order to tackle the more general situation where one is interested in the minimal achievable rate of growth of the system, we now introduce numerical values that quantify the notion of stabilizability.

Definition 2. *We introduce the following stabilizability indices:*

- For a given $x \in \mathbb{R}^d$ let us consider the quantity

$$\tilde{\rho}_x(\mathcal{M}) = \inf\{\lambda \geq 0 \mid \exists \sigma(\cdot) \text{ switching sequence, } M > 0 \text{ s.t. } |x_{\sigma, x}(t)| \leq M\lambda^t|x|, \forall t \geq 0\}$$

and define the pointwise stabilization radius of the switched system as

$$\tilde{\rho}(\mathcal{M}) = \sup_{x \in \mathbb{R}^d} \tilde{\rho}_x(\mathcal{M}).$$

¹A function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is 0-homogeneous if $f(\lambda x) = f(x) \quad \forall \lambda \neq 0, \forall x$.

- In the previous definition the constant M may depend on the initial condition. This is no more the case if we consider the following definition:

$$\tilde{\rho}'(\mathcal{M}) = \inf \left\{ \lambda \geq 0 \mid \begin{array}{l} \exists M > 0 \text{ s.t. } |x_{\sigma,x}(t)| \leq M\lambda^t|x| \text{ for any } x \in \mathbb{R}^d, \\ t \geq 0 \text{ and some switching signal } \sigma \text{ depending on } x \end{array} \right\}.$$

A closely related quantity, that has received more attention in the literature, is the *joint spectral subradius*

$$\check{\rho}(\mathcal{M}) := \liminf_{t \rightarrow \infty} \inf_{\sigma(\cdot)} \|A_{\sigma(t)} \dots A_{\sigma(0)}\|^{1/t}.$$

In terms of control theory, this latter quantity represents the open loop stabilizability of a switched system by a switching law which is independent on the initial point, unlike the ones involved in the computation of $\tilde{\rho}(\mathcal{M})$. See [20] for more on the joint spectral subradius. To the best of our knowledge, Stanford and Urbano [36] are the first ones to observe that $\check{\rho}(\mathcal{M}) \neq \tilde{\rho}(\mathcal{M})$. They provide an example for which the two quantities are different, which we reproduce in a slightly different form here below. See [4] for a recent study of a very similar example (though not concerned with feedback stabilizability).

Example 1. Let us consider $\mathcal{M} = \{A_1, A_2\}$, where

$$A_1 = \begin{pmatrix} \cos \frac{\pi}{4} & \sin \frac{\pi}{4} \\ -\sin \frac{\pi}{4} & \cos \frac{\pi}{4} \end{pmatrix} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}, \quad A_2 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & 2 \end{pmatrix}.$$

In this case it is easy to see that $\check{\rho}(\mathcal{M}) = 1$. Indeed, $\det(A_{\sigma(t)} \dots A_{\sigma(0)}) = 1$ independently on the switching sequence, which implies that $\|A_{\sigma(t)} \dots A_{\sigma(0)}\| \geq 1$ and thus $\check{\rho}(\mathcal{M}) \geq 1$. On the other hand we have $\tilde{\rho}(\mathcal{M}) \leq \|A_1\| = 1$ by submultiplicativity, and thus $\tilde{\rho}(\mathcal{M}) = 1$.

If one is allowed to tune the switching sequence depending on the value of $x(t)$, the situation is different: indeed, for any $x \in \mathbb{R}^2$ there always exists a natural number $n_x \leq 3$ such that the absolute value of the angle formed by the vector $A_1^{n_x}x$ and the x_1 axis is smaller or equal than $\pi/8$. As a consequence it is easy to obtain the estimate $|A_2 A_1^{n_x} x| < 0.9|x|$ and thus $\tilde{\rho}(\mathcal{M}) < 0.9^{1/4} \sim 0.974$. For similar examples and a more detailed exposition, see e.g. [36].

In the next lemma, we state properties that we will need in the rest of the paper.

Lemma 1. The pointwise stabilization radius satisfies the following basic properties:

- (i) Homogeneity: For any compact set of matrices \mathcal{M} , $\forall \gamma > 0$, $\tilde{\rho}(\gamma\mathcal{M}) = \gamma\tilde{\rho}(\mathcal{M})$,
- (ii) For any compact set of matrices \mathcal{M} , $\forall t \in \mathbb{N}$, $\tilde{\rho}(\mathcal{M}^t) = \tilde{\rho}(\mathcal{M})^t$.

The same properties hold for $\tilde{\rho}'$.

Proof. It is easy to see that there is a bijection between trajectories of the system defined by \mathcal{M} and the one defined by $\gamma\mathcal{M}$ (resp. \mathcal{M}^t) such that the number λ in the definition of $\tilde{\rho}$ and $\tilde{\rho}'$ becomes $\gamma\lambda$ (resp. λ^t). The thesis follows immediately. \square

We now establish the equivalence of the two quantities introduced in Definition 2 and we characterize the notion of pointwise stabilizability in terms of them. The following proposition may be easily obtained by adapting the proof of the similar result [37, Theorem 3.9].

Proposition 1. The following conditions are equivalent:

- (i) $\tilde{\rho}'(\mathcal{M}) < 1$,
- (ii) $\tilde{\rho}(\mathcal{M}) < 1$,
- (iii) \mathcal{M} is pointwise stabilizable.

Remark 1. By Lemma 1 and Proposition 1 we deduce that $\tilde{\rho}(\mathcal{M})$ and $\tilde{\rho}'(\mathcal{M})$ coincide.

We now present a rather surprising result, which shows the subtlety of the stabilization radius: Obviously, linear switched systems are homogeneous, that is, if one multiplies the initial condition by $\lambda > 0$, the whole trajectory is multiplied by λ . Furthermore, if a switching signal σ makes a trajectory starting at x_0 converge to zero, the same switching signal makes the trajectory starting at λx_0 converge to zero, for any $\lambda > 0$.

For this reason, one might think that it is enough to develop a switching strategy based on the ‘direction’ of x in the state space (i.e. $x/|x|$), but independent on its norm. A controller could then work by tracking the value $x(t)/|x(t)|$, as a dynamical system evolving on the unit ball, and would decide ‘in real time’ which switching signal to apply, depending on this point on the unit ball.

However, we show in the second part of the proposition that it is not true: there are systems, which are stabilizable, but where no such controller (i.e. an homogeneous function of x) can achieve the stabilization radius.

Proposition 2. *For any $x_0 \in \mathbb{R}^d$, there exists a switching law $\sigma_{x_0}(\cdot)$ such that*

$$\limsup_{t \rightarrow \infty} |x_{\sigma_{x_0}, x_0}(t)|^{1/t} \leq \tilde{\rho}(\mathcal{M}). \quad (1)$$

On the other hand there does not always exist a 0-homogeneous feedback such that each solution $x(\cdot)$ of the corresponding closed loop system satisfies $\limsup_{t \rightarrow \infty} |x(t)|^{1/t} \leq \tilde{\rho}(\mathcal{M})$.

Proof. Let us prove the first part of the statement. First, if $\tilde{\rho}_{x_0}(\mathcal{M}) < \tilde{\rho}(\mathcal{M})$, the proof is obvious. So, let us suppose that $\tilde{\rho}_{x_0}(\mathcal{M}) = \tilde{\rho}(\mathcal{M})$. We construct a suitable solution $x_{\sigma_{x_0}, x_0}(\cdot)$ of the system such that $\limsup_{t \rightarrow \infty} |x_{\sigma_{x_0}, x_0}(t)|^{1/t} = \tilde{\rho}(\mathcal{M})$. Let us denote by M_n the value of M in the definition of $\tilde{\rho}'$ corresponding to $\lambda = \tilde{\rho}(\mathcal{M}) + 2^{-n}$. By definition of $\tilde{\rho}'$ we know that it is possible to construct $x_{\sigma_{x_0}, x_0}(\cdot)$ such that there exists $t_1 \in \mathbb{N}$ large enough satisfying $|x_{\sigma_{x_0}, x_0}(t_1)| \leq \frac{1}{M_1}(\tilde{\rho}(\mathcal{M}) + 1)^{t_1}|x_0|$. Define $x_1 = x_{\sigma_{x_0}, x_0}(t_1)$. Reasoning as before we can prolong $x_{\sigma_{x_0}, x_0}(\cdot)$ on $[0, t_1 + t_2]$, where $t_2 \in \mathbb{N}$ is such that $|x_{\sigma_{x_0}, x_0}(t_1 + t_2)| \leq \frac{M_1}{M_2}(\tilde{\rho}(\mathcal{M}) + \frac{1}{2})^{t_2}|x_1| \leq \frac{1}{M_2}(\tilde{\rho}(\mathcal{M}) + \frac{1}{2})^{t_2}(\tilde{\rho}(\mathcal{M}) + 1)^{t_1}|x_0|$ and in addition

$$|x_{\sigma_{x_0}, x_0}(t_1 + s)| \leq (\tilde{\rho}(\mathcal{M}) + \frac{1}{2})^s (\tilde{\rho}(\mathcal{M}) + 1)^{t_1} |x_0|$$

for $s \in [0, t_2]$. Following the same lines one can prolong indefinitely the trajectory adding at each time an interval of length $t_n \in \mathbb{N}$ to its domain in such a way that

$$|x_{\sigma_{x_0}, x_0}(t)| \leq (\tilde{\rho}(\mathcal{M}) + 2^{-N})^{t - T_N} \prod_{n=1}^N (\tilde{\rho}(\mathcal{M}) + 2^{-n+1})^{t_n} |x_0|,$$

where $T_N = \sum_{n=1}^N t_n$ and $t \in [T_N, T_{N+1}]$. Without loss of generality we assume that T_N goes to infinity with N , so that t can be arbitrarily large. We have

$$\log(|x_{\sigma_{x_0}, x_0}(t)|^{1/t}) \leq \frac{t - T_N}{t} \log(\tilde{\rho}(\mathcal{M}) + 2^{-N}) + \sum_{n=1}^N \frac{t_n}{t} \log(\tilde{\rho}(\mathcal{M}) + 2^{-n+1}) + \frac{1}{t} \log(|x_0|).$$

which converges to $\log(\tilde{\rho}(\mathcal{M}))$ as t goes to infinity, concluding the proof of the first part.

We now provide an explicit example to prove the second part of the proposition. Let us consider the discrete-time system corresponding to the following set of matrices

$$\mathcal{M} = \{A, B\} = \left\{ \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \right\}. \quad (2)$$

The intuition behind this example is as follows: both matrices have spectral radius equal to two, so, one has to make use of both of them in order to keep the trajectory bounded. To do so, one can exploit the fact that $BA^t = BA$ (intuitively, B ‘kills’ the explosion of A^t). However, one cannot

apply B too often, because for instance (taking $e = (1, 1, 1)^T$ as the initial condition) $BA^t e = 2e$. Now, because e is an eigenvector of BA^t , any homogeneous function would lead to a periodic application of BA^t , leading to an exponential growth. Hence, an aperiodic switching strategy is needed, precluding the use of a homogeneous strategy. We now formalize this reasoning.

We claim that for any 0-homogeneous function $\sigma(x)$ the corresponding feedback solution $x_e(\cdot)$ starting at the all-ones vector e is such that $\limsup_{t \rightarrow \infty} |x_e(t)|^{1/t} > 1$. We proceed by contradiction: let us suppose that there exists such a function $\sigma(x)$ with $\limsup_{t \rightarrow \infty} |x_e(t)|^{1/t} \leq 1$. First, certainly $A_{\sigma(e)} \neq B$, because $Be = 2e$, and this would imply that $|x_e(t)|^{1/t} = 2$. Also, certainly this trajectory must contain some multiplication by B , because $|A^t e| \approx 2^t$. Let us call T the first t such that $x(t)$ is multiplied by B . Since $BA^T e = 2e$, the trajectory of x must be of the shape $\dots BA^T BA^T e$, and we see that $\lim_{t \rightarrow \infty} |x_e(t)|^{1/t} > 1$, leading to a contradiction.

Let us now show that $\tilde{\rho}_e(\mathcal{M}) = \tilde{\rho}(\mathcal{M}) = 1$. Since $\|BA^t\|$ is bounded by a constant which is independent on t , and obviously $\tilde{\rho}(\mathcal{M}^{t+1}) \leq \|BA^t\|$, it turns out by item (ii) in Lemma 1 that $\tilde{\rho}(\mathcal{M}) \leq 1$. The inequality $\tilde{\rho}_e(\mathcal{M}) \geq 1$ comes from the fact that the second component is always non-decreasing along trajectories of the system starting at e . \square

Summarizing the above proposition, the intuitive meaning of the quantity $\tilde{\rho}$ is the following: it is the infimum of the asymptotic rates of growth that one can ensure. In the first part of the proposition, we show a reassuring result, namely that this infimum is in fact attained. In the second part however, we show that paradoxically, the optimal switching strategy at time t , in order to ensure this rate of growth, might depend on the norm of the point.

We conclude this section with a result showing a sort of optimality property of the stabilization radius.

Proposition 3. *There always exist $x_0 \in \mathbb{R}^d$ and $\sigma_0(\cdot)$ such that $|x_{\sigma_0, x_0}(t)| \leq \tilde{\rho}(\mathcal{M})^t |x_0|$ for any $t \in \mathbb{N}$.*

Proof. For an arbitrary $\epsilon > 0$, let $\lambda_\epsilon = \tilde{\rho}(\mathcal{M}) + \epsilon$ and consider a trajectory $x_{\sigma_\epsilon, x_\epsilon}(\cdot)$ satisfying $|x_{\sigma_\epsilon, x_\epsilon}(t)| \leq M \lambda_\epsilon^t |x_\epsilon|$ for any $t \in \mathbb{N}$ and some $M > 0$. Let t_ϵ such that

$$\sup_{t \in \mathbb{N}} \frac{|x_{\sigma_\epsilon, x_\epsilon}(t)|}{\lambda_\epsilon^t |x_\epsilon|} \leq (1 + \epsilon) \frac{|x_{\sigma_\epsilon, x_\epsilon}(t_\epsilon)|}{\lambda_\epsilon^{t_\epsilon} |x_\epsilon|}. \quad (3)$$

Define $\hat{x}_\epsilon = \frac{x_{\sigma_\epsilon, x_\epsilon}(t_\epsilon)}{|x_{\sigma_\epsilon, x_\epsilon}(t_\epsilon)|}$ and $\hat{\sigma}_\epsilon(\cdot) = \sigma_\epsilon(t_\epsilon + \cdot)$. Then from (3) we have

$$\sup_{t \in \mathbb{N}} \frac{|x_{\hat{\sigma}_\epsilon, \hat{x}_\epsilon}(t)|}{\lambda_\epsilon^t} \leq 1 + \epsilon. \quad (4)$$

Passing to the limit as ϵ goes to 0, we have (up to subsequences) that \hat{x}_ϵ converges to some unit vector x_0 and $x_{\hat{\sigma}_\epsilon, \hat{x}_\epsilon}$ converges to a trajectory x_{σ_0, x_0} uniformly on compact subsets of \mathbb{N} . Thus, passing to the limit at the left- and right-hand side of (4) we get the thesis. \square

3 Two tools from control theory

In this section we analyze the stabilizability notions introduced above in the discrete-time setting by taking advantage of two classical tools from control theory: control-Lyapunov functions and the joint spectral subradius.

3.1 Control-Lyapunov functions and their mathematical properties

Let us introduce the following adapted definition of control-Lyapunov function.

Definition 3. *We say that a positively homogeneous² function $V : \mathbb{R}^d \rightarrow \mathbb{R}_+$ is a control-Lyapunov function for \mathcal{M} if*

²A function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is said *positively homogeneous* if $f(\lambda x) = |\lambda|f(x) \quad \forall \lambda \neq 0, \forall x$.

- (1) there exist two constants $0 < m \leq M$ such that $m|x| \leq V(x) \leq M|x|$,
- (2) there exists $\mu \in (0, 1)$ such that for any $x \in \mathbb{R}^d$ there exists $\sigma_x(\cdot)$ with $V(x_{\sigma_x, x}(t)) < \mu^t V(x)$ for $t \in \mathbb{N}$.

If there exists a control-Lyapunov function for \mathcal{M} and μ is given by item (2) above, then clearly $\tilde{\rho}(\mathcal{M}) \leq \mu < 1$, that is the system is pointwise stabilizable. If in addition the control-Lyapunov function is regular enough, then the system turns out to be (uniformly) feedback stabilizable, as stated by the result here below.

Proposition 4. *If \mathcal{M} admits a Lipschitz continuous control-Lyapunov function V then \mathcal{M} is uniformly feedback stabilizable, and the stabilizing feedback $\sigma(\cdot)$ can be taken piecewise constant.*

Proof. The existence of a stabilizing feedback is trivial, as it is enough to associate with any $x \in \mathbb{R}^d$ such that $|x| = 1$ the value $\sigma_x(0)$, where σ_x is as in Definition 3, and then extends by homogeneity on the whole space $\mathbb{R}^d \setminus \{0\}$.

The fact that the stabilizing feedback can be taken piecewise constant follows from the continuity of V and the continuous dependence of the solutions with respect to the initial data. Indeed these properties imply that a switching law that produces a certain rate of decrease of V at time 1 starting at x_0 provides the same rate of decrease starting from any point of a small enough neighborhood of x_0 , which can be extended by homogeneity to a cone. By a simple compactness argument there exists a finite number of cones covering $\mathbb{R}^d \setminus \{0\}$ in each of which the stabilizing feedback law may be taken constant. This concludes the proof of the proposition. \square

We now provide a converse Lyapunov theorem for our class of systems. We skip the proof, since our result is a quite straightforward adaptation of the one presented in [38, Section 4.3.1].

Proposition 5. *For any $\lambda > \tilde{\rho}(\mathcal{M})$ the function $V_\lambda : \mathbb{R}^d \rightarrow \mathbb{R}_+$*

$$V_\lambda(x) = \sup_{t \geq 0} \inf_{\sigma(\cdot)} \frac{|x_{\sigma, x}(t)|}{\lambda^t} \quad (5)$$

is well-defined, absolutely homogeneous, Lipschitz continuous and satisfies $V_\lambda(x_{\sigma, x}(t)) \leq \lambda^t V_\lambda(x)$ for any $x \in \mathbb{R}^d$, $t \in \mathbb{N}$ and some $\sigma(\cdot)$, depending on x . In particular \mathcal{M} is pointwise stabilizable if and only if it admits an absolutely homogeneous control-Lyapunov function.

An immediate consequence of the above result is the equivalence between the pointwise stabilizability and the uniform feedback stabilizability as stated in the following corollary. Notice that this result does not contradict Proposition 2, even though they are opposite in spirit: the corollary below indicates that feedback stabilizability and pointwise stabilizability are equivalent, even though, for feedback stabilizability, the ‘most stabilizing’ switching behavior is not always attainable by a homogeneous feedback.

Corollary 1. *The three equivalent conditions in Proposition 1 are also equivalent to the following:*

- (iv) \mathcal{M} is uniformly feedback stabilizable.

Proof. Clearly, if \mathcal{M} is uniformly feedback stabilizable then it is also pointwise stabilizable. The other implication trivially follows from Proposition 4 and Proposition 5. \square

Remark 2. $V_\lambda(x)$ might not be convex. This makes $\tilde{\rho}$ hard to compute. For instance, it hampers our ability to provide a guaranteed accuracy for our algorithms, while this is possible for, for instance, the joint spectral radius (see [20, Theorem 2.12]). Indeed, the joint spectral radius admits a similar notion of Lyapunov function, but in that case it is convex.

Remark 3. *The function*

$$\hat{V}_\lambda(x) = \inf_{\sigma(\cdot)} \sup_{t \geq 0} \frac{|x_{\sigma, x}(t)|}{\lambda^t}$$

is also a control-Lyapunov function for \mathcal{M} . This function can be shown to be lower semi-continuous but in general there is no clear indication that it is also Lipschitz.

3.2 The joint spectral subradius

As already observed (see Example 1), in general the joint spectral subradius $\check{\rho}(\mathcal{M})$ can be strictly larger than the stabilization radius. We now show that if all the matrices in \mathcal{M} are positive (i.e. they map a cone inside itself), then $\tilde{\rho}(\mathcal{M}) = \check{\rho}(\mathcal{M})$. Positive systems have attracted much attention since the work of Perron and Frobenius; more recently, in particular in the control community, they have regained interest, both thanks to their strong properties, but also to the variety of systems that can be modeled (or are equivalent) to positive systems (see e.g. [12, 31]).

We say that a matrix A possesses an invariant (proper) cone K if $AK \subset K$. We say that a convex closed cone K' is *embedded in* K if $(K' \setminus \{0\}) \subset \text{int}K$. In this case we call $\{K, K'\}$ an *embedded pair*. (The embedded cone K' may be degenerate, i.e., may have an empty interior.) An embedded pair $\{K, K'\}$ is called an *invariant pair* for a matrix A (resp. a set of matrices \mathcal{M}) if the cones K and K' are both invariant for A (resp. for all the matrices in \mathcal{M}). Finally, given a particular cone K , for two vectors $x, y \in \mathbb{R}^d$, we note $x \geq_K y$ for $x - y \in K$. Given $v \in \mathbb{R}^d \setminus \{0\}$ we define \mathbb{R}_v^d as the half-space $\{x \in \mathbb{R}^d | v^T x \geq 0\}$. We say that a cone is *proper* if it is contained in some half-space.

Lemma 2. *If a convex closed cone K is proper, then there exists $C > 0$ such that $w_1 \geq_K w_2 \geq_K 0$ implies $|w_1| \geq C|w_2|$.*

Proof. By contradiction assume there exist two sequences $\{w_1^k\}_{k \geq 1}, \{w_2^k\}_{k \geq 1}$ of vectors in K such that, for $k \geq 1$, $w_1^k \geq_K w_2^k \geq_K 0$, $|w_2^k| = 1$ and $\lim_{k \rightarrow \infty} w_1^k = 0$. Without loss of generality we assume that w_2^k converges to a unit vector $w^* \in K$. Then one has that $w_1^k - w_2^k \in K$ converges to $-w^* \in K$, which contradicts the fact that K is embedded in \mathbb{R}_v^d . \square

Theorem 1. *If the matrices in \mathcal{M} have a common invariant embedded pair of proper cones $\{K, K'\}$, then $\tilde{\rho}(\mathcal{M}) = \check{\rho}(\mathcal{M})$.*

Proof. We make use of results from [30] for approximating the joint spectral subradius. In that paper, it is shown that for any set of matrices \mathcal{M} with an invariant cone K , the following quantity is a lower bound for $\check{\rho}(\mathcal{M})$:

$$\check{\sigma}_K(\mathcal{M}) = \sup \left\{ \lambda \geq 0 \mid \exists v \geq_K 0, v \neq 0 \quad Av \geq_K \lambda v \quad \forall A \in \mathcal{M} \right\}. \quad (6)$$

It turns out that $\check{\sigma}_K(\mathcal{M})$ is actually also a valid lower bound for $\tilde{\rho}(\mathcal{M})$. Indeed, it is easy to see that if $Av \geq_K \lambda v$, then $BAv \geq_K \lambda Bv$, provided that B leaves K invariant. Thus, the existence of v as in (6) for each $\lambda < \check{\sigma}_K(\mathcal{M})$ implies that

$$Av \geq_K \lambda^t v, \quad \forall A \in \mathcal{M}^t,$$

and thus, by Lemma 2, $|Av| \geq C\lambda^t |v| \quad \forall A \in \mathcal{M}^t$, which implies $\tilde{\rho}(\mathcal{M}) \geq \check{\sigma}_K(\mathcal{M})$. Similarly, one has that $\tilde{\rho}(\mathcal{M}) \geq \check{\sigma}_K(\mathcal{M}^t)^{1/t}$.

Now, if the set \mathcal{M} has an embedded pair of invariant cones, we have that ([30, Corollary 3])

$$\lim_{t \rightarrow \infty} \check{\sigma}_K(\mathcal{M}^t)^{1/t} = \check{\rho}(\mathcal{M}).$$

Combining the above equations, one gets $\tilde{\rho}(\mathcal{M}) = \check{\rho}(\mathcal{M})$. \square

Corollary 2. *If all the matrices in \mathcal{M} have only positive entries, then $\tilde{\rho}(\mathcal{M}) = \check{\rho}(\mathcal{M})$.*

Proof. As shown in [30, Corollary 1], sets of positive matrices share an invariant embedded pair of proper cones. \square

We remark that matrices with positive entries constitute the simplest example of matrices satisfying Theorem 1, but recent work shows that there are much more general sets of matrices with this property (see [13]). Thus, in the case where the matrices share an embedded pair of cones, one can use algorithms for providing a lower bound to the joint spectral subradius $\tilde{\rho}(\mathcal{M})$ in order to obtain a lower bound on $\check{\rho}(\mathcal{M})$. See for example [16, 30] for methods that perform well in practice (even sometimes terminate in finite time).

4 Algorithms

We explore here the algorithmic computation of $\tilde{\rho}$ and focus on the discrete-time setting. We first consider basic complexity questions, which were untouched in the literature, to the best of our knowledge. We then provide upper and lower bounds on the quantity, that at least in theory, perform as well as can be hoped in view of the complexity results.

4.1 Complexity

Let us show that one should not expect a polynomial time algorithm for the problem. We add that the proofs in this subsection use the same technique as in [21].

Proposition 6. *Unless $P = NP$, there is no polynomial time algorithm for solving the stabilizability problem, even for nonnegative matrices. More precisely, there is no algorithm that receives a pair of nonnegative integer matrices \mathcal{M} , and that decides in polynomial time (w.r.t. the size and number of matrices) whether $\tilde{\rho}(\mathcal{M}) < 1$.*

Proof. Our proof is by reduction from the *mortality problem* which is known to be NP-hard, even for nonnegative integer matrices [3, p. 286]. In this problem, one is given a set of matrices \mathcal{M} , and it is asked whether there exists a product of matrices in \mathcal{M} which is equal to the zero matrix.

We now construct an instance \mathcal{M}' such that $\tilde{\rho}(\mathcal{M}') < 1$ if and only if the set \mathcal{M} is mortal: simply take $\mathcal{M}' = \{A' = 2A \mid A \in \mathcal{M}\}$.

Suppose first that \mathcal{M} is mortal. Then, the corresponding product in \mathcal{M}' is also equal to zero, and thus $\tilde{\rho}(\mathcal{M}') = 0$.

If, on the other hand, \mathcal{M} is not mortal, then all products of matrices in \mathcal{M} have nonnegative integer entries, so that we have

$$\forall A \in \mathcal{M}^t, \quad |Ae| \geq 1,$$

and thus,

$$\forall A' \in \mathcal{M}'^t, \quad |A'e| \geq 2^t$$

(we recall that we note $e = (1, \dots, 1)^T$). This implies that $\tilde{\rho}(\mathcal{M}') \geq 2$. □

If one relaxes the requirement that the matrices and the vectors are nonnegative, then the problem becomes even harder, and can be undecidable, as shown in the next proposition.

Proposition 7. *The stabilizability problem is undecidable.*

Proof. It is known that the mortality problem with matrices having entries in \mathbb{Z} is undecidable [20, Corollary 2.1]. We reduce this problem to the stabilizability problem in a way similar as in Proposition 6, except that we build larger matrices of size n^2 (where n is the size of the initial matrices). The matrices in the set \mathcal{M}' are of the shape $\mathcal{M}' = \{\text{diag}(2A, \dots, 2A) \mid A \in \mathcal{M}\}$, with n copies of the same matrix on the diagonal.

Now if \mathcal{M} is mortal, again $\tilde{\rho} = 0$.

If, on the other hand, \mathcal{M} is not mortal, then every product A of matrices in \mathcal{M} has a nonzero column (say, the i th one), and thus, the product Ae_i has a norm larger than one, where e_i is the i th standard basis vector (recall that the entries have integer values). Thus, defining $v = (e_1, \dots, e_n)$ (that is, the concatenation of the n standard basis vectors), for any product A' in \mathcal{M}'^t , we have that $|A'v| \geq 2^t$, and $\tilde{\rho}(\mathcal{M}') \geq 2$. □

4.2 Upper bound

In this section we provide an algorithm, which not only derives successive upper bounds on $\tilde{\rho}$, but also has the property that these bounds asymptotically converge to the true value $\tilde{\rho}$. Note that algorithms with such a property were recently proposed in [24, 43]. Our algorithm is more elementary, and this might be a good property for scaling issues. However, each iteration is very simple, and it is expected that for practical applications, where only a few iterations are done,

other methods, relying on powerful optimization technologies like Semidefinite Programming, will outperform the technique developed here. We then describe how to adapt the algorithm in order to make it even more scalable, at the price of losing the guarantee of convergence. Our algorithm is based on the following simple proposition:

Proposition 8. *If \mathcal{M} is pointwise stabilizable then for any $\varepsilon > 0$ there exists a large enough time t_ε such that every point of the unit ball can be mapped at time t_ε , with a suitable feedback switching strategy, into the ball centered at the origin and of radius ε .*

Proof. Since $\tilde{\rho}' < 1$ there exists $\lambda < 1$ and $M > 0$ such that $|x_{\sigma,x}(t)| < M\lambda^t|x|$ for any $x \in \mathbb{R}^d$ and some $\sigma(\cdot)$ depending a priori on x . It is thus enough to take $t \geq \frac{\ln(\varepsilon/M)}{\ln \lambda}$. \square

This proposition allows us to propose an asymptotically tight upper bound on the stabilization radius. In other words, we provide a sequence of sufficient conditions for stabilizability, which are asymptotically necessary. So, the theoretical efficiency of our method is similar to the one from [43] (see the introduction for more details). We think that the simplicity of the method (compute products of increasing length t and iterate) offers a valuable alternative to [43], which uses more complex optimization methods. Also, we think that this approach bears advantages in comparison with [14] in that one has a clear procedure to follow, which is guaranteed to terminate after a finite amount of time if the system is stabilizable. On the contrary, the matrix inequalities proposed in [14] are non-convex, and thus there is no clear procedure to efficiently find the optimal upper bound provided by these inequalities. On top of that, in [14], if the optimal upper bound is found, one has no guarantee that this bound is close to the actual value of $\tilde{\rho}$, while our Algorithm 1 is guaranteed to converge asymptotically. However, as said above, for a fixed number of iterations (or, say, when computational time is a strong constraint), it is expectable that the previous algorithms, which are based on more involved optimization techniques, outperform the one presented here.

Algorithm 1: An asymptotically tight algorithm

Data: A set of matrices \mathcal{M} and an integer T

Result: Outputs a sequence of upper bounds r_t , $t = 1, \dots, T$;

begin

```

1 | t := 0;
2 | while t ≤ T do
3 |   t := t + 1;
4 |   Compute  $\mathcal{M}^t$  ;
5 |   Minimize  $\gamma$  such that
   |
   |  $\forall x : |x| = 1, \exists A \in \mathcal{M}^t \text{ s.t. } |Ax| \leq \gamma$  (7)
6 |   Output  $r_t := \gamma$ ;

```

Theorem 2. *Algorithm 1 iteratively provides an upper bound $r_t \geq \tilde{\rho}$ which asymptotically converges towards $\tilde{\rho}$.*

Proof. At every step of the algorithm, one must solve the optimization problem 7, which can be solved by bisection on γ . Indeed, for a fixed γ , the (in)equalities in Problem (7) are algebraic, and can be solved with standard procedures, like Tarski-Seidenberg elimination [33, 39] \square

Remark 4. *Theorem 2 provides a theoretical procedure that asymptotically converges towards the true value. It is well known that algebraic equation solvers are very slow in practice, so, practical implementations of algorithm 2 would advantageously make use of heuristics in order to approximately solve (7).*

We conclude this section by establishing a result that will be applied in the next section and which provides an alternative way to estimate the stabilization radius.

Proposition 9. *Let $\mu > 0$ and $\bar{t} \in \mathbb{N}$. Suppose that for every $z \in \mathbb{R}^n$ of norm 1 there exists a product $A_z \in \mathcal{M}^{t_z}$ for some $t_z \leq \bar{t}$ such that $|A_z z|^{1/t_z} \leq \mu$, then,*

$$\tilde{\rho}(\mathcal{M}) \leq \mu.$$

Proof. Let us extend A_z, t_z by homogeneity: $A_z := A_{z/|z|}, t_z = t_{z/|z|}$. Then for any $z_0 \in \mathbb{R}^n$ we can construct a trajectory $z(\cdot)$ by concatenation in such a way that $z(s_k) = A^{(k)} A^{(k-1)} \dots A^{(1)} z_0$ for some $s_k > 0$, where the matrices $A^{(i)}$ are defined recursively starting from $i = 1$ as $A^{(i)} = A_{z(s_{i-1})}$ (assuming $s_0 = 0$). In particular $s_i = \sum_{j=0}^{i-1} t_{z(s_j)}$ and $|z(s_i)| \leq \mu^{s_i} |z_0|$. By compactness of \mathcal{M} we have that $\|M\|$ is uniformly bounded for $M \in \cup_{t \leq \bar{t}} \mathcal{M}^t$ by a constant $C > 0$, from which we deduce that $|z(t)| \leq C \max\{1, \mu^{-\bar{t}}\} \mu^t |z_0|$. The thesis follows. \square

4.3 Lower bound

In the previous section we have presented an algorithm providing an upper bound on $\tilde{\rho}(\mathcal{M})$. The lower bound appears to be much more complicated to analyze numerically, and the purpose of this section is to provide an analytic criterion to estimate it. This is, to our knowledge, the first algorithm providing a lower bound on $\tilde{\rho}(\mathcal{M})$ in the literature.

Theorem 3. *One has $\tilde{\rho}(\mathcal{M}) \geq \min_{A \in \mathcal{M}} \sigma_m(A)$ where $\sigma_m(A)$ is the smallest singular value of the matrix A .*

Proof. Recall that $\sigma_m(A)$, the square root of the smallest eigenvalue of the non-negative definite symmetric matrix $A^T A$, satisfies

$$\sigma_m(A) = \min_{x \in \mathbb{R}^d \setminus \{0\}} \frac{|Ax|}{|x|}.$$

We thus have

$$\frac{|A_{\sigma(t)} \dots A_{\sigma(1)} x|}{|x|} = \frac{|A_{\sigma(t)} \dots A_{\sigma(1)} x|}{|A_{\sigma(t-1)} \dots A_{\sigma(1)} x|} \dots \frac{|A_{\sigma(1)} x|}{|x|} \geq \left(\min_{A \in \mathcal{M}} \sigma_m(A) \right)^t$$

for any $\sigma(\cdot)$. Thus, by definition of $\tilde{\rho}'(\mathcal{M}) = \tilde{\rho}(\mathcal{M})$ for any $\lambda > \tilde{\rho}(\mathcal{M})$ there exists $M > 0$ such that $\frac{\lambda^t}{M} > (\min_{A \in \mathcal{M}} \sigma_m(A))^t$ for every $t > 0$, that is $\lambda \geq \min_{A \in \mathcal{M}} \sigma_m(A)$, proving the thesis. \square

One can iterate this lower bound for longer and longer products of matrices in \mathcal{M} , thanks to Lemma 1. It can be seen on simple examples (even on a single matrix) that this can improve the lower bound. Unfortunately, the method may also fail to converge to the true value of $\tilde{\rho}(\mathcal{M})$. Indeed, in the proof of the theorem above, we could actually replace $\tilde{\rho}(\mathcal{M})$ by $\tilde{\rho}_x(\mathcal{M})$. In other words, $\min_{x \in \mathbb{R}^d \setminus \{0\}} \tilde{\rho}_x(\mathcal{M}) \geq \min_{A \in \mathcal{M}} \sigma_m(A)$, but the left-hand side of the previous equality coincides with $\tilde{\rho}(\mathcal{M})$ only if $\tilde{\rho}_x(\mathcal{M})$ is constant outside the origin. This is not the case for instance if \mathcal{M} is given by a single matrix with at least two eigenvalues with different modulus.

5 A case study: the Stanford–Urbano example

In this section we return to Example 1, and apply our results. We show that already for this simple example, the issue of estimating the stabilization radius appears to be very difficult. We will see in particular that although numerical algorithms allow to compute upper bounds for the stabilization radius, its computation is complicated by the fact that lower bounds appear to be much more difficult to obtain. Indeed, a simple application of Theorem 3 to the matrices from Example 1 gives $\tilde{\rho}(\mathcal{M}) \geq 1/2$, and applying the same result to \mathcal{M}^t , together with Item (ii) in

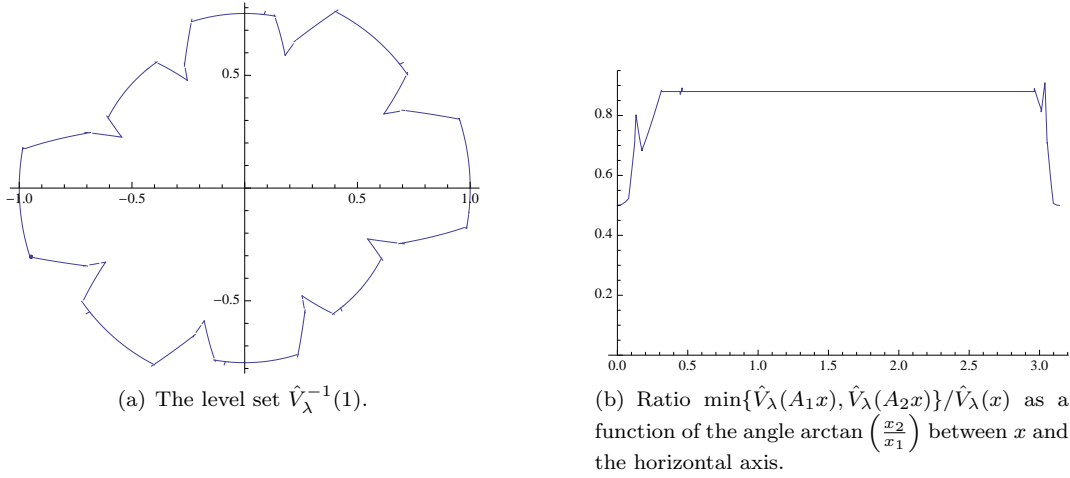


Figure 1: The function \hat{V}_λ for $\lambda = 0.88$.

Lemma 1, does not improve the bound. Worse than that, we will show that $\tilde{\rho}_z(\mathcal{M}) = \frac{1}{2}$ for z belonging to a dense subset of \mathbb{R}^2 , which suggests that improvements of the trivial lower bound could possibly be obtained only with much more subtle techniques.

Let us now derive an upper bound on $\tilde{\rho}(\mathcal{M})$ better than the one already obtained in Example 1. For this purpose we apply Proposition 9 and, in particular, it is possible to check numerically, by taking $\bar{t} = 9$ as maximal value for the times t_z , that one can improve the upper bound to $\tilde{\rho}(\mathcal{M}) \leq 0.886$. Considering longer matrix products appears to be computationally very costly and does not seem to improve significantly the obtained estimate of $\tilde{\rho}(\mathcal{M})$.

As an alternative method, estimates of the Lyapunov functions in Section 3.1 can also be computed numerically, providing a heuristic estimate of $\tilde{\rho}(\mathcal{M})$. Figure 1(a) shows an estimate of the candidate Lyapunov function described in Remark 3 taking $\lambda = 0.88$. This estimate is obtained by computing an approximation of \hat{V}_λ on a large number of points and by interpolating such values. One can see that the corresponding ratio $\min\{\hat{V}_\lambda(A_1 x), \hat{V}_\lambda(A_2 x)\} / \hat{V}_\lambda(x)$, which represents the largest decrease rate of \hat{V}_λ along trajectories and is depicted in Figure 1(b), slightly exceeds the value 0.88 only on very small intervals. Such a heuristic numerical analysis suggests that $\tilde{\rho}(\mathcal{M}) < 0.88$. For smaller values of λ the function \hat{V}_λ appears to be much more irregular and the condition $\min\{\hat{V}_\lambda(A_1 x), \hat{V}_\lambda(A_2 x)\} / \hat{V}_\lambda(x) \leq \lambda$ is violated on larger intervals, so that there is no clear indication that $\tilde{\rho}(\mathcal{M}) < \lambda$.

In the following we will derive a result highlighting a technical obstruction to further improving the trivial lower bound $\tilde{\rho}(\mathcal{M}) > \frac{1}{2}$. In order to do so, we introduce the notion of forward orbit of a point z_0 , defined as

$$\{x_{\sigma, z_0}(t) \in \mathbb{R}^2 \mid \sigma : \mathbb{N} \rightarrow \{1, 2\}, t \in \mathbb{N}\}.$$

For our particular matrices, it turns out that the forward orbit of z_0 actually coincides with the complete orbit of z_0 , i.e.

$$\{A_{\sigma(t)}^{\nu(t)} \dots A_{\sigma(0)}^{\nu(0)} z_0 \mid \sigma : \mathbb{N} \rightarrow \{1, 2\}, \nu : \mathbb{N} \rightarrow \{-1, 1\}, t \in \mathbb{N}\}.$$

This is an immediate consequence of the fact that $A_1^{-1} = A_1^7$ and $A_2^{-1} = A_1^2 A_2 A_1^{-2} = A_1^2 A_2 A_1^6$. Looking at the union of the orbits starting from points on the x_1 axis we can say something more, namely that it is symmetric with respect to the x_1 axis and coincides with the set of points from which there exists a trajectory reaching the x_1 axis in finite time:

$$\mathcal{R} = \left\{ z \in \mathbb{R}^2 \setminus \{0\} \mid (1, 0)^T = \frac{Az}{|Az|}, \text{ for some } A \in \mathcal{M}^t, t \in \mathbb{N} \right\} \cup \{0\}.$$

Proposition 10. *The set \mathcal{R} is a countable union of straight lines passing through the origin. The corresponding angles with the x_1 axis form a strict subset of the set of angles having rational tangent and this subset is dense in \mathbb{R} . As a consequence \mathcal{R} is dense in \mathbb{R}^2 .*

Proof. The set \mathcal{R} is made of a countable union of straight lines passing through the origin since it contains all those for which the tangent of the angle with the x_1 axis is 4^k , $k \in \mathbb{Z}$. Also, the tangent of the angle formed by each line in \mathcal{R} with the x_1 axis is a rational number. Indeed the maps A_1 and A_2 act on the angles in such a way that a rational tangent $\frac{p}{q}$ becomes $\frac{4p}{q}$ and $\frac{p-q}{p+q}$, respectively, and the set \mathcal{R} can be identified with the (integer) couples $\{p, q\}$ obtained by applying iteratively these two operations starting from $\{p, q\} = \{0, 1\}$. We can also assume that at each iteration the numbers p, q are positive (the orbit is symmetric with respect to the x_1 axis) and relatively prime. In particular, applying A_1 or A_1^{-1} leads, up to the sign, to the couple $\{\frac{p+q}{2}, \frac{|p-q|}{2}\}$, when both p, q are odd, while it leads to $\{p+q, |p-q|\}$ if one among p, q is even. On the other hand, applying k times A_2 or A_2^{-1} leads to couples of the form $\{4^k p, q\}$ or $\{4^k q, p\}$, up to a common divisor.

We claim that all the couples $\{p, q\}$ that can be generated with these operations starting from the x_1 axis are such that both p, q are different from 2 in $\mathbb{Z}/4\mathbb{Z}$.

Since applying A_2 or A_2^{-1} to the x_1 axis keeps it invariant, we start by applying A_1 or A_1^{-1} , and we get the couple $\{1, 1\}$. We now apply A_2^k or A_2^{-k} and we get a couple which, in $\mathbb{Z}/4\mathbb{Z}$, is equal to $\{0, 1\}$. We apply again A_1 or A_1^{-1} and we get a couple of the form $\{\pm 1, \pm 1\}$ in $\mathbb{Z}/4\mathbb{Z}$. A further application of A_2^k or A_2^{-k} leads to $\{0, \pm 1\}$ in $\mathbb{Z}/4\mathbb{Z}$. In particular we get that all the couples that can be obtained by iterating this procedure must be of the form $\{0, \pm 1\}$ or $\{\pm 1, \pm 1\}$ in $\mathbb{Z}/4\mathbb{Z}$, proving the claim.

We deduce that \mathcal{R} cannot be identified with the set of all angles having rational tangents. For instance it does not include the angle of tangent $\frac{1}{2}$, as well as all the elements of the corresponding orbit.

Let us now show that \mathcal{R} corresponds to a set of angles which is dense in $[0, 2\pi]$. For this purpose, let us consider the matrix product $A_2 A_1$. One can check that this matrix has complex non real eigenvalues and that it is similar to the rotation

$$R = \begin{pmatrix} \frac{\sqrt{5}}{2\sqrt{2}} & -\frac{\sqrt{7}}{2\sqrt{2}} \\ \frac{\sqrt{7}}{2\sqrt{2}} & \frac{\sqrt{5}}{2\sqrt{2}} \end{pmatrix}.$$

The angle θ of the rotation is such that $\cos 2\theta = \frac{1}{4}$, from which one easily deduces that this angle is incommensurable with π (see e.g. [19]). We thus deduce that the set of points obtained applying iteratively $A_2 A_1$ and starting from the x_1 axis correspond to a set of angles which is dense in $[0, 2\pi]$. This concludes the proof of the proposition. \square

Note that the property that the orbit corresponds to a set of angles dense in $[0, 2\pi]$ remains true if one replaces a point of the x_1 axis with any other starting point.

Since the value $\tilde{\rho}_z(\mathcal{M})$ is constant on each orbit, and thus on \mathcal{R} , we have the following important consequence of Proposition 10.

Corollary 3. *There exists a dense subset Ω of \mathbb{R}^2 , with $\mathcal{R} \subseteq \Omega$, such that $\tilde{\rho}_z(\mathcal{M}) = \frac{1}{2}$ for each $z \in \Omega$.*

At the light of the above result, the following question is very natural:

Is there a point $z \in \mathbb{R}^2$ (and thus a dense subset of \mathbb{R}^2) such that $\tilde{\rho}_z(\mathcal{M}) > \frac{1}{2}$?

A positive answer to this question would reveal interesting discontinuity properties of the function $z \mapsto \tilde{\rho}_z(\mathcal{M})$. However, due to the complexity of the set of trajectories starting from a given point z , giving an estimate of the value $\tilde{\rho}_z(\mathcal{M})$ appears to be very difficult. Summing up, for the considered example the issue of approximating with arbitrary precision the value $\tilde{\rho}(\mathcal{M})$ by simple numerical methods appears to be a very hard, if not intractable, task.

In conclusion, the analysis performed in this section suggests that any accurate computation of the stabilization radius cannot, in general and even for very simple examples, be obtained by usual algorithmic methods, but should instead rely on *ad hoc* arguments.

6 The continuous-time case

Given a compact and convex set of matrices $\mathcal{M} = \{A_\sigma \mid \sigma \in \Sigma\} \subset \mathbb{R}^{d \times d}$, we consider in this section the continuous-time switched system defined by the differential equations of the form

$$\dot{x}(s) = A_{\sigma(s)}x(s), \quad x(0) = x_0,$$

where $\sigma(\cdot)$ is a switching sequence. As in the discrete-time setting, we denote by $x_{\sigma, x_0}(t)$ the corresponding solution, that is $x_{\sigma, x_0}(s) = \Phi_\sigma(s)x_0$ for $s \geq 0$, where $\Phi_\sigma(s)$ is the fundamental matrix associated with the differential equation. Several notions and results in Sections 2 and 3 are still valid in the continuous-time case without need of substantial changes. In particular the notion of pointwise stabilizability given in Definition 1 is still meaningful. Similarly, the stabilizability indices introduced in Definition 2 are still well-defined, and the continuous-time counterparts of Proposition 1 and Proposition 3 hold true. A partial counterpart of Lemma 1 also holds, as it is straightforward to show that $\tilde{\rho}(\mathcal{M} + \gamma\text{Id}) = e^\gamma \tilde{\rho}(\mathcal{M})$, where $\mathcal{M} + \gamma\text{Id}$ is the set of matrices of the form $M + \gamma\text{Id}$ with $M \in \mathcal{M}$.

On the other hand, in the continuous-time case one has to be careful in the definition of feedback stabilizability in order to avoid existence and uniqueness issues for ordinary differential equations with possibly discontinuous right-hand side. In particular, several non-equivalent and meaningful notions of solutions exist in the case of a discontinuous feedback $\sigma(\cdot)$. To circumvent this problem, we choose here to deal with an adapted notion of (uniform) feedback stabilizability based on a ‘sample-and-hold’ scheme. Note that the use of this kind of schemes is consistent with the sampling process used in computer control. Moreover the use of other notions of solutions, such as Filippov solutions, appears to be less practical and may prevent to obtain fully general stabilization results (see e.g. [8]). Even so, we think that the study of the feedback stabilization properties for linear switched systems, in relation with other notions of solutions, is worth further investigation.

Definition 4. For $\delta > 0$ we say that a 0-homogeneous function $\sigma : \mathbb{R}^d \rightarrow \mathcal{M}$ is a δ -stabilizing feedback if for any $x_0 \in \mathbb{R}^d$, the trajectory starting from x_0 and defined by $\dot{x}(t) = A_{\sigma(x(\delta k))}x(t)$ on each interval $[\delta k, \delta(k+1))$, $k \in \mathbb{N}$, converges to 0.

In addition we say that \mathcal{M} is uniformly feedback stabilizable if for any positive ϵ_1, ϵ_2 there exists $T \in \mathbb{R}_+$ such that for any $\delta > 0$ small enough there exists a δ -stabilizing feedback such that if $|x_0| < \epsilon_1$, the trajectory $x(\cdot)$ starting from x_0 satisfies $|x(t)| < \epsilon_2$ if $t \geq T$.

Note that, for the stabilization of continuous-time nonlinear control systems, the use of sample-and-hold feedback controls in combination with control-Lyapunov functions is not new. It has been investigated for instance in [8]. In that paper the asymptotic controllability property for nonlinear control systems is proved under the assumption that there exists a continuous control-Lyapunov function, and allowing more general sampling times compared to our Definition 4 (namely, the differences among subsequent sampling times are small enough but not necessarily equal). The following result is the continuous-time counterpart of Proposition 4. Here the notion of control-Lyapunov function is that of Definition 3 with the exception that $t \in \mathbb{R}_+$ in Item (2).

Proposition 11. If \mathcal{M} admits a Lipschitz continuous control-Lyapunov function V then \mathcal{M} is uniformly feedback stabilizable, and, for each $\delta > 0$ small enough, the δ -stabilizing feedback $\sigma(\cdot)$ can be taken piecewise constant.

The previous result may not be seen as a particular case of the main theorem in [8], as the latter does not guarantee any regularity property of the feedback law. Also, opposed to [8], our result can be proved without resorting to sophisticated tools from nonsmooth analysis, as shown below.

Proof of Proposition 11. Given an initial point $x_0 \in \mathbb{R}^d$ let us consider the switching law $\sigma_{x_0}(\cdot)$ as given by Definition 3. For $\delta > 0$ let us consider the matrix $\bar{A} = \frac{1}{\delta} \int_0^\delta A_{\sigma_{x_0}(s)} ds$. Since \mathcal{M} is convex we have that $\bar{A} \in \mathcal{M}$, that is there exists $\bar{\sigma}(x_0)$ such that $A_{\bar{\sigma}(x_0)} = \bar{A}$. Consider now the solution

$\bar{x}(\cdot)$ of the equation $\dot{x} = \bar{A}x$ with initial condition x_0 and set $y(\cdot) = x_{\sigma_{x_0}, x_0}(\cdot) - \bar{x}(\cdot)$. Then $y(\cdot)$ satisfies the equation

$$\dot{y}(t) = \bar{A}y(t) + (A_{\sigma(t)} - \bar{A})x_{\sigma_{x_0}, x_0}(t)$$

with initial condition $y(0) = 0$. The corresponding solution at time δ , given by the variation of constants formula, is

$$y(\delta) = e^{\delta \bar{A}} \int_0^\delta e^{-t \bar{A}} (A_{\sigma(t)} - \bar{A}) x_{\sigma_{x_0}, x_0}(t) dt.$$

Since

$$\begin{aligned} e^{-t \bar{A}} (A_{\sigma(t)} - \bar{A}) x_{\sigma_{x_0}, x_0}(t) &= (e^{-t \bar{A}} - \text{Id})(A_{\sigma(t)} - \bar{A}) x_{\sigma_{x_0}, x_0}(t) + \\ &\quad + (A_{\sigma(t)} - \bar{A})(x_{\sigma_{x_0}, x_0}(t) - x_0) + (A_{\sigma(t)} - \bar{A})x_0 \end{aligned}$$

and by compacity of \mathcal{M} , and since moreover the integral on $[0, \delta]$ of the last term is zero, we have that (for δ small) there exists a constant $C > 0$ such that

$$|y(\delta)| \leq C \delta^2 |x_0|.$$

Let L be the Lipschitz constant of the Lyapunov function V . Then

$$\begin{aligned} V(x_0) - V(\bar{x}(\delta)) &\geq V(x_0) - V(x_{\sigma_{x_0}, x_0}(\delta)) - |V(\bar{x}(\delta)) - V(x_{\sigma_{x_0}, x_0}(\delta))| \\ &\geq m(1 - \mu^\delta) |x_0| - C \delta^2 |x_0| \\ &\geq \frac{1}{2} m \left(\log \frac{1}{\mu} \right) \delta |x_0| \end{aligned}$$

for δ small enough, which immediately shows that $\bar{\sigma}(\cdot)$ is a δ -stabilizing feedback.

Finally, exactly as in the discrete-time case, the fact that the δ -stabilizing feedback can be taken piecewise constant follows from the continuity of V and the continuous dependence of the solutions with respect to the initial data. \square

Remark 5. *The feedback constructed in the proof of Proposition 11 actually stabilizes the system robustly with respect to small perturbations of the sampling times.*

Proposition 5 and Corollary 1 still hold in the continuous-time case (in particular a proof of the continuous-time counterpart of Proposition 5 may be adapted from an analogous result in [38, Section 4.3.1]) and they are summarized by the following result. Note that converse Lyapunov theorems for the stabilization (or asymptotic controllability) of more general classes of nonlinear control systems may be found for instance in [22, 32].

Proposition 12. *For any $\lambda > \tilde{\rho}(\mathcal{M})$ the function $V_\lambda : \mathbb{R}^d \rightarrow \mathbb{R}_+$*

$$V_\lambda(x) = \sup_{t \geq 0} \inf_{\sigma(\cdot)} \frac{|x_{\sigma, x}(t)|}{\lambda^t} \tag{8}$$

is well-defined, absolutely homogeneous, Lipschitz continuous and satisfies $V_\lambda(x_{\sigma, x}(t)) \leq \lambda^t V_\lambda(x)$ for any $x \in \mathbb{R}^d$, $t \in \mathbb{R}_+$ and some $\sigma(\cdot)$, depending on x . In particular \mathcal{M} is pointwise stabilizable if and only if it admits an absolutely homogeneous control-Lyapunov function. As a consequence the following conditions are equivalent:

- (i) $\tilde{\rho}'(\mathcal{M}) < 1$,
- (ii) $\tilde{\rho}(\mathcal{M}) < 1$,
- (iii) \mathcal{M} is pointwise stabilizable,
- (iv) \mathcal{M} is uniformly feedback stabilizable.

7 Conclusion and open questions

Stabilizability of switched systems is a natural goal for control theorists, with several promising applications. The problem raises many subtle mathematical questions. In this paper, we tried to motivate a thorough analysis of these questions. We showed that many counterintuitive phenomena occur, which one must pay attention to for a proper treatment of the problem. As a striking example, we showed that (Proposition 2), even though linear switched systems are homogeneous systems, the optimal switching law might not be a homogeneous function of the statespace. From a numerical point of view, the stabilization problem appears to be extremely challenging. The discussion and the results obtained throughout the paper raise several open question. Among them we wish to recall the next two ones:

Open Question 1. *Is there an algorithm, or a characterization, to decide whether $\tilde{\rho}(\mathcal{M}) = \check{\rho}(\mathcal{M})$?*

Open Question 2. *Is it possible to improve Theorem 3 and provide a generally better formula for a lower bound? In particular, how to find a better lower bound on $\tilde{\rho}(\mathcal{M})$ in Example 1?*

As for the first question, let us mention that sufficient conditions for this equivalence, for a slightly different notion of feedback stabilizability (stronger than ours, motivated by applications involving supervisory control and measurement scheduling), have been shown in [24]. Concerning the second question, let us notice that a result improving Theorem 3 could for instance lead to a better estimate (from below) of $\tilde{\rho}(\mathcal{M})$ in Example 1. Nevertheless the analysis in Section 5 seems to suggest that an exact computation of such a quantity should rely on *ad hoc* methods.

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