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# Ordering Garside groups

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We introduce a condition on Garside groups that we call Dehornoy structure. An iteration of such a structure leads to a left order on the group. We show conditions for a Garside group to admit a Dehornoy structure, and we apply these criteria to prove that the Artin groups of type  $A$  and  $I_2(m)$ ,  $m \geq 4$ , have Dehornoy structures. We show that the left orders on the Artin groups of type  $A$  obtained from their Dehornoy structures are the Dehornoy orders. In the case of the Artin groups of type  $I_2(m)$ ,  $m \geq 4$ , we show that the left orders derived from their Dehornoy structures coincide with the orders obtained from embeddings of the groups into braid groups.

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## 1 Introduction

In this paper the *braid group on  $n$  strands* is defined by the presentation:

$$\mathcal{B}_n = \left\langle x_1, \dots, x_{n-1} \mid \begin{array}{ll} x_i x_j = x_j x_i & \text{if } |i - j| \geq 2 \\ x_i x_j x_i = x_j x_i x_j & \text{if } |i - j| = 1 \end{array} \right\rangle.$$

A group  $G$  is called *left orderable* if there exists a total order  $<$  on  $G$  which is invariant under left multiplication, that is, such that  $g_1 g_2 < g_1 g_3$  if  $g_2 < g_3$ , for all  $g_1, g_2, g_3 \in G$ . Given such an order  $<$  on  $G$ , we define the *positive cone* of  $<$  to be the set  $P = P_{<} = \{g \in G \mid 1 < g\}$ . This satisfies the following properties:

- (1)  $PP \subset P$  (i.e.  $P$  is a subsemigroup).
- (2)  $G = P \sqcup P^{-1} \sqcup \{1\}$ .

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Conversely, a subset  $P$  of  $G$  which satisfies (1) and (2) determines a left order  $<$  on  $G$  defined by  $g_1 < g_2$  if  $g_1^{-1}g_2 \in P$ , and  $P$  is the positive cone of  $<$ . In the literature the left orders on groups are often defined by their positive cones, and they will be defined in this way in the present paper.

The first explicit left order on the braid group  $\mathcal{B}_n$  was determined by Dehornoy [8]. The fact that the braid group is left orderable is an important result, but, also, the Dehornoy order is interesting by itself, and it is used for several purposes such as in the proof of the faithfulness of some representations of the braid groups (see Shpilrain [26] and Crisp–Paris [7], for example). A complete presentation on left orders on the braid group and, in particular, on the Dehornoy order, is given in Dehornoy–Dyannikov–Rolfsen–Wiest [12]. The definition of the Dehornoy order is based on the following construction:

Let  $G$  be a group, and let  $X = \{x_1, \dots, x_k\}$  be an ordered generating family for  $G$ . Let  $i \in \{1, \dots, k\}$ . We say that a word  $w \in (X \sqcup X^{-1})^*$  is  $x_i$ -positive (resp.  $x_i$ -negative) if it satisfies the following condition:

$$w \in \{x_i, x_{i+1}^{\pm 1}, \dots, x_k^{\pm 1}\}^* \text{ (resp. } w \in \{x_i^{-1}, x_{i+1}^{\pm 1}, \dots, x_k^{\pm 1}\}^*), \text{ but } w \notin \{x_{i+1}^{\pm 1}, \dots, x_k^{\pm 1}\}^*.$$

An element  $g \in G$  is called  $x_i$ -positive (resp.  $x_i$ -negative) if it has an  $x_i$ -positive representative (resp.  $x_i$ -negative representative). We denote by  $G_i$  the subgroup of  $G$  generated by  $\{x_i, x_{i+1}, \dots, x_k\}$ , by  $P_i^+$  the set of  $x_i$ -positive elements of  $G$ , and by  $P_i^-$  the set of  $x_i$ -negative elements. The key of the definition of the Dehornoy order is the following:

**Theorem 1.1** (Dehornoy [8]) *Assume that  $G = \mathcal{B}_n$  is the braid group on  $n$  strands, and that  $\{x_1, \dots, x_{n-1}\}$  is its standard generating set. For all  $i \in \{1, \dots, n-1\}$ , we have the disjoint union  $G_i = P_i^+ \sqcup P_i^- \sqcup G_{i+1}$ , where  $G_n = \{1\}$ .*

Let  $G = \mathcal{B}_n$  be the braid group on  $n$  strands. Set  $P_D = P_1^+ \sqcup P_2^+ \sqcup \dots \sqcup P_{n-1}^+$ . Then, by Theorem 1.1,  $P_D$  is the positive cone of a left order on  $G$ . This order is the *Dehornoy order*. The order relation respective to  $P_D$  shall be denoted by  $<_D$ .

A careful reader may notice that Theorem 1.1 leads to more than one left order on the braid group. Indeed, for  $\epsilon = (\epsilon_1, \dots, \epsilon_{n-1}) \in \{\pm 1\}^{n-1}$ , we set:

$$P^\epsilon = P_1^{\epsilon_1} \sqcup P_2^{\epsilon_2} \sqcup \dots \sqcup P_{n-1}^{\epsilon_{n-1}}.$$

Then, by Theorem 1.1,  $P^\epsilon$  is the positive cone of a left order on  $G = \mathcal{B}_n$ . The case  $\epsilon = (1, -1, 1, -1, \dots)$  is particularly interesting because of the following:

**Theorem 1.2** (Dubrovina–Dubrovin [15]) *Assume that  $G = \mathcal{B}_n$  is the braid group on  $n$  strands, and that  $\{x_1, \dots, x_{n-1}\}$  is its standard generating set. Let  $\epsilon = (1, -1, 1, -1, \dots)$ . Then  $P^\epsilon$  is a finitely generated semigroup. In particular, the left order on  $\mathcal{B}_n$  defined by  $P^\epsilon$  is isolated in the space of left orders on  $\mathcal{B}_n$ .*

The goal of the present paper is to extend the Dehornoy order to some Garside groups.

A first approach would be to adopt the same definition as above. Let  $G$  be a group, and let  $\{x_1, \dots, x_k\}$  be an ordered generating set for  $G$ . We say that  $\{x_1, \dots, x_k\}$  determines a *Dehornoy structure* (in Ito’s sense) if we have the disjoint union  $G_i = P_i^+ \sqcup P_i^- \sqcup G_{i+1}$  for all  $i \in \{1, \dots, k\}$ , where  $G_{k+1} = \{1\}$ . In this case, as for the braid group, if  $\epsilon = (\epsilon_1, \dots, \epsilon_k) \in \{\pm 1\}^k$ , then

$$P^\epsilon = P_1^{\epsilon_1} \sqcup P_2^{\epsilon_2} \sqcup \dots \sqcup P_k^{\epsilon_k}$$

is the positive cone of a left order on  $G$ . This approach is actually used in Ito [25] to construct groups having isolated left orders in their spaces of left orders.

In this paper we adopt another approach based on a definition of the Dehornoy order in terms of Garside groups (see Dehornoy [10], Fromentin [16], Fromentin–Paris [17]). In Section 3 we give a (new and different) definition of a *Dehornoy structure* on a Garside group, and we show a criterion for a Garside group to admit such a structure. We warn the reader that our new notion of a Dehornoy structure is different and, probably, non-equivalent to the one in Ito’s sense given above (see Sibert [27]). In Section 4 we prove that the braid group admits a Dehornoy structure (in the new sense), and that this structure leads to the Dehornoy order. In Sections 5 and 6 we prove that an Artin group of dihedral type admits a Dehornoy structure which leads to a left order on the group. Then, we compare the left order obtained in this way with the one obtained using some embedding of the group in a braid group defined by Crisp [5]. For technical reasons, the study of the Artin groups of dihedral type is divided into two sections, depending on the parity of the length of the Garside element. In Section 2 we give some preliminaries on Garside groups and Artin groups.

## 2 Preliminaries

### 2.1 Garside groups

The Garside groups are a generalization of the braid groups on which one can apply Garside’s ideas [18]. They were introduced in Dehornoy–Paris [13]. We refer to

Dehornoy et al. [11] for a complete presentation on this theory. Here, we will just give some basic definitions and properties that will be needed for our purpose.

Let  $M$  be a monoid. We say that  $M$  is *atomic* if there exists a map  $\nu : M \rightarrow \mathbb{N}$  such that:

- (a)  $\nu(g) = 0$  if and only if  $g = 1$ ;
- (b)  $\nu(gg') \geq \nu(g) + \nu(g')$  for all  $g, g' \in M$ .

Such a map  $\nu : M \rightarrow \mathbb{N}$  is called a *norm* on  $M$ . A non-trivial element  $g \in M$  is called an *atom* if it is indecomposable in the sense that, if  $g = g_1g_2$ , then either  $g_1 = 1$  or  $g_2 = 1$ . We denote by  $\mathcal{A} = \mathcal{A}_M$  the set of atoms.

Assume that  $M$  is atomic. Then any generating set of  $M$  contains  $\mathcal{A}$ , and  $\mathcal{A}$  generates  $M$ . In particular,  $M$  is finitely generated if and only if  $\mathcal{A}$  is finite. We can define two partial orders  $\leq_L$  and  $\leq_R$  on  $M$  as follows:

- We set  $g_1 \leq_L g_2$  if there exists  $g_3 \in M$  such that  $g_1g_3 = g_2$ .
- We set  $g_1 \leq_R g_2$  if there exists  $g_3 \in M$  such that  $g_3g_1 = g_2$ .

These orders are called *left-divisibility order* and *right-divisibility order*, respectively. For  $g \in M$  we set:

$$\text{Div}_L(g) = \{g' \in M \mid g' \leq_L g\}, \quad \text{Div}_R(g) = \{g' \in M \mid g' \leq_R g\}.$$

We say that  $g$  is *balanced* if  $\text{Div}_L(g) = \text{Div}_R(g)$ .

A *Garside monoid* is a monoid  $M$  such that:

- (a)  $M$  is atomic;
- (b)  $M$  is *cancellative*, that is,  $g_1gg_2 = g_1g'g_2$  implies  $g = g'$ , for all  $g, g', g_1, g_2 \in M$ ;
- (c)  $(M, \leq_L)$  and  $(M, \leq_R)$  are lattices;
- (d)  $M$  contains a *Garside element*, that is, a balanced element  $\Delta$  such that  $\text{Div}_L(\Delta) = \text{Div}_R(\Delta)$  is finite and generates  $M$ .

The lattice operations in  $(M, \leq_L)$  (resp.  $(M, \leq_R)$ ) are denoted by  $\wedge_L$  and  $\vee_L$  (resp.  $\wedge_R$  and  $\vee_R$ ). For every  $g, g' \in M$  we denote by  $g' \setminus g$  and  $g/g'$  the unique elements in  $M$  such that  $g \vee_L g' = g'(g' \setminus g)$  and  $g \vee_R g' = (g/g')g'$ .

**Proposition 2.1** (Dehornoy [11]) *For every  $g, g_1, g_2$  in a Garside monoid, we have:*

- (1)  $g \vee_R (g_2g_1) = ((g/g_1) \vee_R g_2)g_1$ .

- (2)  $(g_1 g_2) \vee_L g = g_1 (g_2 \vee_L (g_1 \setminus g))$ .
- (3)  $(g_2 g_1) / g = (g_2 / (g / g_1)) (g_1 / g)$  and  $g / (g_2 g_1) = (g / g_1) / g_2$ .
- (4)  $g \setminus (g_1 g_2) = (g \setminus g_1) ((g_1 \setminus g) \setminus g_2)$  and  $(g_1 g_2) \setminus g = g_2 \setminus (g_1 \setminus g)$ .

Let  $M$  be a monoid. The *enveloping group* of  $M$ , denoted by  $G = G(M)$ , is the group presented by the generating set  $M$  and the relations  $g_1 g_2 = g$  if  $g_1 g_2 = g$  holds in  $M$ . There is a canonical homomorphism  $M \rightarrow G(M)$  which is not injective in general. A *Garside group* is the enveloping group of a Garside monoid.

**Remark** (1) A Garside monoid  $M$  satisfies the Öre conditions, hence the canonical homomorphism  $M \rightarrow G(M)$  is injective (see Dehornoy–Paris [13]). Furthermore, the orders  $\leq_L$  and  $\leq_R$  extend to lattice orders on  $G(M)$  with positive cone  $M$ . These are defined by  $g \leq_L g'$  (resp.  $g \leq_R g'$ ) if  $g^{-1}g' \in M$  (resp.  $g'g^{-1} \in M$ ).

- (2) A Garside element is never unique. For example, if  $\Delta$  is a Garside element, then  $\Delta^k$  is a Garside element for all  $k \geq 1$  (see Dehornoy [9, Lemma 2.2]). We will talk about a *Garside system*  $(M, \Delta)$  whenever we will need to indicate the Garside element.

Let  $M$  be a Garside monoid, and let  $\Delta$  be a fixed Garside element. The elements of  $\text{Div}_L(\Delta) = \text{Div}_R(\Delta)$  are called the *simple elements* of  $M$ , and the set of simple elements is denoted by  $\text{Div}(\Delta)$ . By Dehornoy–Paris [13], there is an automorphism  $\Phi : M \rightarrow M$ , called the *flip automorphism*, such that  $\Delta g = \Phi(g) \Delta$  for all  $g \in M$ . On the other hand, there is a one-to-one correspondence  $\partial : \text{Div}(\Delta) \rightarrow \text{Div}(\Delta)$  such that  $\partial(s)s = \Delta$  for all  $s \in \text{Div}(\Delta)$ . It is easily seen that  $\partial^2(s) = \Phi(s)$  and  $s \partial^{-1}(s) = \Delta$  for all  $s \in \text{Div}(\Delta)$ .

A pair  $(s, s')$  of simple elements is called *right-weighted* (resp. *left-weighted*) if  $ss' \wedge_R \Delta = s'$  (resp.  $ss' \wedge_L \Delta = s$ ).

**Proposition 2.2** (Dehornoy–Paris [13]) *Every element  $g \in G$  admits unique decompositions  $g = s_p \cdots s_1 \Delta^d$  and  $g = \Delta^d s'_1 \cdots s'_p$  such that:*

- (a)  $s_1, \dots, s_p, s'_1, \dots, s'_p \in \text{Div}(\Delta) \setminus \{1, \Delta\}$ ;
- (b)  $(s_{i+1}, s_i)$  is right-weighted for all  $i \in \{1, \dots, p-1\}$ ;
- (c)  $(s'_i, s'_{i+1})$  is left-weighted for all  $i \in \{1, \dots, p-1\}$ .

The expressions  $g = s_p \cdots s_1 \Delta^d$  and  $g = \Delta^d s'_1 \cdots s'_p$  of Proposition 2.2 are called the *right greedy normal form* and the *left greedy normal form* of  $g$ , respectively. The integer  $d$  is called the *infimum* of  $g$ , and is denoted by  $\inf(g)$ , the integer  $p$  is called the *canonical length*, and is denoted by  $\ell(g)$ , and the integer  $d + p$  is called the *supremum*, and is denoted by  $\sup(g)$ . These invariants of the greedy normal forms are frequently used in the theory of Garside groups, especially in the study of the conjugacy problem (see Gebhardt–González-Meneses [19], for example). In this paper we will need a fourth invariant, that we call the *negative-infimum* of  $g$ , and which is defined by:

$$\text{Ninf}(g) = \begin{cases} 0 & \text{if } d \geq 0, \\ -d & \text{if } d < 0. \end{cases}$$

Moreover, we set  $g_R = s_p \cdots s_1$  and  $g_L = s'_1 \cdots s'_p$ . Note that  $g_L = g_R$  if the flip automorphism  $\Phi$  is trivial.

Note that  $M = \{g \in G(M) \mid \text{Ninf}(g) = 0\}$ .

**Proposition 2.3** (Gebhardt–González-Meneses [19]) *Let  $g \in G$ .*

- (1) *Then  $\inf(g)$  is the greatest integer  $d \in \mathbb{Z}$  such that  $\Delta^d \leq_R g$ , and  $\sup(g)$  is the least integer  $k \in \mathbb{Z}$  such that  $g \leq_R \Delta^k$ .*
- (2) *We have  $\inf(g^{-1}) = -\sup(g)$ ,  $\sup(g^{-1}) = -\inf(g)$ , and  $\ell(g^{-1}) = \ell(g)$ .*

We define the *left complement* (resp. the *right complement*) of an element  $g \in G$  to be  $\text{com}_L(g) = \Delta^k g^{-1}$  (resp.  $\text{com}_R(g) = g^{-1} \Delta^k$ ), where  $k = \sup(g)$ . Note that, if  $s \in \text{Div}(\Delta) \setminus \{1, \Delta\}$ , then  $\text{com}_L(s) = \partial(s)$  and  $\text{com}_R(s) = \partial^{-1}(s)$ . Moreover, if  $\Phi = 1$ , then  $\text{com}_L(g) = \text{com}_R(g)$ . The proof of the following is left to the reader:

**Lemma 2.4** *Let  $g \in G$ , let  $g = s_p \cdots s_1 \Delta^d$  be the right greedy normal form of  $g$ , and let  $g = \Delta^d s'_1 \cdots s'_p$  be its left greedy normal form. Then  $\inf(\text{com}_L(g)) = \inf(\text{com}_R(g)) = 0$ , and  $\ell(\text{com}_L(g)) = \ell(\text{com}_R(g)) = p = \ell(g)$ . Moreover,*

$$\begin{aligned} \text{com}_L(g) &= \Phi^{p-1}(\partial(s_1)) \cdots \Phi(\partial(s_{p-1})) \partial(s_p), \\ \text{com}_R(g) &= \partial^{-1}(s'_p) \Phi^{-1}(\partial^{-1}(s'_{p-1})) \cdots \Phi^{-p+1}(\partial^{-1}(s'_1)). \end{aligned}$$

Furthermore,  $\text{com}_L(g) = \Phi^{d+p}(\text{com}_R(g))$ .

**Proposition 2.5** (Dehornoy–Paris [13]) *For every  $h \in G$  there exists a unique pair  $(g, g')$  (resp.  $(g', g)$ ) of elements in  $M$ , such that  $h = g^{-1}g'$  (resp.  $h = g'g^{-1}$ ) and  $g \wedge_L g' = 1$  (resp.  $g \wedge_R g' = 1$ ).*

The pair  $(g, g')$  (resp.  $(g', g)$ ) in Proposition 2.5 is called the *left orthogonal splitting* (resp. *right orthogonal splitting*) of  $h$ .

## 2.2 Parabolic subgroups

We keep the above hypothesis and notations, namely,  $M$  is a Garside monoid,  $\Delta$  is a fixed Garside element, and  $G$  is the enveloping group of  $M$ . Recall that an element  $\delta \in M$  is *balanced* if  $\text{Div}_L(\delta) = \text{Div}_R(\delta)$ . In this case, this set is denoted by  $\text{Div}(\delta)$ . Let  $\delta$  be a balanced element. We denote by  $\mathcal{S}(\delta)$  the set of atoms of  $M$  that belong to  $\text{Div}(\delta)$ , and by  $G_\delta$  the subgroup of  $G$  generated by  $\mathcal{S}(\delta)$ . We say that  $G_\delta$  is a *parabolic subgroup* of  $G$  (or of  $(G, \Delta)$ , when the Garside element needs to be specified) *associated with*  $\delta$  if  $\text{Div}(\delta) = \text{Div}(\Delta) \cap G_\delta$ . We list in the following proposition some results on parabolic subgroups that we will need, and refer to Godelle [20] for the proofs.

**Proposition 2.6** (Godelle [20]) *Let  $\delta \in M$  be a balanced element such that  $G_\delta$  is a parabolic subgroup of  $G$ .*

- (1)  $M_\delta = M \cap G_\delta$  is a Garside monoid, and  $\delta$  is a Garside element for  $M_\delta$ . Moreover,  $G_\delta$  is the enveloping group of  $M_\delta$ .
- (2) Let  $\vee_{\delta,L}$  and  $\wedge_{\delta,L}$  (resp.  $\vee_{\delta,R}$  and  $\wedge_{\delta,R}$ ) be the lattice operations on  $(M_\delta, \leq_L)$  (resp. on  $(M_\delta, \leq_R)$ ). We have  $g_1 \vee_L g_2, g_1 \wedge_L g_2 \in M_\delta$ ,  $g_1 \vee_L g_2 = g_1 \vee_{\delta,L} g_2$ , and  $g_1 \wedge_L g_2 = g_1 \wedge_{\delta,L} g_2$ , for all  $g_1, g_2 \in M_\delta$ . Similarly, we have  $g_1 \vee_R g_2, g_1 \wedge_R g_2 \in M_\delta$ ,  $g_1 \vee_R g_2 = g_1 \vee_{\delta,R} g_2$ , and  $g_1 \wedge_R g_2 = g_1 \wedge_{\delta,R} g_2$ , for all  $g_1, g_2 \in M_\delta$ .
- (3) We have  $\mathcal{A} \cap M_\delta = \mathcal{S}(\delta)$ , and this is the set of atoms of  $M_\delta$ .
- (4) If  $g_1 \leq_L g_2$  (resp.  $g_1 \leq_R g_2$ ) and  $g_2 \in M_\delta$ , then  $g_1 \in M_\delta$ , for all  $g_1, g_2 \in M$ .

The following lemmas in this subsection will be needed in our study:

**Lemma 2.7** *Let  $\delta_1, \delta_2 \in M$  be two balanced elements such that  $G_{\delta_1}$  and  $G_{\delta_2}$  are parabolic subgroups, and  $\mathcal{S}(\delta_1) \subseteq \mathcal{S}(\delta_2)$ . Then  $\delta_1$  is a balanced element of  $M_{\delta_2}$ , and  $G_{\delta_1} = (G_{\delta_2})_{\delta_1}$  is a parabolic subgroup of  $G_{\delta_2}$ .*

**Proof** Clearly  $\delta_1$  is balanced in  $M_{\delta_2}$ . Since  $\mathcal{S}(\delta_1) \subseteq \mathcal{S}(\delta_2)$  then  $M_{\delta_1}$  is a submonoid of  $M_{\delta_2}$ . Hence  $\text{Div}(\delta_1) = \text{Div}(\Delta) \cap M_{\delta_1} \subseteq \text{Div}(\Delta) \cap M_{\delta_2} = \text{Div}(\delta_2)$ . Therefore:

$$\text{Div}(\delta_2) \cap M_{\delta_1} \subseteq \text{Div}(\Delta) \cap M_{\delta_1} = \text{Div}(\delta_1) \subseteq \text{Div}(\delta_2) \cap M_{\delta_1}.$$

Further  $\mathcal{S}(\Delta) \supseteq \mathcal{S}(\delta_2) \supseteq \mathcal{S}(\delta_1) = \mathcal{S}(\Delta) \cap \text{Div}(\delta_1) \subseteq \text{Div}(\delta_1)$ . Hence  $\mathcal{S}(\delta_1) = \mathcal{S}(\delta_2) \cap \text{Div}(\delta_1)$ . Therefore  $G_{\delta_1} = (G_{\delta_2})_{\delta_1}$  is a parabolic subgroup of  $G_{\delta_2}$ .  $\square$

Recall that, by Dehornoy [9],  $\Delta^k$  is a Garside element for all  $k \geq 1$ .

**Lemma 2.8** *Let  $\delta$  be a balanced element such that  $G_\delta$  is a parabolic subgroup of  $(G, \Delta)$ . Let  $k \in \mathbb{N}$ ,  $k \geq 1$ . Then  $\delta^k$  is balanced,  $G_{\delta^k} = G_\delta$ , and  $G_{\delta^k}$  is a parabolic subgroup of  $(G, \Delta^k)$  associated with  $\delta^k$ .*

**Proof** Let  $s \in \text{Div}_L(\delta^k)$ . Then  $s \in M_\delta$  and  $\text{sup}(s) \leq k$ , i.e.  $s = s_1 \cdots s_t$ ,  $t \leq k$ , for some  $s_1, \dots, s_t \in \text{Div}(\delta)$ . Hence  $s \leq_R \delta^t \leq_R \delta^k$  because  $\delta$  is balanced. Analogously, we obtain that if  $s \in \text{Div}_R(\delta^k)$ , then  $s \leq_L \delta^k$ . Therefore  $\delta^k$  is balanced in  $M$ . Since divisors of  $\delta^k$  are products of divisors of  $\delta$ , then  $G_{\delta^k} = G_\delta$ . Clearly  $\text{Div}(\delta^k) \subseteq \text{Div}(\Delta^k) \cap G_{\delta^k}$ . Let  $s \in \text{Div}(\Delta^k) \cap G_{\delta^k}$ ,  $s \neq 1$ . Then  $s = s_1 \cdots s_t$  for some  $s_1, \dots, s_t \in \text{Div}(\Delta)$ ,  $t \leq k$ . If  $k = 1$ , then  $s \in \text{Div}(\Delta) \cap G_\delta = \text{Div}(\delta)$ . If  $k > 1$  we suppose that the statement is true for values less than  $k$ . If  $t < k$ , i.e.  $s \in \text{Div}(\Delta^t) \cap G_{\delta^t}$ , then, the inductive hypothesis implies that  $s \in \text{Div}(\delta^t) \subseteq \text{Div}(\delta^k)$ . If  $t = k$ , Proposition 2.6.4 implies that  $s_1 \cdots s_{k-1} \in \text{Div}(\Delta^{k-1}) \cap G_{\delta^{k-1}}$  because  $s \in G_{\delta^k} = G_\delta$ . So, by the inductive hypothesis, we have  $s_1 \cdots s_{k-1} \in \text{Div}(\delta^{k-1})$ . Hence  $s_k = (s_1 \cdots s_{k-1})^{-1}s \in G_{\delta^k}$ , i.e.  $s_k \in \text{Div}(\Delta) \cap G_\delta = \text{Div}(\delta)$ , thus  $s \in \text{Div}(\delta^k)$ . Therefore  $\text{Div}(\Delta^k) \cap G_{\delta^k} = \text{Div}(\delta^k)$  which implies that  $G_{\delta^k}$  is a parabolic subgroup of  $(G, \Delta^k)$ .  $\square$

**Lemma 2.9** *Let  $g, g' \in M$  such that  $\Delta \not\leq_R g, g'$  and  $\Delta^k \leq_R gg'$  for some  $k \geq 1$ . Then  $g = g_1h$  and  $g' = \text{com}_R(h)g'_1$  for some  $g_1, g'_1 \in A^+$  and  $h \in \text{Div}(\Delta^k)$  with  $\ell(h) = k$ .*

**Proof** Let  $g_0 \in M$  such that  $gg' = g_0\Delta^k$ . Note that  $\ell(g) \geq k$ . Otherwise  $\Delta \leq_L g'$  which is a contradiction. Similarly  $\ell(g') \geq k$ . Let  $s_1, \dots, s_k \in \text{Div}(\Delta)$  be the first  $k$  elements of the right greedy normal form of  $g$ . Then  $g = g_1h$  where  $h = s_k \cdots s_1$ . We have  $g_0 = g_1h\Delta^{-k}\Phi^k(g') = g_1\text{com}_L(h)^{-1}\Phi^k(g')$ . Further  $g_1 \wedge_R \text{com}_L(h) = 1$  because  $h = g \wedge_R \Delta^k$ . Therefore  $\text{com}_L(h) \leq_L \Phi^k(g')$ . So, by Lemma 2.4,  $\text{com}_R(h) = \Phi^{-k}(\text{com}_L(h)) \leq_L g'$ . Hence, there exists  $g'_1 \in M$  such that  $g' = \text{com}_R(h)g'_1$ .  $\square$

### 2.3 Artin groups

Let  $S$  be a finite set. A *Coxeter matrix* over  $S$  is a square matrix  $M = (m_{s,t})_{s,t \in S}$ , indexed by the elements of  $S$ , with coefficients in  $\mathbb{N} \cup \{\infty\}$ , and satisfying:

- (a)  $m_{s,s} = 1$  for all  $s \in S$ ;
- (b)  $m_{s,t} = m_{t,s} \geq 2$  for all  $s, t \in S$ ,  $s \neq t$ .

A Coxeter matrix  $M$  as above is usually represented by a labeled graph,  $\Gamma$ , called *Coxeter graph*. This is defined as follows:

- (a) The set of vertices of  $\Gamma$  is  $S$ .
- (b) Two vertices  $s, t \in S$  are connected by an edge if  $m_{s,t} \geq 3$ . This edge is labeled by  $m_{s,t}$  if  $m_{s,t} \geq 4$ .

If  $a, b$  are two letters and  $m$  is an integer  $\geq 2$ , we set:

$$\text{prod}(a, b : m) = \begin{cases} (ab)^{\frac{m}{2}} & \text{if } m \text{ is even,} \\ (ab)^{\frac{m-1}{2}} a & \text{if } m \text{ is odd.} \end{cases}$$

In other words,  $\text{prod}(a, b : m)$  denotes the word  $aba \cdots$  of length  $m$ . Let  $\Gamma$  be a Coxeter graph as above. The *Artin group* associated with  $\Gamma$  is the group  $A = A_\Gamma$  defined by the presentation:

$$A = \langle S \mid \text{prod}(s, t : m_{s,t}) = \text{prod}(t, s : m_{s,t}) \text{ for all } s, t \in S, s \neq t \text{ and } m_{s,t} \neq \infty \rangle.$$

The monoid  $A^+ = A_\Gamma^+$  having the monoid presentation:

$$A^+ = \langle S \mid \text{prod}(s, t : m_{s,t}) = \text{prod}(t, s : m_{s,t}) \text{ for all } s, t \in S, s \neq t \text{ and } m_{s,t} \neq \infty \rangle^+$$

is called the *Artin monoid* associated with  $\Gamma$ . By Paris [24], the natural homomorphism  $A^+ \rightarrow A$  is injective. The *Coxeter group* associated with  $\Gamma$ , denoted by  $W = W_\Gamma$ , is the quotient of  $A = A_\Gamma$  by the relations  $s^2 = 1, s \in S$ .

The *reversed element* of  $g \in A^+$ , denoted by  $\text{rev}(g)$ , is the element represented by a word  $x_{i_k} \cdots x_{i_1} \in S^*$ , where  $x_{i_1} \cdots x_{i_k}$  is a word representative of  $g$ . Note that  $\text{rev}(g)$  is well defined because relations of Artin monoids are symmetrical. Further,  $\text{rev}^2(g) = g$  for all  $g \in M$ .

The Artin groups were introduced by Tits [28] as extensions of Coxeter groups. There is an extensive literature on these groups, but most of the results concern only special classes. One of the most popular classes is the one of spherical type Artin groups, which is the class that concerns the present paper. We say that a Coxeter graph  $\Gamma$  (or an Artin group  $A = A_\Gamma$ ) is of *spherical type* if the Coxeter group  $W = W_\Gamma$  is finite. A classification of these groups is given in the following theorem:

**Theorem 2.10** (Coxeter [4]) (1) *A Coxeter graph is of spherical type if and only if all its connected components are of spherical type.*

- (2) *A connected Coxeter graph is of spherical type if and only if it is isomorphic to one of the graphs depicted in Figure 2.1.*

Note that the braid group  $\mathcal{B}_{n+1}$  is the Artin group of type  $A_n$ .

Let  $\Gamma$  be a Coxeter graph, let  $S$  be its set of vertices, let  $A$  be its associated Artin group, and let  $A^+$  be its associated Artin monoid. For  $X \subset S$ , we denote by  $\Gamma_X$  the

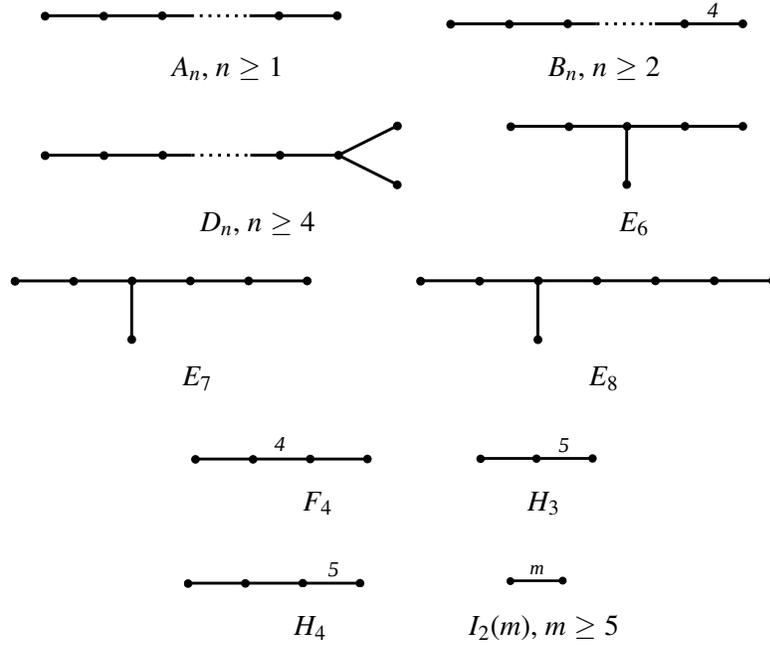


Figure 2.1: Spherical type Coxeter graphs.

full subgraph of  $\Gamma$  spanned by  $X$ , by  $A_X$  the subgroup of  $A$  generated by  $X$ , and by  $A_X^+$  the submonoid of  $A^+$  generated by  $X$ . By Van der Lek [22], the group  $A_X$  (resp. the monoid  $A_X^+$ ) is the Artin group (resp. Artin monoid) associated with  $\Gamma_X$ . Now, Artin groups and Garside groups are related by the following:

**Theorem 2.11** (Brieskorn–Saito [2], Deligne [14]) *Assume that  $\Gamma$  is of spherical type.*

- (1) *The monoid  $A^+$  is a Garside monoid,  $\Delta = \vee_L(S) = \vee_R(S)$  is a Garside element of  $A^+$ , and  $A$  is the enveloping group of  $A^+$ .*
- (2) *Let  $X$  be a subset of  $S$ , and let  $\Delta_X = \vee_L(X)$ . Then  $\Delta_X = \vee_R(X)$ ,  $\Delta_X$  is balanced,  $A_X$  is a parabolic subgroup associated with  $\Delta_X$ , and  $A_X^+ = A^+ \cap A_X$ . In particular,  $A_X = A_{\Delta_X}$ .*
- (3) *We have  $|\Phi| \in \{1, 2\}$ , and  $Z(A)$  is the infinite cyclic group generated by  $\Delta^{|\Phi|}$ . In particular,  $\Delta^2$  is central. Furthermore  $\text{rev}(\Delta) = \Delta$ .*

In particular, by Garside [18] and Brieskorn–Saito [2],  $\mathcal{B}_{n+1}$  is a Garside group with

the following Garside element:

$$\Omega = (x_n \cdots x_2 x_1) \underbrace{(x_n \cdots x_2) \cdots (x_n x_{n-1}) x_n}_{x_2 \vee_R \cdots \vee_R x_n}.$$

## 2.4 Alternating forms and the Dehornoy order

In this subsection we recall some definitions and results on alternating forms from Dehornoy [10] and Fromentin–Paris [17], and how one can define the Dehornoy order from this alternating forms. This is the point of view that we intend to extend in the following sections.

**Proposition 2.12** (Dehornoy [10]) *Let  $M$  be a Garside monoid, let  $N$  be a parabolic submonoid, and let  $g$  be an element of  $M$ . Then there exists a unique element  $R_N(g)$  in  $N$  such that  $\{h \in N \mid h \leq_R g\} = \{h \in N \mid h \leq_R R_N(g)\}$ .*

The element  $R_N(g)$  is called the  $N$ -tale of  $g$ .

Now, we suppose given a Garside monoid  $M$  and two parabolic submonoids  $N_1$  and  $N_2$  such that  $N_1 \cup N_2$  generates  $M$ . Then every non-trivial element  $g \in M$  can be uniquely written in the form  $g = g_p \cdots g_2 g_1$ , where:

$$g_i = \begin{cases} R_{N_1}(g_p \cdots g_i) & \text{if } i \text{ is odd,} \\ R_{N_2}(g_p \cdots g_i) & \text{if } i \text{ is even,} \end{cases}$$

and  $g_p \neq 1$ . This form is called the *alternating form* of  $g$  (with respect to  $(N_2, N_1)$ ). Note that we may have  $g_1 = 1$ , but we have  $g_i \neq 1$  for all  $i \in \{2, \dots, p\}$ . The number  $p$  is called the  $(N_2, N_1)$ -*breadth* of  $g$  and is denoted by  $p = \text{bh}(g)$  (or by  $p = \text{bh}_{N_2, N_1}(g)$ ) if one needs to specify the submonoids  $N_1$  and  $N_2$ ).

Assume that  $M = \mathcal{B}_{n+1}^+ = A_{A_n}^+$  is the positive braid monoid on  $(n+1)$  strands, and that  $S = \{x_1, \dots, x_n\}$  is the standard generating set of  $M$ . Let  $N_1$  be the submonoid of  $M$  generated by  $\{x_2, \dots, x_n\}$ , and let  $N_2$  be the submonoid generated by  $\{x_1, \dots, x_{n-1}\}$ . Note that, by Theorem 2.11,  $N_1$  and  $N_2$  are both parabolic submonoids of  $M$ , and  $N_1 \simeq \mathcal{B}_n^+ \simeq N_2$ . On the other hand, it is easily seen that  $N_1 \cup N_2$  generates  $M$ . In this context we have the following results:

**Theorem 2.13** (Fromentin–Paris [17]) *Let  $g \in \mathcal{B}_{n+1}^+$ , and let  $k$  be a positive integer. Then  $\Omega^{-k}g$  is  $x_1$ -negative if and only if  $k \geq \max\{1, \text{bh}(g) - 1\}$ . In particular, if  $g \in \mathcal{B}_{n+1}$ , then  $g$  is  $x_1$ -negative if and only if  $\text{Ninf}(g) \geq \max\{1, \text{bh}(g_L) - 1\}$ .*

**Theorem 2.14** (Dehornoy [10], Burckel [3]) *Let  $g, g' \in M$ , and let  $(g_p, \dots, g_1), (g'_q, \dots, g'_1)$  be their respective alternating forms. We have  $g <_D g'$ , if and only if, either  $p < q$ , or  $p = q$  and  $\Phi^{k-1}(g_k) <_D \Phi^{k-1}(g'_k)$ , where  $k = \min\{i \mid g_i \neq g'_i\}$ .*

### 3 Orders on Garside groups

Let  $M$  be a Garside monoid, let  $G(M)$  be the enveloping group of  $M$ , and let  $\Delta$  be a Garside element. We assume that the flip automorphism  $\Phi$  is trivial, that is,  $\Delta$  is central. We take two balanced elements  $\Delta_1, \Lambda \in \text{Div}(\Delta) \setminus \{1, \Delta\}$ , and we assume that  $G_1 = G_{\Delta_1}$  is a parabolic subgroup associated with  $\Delta_1$ ,  $H = G_{\Lambda}$  is a parabolic subgroup associated with  $\Lambda$ , and  $\Delta_1$  is central in  $G_1$ . We set  $M_1 = G_1 \cap M$  and  $N = H \cap M$ , and we assume that  $M_1 \cup N$  generates  $M$ . Such a data will be called a *Dehornoy pre-structure*. For convenience, we will often say that the pair  $(H, G_1)$  is the Dehornoy pre-structure, without specifying any other data such as  $\Delta_1$  and  $\Lambda$ .

We will consider alternating forms in  $M$  with respect to  $(N, M_1)$ . Note also that  $M_1$  will play a different role in our study from that of  $N$ .

An element  $g \in M$  is called *unmovable* if  $\Delta \not\leq_R g$ . Note that  $g$  is unmovable if and only if  $\Delta \not\leq_L g$ . Moreover, by Proposition 2.2, for all  $g \in G$ , there exists a unique  $d \in \mathbb{Z}$  and a unique unmovable element  $g_0 \in M$  such that  $g = g_0 \Delta^d$ . In that case we also have  $g = \Delta^d g_0$ , since  $\Delta$  is central. Let  $g \in M$  be an unmovable element. The *depth* of  $g$ , denoted by  $\text{dpt}(g)$ , is defined by

$$\text{dpt}(g) = \begin{cases} \frac{\text{bh}(g)-1}{2} & \text{if } \text{bh}(g) \text{ is odd,} \\ \frac{\text{bh}(g)}{2} & \text{if } \text{bh}(g) \text{ is even.} \end{cases}$$

In other words, if  $g = g_p \cdots g_2 g_1$  is the alternating form of  $g$ , then  $\text{dpt}(g) = |\{i \in \{1, \dots, p\} \mid g_i \in N\}|$ . Now, take any  $g \in G$ , and write  $g = \Delta^d g_0$ , where  $d \in \mathbb{Z}$  and  $g_0 \in M$  is unmovable. Then the *depth* of  $g$  is defined to be  $\text{dpt}(g) = \text{dpt}(g_0)$ .

Set  $\theta = \Delta_1^{-1} \Delta = \Delta \Delta_1^{-1}$ , and, for  $k \geq 0$ , set  $q_k = \text{dpt}(\theta^k)$ . We say that an element  $g \in G$  is  $(H, G_1)$ -*positive* if  $g \notin G_1$  and  $\text{dpt}(g) \geq q_k$ , where  $k = \text{Ninf}(g)$ . We denote by  $P = P_{H, G_1}$  the set of  $(H, G_1)$ -positive elements.

Note that  $M \setminus G_1 \subset P$  because  $\text{Ninf}(g) = 0$  for all  $g \in M$ . Hence, every non  $(H, G_1)$ -positive element has a positive negative-infimum.

We say that  $(H, G_1)$  is a *Dehornoy structure* if  $P$  satisfies the following conditions:

- (a)  $P^2 \subset P$ .
- (b)  $G_1 P G_1 \subset P$ .
- (c) We have the disjoint union  $G = P \sqcup P^{-1} \sqcup G_1$ .

Our goal in this section is to prove a condition on  $(H, G_1)$  that implies that it is a Dehornoy structure (see Theorem 3.2). We will apply in the next sections this criterion

to the Artin groups of type  $A$ ,  $B$ , and  $I_2(m)$  ( $m \geq 5$ ). But, firstly, we show how left orders appear in this way.

Assume given two sequences of parabolic subgroups  $G_0 = G, G_1, \dots, G_n$  and  $H_1, \dots, H_n$  such that  $G_i, H_i \subset G_{i-1}$  and  $(H_i, G_i)$  is a Dehornoy structure of  $G_{i-1}$ , for all  $i \in \{1, \dots, n\}$ , and  $G_n \simeq \mathbb{Z}$ . For  $i \in \{1, \dots, n\}$ , we denote by  $P_i$  the set of  $(H_i, G_i)$ -positive elements in  $G_{i-1}$ . Furthermore, we choose a generator  $g_n$  of  $G_n$  and we set  $P_{n+1} = \{g_n^k \mid k \geq 1\}$ . For  $i \in \{1, \dots, n+1\}$  we set  $P_i^{-1} = \{g^{-1} \mid g \in P_i\}$ , and for  $\epsilon = (\epsilon_1, \dots, \epsilon_n, \epsilon_{n+1}) \in \{\pm 1\}^{n+1}$  we set:

$$P^\epsilon = P_1^{\epsilon_1} \cup \dots \cup P_n^{\epsilon_n} \cup P_{n+1}^{\epsilon_{n+1}}.$$

**Proposition 3.1** *Under the above hypothesis, the set  $P^\epsilon$  is the positive cone for a left order on  $G$ .*

**Proof** Let  $g, g' \in P^\epsilon$ . Then  $g \in P_i^{\epsilon_i}$  and  $g' \in P_j^{\epsilon_j}$  for some  $i, j \in \{1, \dots, n+1\}$ . If  $i = j$ , (a) implies that  $gg' \in P_i^{\epsilon_i} \subset P^\epsilon$ . If  $i \neq j$ , without loss of generality we suppose that  $i < j$ . Hence  $g' \in G_{i+1}$ . By (b),  $gg', g'g \in P_i^{\epsilon_i} \subset P^\epsilon$ . Therefore  $P^\epsilon$  is a subsemigroup of  $G$ . Since  $1 \in G_n$ , (c) implies that  $1 \notin P_i^{\pm 1}$  for all  $i \in \{1, \dots, n\}$ . Further  $1 \notin P_{n+1}^{\pm 1}$  because  $g_n \neq 1$  and  $G_n \simeq \mathbb{Z}$ . Hence  $G = (P^\epsilon \cup P^{-\epsilon}) \sqcup \{1\}$ . Since  $P_i \subset G_{i-1}$  for all  $i \in \{1, \dots, n+1\}$ , then, by (c),  $(P_i \cup P_i^{-1}) \cap (P_j \cup P_j^{-1}) = \emptyset$  for all  $i \neq j$ . Finally, (c) implies that  $P_i \cap P_i^{-1} = \emptyset$ . Thus  $G = P^\epsilon \sqcup (P^\epsilon)^{-1} \sqcup \{1\}$ . Therefore  $P^\epsilon$  is the positive cone for a left order on  $G$ .  $\square$

Let  $(H, G_1)$  be a Dehornoy pre-structure, with the above notations. We say that  $(H, G_1)$  satisfies *Condition A* if there exists a constant  $c \in \mathbb{N}$ ,  $c \geq 1$ , such that  $q_k = kc + 1$  for all  $k \geq 1$ .

Note that  $q_0 = 0$ , so the equality  $q_k = kc + 1$  is never true for  $k = 0$ . Since in our study we will need sometimes to replace 0 by  $1 = 0c + 1$  when  $k = 0$ , we may use the sequence  $\tilde{q}_k$  defined as follows in place of  $q_k$ :

$$\tilde{q}_k = \begin{cases} 1 & \text{if } k = 0, \\ q_k & \text{if } k \geq 1. \end{cases}$$

We say that an element  $g \in M$  is a *theta element* if there exist  $g_0 \in M_1$  and  $k \geq 1$  such that  $g = \theta^k g_0$ , where  $\theta = \Delta_1^{-1} \Delta = \Delta \Delta_1^{-1}$ . The set of theta elements is denoted by  $\Theta = \Theta(G)$ .

Let  $g, g' \in M$  be unmovable elements such that  $\{g, g'\} \not\subset M_1 \cup \Theta$ . We say that the pair  $(g, g')$  satisfies *Condition B* if there exists  $\epsilon \in \{0, 1\}$  such that:

- (a)  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k \geq 0$ , where  $k = \text{inf}(gg')$ ,
- (b)  $\epsilon = 0$  if  $\{g, g'\} \cap \Theta \neq \emptyset$ .

We say that  $(H, G_1)$  satisfies *Condition B* if, for all  $g, g' \in M$  unmovable such that  $\{g, g'\} \not\subset M_1 \cup \Theta$ , the pair  $(g, g')$  satisfies Condition B.

The purpose of this section is to prove the following:

**Theorem 3.2** *Let  $(H, G_1)$  be a Dehornoy pre-structure. If  $(H, G_1)$  satisfies Conditions A and B, then  $(H, G_1)$  is a Dehornoy structure.*

The rest of the section is dedicated to the proof of Theorem 3.2.

**Lemma 3.3** *We have  $\theta g = g\theta$  for all  $g \in G_1$ . Hence  $\Delta^k = \theta^k \Delta_1^k$  for all  $k \geq 1$ . Furthermore  $g \not\leq_L \theta$  and  $g \not\leq_R \theta$  for all  $g \in M_1 \setminus \{1\}$ .*

**Proof** Let  $g \in G_1$ . Then  $g = \Delta g \Delta^{-1} = \theta g \theta^{-1}$  because  $\Delta$  is central in  $G$ , and  $\Delta_1$  is central in  $G_1$ . Hence  $\theta g = g\theta$ . Let  $g \in M_1$  be a right divisor of  $\theta$ . Then  $g\Delta_1$  lies in  $\text{Div}(\Delta) \cap M_1 = \text{Div}(\Delta_1)$ , therefore  $g = 1$ . Similarly we prove that  $g = 1$  whenever it left divides  $\theta$ .  $\square$

**Corollary 3.4** *Let  $g = \theta^k g_0$  be a theta element. Then  $R_{M_1}(g) = g_0$ . Furthermore, if  $g$  is unmovable, then  $\text{dpt}(g) = q_k$ .*

**Proof** Lemma 3.3 implies that  $g_1 \not\leq_R \theta^k$  for all  $g_1 \in M_1$ , thus  $R_{M_1}(g) = g_0$ . Hence  $\text{bh}(g) = \text{bh}(\theta^k)$ . Therefore  $\text{dpt}(g) = q_k$  if  $g$  is unmovable.  $\square$

In what follows of this section we assume that Condition A holds.

**Lemma 3.5** *Let  $g = \theta^k g_0$  be an unmovable theta element of  $G$ , and let  $p = \ell(g_0)$ . Then  $\ell(g) = \max\{k, p\}$ ,  $\text{com}_L(g) = \theta^{\ell(g)-k} \Delta_1^{\ell(g)-p} g_1$ , and:*

$$\text{dpt}(g) + \text{dpt}(\text{com}_L(g)) = \begin{cases} q_k & \text{if } p \leq k \\ q_p + 1 & \text{if } p > k \end{cases}$$

where  $g_1 = \Delta_1^p g_0^{-1}$  is the left complement of  $g_0$  with respect to  $\Delta_1$ .

**Proof** Let  $s_p \cdots s_1$  be the right greedy normal form of  $g_0$ . Note that  $s_i \neq \Delta$  for all  $i$  because  $g$  is unmovable. If  $p \leq k$ , Lemma 3.3 implies that  $g = \theta \cdots \theta \theta s_p \cdots \theta s_1$ , where  $\theta \cdots \theta = \theta^{k-p}$ , is the right greedy normal form of  $g$ . Hence  $\ell(g) = k$ . Therefore  $\text{com}_L(g) = \Delta_1^{k-p} g_1$ , and  $\text{dpt}(g) + \text{dpt}(\text{com}_L(g)) = q_k$ . If  $p > k$ , Lemma 3.3 implies that  $g = s_p \cdots s_{k+1} \theta s_k \cdots \theta s_1$  is the right greedy normal form of  $g$ . Hence  $\ell(g) = p$ . Therefore  $\text{com}_L(g) = \theta^{p-k} g_1$ , and  $\text{dpt}(g) + \text{dpt}(\text{com}_L(g)) = q_k + q_{p-k} = q_p + 1$ .  $\square$

**Corollary 3.6** *Let  $g = \theta^k g_0$  be an unmovable theta element of  $G$ , and let  $t > 0$  be an integer. Then:*

$$\begin{cases} \Delta^{-t}g \in P \text{ and } (\Delta^{-t}g)^{-1} \notin P & \text{if } t < k \\ \Delta^{-t}g \in G_1 & \text{if } t = k \\ \Delta^{-t}g \notin P \text{ and } (\Delta^{-t}g)^{-1} \in P & \text{if } t > k \end{cases}$$

Furthermore  $\Delta^{t'}g \in P$  and  $(\Delta^{t'}g)^{-1} \notin P$  for all integer  $t' \geq 0$ .

**Proof** We have  $(\Delta^{-t}g)^{-1} = \Delta^t g^{-1} = \Delta^{t-d} \text{com}_L(g)$ , where  $d = \ell(g)$ . If  $t < k$ , then  $\text{dpt}(\theta^k g_0) = q_k > q_t$ . Hence  $\Delta^{-t}g \in P$ . Lemma 3.5 implies that  $d \geq k > t$ , and that  $\text{dpt}(\text{com}_L(g)) \leq q_d + 1 - \text{dpt}(g) = q_d - kc = q_{d-k} < q_{d-t}$ . Thus  $(\Delta^{-t}g)^{-1} \notin P$ . If  $t = k$ , then  $\Delta^{-t}g = \Delta_1^{-t}g_0 \in G_1$ . If  $t > k$ , then  $\text{dpt}(\theta^k g_0) = q_k < q_t$ . Therefore  $\Delta^{-t}g \notin P$ . If  $t \geq d$ , then  $(\Delta^{-t}g)^{-1} \in P$ . Otherwise, Lemma 3.5 implies that  $\text{dpt}(\text{com}_L(g)) \geq q_d - \text{dpt}(g) = (d-k)c \geq q_{d-t}$ . Hence  $(\Delta^{-t}g)^{-1} \in P$ . We have  $(\Delta^{t'}g)^{-1} = \Delta^{-t'-d} \text{com}_L(g)$ . Lemma 3.5 implies that  $\text{dpt}(\text{com}_L(g)) \leq q_d + 1 - \text{dpt}(g) = q_d - kc = q_{d-k} < q_{d+t'}$ . Therefore  $(\Delta^{t'}g)^{-1} \notin P$ .  $\square$

**Lemma 3.7** *Suppose that  $\text{dpt}(g) + \text{dpt}(\text{com}_L(g)) = q_k$  for all unmovable element  $g \in M \setminus \Theta$ , where  $k = \ell(g)$ . Then  $G = P \sqcup P^{-1} \sqcup G_1$ .*

**Proof** By definition, no  $(H, G_1)$ -positive element belongs to  $G_1$ , hence  $(P \cup P^{-1}) \cap G_1 = \emptyset$ . We shall prove that  $P^{-1} = (G \setminus G_1) \setminus P$ . Let  $g^{-1} \in P^{-1}$  for some  $g = \Delta^d g_0 \in P$ , where  $d = \inf(g)$ , and let  $k = \ell(g_0)$ . Suppose that  $g_0 = \theta^t g_1 \in \Theta$  for some  $g_1 \in M_1$ . If  $d \geq 0$ , Corollary 3.6 implies that  $g^{-1} \notin P$ . If  $d < 0$ , then  $t > -d$  because  $g$  is  $(H, G_1)$ -positive, hence  $\text{dpt}(g) = q_t > q_{-d}$ . Then, by Corollary 3.6,  $g^{-1} \notin P$ . Suppose that  $g_0 \in M_1$ . Then  $d > 0$  because  $g$  is  $(H, G_1)$ -positive. Hence  $g^{-1} = \Delta^{-d-k} \text{com}_L(g_0)$ . By hypothesis, we have  $\text{dpt}(\text{com}_L(g_0)) = q_k < q_{d+k}$ . Therefore  $g^{-1} \notin P$ . Suppose that  $g_0 \in M \setminus (\Theta \cup M_1)$ . In particular  $\text{dpt}(g_0) \geq 1$ . If  $d \geq 0$ , then  $g^{-1} = \Delta^{-d-k} \text{com}_L(g_0)$ . Hence, by hypothesis,  $\text{dpt}(\text{com}_L(g_0)) = q_k - \text{dpt}(g_0) < q_k \leq q_{d+k}$ . In particular, we have proved that  $M^{-1} \cap P = \emptyset$ . If  $d < 0$ , i.e.  $d = -t$  for some  $t \geq 1$ , then  $g^{-1} = \Delta^{t-k} \text{com}_L(g_0)$ . We have  $t < k$ . Otherwise  $g^{-1} \in M$ , and so  $g \notin P$ , which is a contradiction. By hypothesis, we have  $\text{dpt}(\text{com}_L(g_0)) = q_k - \text{dpt}(g_0) \leq q_k - q_t < q_{k-t}$ . Therefore  $P^{-1} \subseteq (G \setminus G_1) \setminus P$ . Let  $g = \Delta^{-t}g_0 \in (G \setminus G_1) \setminus P$  with  $\inf(g) = -t$ , and  $\text{dpt}(g_0) < q_t$ . Note that  $\inf(g) < 0$  for all  $g \in (G \setminus G_1) \setminus P$ , because  $M \subseteq P$ . Suppose that  $g_0 = \theta^d g_1 \in \Theta$  for some  $g_1 \in M_1$ . Hence,  $d < t$  because  $q_d = \text{dpt}(g_0) < q_t$ . Corollary 3.6 implies that  $g^{-1} \in P$ . Suppose that  $g_0 \in M \setminus \Theta$ . We have  $g^{-1} = \Delta^{t-k} \text{com}_L(g_0)$ . If  $t \geq k$ , then  $g^{-1} \in M \subseteq P$ . If  $t < k$ , then, by hypothesis,  $\text{dpt}(\text{com}_L(g_0)) = q_k - \text{dpt}(g_0) \geq q_k - q_t + 1 = q_{k-t}$ . Therefore  $(G \setminus G_1) \setminus P \subseteq P^{-1}$ .  $\square$

**Lemma 3.8** *Condition B implies that  $\text{dpt}(g) + \text{dpt}(\text{com}_L(g)) = q_k$  for all unmovable elements  $g \in M \setminus \Theta$ ,  $g \neq 1$ , where  $k = \ell(g)$ .*

**Proof** Let  $g \in M \setminus \Theta$  be an unmovable element, and let  $k = \ell(g)$ . Note that  $\text{com}_L(g) \notin M_1$  because  $g \notin \Theta$ . If  $g \in M_1$ , then  $\text{com}_L(g) = \theta^k(\Delta_1^k g^{-1})$ . Therefore  $\text{dpt}(g) + \text{dpt}(\text{com}_L(g)) = q_k$ . If  $g \in M \setminus M_1$ , Lemma 3.5 implies that  $\text{com}_L(g) \notin \Theta$ . So, by Condition B, we have  $0 = \text{dpt}(\Delta^k) = \text{dpt}(\text{com}_L(g)g) \geq \text{dpt}(\text{com}_L(g)) + \text{dpt}(g) - \tilde{q}_k \geq 0$ . Therefore  $\text{dpt}(\text{com}_L(g)) + \text{dpt}(g) = \tilde{q}_k = q_k$ , because  $k \geq 1$ .  $\square$

**Lemma 3.9** *Let  $g \in G_1 \setminus M_1$ . Then  $g_L \in \Theta$  and  $\text{dpt}(g) = q_k$ , where  $k = \text{Ninf}(g)$ .*

**Proof** We have  $g = \Delta_1^{-k} g_0$  for some  $k \geq 1$  and  $g_0 \in M_1$  such that  $\Delta_1 \not\leq_R g_0$ . Note that  $g = \Delta_1^{-k} \theta^{-k} \theta^k g_0 = \Delta^{-k} \theta^k g_0$ . If  $\Delta \leq_L \theta^k g_0$ , Corollary 3.4 implies that  $\Delta_1 \leq_L g_0$ , which is a contradiction. Therefore  $g_L = \theta^k g_0 \in \Theta$  and  $\text{inf}(g) = -k$ . By Corollary 3.4,  $\text{bh}(g_L) = \text{bh}(\theta^k)$ . Hence  $\text{dpt}(g) = q_k$ .  $\square$

Lemmas 3.7, 3.8 and 3.9 imply that, if Condition B holds, then:

$$P^{-1} = \{g \in G \mid \text{dpt}(g) < q_k \text{ where } k = \text{Ninf}(g)\}.$$

**Lemma 3.10** *Condition B implies that  $P^{-1}$  is a subsemigroup of  $G$ .*

**Proof** Let  $g, g' \in P^{-1}$ , that is,  $g = \Delta^{-t} g_0$  and  $g' = \Delta^{-t'} g'_0$ , where  $t = \text{Ninf}(g)$  and  $t' = \text{Ninf}(g')$ , such that  $\text{dpt}(g_0) < q_t$  and  $\text{dpt}(g'_0) < q_{t'}$ . We have  $gg' = \Delta^{-t-t'+k} h$ , where  $k = \text{inf}(g_0 g'_0)$ , and  $h = (g_0 g'_0)_L$ , i.e.  $h = \Delta^{-k} g_0 g'_0$ . If  $g_0, g'_0 \in M_1$ , then  $k = 0$ , and  $\text{dpt}(h) < q_{t+t'}$ . Hence  $gg' \in P^{-1}$ . If  $g_0, g'_0 \in \Theta$ , that is,  $g_0 = \theta^d g_1$  and  $g'_0 = \theta^{d'} g'_1$ . Then  $g_0 g'_0 = \theta^{d+d'} g_1 g'_1$  and  $h = \theta^{d+d'-k} h_0$  for some  $h_0 \in M_1$ . Notice that  $d < t$  and  $d' < t'$  because  $g, g' \in P^{-1}$ , therefore  $k \leq d + d' < t + t'$ . We have  $\text{dpt}(h) = q_{d+d'-k} \leq \tilde{q}_{d+d'-k} = \tilde{q}_d + \tilde{q}_{d'} - \tilde{q}_k < \tilde{q}_t + \tilde{q}_{t'} - \tilde{q}_k = \tilde{q}_{t+t'-k} = q_{t+t'-k}$ . Thus  $gg' \in P^{-1}$ . If  $g_0 = \theta^d g_1 \in \Theta$  and  $g'_0 \in M_1$ , then  $g_0 g'_0 = \theta^d g_1 g'_0 \in \Theta$  and  $h = \theta^{d-k} h_0$  for some  $h_0 \in M_1$ . Note that  $d < t$  because  $g \in P^{-1}$ , therefore  $k \leq d < t$ . We have  $\text{dpt}(h) = q_{d-k} < q_{t-k} \leq q_{t+t'-k}$ . Thus  $gg' \in P^{-1}$ . Similarly, we prove that  $gg' \in P^{-1}$  whenever  $g_0 \in M_1$  and  $g'_0 \in \Theta$ . If  $\{g_0, g'_0\} \not\subseteq M_1 \cup \Theta$ , Condition B implies that  $\tilde{q}_k \leq \text{dpt}(g_0) + \text{dpt}(g'_0) \leq q_t + q_{t'} - 2 = q_{t+t'} - 1 < q_{t+t'}$ , therefore  $k < t + t'$ . By Condition B, we have  $\text{dpt}(h) \leq \text{dpt}(g_0) + \text{dpt}(g'_0) - \tilde{q}_k + 1 \leq q_t + q_{t'} - \tilde{q}_k - 1 = q_{t+t'-k} - 1 < q_{t+t'-k}$ . Thus  $gg' \in P^{-1}$ . Therefore  $P^{-1}$  is a subsemigroup of  $G$ .  $\square$

Lemma 3.10 implies that, if Condition B holds, then  $P$  is a semigroup.

**Lemma 3.11** *Condition B implies that  $G_1P^{-1}G_1 \subseteq P^{-1}$ .*

**Proof** Let  $g = \Delta^{-t}g_0 \in P^{-1}$ , and let  $g' \in G_1$ . If  $g' \in M_1$ , then  $gg' = \Delta^{-t}g_0g' = \Delta^{-t+k}h$ , where  $k = \inf(g_0g')$  and  $h = \Delta^{-k}g_0g'$ . If  $g_0 \in M_1$ , then  $k = 0$ . Hence  $gg' \in P^{-1}$  because  $\text{dpt}(h) = 0 < q_t$ . If  $g_0 = \theta^p g_1 \in \Theta$ , then  $h = \theta^{p-k}h'$ . Note that  $k \leq p < t$  because  $g$  is  $(H, G_1)$ -negative. Then  $\text{dpt}(h) = q_{p-k} \leq \tilde{q}_{p-k} < \tilde{q}_{t-k}$ . Hence  $gg' \in P^{-1}$ . If  $g_0 \in M \setminus (M_1 \cup \Theta)$ , Condition B implies that  $\tilde{q}_k \leq \text{dpt}(g_0) + \text{dpt}(g') < q_t$ . Hence  $k < t$ . Further, by Condition B, we have  $\text{dpt}(h) \leq \text{dpt}(g_0) + \text{dpt}(g') - \tilde{q}_k + 1 \leq q_t - \tilde{q}_k < q_{t-k}$ . Therefore  $gg' \in P^{-1}$ . Similarly, we can prove that  $g'g \in P^{-1}$ . If  $g' \in G_1 \setminus M_1$ , Lemma 3.9 implies that  $g' = \Delta_1^{-t'}g'_0 = \Delta^{-t'}\theta^{t'}g'_0$ , where  $\inf(g') = -t'$ . Hence  $gg' = \Delta^{-t-t'}g_0\theta^{t'}g'_0 = \Delta^{-t-t'+k}h$ , where  $k = \inf(g_0\theta^{t'}g'_0)$  and  $h = \Delta^{-k}g_0\theta^{t'}g'_0$ . If  $g_0 \in M_1$ , then  $h = \theta^{t'-k}h'$  for some  $h' \in M_1$ . Note that  $k \leq t' < t + t'$ . Further  $\text{dpt}(h) = q_{t'-k} \leq \tilde{q}_{t'-k} < q_{t+t'-k}$ . Hence  $gg' \in P^{-1}$ . If  $g_0 = \theta^p g_1 \in \Theta$ , then  $h = \theta^{p+t'-k}h'$  for some  $h' \in M_1$ . Note that  $p < t$  because  $g$  is  $(H, G_1)$ -negative. Hence  $k \leq p + t' < t + t'$ . Further  $\text{dpt}(h) = q_{p+t'-k} \leq \tilde{q}_{p+t'-k} < q_{t+t'-k}$ . Therefore  $gg' \in P^{-1}$ . If  $g_0 \in M \setminus (M_1 \cup \Theta)$ , Condition B implies that  $\tilde{q}_k \leq \text{dpt}(g_0) + q_{t'} \leq q_t + q_{t'} - 1 = q_{t+t'}$ . Hence  $k \leq t + t'$ . If  $k = t + t'$ , then  $gg' = h \in M$ . If  $h \in M \setminus M_1$ , Lemma 3.10 implies that  $g' = g^{-1}h \in P$  since  $g^{-1} \in P$ , which is a contradiction because  $g' \in G_1$ . If  $h \in M_1$ , then  $g_0\theta^{t'}g'_0 = \Delta^{t+t'}h = \theta^{t+t'}\Delta_1^{t+t'}h = \theta^{t+t'}g_1g'_0 = \theta^t g_1\theta^{t'}g'_0$  for some  $g_1 \in M_1$ . Then  $g_0 = \theta^t g_1$ , which is a contradiction because  $g_0 \notin \Theta$  and  $g$  is  $(H, G_1)$ -negative. Hence  $k < t + t'$ . By Condition B,  $\text{dpt}(h) \leq \text{dpt}(g_0) + q_{t'} - \tilde{q}_k \leq q_t + q_{t'} - \tilde{q}_k - 1 = q_{t+t'-k} - 1 < q_{t+t'-k}$  because  $g'_L \in \Theta$ . Thus  $gg' \in P^{-1}$ . Similarly, we have  $g'g \in P^{-1}$ . Hence  $P^{-1}G_1 \subseteq P^{-1}$  and  $G_1P^{-1} \subseteq P^{-1}$ . Therefore  $G_1P^{-1}G_1 \subseteq P^{-1}$ .  $\square$

Lemma 3.11 implies that, if Condition B holds, then  $G_1PG_1 \subseteq P$ .

**Remark** Assume that  $(H, G_1)$  is a Dehornoy pre-structure of  $G$ . Let  $g \in G$  with  $\inf(g) < 0$ , and set  $t = -\inf(g) \geq 1$ . Then  $g$  is  $(H, G_1)$ -positive, if and only if,  $\text{bh}(g) \geq 2\tilde{q}_t$ . Indeed, if  $\text{dpt}(g) \geq \tilde{q}_t$ , then  $\text{bh}(g) \geq 2\text{dpt}(g) \geq 2\tilde{q}_t$ . Suppose that  $\text{bh}(g) \geq 2\tilde{q}_t$ . If  $\text{bh}(g)$  is even, then  $\text{dpt}(g) \geq \tilde{q}_t$ . If  $\text{bh}(g)$  is odd, then  $\text{bh}(g) > 2\tilde{q}_t$ , hence  $\text{dpt}(g) \geq \tilde{q}_t$ .

## 4 Artin groups of type A

In this section  $G$  and  $M$  denote the Artin group and the Artin monoid of type  $A_n$ , respectively, where  $n \geq 2$ . Recall that  $M$  is defined by the monoid presentation with

generating set  $S = \{x_1, \dots, x_n\}$  and relations:

$$\begin{aligned} x_i x_{i+1} x_i &= x_{i+1} x_i x_{i+1} & \text{if } 1 \leq i \leq n-1, \\ x_i x_j &= x_j x_i & \text{if } |i-j| \geq 2. \end{aligned}$$

The group  $G$  is the enveloping group of  $M$ , and it is isomorphic to the braid group  $\mathcal{B}_{n+1}$  on  $n+1$  strands. By Theorem 2.11,  $M$  is a Garside monoid, and  $\Omega = x_1 \vee_R \cdots \vee_R x_n$  is a Garside element of  $M$ .

We have  $\Omega x_i = x_{n+1-i} \Omega$  for all  $i \in \{1, \dots, n\}$ . In particular,  $\Omega$  is not central, but  $\Delta = \Omega^2$  is central. Note that, by Dehornoy [9],  $\Delta$  is a Garside element, and, by Brieskorn–Saito [2],  $\Delta = (x_1 \cdots x_n)^{n+1}$ .

Let  $G_1$  be the subgroup of  $G$  generated by  $\{x_2, \dots, x_n\}$ , and let  $H$  be the subgroup generated by  $\{x_1, \dots, x_{n-1}\}$ . By Theorem 2.11 and Lemma 2.8,  $G_1$  and  $H$  are parabolic subgroups of  $(G, \Delta)$ . The subgroup  $G_1$  is associated with  $\Delta_1 = (x_2 \vee_R \cdots \vee_R x_n)^2 = (x_2 \cdots x_n)^n$ , and  $H$  is associated with  $\Lambda = (x_1 \vee_R \cdots \vee_R x_{n-1})^2 = (x_1 \cdots x_{n-1})^n$ . As ever, we set  $M_1 = G_1 \cap M$  and  $N = H \cap M$ . Obviously,  $M_1 \cup N$  generates  $M$ , hence  $(H, G_1)$  is a Dehornoy pre-structure. Note that  $\Delta$  is decomposed as follows:

$$\Delta = \underbrace{(x_1 x_2 \cdots x_{n-1} x_n^2 x_{n-1} \cdots x_2 x_1)}_{\theta = \Delta \Delta_1^{-1} = \Delta_1^{-1} \Delta} \underbrace{(x_2 \cdots x_{n-1} x_n^2 x_{n-1} \cdots x_2) \cdots (x_{n-1} x_n^2 x_{n-1} x_n^2)}_{\Delta_1 = (x_2 \vee_R \cdots \vee_R x_n)^2}.$$

The main result of this section is the following:

**Theorem 4.1** *The pair  $(H, G_1)$  satisfies Conditions A and B.*

Applying Theorem 3.2 we get:

**Corollary 4.2** *The pair  $(H, G_1)$  is a Dehornoy structure.*

Since  $G_1$  is an Artin group of type  $A_{n-1}$  (see Theorem 2.11), iterating Corollary 4.2 and applying Proposition 3.1 we get:

**Corollary 4.3** *Set:*

$$G_i = \langle x_{i+1}, \dots, x_n \rangle, H_i = \langle x_i, \dots, x_{n-1} \rangle, \Delta_i = (x_{i+1} \cdots x_n)^{n+1-i},$$

for  $i \in \{1, \dots, n-1\}$ .

- (1) *The group  $G_i$  is a Garside group, and  $\Delta_i$  is a Garside element of  $G_i$ , for all  $i \in \{1, \dots, n-1\}$ .*

- (2) The pair  $(H_i, G_i)$  is a Dehornoy structure of  $G_{i-1}$  for all  $i \in \{1, \dots, n-1\}$ , where  $G_0 = G$ .
- (3) For  $i \in \{1, \dots, n-1\}$ , we denote by  $P_i$  the set of  $(H_i, G_i)$ -positive elements of  $G_{i-1}$ . Furthermore, we set  $P_n = \{x_n^k \mid k \geq 1\}$ . Take  $\epsilon = (\epsilon_1, \dots, \epsilon_n) \in \{\pm 1\}^n$ , and set:

$$P^\epsilon = P_1^{\epsilon_1} \sqcup \dots \sqcup P_n^{\epsilon_n}.$$

Then  $P^\epsilon$  is the positive cone of a left order on  $G$ .

Before proving Theorem 4.1, we show that the left order obtained from it coincides with that obtained from Theorem 1.1. More precisely, we prove the following:

Recall that the definition of an  $x_1$ -positive element is given in the introduction.

**Proposition 4.4** *The set  $P = P_{(H, G_1)}$  is the set of  $x_1$ -positive elements of  $G = \mathcal{B}_{n+1}$ .*

**Proof** Let  $g$  be a  $x_1$ -positive element of  $G$ . Then  $g = g_0 x_1^{k_1} g_1 \cdots g_{p-1} x_1^{k_p} g_p$  for some  $k_1, \dots, k_p \geq 1$  and  $g_0, \dots, g_p \in G_1$ . Note that  $x_1^{k_i} \in P$  for all  $i \in \{1, \dots, p\}$ . By Theorem 4.1,  $G_1 P G_1 \subseteq P$ . Therefore  $g \in P$ . Conversely, let  $g \in P$ . If  $g \in M$ , then  $g$  is  $x_1$ -positive. Otherwise  $g = \Delta^{-k} g_0$ , where  $k = \text{Ninf}(g)$ , such that  $\text{dpt}(g_0) \geq q_k = ck + 1 \geq k + 1$ . (Actually, as we will see soon, we have  $q_k = k + 1$ .) Hence  $\text{bh}(g_0) \geq 2\text{dpt}(g_0) \geq 2k + 2$ . Since  $g \notin G_1$  and  $g = \Omega^{-2k} g_0$ , Theorem 2.13 implies that  $g$  is  $x_1$ -positive.  $\square$

The rest of the section is dedicated to the proof of Theorem 4.1.

By Dehornoy [10, Lemma 4.19], the  $(N, M_1)$ -breadth of  $\Delta^k$  is  $2k + 2$ , for all  $k \geq 1$ . Lemma 3.3 implies that  $\text{bh}(\theta^k) = \text{bh}(\Delta^k)$  because  $\Delta_1^k \in M_1$ . Hence  $\tilde{q}_k = k + 1$  for all  $k \geq 0$ . Therefore  $(H, G_1)$  satisfies Condition A.

In what follows we will prove that Condition B holds:

**Lemma 4.5** *Let  $g \in M \setminus M_1$ , and let  $k \geq 1$  be an integer. Then  $\text{bh}(g\theta^k) = \text{bh}(g) + 2k$ .*

**Proof** Let  $(g_p, \dots, g_1)$  be the alternating form of  $g$ . Note that  $gg_1^{-1} \neq 1$  because  $g \notin M_1$ . Hence  $x_1 \leq_R g_2$  and  $x_t \not\leq_R gg_1^{-1}$  for all  $t \geq 2$ . By Lemma 3.3,  $\theta g_1 = g_1 \theta$ . Then  $(g_p, \dots, g_3, g_2 x_1, x_2 \cdots x_{n-1} x_n^2, x_{n-1} \cdots x_1, g_1)$  is an  $(N, M_1)$ -sequence of  $g\theta$ . If  $x_t \leq_R g\theta g_1^{-1} = gg_1^{-1} \theta$  for some  $t \in \{2, \dots, n\}$ , Lemma 3.3 implies that  $x_t \leq_R gg_1^{-1}$  which is a contradiction. Hence  $g_1 = R_{M_1}(g\theta)$ . If  $x_{n-1} \leq_R gg_1^{-1} x_1 \cdots x_{n-1} x_n^2$ , then  $x_n x_{n-1} \leq_R gg_1^{-1} x_1 \cdots x_{n-1} x_n$ . In particular,  $x_{n-1} \leq_R gg_1^{-1} x_1 \cdots x_{n-1} x_n$ . Then  $x_n \leq_R gg_1^{-1}$  which is a contradiction. If  $x_k \leq_R gg_1^{-1} x_1 \cdots x_{n-1} x_n^2$  for some  $k \in \{1, \dots, n-2\}$ , then  $x_{k+1} \leq_R gg_1^{-1}$  which is a contradiction as well. Therefore  $\text{bh}(g\theta) = \text{bh}(g) + 2$ . Inductively, we obtain that  $\text{bh}(g\theta^k) = \text{bh}(g) + 2k$  for all  $k \geq 2$ .  $\square$

In general, if  $M$  is a monoid, and  $N_1, N_2$  are two submonoids of  $M$ . A  $(N_2, N_1)$ -sequence of an element  $g \in M$  is a tuple  $(g_p, \dots, g_1)$  such that,  $g = g_p \cdots g_1$ ,  $g_i \in N_1$  when  $i$  is odd, and  $g_i \in N_2$  when  $i$  is even. If  $M$  is Garside and it allows alternating decompositions with respect to  $(N_2, N_1)$ , we say that  $(N_2, N_1)$ -breadths are *minimal* if for all  $g \in M$ , its  $(N_2, N_1)$ -breadth is minimal in the set of lengths of all  $(N_2, N_1)$ -sequences of  $g$ .

**Proposition 4.6** (Dehornoy [10], Burckel [3])  *$(N, M_1)$ -breadths are minimal.*

In general, if  $(M, \Delta)$  is a Garside monoid,  $(H, G_1)$  is a Dehornoy pre-structure, and  $g \in M$ , then we denote by  $\text{dpt}^*(g)$  the depth of the element  $gR_{M_1}(g)^{-1}$ . More specifically, we have the following result:

**Lemma 4.7** *Let  $(M, \Delta)$  be a Garside monoid, let  $(H, G_1)$  be a Dehornoy pre-structure, and let  $g \in M$ . Then:*

$$\text{dpt}^*(g) = \begin{cases} \frac{\text{bh}(g)-1}{2} & \text{if } \text{bh}(g) \text{ is odd,} \\ \frac{\text{bh}(g)}{2} & \text{if } \text{bh}(g) \text{ is even.} \end{cases}$$

**Proof** Let  $g' = gR_{M_1}(g)^{-1}$ . If  $\Delta \leq_R g'$ , then  $\Delta_1 \leq_R g'$  because  $\Delta_1 \in \text{Div}(\Delta) \setminus \{1\}$ . This is impossible because  $\Delta_1 \in M_1$  and  $h \not\leq_R g'$  for all  $h \in M_1 \setminus \{1\}$ . Therefore  $g'$  is unmovable. Since  $R_{M_1}(g) \in M_1$ , then  $\text{bh}(g) = \text{bh}(g')$ . This implies the equality.  $\square$

We come back to the hypothesis given at the beginning of the section. In particular, now  $G$  denotes the Artin group of type  $A_n$ . Note that  $\text{rev}(g\Delta) = \text{rev}(g)\Delta$  for all  $g \in M$  because  $\text{rev}(\Delta) = \Delta$ . Thus  $\text{inf}(g) = \text{inf}(\text{rev}(g))$ .

**Lemma 4.8** *Let  $g \in M \setminus M_1$ , and let  $k$  be its infimum with respect to  $\Delta$ . Let  $g_0 \in M_1$ , and let  $t \geq 1$  be an integer. Then:*

- (1)  $\text{dpt}^*(g_0\Delta^t) = \text{dpt}^*(g_0\theta^t) = \text{dpt}^*(\theta^t g_0) = \text{dpt}^*(\Delta^t g_0) = \tilde{q}_t$ .
- (2)  $\text{dpt}^*(g) = \text{dpt}(g) + k$ .
- (3)  $\text{dpt}^*(g) = \text{dpt}^*(\text{rev}(g))$ .
- (4)  $\text{dpt}^*(gg_0) = \text{dpt}^*(g_0g) = \text{dpt}^*(g)$ .
- (5)  $\text{dpt}^*(\Delta^t g) = \text{dpt}^*(\theta^t g) = \text{dpt}^*(g\theta^t) = \text{dpt}^*(g\Delta^t) = \text{dpt}^*(g) + \tilde{q}_t - 1$ .

**Proof** (1) It is a consequence of the fact that  $g_0, \Delta'_1 \in M_1$  and Lemma 3.3. (2) Lemma 4.5 implies that  $\text{bh}(g) = \text{bh}(g_L \Delta^k) = \text{bh}(g_L \theta^k) = \text{bh}(g_L) + 2k$ . Therefore  $\text{dpt}^*(g) = \text{dpt}^*(g_L) + k = \text{dpt}(g) + k$ . (3) Let  $(g_p, \dots, g_1)$  be the alternating form of  $g' = g \Delta^{-k}$ . If  $p$  is odd, then  $(\text{rev}(g_1), \dots, \text{rev}(g_p))$  is an  $(N, M_1)$ -sequence of  $\text{rev}(g')$ . Proposition 4.6 implies that  $\text{bh}(\text{rev}(g')) \leq p$ . Therefore  $\text{dpt}(\text{rev}(g')) \leq (p-1)/2 = \text{dpt}(g')$ . If  $p$  is even, then  $(\text{rev}(g_1), \dots, \text{rev}(g_p), 1)$  is an  $(N, M_1)$ -sequence of  $\text{rev}(g')$ . Proposition 4.6 implies that  $\text{bh}(\text{rev}(g')) \leq p+1$ . Hence  $\text{dpt}(\text{rev}(g')) \leq p/2 = \text{dpt}(g')$ . Analogously we prove that  $\text{dpt}(g') \leq \text{dpt}(\text{rev}(g'))$  because  $\text{rev}^2(g') = g'$ . Hence, (2) implies that  $\text{dpt}^*(g) = \text{dpt}(g') + k = \text{dpt}(\text{rev}(g')) + k = \text{dpt}^*(\text{rev}(g))$ . (4) Since  $g_0 \in M_1$ , then  $\text{dpt}^*(gg_0) = \text{dpt}^*(g)$ . Further, by (3),  $\text{dpt}^*(g_0g) = \text{dpt}^*(\text{rev}(g_0g)) = \text{dpt}^*(\text{rev}(g)\text{rev}(g_0)) = \text{dpt}^*(\text{rev}(g)) = \text{dpt}^*(g)$ . (5) Lemma 4.5 implies that  $\text{dpt}^*(g\Delta^t) = \text{dpt}^*(g\theta^t) = \text{dpt}^*(g) + \tilde{q}_t - 1$  and  $\text{dpt}^*(\text{rev}(g)\theta^t) = \text{dpt}^*(\text{rev}(g)) + \tilde{q}_t - 1$ . By (4),  $\text{dpt}^*(\Delta^t g) = \text{dpt}^*(\theta^t g)$ . Finally, (3) implies that  $\text{dpt}^*(\theta^t g) = \text{dpt}^*(\text{rev}(g)\theta^t) = \text{dpt}^*(\text{rev}(g)) + \tilde{q}_t - 1 = \text{dpt}^*(g) + \tilde{q}_t - 1$ .  $\square$

**Lemma 4.9** *Let  $g \in M_1 \cup \Theta$ , and let  $g' \in M$  such that either  $gg' \in \Theta$  or  $g'g \in \Theta$ . Then  $g' \in M_1 \cup \Theta$  as well.*

**Proof** Assume that  $gg' \in \Theta$ . Let  $k \geq 1$  be such that  $gg' = \theta^k h$  for some  $h \in M_1$ . If  $g \in M_1$ , Lemma 3.3 implies that  $g \leq_L h$  and  $g' = \theta^k(g^{-1}h) \in \Theta$ . If  $g \in \Theta$ , namely  $g = \theta^p g_0$  for some  $p \geq 1$  and  $g_0 \in M_1$ , Lemma 3.3 implies that  $g_0 \leq_L h$ . Further  $p \leq k$ . Hence  $g' = \theta^{k-p}(g_0^{-1}h) \in M_1 \cup \Theta$ . If  $g'g \in \Theta$ , then we show in the same way that  $g' \in M_1 \cup \Theta$ .  $\square$

**Lemma 4.10** *Let  $g \in M \setminus (M_1 \cup \Theta)$  be an unmovable element, let  $g' \in \Theta$  be another unmovable element, and let  $g_0 \in M_1$ . Then:*

- (1)  $\text{dpt}(gg_0) - 1 = \text{dpt}(g) - \tilde{q}_k \geq 0$ , where  $k = \inf(gg_0)$ .
- (2)  $\text{dpt}(g_0g) - 1 = \text{dpt}(g) - \tilde{q}_k \geq 0$ , where  $k = \inf(g_0g)$ .
- (3)  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k$ , where  $k = \inf(gg')$ .
- (4)  $\text{dpt}(g'g) = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k$ , where  $k = \inf(g'g)$ .

**Proof** (1) We have  $gg_0 = \Delta^k h$  for some  $h \in M$  with  $\text{dpt}(gg_0) = \text{dpt}(h)$ . Lemma 4.9 implies that  $h \notin M_1$  because  $g \notin M_1 \cup \Theta$ . Hence  $\text{dpt}(h) \geq 1$ . Lemma 4.8.4 implies that  $\text{dpt}^*(gg_0) = \text{dpt}^*(g) = \text{dpt}(g)$ , and Lemma 4.8.5 implies that  $\text{dpt}^*(gg_0) = \text{dpt}^*(\Delta^k h) = \text{dpt}^*(h) + \tilde{q}_k - 1 = \text{dpt}(gg_0) + \tilde{q}_k - 1$ . Therefore  $\text{dpt}(gg_0) - 1 = \text{dpt}(g) - \tilde{q}_k$ . (2) By Lemma 4.8.2-3 and (1),  $\text{dpt}(g_0g) = \text{dpt}^*(g_0g) - k = \text{dpt}^*(\text{rev}(g_0g)) - k = \text{dpt}(\text{rev}(g_0g)) = \text{dpt}(\text{rev}(g)) - \tilde{q}_k + 1 = \text{dpt}(g) - \tilde{q}_k + 1$ . Further, Lemma 4.9 implies that  $\text{dpt}(g_0g) \geq 1$ .

(3) Let  $p \geq 1$  be such that  $g' = \theta^p h'$  for some  $h' \in M_1$ , and let  $h \in M$  be an unmovable element such that  $gg' = \Delta^k h$ . By Lemma 4.9,  $h \notin M_1$  because  $g \notin M_1 \cup \Theta$ . Lemma 4.8.4-5 implies that  $\text{dpt}^*(gg') = \text{dpt}^*(gh'\theta^p) = \text{dpt}^*(gh') + \tilde{q}_p - 1 = \text{dpt}(g) + \tilde{q}_p - 1 = \text{dpt}(g) + \text{dpt}(g') - 1$  and  $\text{dpt}^*(gg') = \text{dpt}(h) + \tilde{q}_k - 1 = \text{dpt}(gg') + \tilde{q}_k - 1$ . Therefore  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k$ . (4) We have  $\text{rev}(\Theta) \subseteq \Theta$  because  $\text{rev}(\theta) = \theta$ . Lemma 4.8.2-3 and (3) imply that  $\text{dpt}(g'g) = \text{dpt}^*(g'g) - k = \text{dpt}^*(\text{rev}(g'g)) - k = \text{dpt}(\text{rev}(g'g)) = \text{dpt}(\text{rev}(g)) + \text{dpt}(\text{rev}(g')) - \tilde{q}_k = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k$ .  $\square$

**Lemma 4.11** *Let  $g, g' \in M \setminus M_1$  be two unmovable elements such that  $gg' = \Delta^k$  for some integer  $k \geq 1$ . If  $\text{dpt}(g) + \text{dpt}(g') = k + 2$ . Then  $g, g' \in \Theta$ .*

**Proof** Let  $p = \text{dpt}(g) - 1$ , and let  $q = \text{dpt}(g') - 1$ . Then  $1 = (g^{-1}\Delta^p)(\Delta^q g'^{-1})$  because  $gg' = \Delta^{p+q}$ . Note that  $g^{-1}\Delta^p = \Omega^{-2q}g'$  and  $\Delta^q g'^{-1} = \Omega^{-2p}g$ . Further,  $2q < 2\text{dpt}(g') - 1 \leq \text{bh}(g') - 1$  and  $2p < 2\text{dpt}(g) - 1 \leq \text{bh}(g) - 1$ . So, Theorem 2.13 implies that  $g^{-1}\Delta^p$  (resp.  $\Delta^q g'^{-1}$ ) is either  $x_1$ -positive or belongs to  $G_1$ . Note that, if at least one of them is  $x_1$ -positive, then 1 is  $x_1$ -positive, which is a contradiction. Therefore  $g^{-1}\Delta^p = h$  and  $\Delta^q g'^{-1} = h'$  for some  $h, h' \in G_1$ . Thus  $g = \theta^p(\Delta_1^p h^{-1})$  and  $g' = (h'^{-1}\Delta_1^q)\theta^q$ . Let  $(h_0, h'_0)$  be the left orthogonal splitting of  $\Delta_1^p h^{-1}$ , and let  $(h'_1, h_1)$  be the right orthogonal splitting of  $h'^{-1}\Delta_1^q$ . Then  $g = \theta^p h_0^{-1} h'_0$  and  $g' = h'_1 h_1^{-1} \theta^q$ . Since  $h_0 \wedge_L h'_0 = 1$  and  $h_1 \wedge_R h'_1 = 1$ , then  $h_0 \leq_R \theta^p$  and  $h_1 \leq_L \theta^q$ , because  $g, g' \in M$ . Lemma 3.3 implies that  $h_0 = h_1 = 1$ . Hence  $g = \theta^p h'_0$  and  $g' = h'_1 \theta^q = \theta^q h'_1$ . Note that  $\Delta_1 \not\leq_R h'_0, h'_1$  because  $\Delta \not\leq_R g, g'$ . Therefore  $g, g' \in \Theta$ .  $\square$

**Lemma 4.12** *Let  $h, h' \in M \setminus M_1$  be two unmovable elements, and let  $k = \inf(hh')$ . If  $\Delta^{-k}hh' \in M_1$  and  $\text{dpt}(h) + \text{dpt}(h') = k + 2$ . Then  $h, h' \in \Theta$ .*

**Proof** By Lemma 2.9, there exist  $g, g', g_0, g'_0 \in M$  such that  $h = g_0 g$ ,  $h' = g' g'_0$ , and  $gg' = \Delta^k$ . Then  $\Delta^{-k}hh' = g_0 g'_0$  because  $\Delta$  is central, hence  $g_0, g'_0 \in M_1$ . We have  $\text{dpt}(g') = \text{dpt}(h')$ , and, by Lemma 4.8.4,  $\text{dpt}(g) = \text{dpt}(h)$ . Therefore  $\text{dpt}(g) + \text{dpt}(g') = k + 2$ . Lemma 4.11 implies that  $g, g' \in \Theta$ , i.e.  $g = \theta^p g_1$  and  $g' = \theta^q g'_1$  for some  $p, q \geq 1$  and  $g_1, g'_1 \in M_1$ . Hence  $h = \theta^p(g_0 g_1)$  and  $h' = \theta^q(g'_1 g'_0)$ . Note that  $\Delta_1 \not\leq_R g_0 g_1, g'_1 g'_0$  because  $\Delta \not\leq_R h, h'$ . Therefore  $h, h' \in \Theta$ .  $\square$

**Lemma 4.13** *Let  $g, g' \in M \setminus (M_1 \cup \Theta)$  be two unmovable elements, and let  $k = \inf(gg')$ . Then  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_k \geq 0$  for some  $\epsilon \in \{0, 1\}$ .*

**Proof** Let  $p, q \geq 1$  be the respective depths of  $g, g'$ . Then  $\text{bh}(g) \in \{2p, 2p + 1\}$  and  $\text{bh}(g') \in \{2q, 2q + 1\}$ . Theorem 2.13 implies that  $\Omega^{-2p}g$  and  $\Omega^{-2q}g'$  are  $x_1$ -negative, and that  $\Omega^{-2p+2}g$  and  $\Omega^{-2q+2}g'$  are either  $x_1$ -positive or belong to  $G_1$ .

Then  $\Omega^{-2p-2q}gg' = \Delta^{-p-q+k}g_0$  is  $x_1$ -negative, and  $\Omega^{-2p-2q+4}gg' = \Delta^{-p-q+2+k}g_0$  is either  $x_1$ -positive or belongs to  $G_1$ , where  $g_0 = \Delta^{-k}gg'$ . Note that  $k < p + q$  because  $\Delta^{-p-q+k}g_0 \notin M$ . Hence  $p + q - \tilde{q}_k \geq 0$ . Theorem 2.13 implies that  $2p+2q-2 \leq \text{bh}(gg') \leq 2p+2q+1$ . Therefore  $p+q-1 \leq \text{dpt}^*(gg') \leq p+q$ . Assume that  $g_0 \in M_1$ , then  $\text{dpt}^*(gg') = q_k$ , and  $k \geq 1$  because  $g, g' \notin M_1$ . If  $q_k = p + q - 1$ , i.e.  $\text{dpt}(g) + \text{dpt}(g') = k + 2$ , Lemma 4.12 implies that  $g, g' \in \Theta$  which contradicts the enunciate. Therefore  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - q_k$ . Assume that  $g_0 \notin M_1$ . Then, by Lemma 4.8.2,  $p+q-\tilde{q}_k = p+q-1-k \leq \text{dpt}(gg') \leq p+q-k = p+q-\tilde{q}_k+1$ .  $\square$

Thus, Lemmas 4.10 and 4.13 have proved that Condition B holds.

## 5 Artin groups of dihedral type, the even case

In this section  $G$  and  $M$  denote the Artin group and the Artin monoid of type  $I_2(m)$ , respectively, where  $m = 2k$  is even and  $k \geq 2$ . Recall that  $M$  is defined by the monoid presentation:

$$M = \langle x, y \mid (xy)^k = (yx)^k \rangle^+.$$

The group  $G$  is the enveloping group of  $M$ . By Theorem 2.11,  $M$  is a Garside monoid and  $\Delta = x \vee_R y$  is a Garside element of  $M$ . By Brieskorn–Saito [2],  $\Delta$  is central, and  $\Delta = (xy)^k = (yx)^k$ .

Let  $G_1 = \langle y \rangle \simeq \mathbb{Z}$  and  $H = \langle x \rangle \simeq \mathbb{Z}$ . By Theorem 2.11,  $G_1$  and  $H$  are parabolic subgroups of  $(G, \Delta)$ . The subgroup  $G_1$  is associated with  $\Delta_1 = y$ , and  $H$  is associated with  $\Delta_2 = x$ . Once again, we set  $M_1 = G_1 \cap M \simeq \mathbb{N}$  and  $N = H \cap M \simeq \mathbb{N}$ . Obviously,  $M_1 \cup N$  generates  $M$ , hence  $(H, G_1)$  is a Dehornoy pre-structure.

The main result of the section is the following:

**Theorem 5.1** *The pair  $(H, G_1)$  satisfies Conditions A and B.*

By Theorem 3.2, this implies the following:

**Corollary 5.2** *The pair  $(H, G_1)$  is a Dehornoy structure.*

We denote by  $P_1$  the set of  $(H, G_1)$ -positive elements of  $G$ , and we set  $P_2 = \{y^n \mid n \geq 1\}$ . For  $\epsilon = (\epsilon_1, \epsilon_2) \in \{\pm 1\}^2$  we set  $P^\epsilon = P_1^{\epsilon_1} \cup P_2^{\epsilon_2}$ . Then, by Proposition 3.1:

**Corollary 5.3**  *$P^\epsilon$  is the positive cone of a left order on  $G$ .*

We denote by  $z_1, \dots, z_{2k-1}$  the standard generators of the braid group  $\mathcal{B}_{2k}$ . By Crisp [5], the map  $\phi : S \rightarrow \mathcal{B}_{2k}$  defined by:

$$\phi(x) = \prod_{i=0}^{k-1} z_{2i+1}, \quad \phi(y) = \prod_{i=1}^{k-1} z_{2i},$$

induces an embedding  $\phi : G \hookrightarrow \mathcal{B}_{2k}$  satisfying  $\phi(M) \subseteq \mathcal{B}_{2k}^+$ . In a second part of the section we will show how the left orders on  $G$  obtained in Corollary 5.3 can be deduced from  $\phi$ . More precisely, we prove the following:

**Proposition 5.4** *Let  $g \in G$ . Then  $g$  is  $(H, G_1)$ -positive if and only if  $\phi(g)$  is  $z_1$ -positive.*

The rest of the section is dedicated to the proofs of Theorem 5.1 and Proposition 5.4. Theorem 5.1 is proved in Subsection 5.1, and Proposition 5.4 is proved in Subsection 5.2.

## 5.1 Proof of Theorem 5.1

Note that, if  $g \in M$  is unmovable, then it has a unique word representative in  $\{x, y\}^*$ .

**Lemma 5.5** *Let  $g, g' \in M$  be two unmovable elements such that  $\inf(gg') = 0$ . Then:*

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(g) + \text{dpt}(g') - 1 & \text{if } x \leq_R g \text{ and } x \leq_L g' \\ \text{dpt}(g) + \text{dpt}(g') & \text{otherwise} \end{cases}$$

**Proof** If  $x \not\leq_R g$  or  $x \not\leq_L g'$  it is obvious that  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g')$ . If  $x \leq_R g$  and  $x \leq_L g'$ , let  $k$  be the greatest integer such that  $x^k \leq_L g'$ . Then  $g' = x^k g'_0$  for some  $g'_0 \in M$  unmovable where  $x \not\leq_L g'_0$ . We have  $\text{dpt}(gx^k) = \text{dpt}(g)$ , and  $\text{dpt}(g'_0) = \text{dpt}(g') - 1$ . Hence  $\text{dpt}(gg') = \text{dpt}(gx^k) + \text{dpt}(g'_0) = \text{dpt}(g) + \text{dpt}(g') - 1$ .  $\square$

Note that  $\theta = (xy)^{k-1}x = x(yx)^{k-1}$  because  $\Delta_1 = y$ .

**Lemma 5.6** *Let  $g_0 \in M_1$ , and let  $g \in M \setminus M_1$ . Then:*

- (1)  $\text{bh}(g_0\theta) = m$ .
- (2)  $\text{bh}(g\theta) = \text{bh}(g) + m - 2$ .
- (3)  $\text{bh}(\theta^t) = (m - 2)t + 2$  for all  $t \geq 1$ .

**Proof** (1) Note that  $(x, y, \dots, x, 1)$  is the alternating form of  $\theta$ , and  $\text{bh}(\theta) = m$  because  $\text{lg}_S(\theta) = m - 1$ . Lemma 3.3 implies that  $g_0\theta = \theta g_0$ . Hence  $\text{bh}(g_0\theta) = \text{bh}(\theta) = m$ . (2) Let  $g_1$  be the  $M_1$ -tale of  $g$ , and let  $g' = gg_1^{-1}$ . Note that  $g' \neq 1$  because  $g \notin M_1$ . By Lemma 3.3,  $g\theta = g'\theta g_1$ , and  $R_{M_1}(g\theta) = g_1$  because  $y \not\leq_R g'$ . Moreover, if  $h \leq_L \theta$ , and  $a \leq_R h$  for some  $a \in \{x, y\}$ , then  $b \not\leq_R g'h$  because  $y \not\leq_R g'$ , where  $b \in \{x, y\} \setminus \{a\}$ . Therefore  $(g_p, \dots, g_3, g_2x, y, \dots, x, g_1)$  is the alternating form of  $g\theta$ , where  $g' = g_p \cdots g_2$  and  $(g_p, \dots, g_1)$  is the alternating form of  $g$ . So,  $\text{bh}(g\theta) = \text{bh}(g) - 2 + m$ . (3) By (1),  $\text{bh}(\theta) = (m - 2) + 2$ . If  $t > 1$ , we suppose that  $\text{bh}(\theta^{t-1}) = (m - 2)(t - 1) + 2$ . Hence, (2) implies that  $\text{bh}(\theta^t) = \text{bh}(\theta^{t-1}) + m - 2 = (m - 2)(t - 1) + 2 + m - 2 = (m - 2)t + 2$ .  $\square$

Lemma 5.6.3 implies that  $q_t = (k - 1)t + 1$  for all  $k \geq 1$ , because  $\text{bh}(\theta^t)$  is even. Therefore, Condition A holds, where  $\tilde{q}_t = tc + 1$  with  $c = k - 1$  for all  $t \geq 1$ .

Note that, since  $\Delta = (xy)^k$ , then  $\text{dpt}(s) + \text{dpt}(\text{com}_R(s)) = q_1$  for all  $s \in \text{Div}(\Delta) \setminus \{\Delta\}$ .

Since relations of Artin monoids preserve lengths, all words representing an element  $g$  of an Artin monoid  $A$  have the same length. Thus, we may consider the *length* of  $g$ , denoted by  $\text{lg}_S(g)$ , defined as the length of its word representatives in  $S^*$ , where  $S$  is the standard generating set of  $A$ .

**Lemma 5.7** *Let  $g, g' \in M$  be two unmovable elements, and let  $h_1, h_2$  be two simple elements in  $\text{Div}(\Delta) \setminus \{1, \Delta\}$  such that  $h_2h_1 \leq_R g$  and  $\text{com}_R(h_1)\text{com}_R(h_2) \leq_L g'$ . If  $a \leq_L h_1$  and  $b \leq_R \text{com}_R(h_1)$ , where  $\{a, b\} = \{x, y\}$ , then:*

- (1)  $a \leq_R h_2$  and  $b \leq_L \text{com}_R(h_2)$ .
- (2)  $\text{dpt}(gh_1^{-1}) + \text{dpt}(\text{com}_R(h_1)^{-1}g') = \text{dpt}(g) + \text{dpt}(g') - q_1 + 1$ .

**Proof** (1) Since  $h_2\text{com}_R(h_2) = \Delta$ , if  $b \leq_R h_2$ , then  $a \leq_L \text{com}_R(h_2)$ . Similarly, if  $a \leq_R \text{com}_R(h_2)$ , then  $b \leq_R h_2$ . Hence  $h_2h_1$  and  $\text{com}_R(h_1)\text{com}_R(h_2)$  are simple elements. Note that  $\text{lg}_S(h_2h_1) < m$  and  $\text{lg}_S(\text{com}_R(h_1)\text{com}_R(h_2)) < m$  because  $g, g'$  are unmovable. Thus,  $\text{lg}_S(\Delta^2) = \text{lg}_S(h_2h_1\text{com}_R(h_1)\text{com}_R(h_2)) < 2m$ , which is a contradiction because  $\text{lg}_S(\Delta^2) = 2m$ . Therefore  $a \leq_R h_2$  and  $b \leq_L \text{com}_R(h_2)$ . (2) By Lemma 5.5, we have the following:

$$\text{dpt}(gh_1^{-1}) = \begin{cases} \text{dpt}(g) - \text{dpt}(h_1) & \text{if } a = y \\ \text{dpt}(g) - \text{dpt}(h_1) + 1 & \text{if } a = x \end{cases}$$

$$\text{dpt}(\text{com}_R(h_1)^{-1}g') = \begin{cases} \text{dpt}(g') - \text{dpt}(\text{com}_R(h_1)) + 1 & b = x \\ \text{dpt}(g') - \text{dpt}(\text{com}_R(h_1)) & b = y \end{cases}$$

Therefore  $\text{dpt}(gh_1^{-1}) + \text{dpt}(\text{com}_R(h_1)^{-1}g') = \text{dpt}(g) + \text{dpt}(g') - q_1 + 1$ .  $\square$

**Lemma 5.8** *Let  $g, g' \in M$  be two unmovable elements such that  $\inf(gg') = 1$ . Then  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - q_1 \geq 0$  for some  $\epsilon \in \{0, 1\}$ .*

**Proof** Let  $h \in M$  be a simple element, such that  $h \leq_R g$  and  $\text{com}_R(h) \leq_L g'$ , let  $g_1 = gh^{-1}$ , and let  $g'_1 = \text{com}_R(h)^{-1}g'$ . By Lemma 5.5, if  $g_1, g'_1 \neq 1$ , we have:

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(g_1) + \text{dpt}(g'_1) - 1 & \text{if } x \leq_R g_1 \text{ and } x \leq_L g'_1 \\ \text{dpt}(g_1) + \text{dpt}(g'_1) & \text{otherwise} \end{cases}$$

and:

$$\text{dpt}(g_1) = \begin{cases} \text{dpt}(g) - \text{dpt}(h) + 1 & \text{if } x \leq_R g_1 \text{ and } x \leq_L h \\ \text{dpt}(g) - \text{dpt}(h) & \text{otherwise} \end{cases}$$

$$\text{dpt}(g'_1) = \begin{cases} \text{dpt}(g') - \text{dpt}(\text{com}_R(h)) + 1 & \text{if } x \leq_R \text{com}_R(h) \text{ and } x \leq_L g'_1 \\ \text{dpt}(g') - \text{dpt}(\text{com}_R(h)) & \text{otherwise} \end{cases}$$

If  $x \leq_L h$  and  $g_1, g'_1 \neq 1$  we obtain the following:

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(g) + \text{dpt}(g') - q_1 + 1 & \text{if } x \leq_R g_1 \text{ and } y \leq_L g'_1 \\ \text{dpt}(g) + \text{dpt}(g') - q_1 & \text{otherwise} \end{cases}$$

If  $y \leq_L h$  and  $g_1, g'_1 \neq 1$  we obtain the following:

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(g) + \text{dpt}(g') - q_1 + 1 & \text{if } y \leq_R g_1 \text{ and } x \leq_L g'_1 \\ \text{dpt}(g) + \text{dpt}(g') - q_1 & \text{otherwise} \end{cases}$$

If  $x \leq_L h$  and  $1 \in \{g_1, g'_1\}$  we obtain the following:

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(g) + \text{dpt}(g') - q_1 + 1 & \text{if } x \leq_R g_1 \\ \text{dpt}(g) + \text{dpt}(g') - q_1 & \text{if } x \not\leq_R g_1 \end{cases}$$

If  $y \leq_L h$  and  $1 \in \{g_1, g'_1\}$  we obtain the following:

$$\text{dpt}(gg') = \begin{cases} \text{dpt}(g) + \text{dpt}(g') - q_1 + 1 & \text{if } x \leq_L g'_1 \\ \text{dpt}(g) + \text{dpt}(g') - q_1 & \text{if } x \not\leq_L g'_1 \end{cases}$$

Therefore  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - q_1$  for some  $\epsilon \in \{0, 1\}$ . If  $\Delta^{-1}gg' \notin M_1$ , then  $\text{dpt}(gg') \geq 1$ . Hence  $\text{dpt}(g) + \text{dpt}(g') - q_1 \geq 0$ . If  $\Delta^{-1}gg' \in M_1$ , then  $\text{dpt}(g) + \text{dpt}(g') = \text{dpt}(h) + \text{dpt}(\text{com}_R(h)) = q_1$ . Thus  $\text{dpt}(g) + \text{dpt}(g') - q_1 \geq 0$ .  $\square$

The following lemma proves Part (a) of Condition B:

**Lemma 5.9** *Let  $g, g' \in M$  be two unmovable elements with  $t = \inf(gg')$ , such that  $\{g, g'\} \not\subseteq M_1 \cup \Theta$ . Then  $\text{dpt}(gg') - \epsilon = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t \geq 0$  for some  $\epsilon \in \{0, 1\}$ .*

**Proof** If  $t = 0$ , Lemma 5.5 implies that  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_0 + \epsilon$  for some  $\epsilon \in \{0, 1\}$ , where  $\epsilon = 0$  if and only if  $x \leq_R g$  and  $x \leq_L g'$ . Note that  $\text{dpt}(gg') = 0$  if and only if  $g, g' \in M_1$ . Therefore  $\text{dpt}(g) + \text{dpt}(g') - \tilde{q}_0 \geq 0$ . If  $t = 1$ , the result is clear by Lemma 5.8. If  $t \geq 2$ , there exist simple elements  $h_1, \dots, h_t$  such that  $g = g_1 h_t \cdots h_1$  and  $g' = \text{com}_R(h_1) \cdots \text{com}_R(h_t) g'_1$ , for some  $g_1, g'_1 \in M$  such that  $\text{inf}(g_1 g'_1) = 0$ . Lemma 5.7 implies that  $\text{dpt}(g_1 h_t) + \text{dpt}(\text{com}_R(h_t) g'_1) = \text{dpt}(g) + \text{dpt}(g') - q_{t-1} + 1$ . So, by Lemma 5.8,  $\text{dpt}(gg') - \epsilon = \text{dpt}(g_1 g'_1) - \epsilon = \text{dpt}(g_1 h_t) + \text{dpt}(\text{com}_R(h_t) g'_1) - q_1 = \text{dpt}(g) + \text{dpt}(g') - q_t$  for some  $\epsilon \in \{0, 1\}$  with  $\text{dpt}(g) + \text{dpt}(g') - q_t \geq 0$ .  $\square$

The following lemma proves Part (b) of Condition B:

**Lemma 5.10** *Let  $g, g' \in M$  be two unmovable elements with  $t = \text{inf}(gg')$ , such that  $\{g, g'\} \not\subseteq M_1 \cup \Theta$  and  $\{g, g'\} \cap \Theta \neq \emptyset$ . Then  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t$ .*

**Proof** Suppose that  $g \in \Theta$ , i.e.  $g = g_0 \theta^s$  for some  $s \geq 1$  and  $g_0 \in M_1$ . Note that  $g_0 = 1$  because  $g$  is unmovable. Hence  $g = \theta^s$ . If  $s < t$ , then  $y^s \Delta^{t-s} \leq_L g'$ , which is impossible because  $g'$  is unmovable. Then  $s \geq t$ , and  $y^t \leq_L g'$ . If  $s = t$ , then  $\text{dpt}(gg') = \text{dpt}(g') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t$ . If  $s > t$ , Lemma 5.5 implies that  $\text{dpt}(gg') = \text{dpt}(\theta^{s-t}) + \text{dpt}(g') - 1 = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_t$  because  $y \not\leq_L y^{-t} g'$ .  $\square$

## 5.2 Proof of Proposition 5.4

We denote by  $\Omega$  the standard Garside element of  $\mathcal{B}_{2k}$ , by  $A_1$  the submonoid of  $\mathcal{B}_{2k}^+$  generated by  $z_1, \dots, z_{2k-2}$ , and by  $A_2$  the submonoid of  $\mathcal{B}_{2k}^+$  generated by  $z_2, \dots, z_{2k-1}$ . Note that  $\phi(G_1) \subseteq A_1 \cap A_2$ .

**Proposition 5.11** (Crisp [5]) *Let  $g, g' \in M$ , and let  $(g_p, \dots, g_1)$  be the greedy normal form of  $g$ . Then:*

- (1)  $g \leq_R g'$  if and only if  $\phi(g) \leq_R \phi(g')$ .
- (2)  $\phi(g \wedge_R g') = \phi(g) \wedge_R \phi(g')$  and  $\phi(g \vee_R g') = \phi(g) \vee_R \phi(g')$ .
- (3)  $(\phi(g_p), \dots, \phi(g_1))$  is the greedy normal form of  $\phi(g)$ .

**Lemma 5.12** *We have  $\phi(g) \in \mathcal{B}_{2k} \setminus \langle z_2, \dots, z_{2k-1} \rangle$  for all  $g \in G \setminus G_1$ .*

**Proof** Let  $h \in \mathcal{B}_{2k}$ , and let  $(h_2, h_1)$  be its right orthogonal splitting, that is  $h = h_1 h_2^{-1}$  and  $h_1 \wedge_R h_2 = 1$ . Since  $G(A_2)$  is a parabolic subgroup, we have  $h \in G(A_2)$  if and only if  $h_1, h_2 \in A_2$ . Let  $g \in G$ , and let  $(g_2, g_1)$  be its right orthogonal splitting. By

Proposition 5.11, we have  $\phi(g) = \phi(g_1)\phi(g_2)^{-1}$  and  $\phi(g_1)\wedge_R\phi(g_2) = 1$ . It follows that  $\phi(g) \in G(A_2)$  if and only if  $\phi(g_1), \phi(g_2) \in A_2$ . Furthermore, we have  $g \in G_1$  if and only if  $g_1, g_2 \in M_1$ . So, we only need to prove that  $\phi(g) \in \mathcal{B}_{2k}^+ \setminus A_2$  for all  $g \in M \setminus M_1$ . Let  $h \in \mathcal{B}_{2k}^+$ , and let  $(h_p, \dots, h_1)$  be its right greedy normal form. Since  $G(A_2)$  is a parabolic subgroup, we have  $h \in A_2$  if and only if  $h_i \in A_2$  for all  $i \in \{1, \dots, p\}$ . Let  $g \in M$ , and let  $(g_p, \dots, g_1)$  be its right greedy normal form. By Proposition 5.11,  $\phi(g) = \phi(g_p) \cdots \phi(g_1)$  is the right greedy normal form of  $\phi(g)$ . It follows that  $\phi(g) \in A_2$  if and only if  $\phi(g_i) \in A_2$  for all  $i \in \{1, \dots, p\}$ . Furthermore, we have  $g \in M_1$  if and only if  $g_i \in M_1$  for all  $i \in \{1, \dots, p\}$ . So, we only need to show that  $\phi(g) \in \text{Div}(\Omega) \setminus A_2$  for  $g \in \text{Div}(\Delta) \setminus M_1 = \text{Div}(\Delta) \setminus \text{Div}(x)$ . Let  $h \in \text{Div}(\Omega)$ , and let  $i_1, \dots, i_l \in \{1, \dots, 2k-1\}$  such that  $h = z_{i_1} \cdots z_{i_l}$ . By Bourbaki [1], Chapter IV, Section 1.8, we have  $h \in A_2$  if and only if  $\{z_{i_1}, \dots, z_{i_l}\} \subset \{z_2, \dots, z_{2k-1}\}$ . This implies that, for  $g \in \text{Div}(\Delta)$ , we have  $\phi(g) \in A_2$  if and only if  $g \in \{1, y\} = \text{Div}(y)$ .  $\square$

For each  $i, j \in \{0, \dots, k-1\}$  with  $j \neq 0$ , and for every  $t \geq 1$ , we define:

$$x_{2i+1}^t = \prod_{l=i}^{k-1} z_{2l+1}^t, \quad y_{2j}^t = \prod_{l=j}^{k-1} z_{2l}^t.$$

In particular  $x_1^t = \phi(x^t) = \phi(x)^t$  and  $y_2^t = \phi(y^t) = \phi(y)^t$ . Note that  $y_{2j}^t \in A_1 \cap A_2$ .

**Lemma 5.13** *Let  $g \in M \setminus M_1$  be an unmovable element such that  $\text{dpt}(g) \leq k-1$ . Then  $\phi(g)$  has a  $(A_1, A_2)$ -sequence and a  $(A_2, A_1)$ -sequence of length 2.*

**Proof** Let  $d = \text{dpt}(g)$ . Then  $g = y^{t_d} x^{s_d} \cdots y^{t_1} x^{s_1} y^{t_0}$ , where  $s_1, \dots, s_d \geq 1$ ,  $t_0, t_d \geq 0$ , and  $t_1, \dots, t_{d-1} \geq 1$  if  $d \geq 2$ . Since  $d \leq k-1$ , then:

$$\begin{aligned} \phi(g) &= y_2^{t_d} x_1^{s_d} \cdots x_1^{s_2} y_2^{t_1} x_1^{s_1} \cdot y_2^{t_0} \\ &= y_2^{t_d} x_1^{s_d} \cdots x_1^{s_2} y_2^{t_1} z_1^{s_1} \cdot x_3^{s_1} y_2^{t_0} \\ &= y_2^{t_d} x_1^{s_d} \cdots x_1^{s_2} z_2^{t_1} z_1^{s_1} \cdot y_4^{t_1} x_3^{s_1} y_2^{t_0} \\ &= y_2^{t_d} x_1^{s_d} \cdots z_3^{s_2} z_1^{s_2} z_2^{t_1} z_1^{s_1} \cdot x_5^{s_2} y_4^{t_1} x_3^{s_1} y_2^{t_0} \\ &\cdots \\ &= y_2^{t_d} z_{2d-1}^{s_d} \cdots z_3^{s_d} z_1^{s_d} \cdots z_3^{s_2} z_1^{s_2} z_2^{t_1} z_1^{s_1} \cdot x_{2d+1}^{s_d} \cdots x_5^{s_2} y_4^{t_1} x_3^{s_1} y_2^{t_0} \end{aligned}$$

Let  $h' = y_2^{t_d} z_{2d-1}^{s_d} \cdots z_3^{s_d} z_1^{s_d} \cdots z_3^{s_2} z_1^{s_2} z_2^{t_1} z_1^{s_1}$ , and let  $h = x_{2d+1}^{s_d} \cdots x_5^{s_2} y_4^{t_1} x_3^{s_1} y_2^{t_0}$ . Note that  $h \in A_2$  and  $h' \in A_1$ . Therefore  $(h', h)$  is a  $(A_1, A_2)$ -sequence of  $\phi(g)$ . Let  $\Phi$  be the flip automorphism of  $\mathcal{B}_{2k}$ . Then  $\phi(g) = \Phi(\phi(g)) = \Phi(h')\Phi(h)$  because  $\phi(\Delta)$  is the Garside element of  $\mathcal{B}_{2k}$ . Note that  $\Phi(h') \in A_2$  and  $\Phi(h) \in A_1$ . Therefore  $(\Phi(h'), \Phi(h))$  is a  $(A_2, A_1)$ -sequence of  $\phi(g)$ . Both sequences are of length 2.  $\square$

The *limit* of an unmovable element  $g \in M \setminus M_1$ , denoted by  $\lim(g)$ , is the unique integer  $t \geq 1$  such that  $\tilde{q}_{t-1} < \text{dpt}(g) + 1 \leq \tilde{q}_t$ . Since  $\tilde{q}_{t-1} - 1 < \text{dpt}(g)$ , then:

$$(5-1) \quad \lim(g) = \frac{2\tilde{q}_{t-1} + m - 4}{m - 2} < \frac{2\text{dpt}(g) + m - 2}{m - 2}.$$

We set  $\lim(g_0) = 0$  for all  $g_0 \in M_1$ . Thus  $\text{bh}_{A_1, A_2}(\phi(g_0)) = \lim(g_0) + 1$ .

**Lemma 5.14** *Let  $g \in M \setminus M_1$  be an unmovable element. Then  $\phi(g)$  has a  $(A_1, A_2)$ -sequence of length  $\lim(g) + 1$ .*

**Proof** We have  $g = y^{t_d} x^{s_d} \dots y^{t_1} x^{s_1} y^{t_0}$  with  $d = \text{dpt}(g)$ , where  $s_1, \dots, s_d \geq 1$ ,  $t_0, t_d \geq 0$ , and  $t_1, \dots, t_{d-1} \geq 1$  if  $d \geq 2$ . Let  $t = \lim(g)$ . For every  $i \in \{1, \dots, t\}$ , we define an element  $g_i \in M$ , as follows:

$$g_i = \begin{cases} y^{i(k-1)} x^{s_i(k-1)} \dots y^{(i-1)(k-1)+1} x^{s_{(i-1)(k-1)+1}} & \text{if } i < t \\ y^{t_d} x^{s_d} \dots y^{(t-1)(k-1)+1} x^{s_{(t-1)(k-1)+1}} & \text{if } i = t \end{cases}$$

Note that  $\text{dpt}(g_i) = k - 1$  for all  $i < t$ , and  $\text{dpt}(g_t) \in \{1, \dots, k - 1\}$ . Lemma 5.13 implies that there are  $h_1, \dots, h_t \in A_2$  and  $h'_1, \dots, h'_t \in A_1$ , such that  $(h'_i, h_i)$  is a  $(A_1, A_2)$ -sequence of  $g_i$  if  $i$  is odd, and  $(h_i, h'_i)$  is a  $(A_2, A_1)$ -sequence of  $g_i$  if  $i$  is even. Therefore, the following is a  $(A_1, A_2)$ -sequence of  $\phi(g)$ :

$$\begin{cases} (h'_1, h_1 h_{t-1}, \dots, h_3 h_2, h'_2 h'_1, h_1 y^{t_0}) & \text{if } t \text{ is odd} \\ (h_t, h'_t h'_{t-1}, \dots, h_3 h_2, h'_2 h'_1, h_1 y^{t_0}) & \text{if } t \text{ is even} \end{cases}$$

Note that, the length of this sequence is  $t + 1$ . □

Proposition 4.6 and Lemma 5.14 imply that  $\text{bh}_{A_1, A_2}(\phi(g)) \leq \lim(g) + 1$  for all unmovable element  $g \in M \setminus M_1$ .

**Lemma 5.15** *Let  $g' \in G$  such that  $\phi(g')$  is  $z_1$ -positive. Then  $g'$  is  $(H, G_1)$ -positive.*

**Proof** Let  $m = \inf(g')$  such that  $g' = \Delta^m g$  for some unmovable element  $g \in M$ . Note that  $g' \notin G_1$  because  $\phi(g')$  is  $z_1$ -positive. We have  $\phi(g') = \phi(\Delta)^m \phi(g)$ . If  $m \geq 0$ , then  $g' \in M \setminus M_1$ . Therefore  $g'$  is  $(H, G_1)$ -positive. If  $m < 0$ , namely  $m = -t$  for some  $t \geq 1$ , Theorem 2.13 and Equation (5-1) imply that  $t + 2 \leq \text{bh}_{A_1, A_2}(\phi(g)) \leq \lim(g) + 1 < (2\text{dpt}(g) + 2(m - 2))/(m - 2)$ . Hence  $(k - 1)t < \text{dpt}(g)$ , i.e.  $\text{dpt}(g) \geq q_t$ . Therefore  $g'$  is  $(H, G_1)$ -positive. □

If  $g \in G$  is  $(H, G_1)$ -positive and  $\phi(g)$  is  $z_1$ -negative. Then  $\phi(g)^{-1}$  is  $z_1$ -positive. So, Lemma 5.15 implies that  $g^{-1}$  is  $(H, G_1)$ -positive as well, which contradicts Corollary 5.2. This proves Proposition 5.4.

We finish this subsection with the following consequence:

**Corollary 5.16** *We have  $\text{bh}_{A_1, A_2}(\phi(g)) = \text{lim}(g) + 1$  for all unmovable  $g \in M \setminus M_1$ .*

**Proof** Let  $g_0 \in M_1$  be the  $M_1$ -tale of  $g$ , and let  $g' = gg_0^{-1}$ . By definition,  $\tilde{q}_{t-1} < \text{dpt}(g) = \text{dpt}(g')$  where  $t = \text{lim}(g)$ . Then  $\Delta^{-t+1}g'$  is  $(H, G_1)$ -positive. So, Proposition 5.4 implies that  $\phi(\Delta^{-t+1}g')$  is  $z_1$ -positive. By Theorem 2.13 and Lemma 5.14,  $t + 1 \leq \text{bh}_{A_1, A_2}(\phi(g')) \leq t + 1$ . Therefore  $\text{bh}_{A_1, A_2}(\phi(g)) = \text{lim}(g) + 1$ .  $\square$

## 6 Artin groups of dihedral type, the odd case

In this section  $G$  and  $M$  denote the Artin group and the Artin monoid of type  $I_2(m)$ , respectively, where  $m = 2k + 1$  is odd and  $k \geq 2$ . Recall that  $M$  is defined by the monoid presentation

$$M = \langle x, y \mid (xy)^k x = (yx)^k y \rangle^+.$$

The group  $G$  is the enveloping group of  $M$ . By Theorem 2.11,  $M$  is a Garside monoid and  $\Omega = x \vee_R y$  is a Garside element of  $M$ .

What makes the difference between the case  $m$  even, treated in Section 5, and the case  $m$  odd, treated in the present section, is that, now, the Garside element  $\Omega = (xy)^k x = (yx)^k y$  is not central. It satisfies  $\Omega x = y\Omega$  and  $\Omega y = x\Omega$ , hence  $\Delta = \Omega^2 = (xy)^m$  is central. In order to apply our criteria, we will use the Garside element  $\Delta$  and not  $\Omega$ , making the proofs more complicated than in Section 5, although the statements are identical.

Let  $G_1 = \langle y \rangle \simeq \mathbb{Z}$ , and let  $H = \langle x \rangle \simeq \mathbb{Z}$ . By Theorem 2.11,  $G_1$  and  $H$  are parabolic subgroups of  $(G, \Delta)$ . The subgroup  $G_1$  is associated with  $\Delta_1 = y^2$ , and  $H$  is associated with  $\Lambda = x^2$ . Once again, we set  $M_1 = G_1 \cap M \simeq \mathbb{N}$  and  $N = H \cap M \simeq \mathbb{N}$ . Obviously,  $M_1 \cup N$  generates  $M$ , hence  $(H, G_1)$  is a Dehornoy pre-structure.

The main result of the section is the following:

**Theorem 6.1** *The pair  $(H, G_1)$  satisfies Conditions A and B.*

Then, applying Theorem 3.2 we obtain:

**Corollary 6.2** *The pair  $(H, G_1)$  is a Dehornoy structure.*

We denote by  $P_1$  the set of  $(H, G_1)$ -positive elements of  $G$ , and we set  $P_2 = \{y^n \mid n \geq 1\}$ . For  $\epsilon = (\epsilon_1, \epsilon_2) \in \{\pm 1\}^2$  we set  $P^\epsilon = P_1^{\epsilon_1} \cup P_2^{\epsilon_2}$ . Then, by Proposition 3.1:

**Corollary 6.3** *The set  $P^e$  is the positive cone of a left order on  $G$ .*

We denote by  $z_1, \dots, z_{2k}$  the standard generators of the braid group  $\mathcal{B}_{2k+1}$ . By Crisp [5], the map  $\phi : S \rightarrow \mathcal{B}_{2k+1}$  defined by

$$\phi(x) = \prod_{i=0}^{k-1} z_{2i+1}, \quad \phi(y) = \prod_{i=1}^k z_{2i},$$

induces an embedding  $\phi : G \hookrightarrow \mathcal{B}_{2k+1}$ . As in the case  $m$  even, we will show how the left orders on  $G$  obtained in Corollary 6.3 can be deduced from  $\phi$ . More precisely, we will prove the following:

**Proposition 6.4** *Let  $g \in G$ . Then  $g$  is  $(H, G_1)$ -positive if and only if  $\phi(g)$  is  $z_1$ -positive.*

The rest of the section is dedicated to the proofs of Theorem 6.1 and Proposition 6.4. Theorem 6.1 is proved in Subsection 6.1, and Proposition 6.4 is proved in Subsection 6.2.

## 6.1 Proof of Theorem 6.1

The proof of Theorem 6.1 is a similar but much more complex proof of Theorem 5.1. Because of that, we will only give a sketch of this proof:

Note that, if  $g \in M$  and  $\Omega \not\leq_R g$ , then  $g$  does not contain  $x \vee_L y$  as a subword. Therefore  $g$  has a unique word representative in  $\{x, y\}^*$ . We will denote by  $\Phi$  the flip automorphism of  $G$  relative to  $\Omega$ . Note that, if  $g \in M \setminus \{1\}$  such that  $\Omega \not\leq_R g$ , then  $\text{bh}(\Phi(g)) = \text{bh}(g) + 1$  when  $y \leq_R g$ , and  $\text{bh}(\Phi(g)) = \text{bh}(g) - 1$  when  $x \leq_R g$ .

Let  $g, g' \in M$  such that  $\Omega \not\leq_R g'$ ,  $\Omega \not\leq_R g$ , and  $\Omega \not\leq_R gg'$ . As in Lemma 5.5, we have:

$$(6-1) \quad \text{bh}(gg') = \begin{cases} \text{bh}(g) + \text{bh}(g') & \text{if } y \leq_R g \text{ and } x \leq_L g' \\ \text{bh}(g) + \text{bh}(g') - 2 & \text{if } x \leq_R g \text{ and } x \leq_L g' \\ \text{bh}(g) + \text{bh}(g') - 1 & \text{otherwise} \end{cases}$$

By using Equation (6-1) it is not hard to prove the following properties:

- (1)  $\text{bh}(g_0\theta) = 2(m-1)$  for all  $g_0 \in M_1$ .
- (2)  $\text{bh}(g\theta) = \text{bh}(g) + 2(m-2)$  for all  $g \in M \setminus M_1$ .
- (3)  $\text{bh}(\theta^t) = 2(m-2)t + 2$  for all  $t \geq 1$ .

So, more generally, for all  $g_0 \in M_1$  and  $g \in M \setminus M_1$  with  $\Omega \not\leq_R g$ , we obtain:

- (4)  $\text{bh}(g_0\Omega^t) = \text{bh}(\Omega^t g_0) = \text{bh}(g_0) + (m-2)t + 1$  for all  $t \geq 2$ .
- (5)  $\text{bh}(g\Omega^t) = \text{bh}(\Omega^t g) = \text{bh}(g) + (m-2)t$  when  $t \geq 2$  is even.
- (6)  $\text{bh}(g\Omega^t) = \text{bh}(g) + (m-2)t + 1$  when  $t \geq 2$  is odd and  $y \leq_R g$ .
- (7)  $\text{bh}(g\Omega^t) = \text{bh}(g) + (m-2)t$  when  $t \geq 2$  is odd and  $g \in N$ .
- (8)  $\text{bh}(g\Omega^t) = \text{bh}(g) + (m-2)t - 1$  when  $t \geq 2$  is odd,  $x \leq_R g$  and  $g \notin N$ .
- (9)  $\text{bh}(\Omega^t g) = \text{bh}(g) + (m-2)t$  when  $t \geq 2$  is odd.

Property (3) implies that  $q_t = (m-2)t + 1$  for all  $k \geq 1$ , because  $\text{bh}(\theta^t)$  is even. Therefore, Condition A holds, where  $\tilde{q}_t = tc + 1$  with  $c = m-2$  for all  $t \geq 0$ .

Let  $g, g' \in M$  such that  $\Omega \not\leq_R g, g'$  and  $gg' \neq \Omega$ , and let  $s, s' \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $s \leq_R g$ ,  $s' \leq_L g'$ , and  $ss' = \Omega$ . Note that,  $x \leq_L s$  if and only if  $x \leq_R s'$ . Further,  $x \leq_R s$  if and only if  $y \leq_L s'$ . By using Equation (6-1) and Properties (1) to (9) we can prove that there is  $\epsilon \in \{0, 1, 2, 3\}$  such that:

$$(6-2) \quad \text{bh}(gs^{-1}) + \text{bh}(s'^{-1}g') = \text{bh}(s) + \text{bh}(s') - m + \epsilon.$$

Similarly to Lemma 5.7, we have the following result:

**Lemma 6.5** *Let  $g, g' \in M$  with  $\Omega \not\leq_R g, g'$ , and let  $s_1, s_2, s'_1, s'_2 \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $s_2s_1 \leq_R g$ ,  $s'_1s'_2 \leq_L g'$ , and  $s_1s'_1 = \Phi(s_2)s'_2 = \Omega$ . Then, there is a unique  $z \in \{x, y\}$  satisfying  $z \leq_L s_1, s'_2$  and  $z \leq_R s'_1, s_2$ .*

**Proof** Let  $z$  be the unique element in  $\{x, y\}$  such that  $z \leq_L s_1$  and  $z \leq_R s'_1$ , and let  $r \in \{x, y\}$  such that  $r \leq_L s'_2$ . Since  $\Phi(s_2)s'_2 = \Omega$ , then  $r \leq_R s_2$ . Suppose that  $r \neq z$ , then  $\text{lg}_S(s_2s_1) < m$  and  $\text{lg}_S(s'_1s'_2) < m$  because  $\Omega \not\leq_R g, g'$ . Hence,  $\text{lg}_S(\Delta) = \text{lg}_S(s_2s_1s'_1s'_2) < 2m$ , which is a contradiction. Therefore  $r = z$ .  $\square$

Let  $g, g' \in M$  with  $\Omega \not\leq_R g, g'$ , and let  $s, s' \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $s \leq_R g$ ,  $s' \leq_L g'$ , and  $ss' = \Omega$ . Let  $z \in \{x, y\}$  such that  $z \leq_L s$  and  $z \leq_R s'$ . Assume that  $z \leq_R gs^{-1}$ ,  $z \leq_L s'^{-1}g'$ , and  $\Omega \not\leq_R \Phi(gs^{-1})s'^{-1}g'$ . Then, as a consequence of Equations (6-1) and (6-2), there is  $k \in \{1, 2\}$  satisfying:

$$(6-3) \quad \text{bh}(\Phi(gs^{-1})s'^{-1}g') = \text{bh}(\Phi^k(g)) + \text{bh}(g') - m + 2.$$

Furthermore, if  $g, g' \in M$  such that  $\Omega \not\leq_R \Phi(g)g'$ , and, if  $s, s' \in \text{Div}(\Omega) \setminus \{1, \Omega\}$  such that  $ss' = \Omega$  and  $\Omega \not\leq_R gs, s'g'$ . Then, by Equation (6-1) and Properties (1) to (9), we obtain that, there are  $k \in \{1, 2\}$  and  $\epsilon \in \{0, 1, 2, 3\}$ , such that:

$$(6-4) \quad \text{bh}(\Phi(g)g') = \text{bh}(\Phi^k(gs)) + \text{bh}(s'g') - m + \epsilon.$$

Consequently, Equations (6-3) and (6-4) imply a kind of Part (a) of Condition B, namely, for breadths instead of depths, as follows:

**Proposition 6.6** *Let  $g, g' \in M$  such that  $\Omega \not\leq_R g, g'$  and  $\Omega \leq_R gg'$ , and let  $t \geq 1$  be the maximal integer satisfying  $\Omega^t \leq_R gg'$ . Then, there exists  $\epsilon' \in \{0, 1, 2\}$  such that:*

$$\text{bh}(\Omega^{-t}gg') - \epsilon' = \text{bh}(u_1(g, g')) - \text{bh}(\Omega^t) \geq 0.$$

We obtain Part (a) of Condition B as a consequence of Proposition 6.6 by passing from breadths to depths. Part (b) of Condition B is a consequence of the following lemma:

**Lemma 6.7** *Let  $g \in M \setminus M_1$ , and let  $g' \in \Theta$  unmovable. Then  $\text{dpt}(gg') = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$  where  $r = \inf(gg')$ .*

**Proof** We have  $g' = \theta^t y^e$  for some  $t \geq 1$  and  $e \in \{0, 1\}$ . Let  $t' \geq 0$  be the maximal integer satisfying  $y^{t'} \leq_R g$ . Then  $g = g_0 y^{t'}$  for some  $g_0 \in M$  satisfying  $y \not\leq_R g_0$  and  $x \leq_R g_0$  because  $g \notin M_1$ . If  $r = t$ , then  $\text{dpt}(gg') = \text{dpt}(g) = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$ . Assume that  $r < t$ . If  $t' = 2r$ , then  $gg' = \Delta^r g_0 \theta^{t-r} y^e$ . If  $t' = 2r + 1$  and  $e = 0$ , then  $gg' = \Delta^r g_0 \theta^{t-r} y$ , and, if  $t' = 2r - 1$  and  $e = 1$ , then  $gg' = \Delta^r g_0 \theta^{t-r}$ . So, Property (2) implies that  $\text{dpt}(gg') = \text{dpt}(g_0) + (m - 2)(t - r) = \text{dpt}(g) + \text{dpt}(g') - \tilde{q}_r$ .  $\square$

## 6.2 Proof of Proposition 6.4

In this subsection, we denote by  $\Omega_B$  the standard Garside element of  $\mathcal{B}_{2k+1}$ , by  $A_1$  the submonoid of  $\mathcal{B}_{2k+1}$  generated by  $z_1, \dots, z_{2k-1}$ , and by  $A_2$  the submonoid of  $\mathcal{B}_{2k+1}$  generated by  $z_2, \dots, z_{2k}$ . Note that  $\phi(H) \subseteq A_1 \setminus A_2$  and  $\phi(G_1) \subseteq A_2 \setminus A_1$ . Further, we denote by  $\Omega$  the standard Garside element of  $G$ .

**Proposition 6.8** (Crisp [5]) *Let  $g, g' \in M$ , and let  $(g_p, \dots, g_1)$  be the greedy normal form of  $g$ . Then:*

- (1)  $g \leq_R g'$  if and only if  $\phi(g) \leq_R \phi(g')$ .
- (2)  $\phi(g \wedge_R g') = \phi(g) \wedge_R \phi(g')$  and  $\phi(g \vee_R g') = \phi(g) \vee_R \phi(g')$ .
- (3)  $(\phi(g_p), \dots, \phi(g_1))$  is the greedy normal form of  $\phi(g)$ .

**Lemma 6.9** *We have  $\phi(g) \in \mathcal{B}_{2k+1} \setminus \langle z_2, \dots, z_{2k} \rangle$  for all  $g \in G \setminus G_1$ .*

**Proof** Let  $h \in \mathcal{B}_{2k+1}$ , and let  $(h_2, h_1)$  be its right orthogonal splitting that is  $h = h_1 h_2^{-1}$  and  $h_1 \wedge_R h_2 = 1$ . Since  $G(A_2)$  is a parabolic subgroup, we have  $h \in G(A_2)$  if and only if  $h_1, h_2 \in A_2$ . Let  $g \in G$ , and let  $(g_2, g_1)$  be its right orthogonal splitting. By Proposition 6.8, we have  $\phi(g) = \phi(g_1) \phi(g_2)^{-1}$  and  $\phi(g_1) \wedge_R \phi(g_2) = 1$ . It follows

that  $\phi(g) \in G(A_2)$  if and only if  $\phi(g_1), \phi(g_2) \in A_2$ . Furthermore, we have  $g \in G_1$  if and only if  $g_1, g_2 \in M_1$ . So, we only need to prove that  $\phi(g) \in \mathcal{B}_{2k+1}^+ \setminus A_2$  for all  $g \in M \setminus M_1$ . Let  $h \in \mathcal{B}_{2k+1}^+$ , and let  $(h_p, \dots, h_1)$  be its greedy normal form. Since  $G(A_2)$  is a parabolic subgroup, we have  $h \in A_2$  if and only if  $h_i \in A_2$  for all  $i \in \{1, \dots, p\}$ . Let  $g \in M$ , and let  $(g_p, \dots, g_1)$  be its right greedy normal form. By Proposition 6.8,  $\phi(g) = \phi(g_p) \cdots \phi(g_1)$  is the right greedy normal form of  $\phi(g)$ . It follows that  $\phi(g) \in A_2$  if and only if  $\phi(g_i) \in A_2$  for all  $i \in \{1, \dots, p\}$ . Furthermore, we have  $g \in M_1$  if and only if  $g_i \in M_1$  for all  $i \in \{1, \dots, p\}$ . So, we only need to show that  $\phi(g) \in \text{Div}(\Omega_{\mathcal{B}}) \setminus A_2$  if  $g \in \text{Div}(\Omega) \setminus M_1$ . Let  $h \in \text{Div}(\Omega_{\mathcal{B}})$ , and let  $i_1, \dots, i_l \in \{1, \dots, 2k\}$  such that  $h = z_{i_1} \cdots z_{i_l}$ . By Bourbaki [1], Chapter IV, Section 1.8, we have  $h \in A_2$  if and only if  $\{z_{i_1}, \dots, z_{i_l}\} \subset \{z_2, \dots, z_{2k}\}$ . This implies that, for  $g \in \text{Div}(\Omega)$ , we have  $\phi(g) \in A_2$  if and only if  $g \in \{1, y\} = \text{Div}(y)$ .  $\square$

For each  $i, j \in \{1, \dots, k\}$  with  $j \neq k$ , and for every  $t \geq 1$ , we define:

$$x_{2i-1}^t = \prod_{l=i}^k z_{2l-1}^t, \quad y_{2j}^t = \prod_{l=j}^k z_{2l}^t.$$

In particular  $x_1^t = \phi(x^t) = \phi(x)^t$  and  $y_2^t = \phi(y^t) = \phi(y)^t$ .

**Lemma 6.10** *Let  $g \in M \setminus M_1$  be an unmovable element such that  $\text{bh}(g) \leq 2k$ . Then  $\phi(g)$  has a  $(A_1, A_2)$ -sequence of length 2. Moreover, if  $\text{bh}(g) \neq 2k$ , then  $\phi(g)$  has a  $(A_2, A_1)$ -sequence of length 2.*

**Proof** Let  $b = \text{bh}(g)$ . Then  $g = a^{t_b} \cdots y^{t_3} x^{t_2} y^{t_1}$ , where  $t_1 \geq 0$ ,  $t_2, \dots, t_b \geq 1$ ,  $a = x$  if  $b$  is even, and  $a = y$  if  $b$  is odd. Since  $b \leq 2k$ , then:

$$\begin{aligned} \phi(g) &= \phi(a)^{t_b} \cdots x_1^{t_4} y_2^{t_3} x_1^{t_2} \cdot y_2^{t_1} \\ &= \phi(a)^{t_b} \cdots x_1^{t_4} y_2^{t_3} z_1^{t_2} \cdot x_3^{t_2} y_2^{t_1} \\ &= \phi(a)^{t_b} \cdots x_1^{t_4} z_2^{t_3} z_1^{t_2} \cdot y_4^{t_3} x_3^{t_2} y_2^{t_1} \\ &= \phi(a)^{t_b} \cdots z_3^{t_4} z_1^{t_4} z_2^{t_3} z_1^{t_2} \cdot x_5^{t_4} y_4^{t_3} x_3^{t_2} y_2^{t_1} \\ &\dots \\ &= (z_{b-1}^{t_b} \cdots z_i^{t_b}) \cdots z_3^{t_4} z_1^{t_4} z_2^{t_3} z_1^{t_2} \cdot a_{b+1}^{t_b} \cdots x_5^{t_4} y_4^{t_3} x_3^{t_2} y_2^{t_1} \end{aligned}$$

where  $i = 1$  if  $a = x$ , and  $i = 2$  if  $a = y$ . Let  $h' = (z_{b-1}^{t_b} \cdots z_i^{t_b}) \cdots z_3^{t_4} z_1^{t_4} z_2^{t_3} z_1^{t_2}$ , and let  $h = a_{b+1}^{t_b} \cdots x_5^{t_4} y_4^{t_3} x_3^{t_2} y_2^{t_1}$ . Note that  $h \in A_2$  and  $h' \in A_1$ . Therefore  $(h', h)$  is a  $(A_1, A_2)$ -sequence of  $\phi(g)$ . If  $\text{bh}(g) \neq 2k$ , then  $\text{bh}(\Phi(g)) \leq 2k$ . As above,  $\Omega_{\mathcal{B}} \phi(g) \Omega_{\mathcal{B}}^{-1}$  has a  $(A_1, A_2)$ -sequence  $(h', h)$ . Then  $(\Omega_{\mathcal{B}} h' \Omega_{\mathcal{B}}^{-1}, \Omega_{\mathcal{B}} h \Omega_{\mathcal{B}}^{-1})$  is a  $(A_2, A_1)$ -sequence of  $g$ .  $\square$

The *limit* of an unmovable element  $g \in M \setminus M_1$ , denoted by  $\lim(g)$ , is the unique integer  $t \geq 1$  such that  $\tilde{q}_{t-1} < \text{bh}(g) \leq \tilde{q}_t$ . Since  $\tilde{q}_{t-1} < \text{bh}(g)$ , then:

$$(6-5) \quad \lim(g) = \frac{\tilde{q}_{t-1} + m - 3}{m - 2} < \frac{\text{bh}(g) + m - 3}{m - 2}.$$

We set  $\lim(g_0) = 0$  for all  $g_0 \in M_1$ . Thus  $\text{bh}_{A_1, A_2}(\phi(g_0)) = \lim(g_0) + 1$ .

**Lemma 6.11** *Let  $g \in M \setminus M_1$  be an unmovable element. Then  $\phi(g)$  has a  $(A_1, A_2)$ -sequence of length  $\lim(g) + 1$ .*

**Proof** Let  $(g'_p, \dots, g'_0)$  be the alternating form of  $g$ , and let  $t = \lim(g)$ . Let  $g_0 = g'_0$ , for each  $i \in \{1, \dots, t-1\}$  let  $g_i = \prod_{j=1}^{m-2} g'_{(m-2)(i-1)+j}$ , and let  $g_t = g'_p \cdots g'_{(m-2)(t-1)+1}$ . Let  $i \in \{1, \dots, t-1\}$ . If  $i$  is odd, then  $R_{M_1}(g_i) = 1$  and  $\text{bh}(g_i) = m-1$ . Similarly, if  $i$  is even, then  $R_N(g_i) = 1$  and  $\text{bh}(g_i) = m-2$ . Furthermore,  $R_{M_1}(g_t) = 1$  and  $\text{bh}(g_t) \leq m-1$  if  $t$  is odd, and  $R_N(g_t) = 1$  and  $\text{bh}(g_t) \leq m-2$  if  $t$  is even. Lemma 6.10 implies that there are  $h_1, \dots, h_t \in A_2$  and  $h'_1, \dots, h'_t \in A_1$ , such that  $(h'_i, h_i)$  is a  $(A_1, A_2)$ -sequence of  $\phi(g_i)$  if  $i$  is odd, and  $(h_i, h'_i)$  is a  $(A_2, A_1)$ -sequence of  $\phi(g_i)$  if  $i$  is even. Therefore, the following is a  $(A_1, A_2)$ -sequence of  $\phi(g)$ :

$$\begin{cases} (h'_t, h_t h_{t-1}, \dots, h_3 h_2, h'_2 h'_1, h_1 g_0) & \text{if } t \text{ is odd} \\ (h_t, h'_t h'_{t-1}, \dots, h_3 h_2, h'_2 h'_1, h_1 g_0) & \text{if } t \text{ is even} \end{cases}$$

Note that the length of this sequence is  $t+1$ . □

**Lemma 6.12** *Let  $g \in G$ , and let  $t, t' \geq 1$  such that  $g = \Delta^{-t} g_0 = \Omega^{-t'} g'_0$  for some  $g_0, g'_0 \in M$ , where  $\Delta \not\prec_R g_0$  and  $\Omega \not\prec_R g'_0$ . Then  $\text{bh}(g_0) \geq 2\tilde{q}_t$  if and only if  $\text{bh}(g'_0) \geq \text{bh}(\Omega^{t'})$ .*

**Proof** If  $\Omega \not\prec_R g_0$  we are done because  $g_0 = g'_0$  and  $t' = 2t$ . Otherwise  $g_0 = \Omega g'_0$  and  $t' = 2t-1$ . Lemma ??-4-8 implies that  $\text{bh}(g_0) = \text{bh}(g'_0) + (m-2)$  when  $g'_0 \notin M_1$ , and  $\text{bh}(g_0) = \text{bh}(g'_0) + (m-1)$  when  $g'_0 \in M_1$ . If  $\text{bh}(g_0) \geq 2\tilde{q}_t = 2(m-2)t + 2$ , then  $\text{bh}(g'_0) \geq (m-2) + 1 \geq 2$ , i.e.  $g'_0 \notin M_1$ . Hence  $\text{bh}(g_0) = \text{bh}(g'_0) + (m-2)$ . So  $\text{bh}(g'_0) \geq (m-2)(2t-1) + 2 = \text{bh}(\Omega^{t'})$ . Conversely, if  $\text{bh}(g'_0) \geq \text{bh}(\Omega^{t'})$ , then  $g'_0 \notin M_1$ . Therefore  $\text{bh}(g_0) = \text{bh}(g'_0) + (m-2) \geq \text{bh}(\Omega^{t'}) + (m-2) = 2\tilde{q}_t$ . □

**Lemma 6.13** *Let  $g' \in G$  such that  $\phi(g')$  is  $z_1$ -positive. Then  $g'$  is  $(H, G_1)$ -positive.*

**Proof** Let  $m \in \mathbb{Z}$  such that  $g' = \Omega^m g$  for some  $g \in M$  with  $\Omega \not\prec_R g$ . Note that  $g' \notin G_1$  because  $\phi(g')$  is  $z_1$ -positive. We have  $\phi(g') = \Omega_B^m \phi(g)$ . If  $m \geq 0$ , then  $g' \in M \setminus M_1$ . Therefore  $g'$  is  $(H, G_1)$ -positive. If  $m < 0$ , namely  $m = -t$  for some  $t \geq 1$ ,

Theorem 2.13, Lemma 6.11 and Equation (6–5) imply that  $t + 2 \leq \text{bh}_{A_1, A_2}(\phi(g)) \leq \text{lim}(g) + 1 < (\text{bh}(g) + 2m - 5)/(m - 2)$ . Hence  $\text{bh}(\Omega') \leq \text{bh}(g)$ . So, Lemma 6.12 implies that  $g'$  is  $(H, G_1)$ -positive.  $\square$

If  $g \in G$  is  $(H, G_1)$ -positive and  $\phi(g)$  is  $z_1$ -negative. Then  $\phi(g)^{-1}$  is  $z_1$ -positive. So, Lemma 6.12 implies that  $g^{-1}$  is  $(H, G_1)$ -positive as well, which contradicts Corollary 6.2. This proves Proposition 6.4.

**Corollary 6.14** We have  $\text{bh}_{A_1, A_2}(\phi(g)) = \text{lim}(g) + 1$  for all unmovable  $g \in M \setminus M_1$ .

**Proof** Let  $g_0 \in M_1$  be the  $M_1$ -tale of  $g$ , and let  $g' = gg_0^{-1}$ . By definition,  $\tilde{q}_{t-1} < \text{bh}(g) = \text{bh}(g')$  where  $t = \text{lim}(g)$ . Lemma 6.12 implies that  $\Omega^{-t+1}g'$  is  $(H, G_1)$ -positive. Proposition 6.4 implies that  $\Omega_B^{-t+1}\phi(g')$  is  $z_1$ -positive. By Theorem 2.13 and Lemma 6.11,  $t+1 \leq \text{bh}_{A_1, A_2}(\phi(g')) \leq t+1$ . Therefore  $\text{bh}_{A_1, A_2}(\phi(g)) = \text{lim}(g)+1$ .  $\square$

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