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# Explicit formulas for $C^{1,1}$ Glaeser-Whitney extensions of 1-fields in Hilbert spaces.

ARIS DANILIDIS, MOUNIR HADDOU, ERWAN LE GRUYER, OLIVIER LEY

**Abstract.** We give a simple alternative proof for the  $C^{1,1}$ -convex extension problem which has been introduced and studied by D. Azagra and C. Mudarra [2]. As an application, we obtain an easy constructive proof for the Glaeser-Whitney problem of  $C^{1,1}$  extensions on a Hilbert space. In both cases we provide explicit formulae for the extensions. For the Glaeser-Whitney problem the obtained extension is almost minimal, that is, minimal up to a factor  $\frac{1+\sqrt{3}}{2}$  in the sense of Le Gruyer [15].

**Key words.** Whitney extension problem, convex extension, sup-inf convolution, semiconvex function.

**AMS Subject Classification** *Primary* 54C20 ; *Secondary* 52A41, 26B05, 26B25, 58C25.

## 1 Introduction

Determining a function (or a class of functions) of a certain regularity fitting to a prescribed set of data is one of the most challenging problems in modern mathematics. The origine of this problem is very old, since this general framework encompasses classical problems of applied analysis. Depending on the requested regularity, it goes from the Tietze extension theorem in normal topological spaces, where the required regularity is minimal (continuity), to results where the requested regularity is progressively increasing: McShane results on uniformly continuous, Hölder or Lipschitz extensions [19], Lipschitz extensions for vector-valued functions (Valentine [20]), differentiable and  $C^k$ -extensions (Whitney [22], Glaeser [12], and more recently Brudnyi-Shvartsman [7], Zobin [23], Fefferman [9]), monotone multi-valued extensions (Bauschke-Wang [5]), definable (in some o-minimal structure) Lipschitz extensions (Aschenbrenner-Fischer [1]), etc. In this work we are interested in the Glaeser-Whitney  $C^{1,1}$ -extension problem, which we describe below.

Let  $S$  be a nonempty subset of a Hilbert space  $\mathcal{H}$  and assume  $\alpha : S \rightarrow \mathbb{R}$  and  $v : S \rightarrow \mathcal{H}$  satisfy the so-called Glaeser-Whitney conditions:

$$\begin{cases} |\alpha(s_2) - \alpha(s_1) - \langle v(s_1), s_2 - s_1 \rangle| \leq K_1 |s_1 - s_2|^2 \\ |v(s_1) - v(s_2)| \leq K_2 |s_1 - s_2|. \end{cases} \quad (1.1)$$

In [12,22] it has been shown that under the above conditions, in case  $\mathcal{H} = \mathbb{R}^n$ , there exists a  $C^{1,1}$ -smooth function  $F : \mathbb{R}^n \rightarrow \mathbb{R}$  such that the prescribed Taylor polynomial  $(\alpha(s), v(s))$  coincides, at every  $s \in S$ , with the 1-Taylor field  $(F(s), \nabla F(s))$  of  $F$ . The above result has been extended to Hilbert spaces in Wells [21] and Le Gruyer [15]. In [15] a constant  $\Gamma^1(S, (\alpha, v))$ , completely determined by the initial data  $\{(\alpha(s), v(s)) : s \in S\}$ , has been introduced in a way that  $\Gamma^1(S, (\alpha, v)) < +\infty$  if and only if conditions (1.1) hold. Under this condition, it has been established the existence of a  $C^{1,1}$  function  $F : \mathcal{H} \rightarrow \mathbb{R}$  such that  $F|_S = \alpha$ ,  $\nabla F|_S = v$  and

$$\Gamma^1(\mathcal{H}, (F, \nabla F)) = \Gamma^1(S, (\alpha, v)). \quad (1.2)$$

Henceforth, every  $C^{1,1}$ -extension satisfying (1.2) will be called a minimal Glaeser-Whitney extension. The terminology is justified by the fact that  $\Gamma^1(S, (\alpha, v))$  eventually coincides with the Lipschitz constant of  $\nabla F$  on  $\mathcal{H}$ , that is,

$$\Gamma^1(\mathcal{H}, (F, \nabla F)) = \text{Lip}(\nabla F) \geq K_2, \quad (1.3)$$

when  $\mathcal{H}$  is finite dimensional and  $S \subset \mathcal{H}$  is a compact subset [15, Theorem 3.2]. An extension which is minimal upon a multiplication factor will be called almost minimal.

Recently, several works are interested in extensions with additional constraints: extensions with preserve positivity [10,11] or convexity [2,3]. In [2], D. Azagra and C. Mudarra considered the problem of finding a *convex*  $C^{1,1}$ -smooth extension over a prescribed Taylor polynomial  $(\alpha(s), v(s))_{s \in S}$  in a Hilbert space  $\mathcal{H}$  and established that the condition

$$\alpha(s_2) \geq \alpha(s_1) + \langle v(s_1), s_2 - s_1 \rangle + \frac{1}{2M} |v(s_1) - v(s_2)|^2, \quad \forall s_1, s_2 \in S, \quad (1.4)$$

is necessary and sufficient for the existence of such an extension.

Inspired from the recent work [2] concerning  $C^{1,1}$ -convex extensions, we revisit the classical Glaeser-Whitney problem. We first provide an alternative shorter proof of the result of [2] concerning  $C^{1,1}$ -convex extensions in Hilbert spaces by giving a simple explicit formula. This formula is heavily based on the regularization via sup-inf convolution in the spirit of Lasry-Lions [14] and can be efficiently computed, see Remark 2.2. As an easy

consequence, we obtain a direct proof for the classical  $C^{1,1}$ -Glaeser-Whitney problem in Hilbert spaces, which goes together with an explicit formula of the same type as for the convex extension problem. Let us mention that the previous proofs are quite involved both in finite dimension [12, 22] and Hilbert spaces [15, 21]. In the finite dimension case, a construction of the extension is proposed in [21] and some explicit formulae can be found in [16] but both are not tractable (see however the work [13] for concrete computations). Our approach also compares favorably to the result of [15], in which the existence of minimal extensions is established. On the other hand, the extension given by our explicit formula we may fail to be minimal —though it is almost minimal up to a multiplication factor.

Before we proceed, we recall that a function  $f : \mathcal{H} \rightarrow \mathbb{R}$  is called  $C_*$ -semiconvex (resp.,  $C^*$ -semiconcave) when, for all  $x, y \in H$ ,

$$f(y) - f(x) - \langle \nabla f(x), y - x \rangle \geq -\frac{C_*}{2}|x - y|^2 \quad (\text{resp.}, \leq \frac{C^*}{2}|x - y|^2).$$

This is equivalent to assert that  $f + \frac{C_*}{2}|x|^2$  is convex (respectively  $f - \frac{C^*}{2}|x|^2$  is concave). When  $f$  is both  $C$ -semiconvex and  $C$ -semiconcave, then  $f$  is  $C^{1,1}$  in  $\mathcal{H}$  with  $\text{Lip}(\nabla f) \leq C$  (for a proof of this latter result in finite dimension, see [8] and use the arguments of [14] to extend the result to Hilbert spaces).

Let us finally notice that the first condition in (1.1) means that the expected extension of  $(\alpha, v)$  shall be both  $2K_1$ -semiconvex and semiconcave, so must have a Lipschitz constant equal to  $2K_1$ . It is therefore natural, in view of the second condition in (1.1), to assume  $K_2 = 2K_1$  and that  $K_2$  is the minimal value for which (1.1) holds.

## 2 Convex $C^{1,1}$ extension of 1-fields

For any  $f : \mathcal{H} \rightarrow \mathbb{R}$  and  $\varepsilon > 0$ , we define respectively the sup and the inf-convolution of  $f$  by

$$f^\varepsilon(x) = \sup_{y \in \mathcal{H}} \left\{ f(y) - \frac{|y - x|^2}{2\varepsilon} \right\}, \quad f_\varepsilon(x) = \inf_{y \in \mathcal{H}} \left\{ f(y) + \frac{|y - x|^2}{2\varepsilon} \right\}.$$

**Theorem 2.1** ( $C^{1,1}$ -convex extension). *Let  $S$  be any nonempty subset of the Hilbert space  $\mathcal{H}$  and  $(\alpha(s), v(s))_{s \in S}$  be a 1-Taylor field on  $S$  satisfying (1.4) for some constant  $M > 0$ . Then*

$$f(x) = \sup_{s \in S} \{ \alpha(s) + \langle v(s), x - s \rangle \} \quad (2.1)$$

is the smallest continuous convex extension of  $(\alpha(s), v(s))_{s \in S}$  and

$$F(x) = \sup_{\varepsilon \in (0, \frac{1}{M})} (f^\varepsilon)_\varepsilon(x) = \sup_{\varepsilon \in (0, \frac{1}{M})} \inf_{z \in \mathcal{H}} \sup_{y \in \mathcal{H}} \left\{ f(z) - \frac{|y - z|^2}{2\varepsilon} + \frac{|z - x|^2}{2\varepsilon} \right\} \quad (2.2)$$

is a  $C^{1,1}$  convex extension of  $(\alpha(s), v(s))_{s \in S}$  in  $\mathcal{H}$ . Moreover,  $\text{Lip}(\nabla F) \leq M$ .

*Remark 2.2.* The inf-convolution corresponds to the well-known Moreau-Yosida regularization in convex analysis. It is also directly linked to the Legendre-Fenchel transform (convex conjugate). More on practical and theoretical properties of this regularization can be found in [17] and references therein. In practice,  $f_\varepsilon$ ,  $f^\varepsilon$  and therefore the formula (2.2) can be very efficiently computed using different techniques and algorithms such as [6] or [18].

*Proof of Theorem 2.1.* For all  $x \in \mathcal{H}$  and  $s_1, s_2 \in S$ , by (1.4), we have

$$\begin{aligned} & \alpha(s_1) + \langle v(s_1), x - s_1 \rangle \\ & \leq \alpha(s_2) + \langle v(s_2), x - s_2 \rangle + \langle v(s_1) - v(s_2), x - s_2 \rangle - \frac{1}{2M} |v(s_1) - v(s_2)|^2 \\ & \leq \alpha(s_2) + \langle v(s_2), x - s_2 \rangle + \sup_{\xi \in \mathcal{H}} \left\{ \langle \xi, x - s_2 \rangle - \frac{1}{2M} |\xi|^2 \right\} \\ & = \alpha(s_2) + \langle v(s_2), x - s_2 \rangle + \frac{M}{2} |x - s_2|^2. \end{aligned}$$

It follows that for all  $x \in \mathcal{H}$  and  $s \in S$

$$\alpha(s) + \langle v(s), x - s \rangle \leq f(x) \leq \alpha(s) + \langle v(s), x - s \rangle + \frac{M}{2} |x - s|^2. \quad (2.3)$$

In particular, the function  $f$  defined by (2.1) is convex, finite in  $\mathcal{H}$  and trapped between affine hyperplanes and quadratics with equality on  $S$ . Therefore, it is differentiable on  $S$  with  $f(s) = \alpha(s)$ ,  $\nabla f(s) = v(s)$  and it is clearly the smallest continuous convex extension of the field.

Setting  $q(x) = \alpha(s) + \langle v, x - s \rangle + \frac{M}{2} |x - s|^2$ , for  $\varepsilon \in (0, M^{-1})$ , straightforward computations lead to the formulae:

$$\begin{aligned} q^\varepsilon(x) &= \alpha(s) + \frac{1}{1 - \varepsilon M} \left( \frac{M}{2} |x - s|^2 + \langle v, x - s \rangle + \frac{\varepsilon}{2} |v|^2 \right), \\ q_\varepsilon(x) &= \alpha(s) + \frac{1}{1 + \varepsilon M} \left( \frac{M}{2} |x - s|^2 + \langle v, x - s \rangle - \frac{\varepsilon}{2} |v|^2 \right). \end{aligned} \quad (2.4)$$

In particular, after a new short computation, we deduce that

$$(q^\varepsilon)_\varepsilon = q, \quad (2.5)$$

and from (2.3), since the sup and inf-convolution are order-preserving operators, we obtain that for every  $\varepsilon \in (0, M^{-1})$ ,  $x \in \mathcal{H}$  and  $s \in S$ ,

$$\alpha(s) + \langle v(s), x - s \rangle \leq (f^\varepsilon)_\varepsilon(x) \leq \alpha(s) + \langle v(s), x - s \rangle + \frac{M}{2}|x - s|^2. \quad (2.6)$$

It follows that  $(f^\varepsilon)_\varepsilon$  is well-defined on  $\mathcal{H}$ . Notice also that

$$f \leq (f^\varepsilon)_\varepsilon \quad \text{in } \mathcal{H} \quad (2.7)$$

and that  $(f^\varepsilon)_\varepsilon$  is differentiable on  $S$  with  $(f^\varepsilon)_\varepsilon(s) = \alpha(s)$  and  $\nabla(f^\varepsilon)_\varepsilon(s) = v(s)$ , for every  $s \in S$ .

Notice that since  $f$  is defined as the supremum of the affine functions  $\ell(x) = \alpha + \langle v, x - s \rangle$  and  $\ell^\varepsilon(x) = \ell(x) + \frac{\varepsilon}{2}|v|^2$  by (2.4), we have

$$f^\varepsilon(x) = \sup_{y \in \mathcal{H}} \sup_{s \in S} \{\ell(x)\} = \sup_{s \in S} \{\ell^\varepsilon(x)\},$$

which proves that  $f^\varepsilon$  is convex. Therefore,  $(f^\varepsilon)_\varepsilon$  is still convex, being the infimum with respect to  $y$  of the jointly convex functions

$$f^\varepsilon(y) + \frac{1}{2\varepsilon}|y - x|^2, \quad (x, y) \in \mathcal{H} \times \mathcal{H}.$$

It is well-known [14] that the sup and inf-convolution satisfy some semigroup properties,

$$f^{\varepsilon+\varepsilon'} = (f^\varepsilon)^{\varepsilon'} \quad \text{and} \quad f_{\varepsilon+\varepsilon'} = (f_\varepsilon)_{\varepsilon'} \quad \text{for all } \varepsilon, \varepsilon' > 0.$$

Therefore, for  $0 < \varepsilon < \varepsilon'$ ,  $f^{\varepsilon'} = (f^\varepsilon)^{\varepsilon'-\varepsilon}$ . By (2.7), we infer  $((f^\varepsilon)^{\varepsilon'-\varepsilon})_{\varepsilon'-\varepsilon} \geq f^\varepsilon$ . It follows

$$((f^{\varepsilon'})_{\varepsilon'-\varepsilon})_\varepsilon = (f^{\varepsilon'})_{\varepsilon'} \geq (f^\varepsilon)_\varepsilon \quad \text{for all } 0 < \varepsilon < \varepsilon'.$$

We conclude that  $\varepsilon \mapsto (f^\varepsilon)_\varepsilon$  is nondecreasing on  $(0, M^{-1})$  so  $F$  is well defined, convex and still satisfies (2.6). Therefore  $F$  is an extension of  $(\alpha(s), v(s))_{s \in S}$  in  $\mathcal{H}$  and is differentiable on  $S$ .

It remains to prove that  $F$  is  $C^{1,1}$  in  $\mathcal{H}$  and to estimate  $\text{Lip}(\nabla F)$ . From [14], we know that the inf-convolution  $(f^\varepsilon)_\varepsilon$  of  $f^\varepsilon$  is  $\varepsilon^{-1}$ -semiconcave. Since  $(f^\varepsilon)_\varepsilon$  is also convex, it means that  $(f^\varepsilon)_\varepsilon$  is both  $\varepsilon^{-1}$ -semiconcave and  $\varepsilon^{-1}$ -semiconvex. Therefore  $(f^\varepsilon)_\varepsilon$  is  $C^{1,1}$  in  $\mathcal{H}$  with  $\text{Lip}(\nabla(f^\varepsilon)_\varepsilon) \leq \varepsilon^{-1}$ . Since  $(f^\varepsilon)_\varepsilon - \frac{1}{2\varepsilon}|x|^2$  is concave for every  $0 < \varepsilon < M^{-1}$ , sending  $\varepsilon \nearrow M^{-1}$ , we conclude that  $F$  is  $M$ -semiconcave. Since  $F$  is also convex, the previous arguments allow to conclude that  $F$  is  $C^{1,1}$  in  $\mathcal{H}$  with  $\text{Lip}(\nabla F) \leq M$ .  $\square$

*Remark 2.3.* In [14], the  $C^{1,1}$  regularization result is stated for  $(f^\varepsilon)_\delta$  with  $0 < \delta < \varepsilon$ . To obtain an extension in our framework, we need to take  $\delta = \varepsilon$ . The fact that we are able to take  $\delta$  up to  $\varepsilon$  without losing the  $C^{1,1}$  regularity relies strongly on the convexity of  $f$ . Since the convexity is preserved under the sup and inf-convolution operations, the inf-convolution does not affect the semiconvexity property of  $f^\varepsilon$  even for  $\delta = \varepsilon$ . For this reason, one cannot reverse the above operations: more precisely, the function  $(f_\varepsilon)^\varepsilon = f$  is not semiconcave.

### 3 $C^{1,1}$ extension of 1-fields: explicit formulae

Let us now apply the previous result to obtain a general  $C^{1,1}$ -extension in the Glaeser-Whitney problem.

**Theorem 3.1** ( $C^{1,1}$ -Glaeser-Whitney almost minimal extension). *Let  $S$  be a nonempty subset of a Hilbert space  $\mathcal{H}$  and  $(\alpha, v)$  be a 1-Taylor field on  $S$  satisfying (1.1) with  $K_2 = 2K_1$ . Then, the function*

$$G(x) = F(x) - \frac{1 + \sqrt{3}}{4}|x|^2,$$

is an explicit  $C^{1,1}$  extension of the 1-field  $(\alpha, v)$  satisfying

$$(\Gamma^1(S, (\alpha, v)) \leq) \Gamma^1(\mathcal{H}, (G, \nabla G)) = \text{Lip}(\nabla G) \leq \left( \frac{1 + \sqrt{3}}{2} \right) \Gamma^1(S, (\alpha, v)),$$

where  $F$  is the convex extension of the 1-Taylor field (c.f. Theorem 2.1)

$$(\alpha(s) + \frac{1 + \sqrt{3}}{4}|s|^2, v(s) + \frac{1 + \sqrt{3}}{2}s)_{s \in S}.$$

*Proof of Theorem 3.1.* We check that for every  $\mu > 2K_1$  the 1-Taylor field

$$(\tilde{\alpha}(s), \tilde{v}(s)) := (\alpha(s) + \frac{\mu}{2}|s|^2, v(s) + \mu s)$$

satisfies (1.4) with  $M = K_2^2(\mu - 2K_1)^{-1}$ . Indeed, for any  $s_1, s_2 \in S$  we obtain, using (1.1),

$$\begin{aligned} & \tilde{\alpha}(s_2) - \tilde{\alpha}(s_1) - \langle \tilde{v}(s_1), s_2 - s_1 \rangle \\ &= \alpha(s_2) - \alpha(s_1) - \langle v(s_1), s_2 - s_1 \rangle + \frac{\mu}{2} (|s_2|^2 - |s_1|^2 - 2\langle s_1, s_2 - s_1 \rangle) \\ &\geq \left( \frac{\mu - 2K_1}{2} \right) |s_1 - s_2|^2 \geq \frac{1}{2} \left( \frac{\mu - 2K_1}{K_2^2} \right) |v(s_1) - v(s_2)|^2. \end{aligned}$$

Thus, the function  $F$  given by Theorem 2.1 is a  $C^{1,1}$ -convex extension of  $(\tilde{\alpha}(s), \tilde{v}(s))$  satisfying  $F|_S = \tilde{\alpha}$ ,  $\nabla F|_S = \tilde{v}$  and  $\text{Lip}(\nabla F) \leq K_2^2(\mu - 2K_1)^{-1}$ . Therefore  $G(x) = F(x) - \frac{\mu}{2}|x|^2$  satisfies  $G|_S = \alpha$ ,  $\nabla G|_S = v$ . Moreover,  $G$  is  $\left(\frac{K_2^2}{\mu - 2K_1} - \mu\right)$ -semiconcave and  $\mu$ -semiconvex (since  $F$  is convex). We deduce that

$$\text{Lip}(\nabla G) \leq \max \left\{ \mu, \frac{K_2^2}{\mu - 2K_1} - \mu \right\}.$$

Minimizing the above quantity on  $\mu \in (2K_1, +\infty)$  yields an optimal bound

$$\text{Lip}(\nabla G) \leq K_1 \left( 1 + \sqrt{1 + 2 \left( \frac{K_2}{2K_1} \right)^2} \right) = \left( \frac{1 + \sqrt{3}}{2} \right) K_2,$$

recalling  $K_2 = 2K_1$ . The result follows in view of (1.3).  $\square$

## 4 Limitations of the sup-inf approach

The main result (Theorem 3.1) is heavily based on the explicit construction of a  $C^{1,1}$ -convex extension of a 1-Taylor field  $(\alpha, v)$  satisfying (1.4), which in turn, is based on the sup-inf convolution approach. The reader might wonder whether our approach can be adapted to include cases where less regularity is required, as for instance  $C^{1,\theta}$ -extensions, that is, extensions to a  $C^1$ -function whose derivative has a Hölder modulus of continuity with exponent  $\theta \in (0, 1)$ . The existence of such *convex* extensions (and even  $C^{1,\omega}$  convex extensions with a general modulus of continuity  $\omega$ ) was established in finite dimensions in Azagra-Mudarra [3] by means of involved arguments. Indeed, it would be natural to endeavor an adaptation of formula (2.2) to treat the problem of  $C^{1,\theta}$ -convex extensions, for  $0 < \theta < 1$ . According to [3], the adequate condition, analogous to (1.4), is that the 1-Taylor field has to satisfy, for some  $M > 0$ ,

$$\alpha(s_2) \geq \alpha(s_1) + \langle v(s_1), s_2 - s_1 \rangle + \frac{\theta}{(1 + \theta)M^{1/\theta}} |v(s_1) - v(s_2)|^{1+\frac{1}{\theta}}. \quad (4.1)$$

Unfortunately, the technique developed in Section 2 is specific to the  $C^{1,1}$ -regularity and cannot be easily adapted to this more general case. Let us briefly explain the reason.

Considering the suitable sup and inf-convolutions

$$f^\varepsilon(x) = \sup_{y \in \mathcal{H}} \left\{ f(y) - \frac{|y - x|^{1+\theta}}{(1 + \theta)\varepsilon^\theta} \right\}, \quad f_\varepsilon(x) = \inf_{y \in \mathcal{H}} \left\{ f(y) + \frac{|y - x|^{1+\theta}}{(1 + \theta)\varepsilon^\theta} \right\},$$



all arguments of the proof of Theorem 2.1 go through except (2.5), which fails to hold in this general case. More precisely, the convex extension  $f$  defined by (2.1) satisfies

$$l(x) \leq f(x) \leq q(x) \quad \text{for all } x \in \mathcal{H} \text{ and } s \in S, \quad (4.2)$$

with equalities for  $x = s$ , where

$$l(x) := \alpha(s) + \langle v(s), x - s \rangle \quad (4.3)$$

$$q(x) := \alpha(s) + \langle v(s), x - s \rangle + \frac{M}{1 + \theta} |x - s|^{1+\theta}. \quad (4.4)$$

Therefore for every  $\varepsilon > 0$  such that  $M\varepsilon^\theta < 1$ , we have

$$l(x) \leq (f^\varepsilon)_\varepsilon(x) \leq (q^\varepsilon)_\varepsilon(x).$$

Nonetheless, we may now possibly have

$$(q^\varepsilon)_\varepsilon(s) \neq q(s), \quad (4.5)$$

yielding that  $(f^\varepsilon)_\varepsilon$  is a  $C^{1,\theta}$ -convex function but may differ from  $f$  on  $S$ , hence it is not an extension of the latter. Let us underline that the problem arises even in dimension 1 and even for small  $\varepsilon$ . In particular, the sup-convolution  $q^\varepsilon$  may develop singularities for arbitrary small  $\varepsilon$  so that  $q^\varepsilon$  is not anymore in the same class as  $q$ , contrary to the quadratic case (see (2.4)).

*Remark 4.1.* Recalling [14] that  $u(x, t) := q^t(x)$  is a viscosity solution to the Hamilton-Jacobi equation  $\partial_t u - \frac{\theta}{1+\theta} |\nabla u|^{1+\frac{1}{\theta}} = 0$  in  $\mathcal{H} \times (0, \varepsilon)$ , we obtain an explicit example where the solutions develop singularities instantaneously, even when starting with a  $C^{1,\theta}$  initial condition  $u(x, 0) = q(x)$ . See [4] for related comments.

*Sketch of proof of the Claim (4.5).* Without loss of generality we may assume that  $\alpha(s) = 0$  and  $s = 0$  in (4.2). Fix  $v \neq 0$ . Assume by contradiction that  $(q^\varepsilon)_\varepsilon(0) = q(0) = l(0) = 0$ . Then, since  $q$  is a  $C^{1,\theta}$  function, necessarily,  $\nabla(q^\varepsilon)_\varepsilon(0) = \nabla l(0) = v$ . Using that  $y \mapsto q^\varepsilon(y) + \frac{|y|^{1+\theta}}{(1+\theta)\varepsilon^\theta}$  is a strictly convex function achieving a unique minimum  $\bar{y}$  in  $\mathcal{H}$ , we obtain that

$$(q^\varepsilon)_\varepsilon(0) = q^\varepsilon(\bar{y}) + \frac{|\bar{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} = \sup_{y \in \mathcal{H}} \left\{ q(y) - \frac{|y - \bar{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} \right\} + \frac{|\bar{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} = 0,$$

$$\nabla(q^\varepsilon)_\varepsilon(0) = -\frac{\bar{y}|\bar{y}|^{\theta-1}}{\varepsilon^\theta} = v,$$

yielding  $\bar{y} = -\varepsilon v|v|^{\frac{1}{\theta}-1} \neq 0$ . To prove the claim, it is enough to find some  $y \in \mathcal{H}$  such that

$$\varphi(y) := q(y) - \frac{|y - \bar{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} + \frac{|\bar{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} > 0.$$

In particular, let us seek for  $y = \lambda\bar{y}$  where  $\lambda \in \mathbb{R}$  is small. (Notice that this guarantees that the computation would also hold when  $\mathcal{H}$  is one dimensional.) We have

$$\begin{aligned} (q^\varepsilon)_\varepsilon(0) &\geq \varphi(y) = \frac{|\bar{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} \left( M\varepsilon^\theta |\lambda|^{1+\theta} - (1+\theta)\lambda - |\lambda - 1|^{1+\theta} + 1 \right) \\ &= \frac{|\bar{y}|^{1+\theta}}{(1+\theta)\varepsilon^\theta} \left( M\varepsilon^\theta |\lambda|^{1+\theta} - \frac{1}{2}(1+\theta)\theta\lambda^2 + o(\lambda^2) \right) > 0 = q(0), \end{aligned}$$

at least for small  $\lambda > 0$ . □

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Aris Daniilidis

DIM-CMM, UMI CNRS 2807

Beauchef 851 (Torre Norte, piso 5), Universidad de Chile, Santiago de Chile.

E-mail: [arisd@dim.uchile.cl](mailto:arisd@dim.uchile.cl)

<http://www.dim.uchile.cl/~arisd>

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Mounir Haddou, Olivier Ley, Erwan Le Gruyer

IRMAR, INSA Rennes, CNRS UMR 6625

20 avenue des Buttes de Coesmes, F-35708 Rennes, France

E-mail: [{mounir.haddou, olivier.ley, erwan.le-gruyer}@insa-rennes.fr](mailto:{mounir.haddou, olivier.ley, erwan.le-gruyer}@insa-rennes.fr)

<http://{haddou, ley}.perso.math.cnrs.fr/>