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A Complete Axiomatisation of the ZX-Calculus for Clifford+T Quantum Mechanics

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Abstract. We introduce the first complete and approximatively universal diagrammatic language for quantum mechanics. We make the ZX-Calculus, a diagrammatic language introduced by Coecke and Duncan, complete for the so-called Clifford+T quantum mechanics by adding four new axioms to the language. The completeness of the ZX-Calculus for Clifford+T quantum mechanics was one of the main open questions in *categorical quantum mechanics*. We prove the completeness of the $\frac{\pi}{4}$ -fragment of the ZX-Calculus using the recently studied ZW-Calculus, a calculus dealing with integer matrices. We also prove that the $\frac{\pi}{4}$ -fragment of the ZX-Calculus represents exactly all the matrices over some finite dimensional extension of the ring of dyadic rationals.

1 Introduction

The ZX-Calculus is a powerful graphical language for quantum reasoning and quantum computing introduced by Bob Coecke and Ross Duncan [5]. The language comes with a way of interpreting any ZX-diagram as a matrix – called the *standard interpretation*. Two diagrams represent the same quantum evolution if and only if they have the same standard interpretation. The language is also equipped with a set of axioms – transformation rules – which are sound, i.e. they preserve the standard interpretation. Their purpose is to explain how a diagram can be transformed into an equivalent one.

The ZX-calculus has several applications in quantum information processing (e.g. MBQC [9], quantum codes [8,4,13,7]), and can be used through the interactive theorem prover Quantomatic [15,16]. However, the main obstacle to the wider use of the ZX-calculus was the absence of a *completeness* result for a *universal* fragment of quantum mechanics, to guarantee that any provable property is provable using the ZX-calculus. More precisely *completeness* is the converse of the soundness: the language would be complete if, given any two diagrams representing the same matrix, one can transform one diagram into the other using the axioms of the language. Completeness is crucial, it means in particular that all the fundamental properties of quantum mechanics are captured by the graphical rules.

ZX-Calculus have been proved to be incomplete in general [19], and despite of the necessary axioms that have since been identified [17,14], there is very little hope that it can be completed with a countable set of axioms. However, several fragments of the language have been proved to be complete ($\frac{\pi}{2}$ -fragment [1]; π -fragment [10]; single-qubit $\frac{\pi}{4}$ -fragment [2]), but none of them is *universal* for quantum mechanics, even approximatively. In particular all quantum algorithms expressible in these fragments are efficiently simulable on a classical computer.

As a consequence, most of the attention has been paid to find a complete axiomatisation of $\frac{\pi}{4}$ -fragment of the ZX-Calculus for the Clifford+T quantum mechanics, the simplest approximatively universal fragment of quantum mechanics, which is widely used in quantum computing. A first result of completeness on a single-qubit has been given [2] but to this day the completeness of the many-qubit case has been an open question. We answer this question in the following thanks to the help of the ZW-Calculus, another graphical language – based on the interactions of the so-called GHZ and W states [6]. The ZW-Calculus has been proven to be complete [12] but its diagrams only represent matrices over \mathbb{Z} , and hence is not approximatively universal.

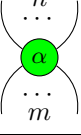
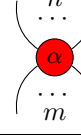





The paper is structured as follows: A ZX-Calculus augmented with four new axioms is presented in section 2. Section 3 gives a general overview of the completeness proof. In section 4, we introduce an extension of the ZW-Calculus that deals with matrices over dyadic rational numbers $\mathbb{D} = \mathbb{Z}[1/2]$ and show its completeness. Sections 5 and 6 are presenting a back and forth translation between

the ZX- and ZW-calculi, from which we deduce the completeness of the ZX-Calculus for Clifford+T quantum mechanics in section 7. Finally, in section 8 we characterise the exact expressive power of the $\frac{\pi}{4}$ -fragment of the ZX-Calculus: the diagrams of this fragment represent exactly the matrices over $\mathbb{D}[e^{i\frac{\pi}{4}}]$.

2 ZX-Calculus

2.1 Diagrams and standard interpretation

A ZX-diagram $D : k \rightarrow l$ with k inputs and l outputs is generated by:

$R_Z^{(n,m)}(\alpha) : n \rightarrow m$		$R_X^{(n,m)}(\alpha) : n \rightarrow m$	
$H : 1 \rightarrow 1$		$e : 0 \rightarrow 0$	
$\mathbb{I} : 1 \rightarrow 1$		$\sigma : 2 \rightarrow 2$	
$\epsilon : 2 \rightarrow 0$		$\eta : 0 \rightarrow 2$	

where $n, m \in \mathbb{N}$ and $\alpha \in \mathbb{R}$

and the two compositions:

- Spacial Composition: for any $D_1 : a \rightarrow b$ and $D_2 : c \rightarrow d$, $D_1 \otimes D_2 : a + c \rightarrow b + d$ consists in placing D_1 and D_2 side by side, D_2 on the right of D_1 .
- Sequential Composition: for any $D_1 : a \rightarrow b$ and $D_2 : b \rightarrow c$, $D_2 \circ D_1 : a \rightarrow c$ consists in placing D_1 on the top of D_2 , connecting the outputs of D_1 to the inputs of D_2 .

The standard interpretation of the ZX-diagrams associates to any diagram $D : n \rightarrow m$ a linear map $\llbracket D \rrbracket : \mathbb{C}^{2^n} \rightarrow \mathbb{C}^{2^m}$ inductively defined as follows:

$$\llbracket D_1 \otimes D_2 \rrbracket := \llbracket D_1 \rrbracket \otimes \llbracket D_2 \rrbracket \quad \llbracket D_2 \circ D_1 \rrbracket := \llbracket D_2 \rrbracket \circ \llbracket D_1 \rrbracket \quad \llbracket \begin{array}{|c|} \hline \square \\ \hline \end{array} \rrbracket := (1) \quad \llbracket \begin{array}{|c|} \hline | \\ \hline \end{array} \rrbracket := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\llbracket \begin{array}{|c|} \hline | \\ \hline \text{yellow square} \\ \hline \end{array} \rrbracket := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \llbracket \begin{array}{|c|} \hline \text{crossing} \\ \hline \end{array} \rrbracket := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \llbracket \begin{array}{|c|} \hline \text{cup} \\ \hline \end{array} \rrbracket := (1 \ 0 \ 0 \ 1) \quad \llbracket \begin{array}{|c|} \hline \text{cap} \\ \hline \end{array} \rrbracket := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\llbracket \begin{array}{|c|} \hline \text{green circle } \alpha \\ \hline \end{array} \rrbracket := (1 + e^{i\alpha}) \quad \llbracket \begin{array}{|c|} \hline \text{red circle } \alpha \\ \hline \end{array} \rrbracket := 2^m \overbrace{\begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & e^{i\alpha} \end{pmatrix}}^{2^n} \quad (n + m > 0)$$

$$\text{For any } n, m \geq 0 \text{ and } \alpha \in \mathbb{R}, \quad \llbracket \begin{array}{|c|} \hline \text{red circle } \alpha \\ \hline \end{array} \rrbracket = \llbracket \begin{array}{|c|} \hline | \\ \hline \text{yellow square} \\ \hline \end{array} \rrbracket^{\otimes m} \circ \llbracket \begin{array}{|c|} \hline \text{green circle } \alpha \\ \hline \end{array} \rrbracket \circ \llbracket \begin{array}{|c|} \hline | \\ \hline \text{yellow square} \\ \hline \end{array} \rrbracket^{\otimes n}$$

(where $M^{\otimes 0} = (1)$ and $M^{\otimes k} = M \otimes M^{\otimes k-1}$ for any $k \in \mathbb{N}^*$).

To simplify, the red and green nodes will be represented empty when holding a 0 angle:



Also in order to make the diagrams a little less heavy, when n copies of the same sub-diagram occur, we will use the notation $(.)^{\otimes n}$.

ZX-Diagrams are universal:

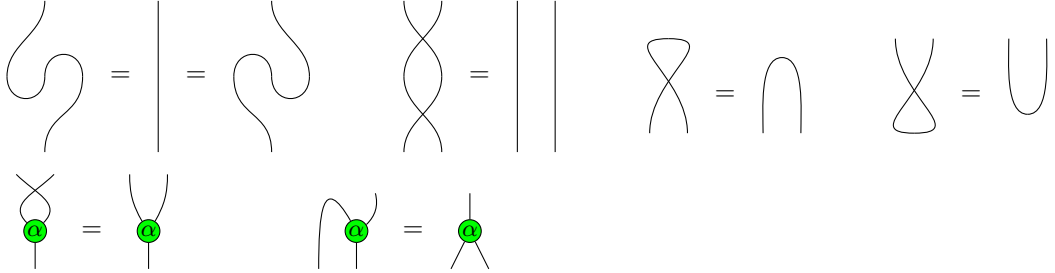
$$\forall A \in \mathbb{C}^{2^n} \times \mathbb{C}^{2^m}, \exists D, \llbracket D \rrbracket = A$$

This implies dealing with an uncountable set of angles, so it is generally preferred to work with *approximate* universality – the ability to approximate any linear map with arbitrary accuracy – in which only a finite set of angles is involved. The $\frac{\pi}{4}$ -fragment – ZX-diagrams where all angles are multiples of $\frac{\pi}{4}$ – is one such approximately universal fragment, whereas the $\frac{\pi}{2}$ -fragment is not.

2.2 Calculus

The diagrammatic representation of a matrix is not unique in the ZX-Calculus. As a consequence the language comes with a set of axioms. Additionally to the main axioms of the language described in Figure 1, one can:

- bend any wire of a ZX-diagram at will, without changing its semantics. This paradigm – the so-called **Only Topology Matters** – can be derived from the following axioms:



- apply the axioms to sub-diagrams. For any D_1, D_2 , and D ,

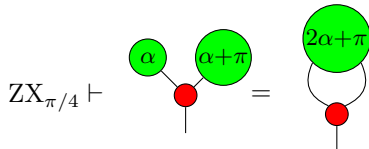
$$(ZX \vdash D_1 = D_2) \Rightarrow \begin{cases} (ZX \vdash D_1 \circ D = D_2 \circ D) \wedge (ZX \vdash D \circ D_1 = D \circ D_2) \\ (ZX \vdash D_1 \otimes D = D_2 \otimes D) \wedge (ZX \vdash D \otimes D_1 = D \otimes D_2) \end{cases}$$

where $ZX \vdash D_1 = D_2$ means that D_1 can be transformed into D_2 using the axioms of the ZX-Calculus.

In the following, $ZX_{\pi/4}$ will denote either the set of ZX-diagrams in the $\frac{\pi}{4}$ -fragment or the set of axioms in figure 1, whenever it makes sense.

2.3 What's new?

We introduce in this paper a new axiomatisation of the ZX-Calculus. We briefly review here the differences with the previous version of the ZX-Calculus. Since we are only interested in the $\frac{\pi}{4}$ -fragment of the ZX-Calculus in this paper, all the axioms which are not expressible with angles multiple of $\frac{\pi}{4}$, like the generalised supplementarity [14], are ignored. The three axioms (C1,C2,C3) given in Figure 1 are new axioms, for which we don't know any derivation using the previous axiomatisations of the language. (SUP) subsumes the previous supplementarity axiom introduced in [17] (proof is given in appendix at page 14):



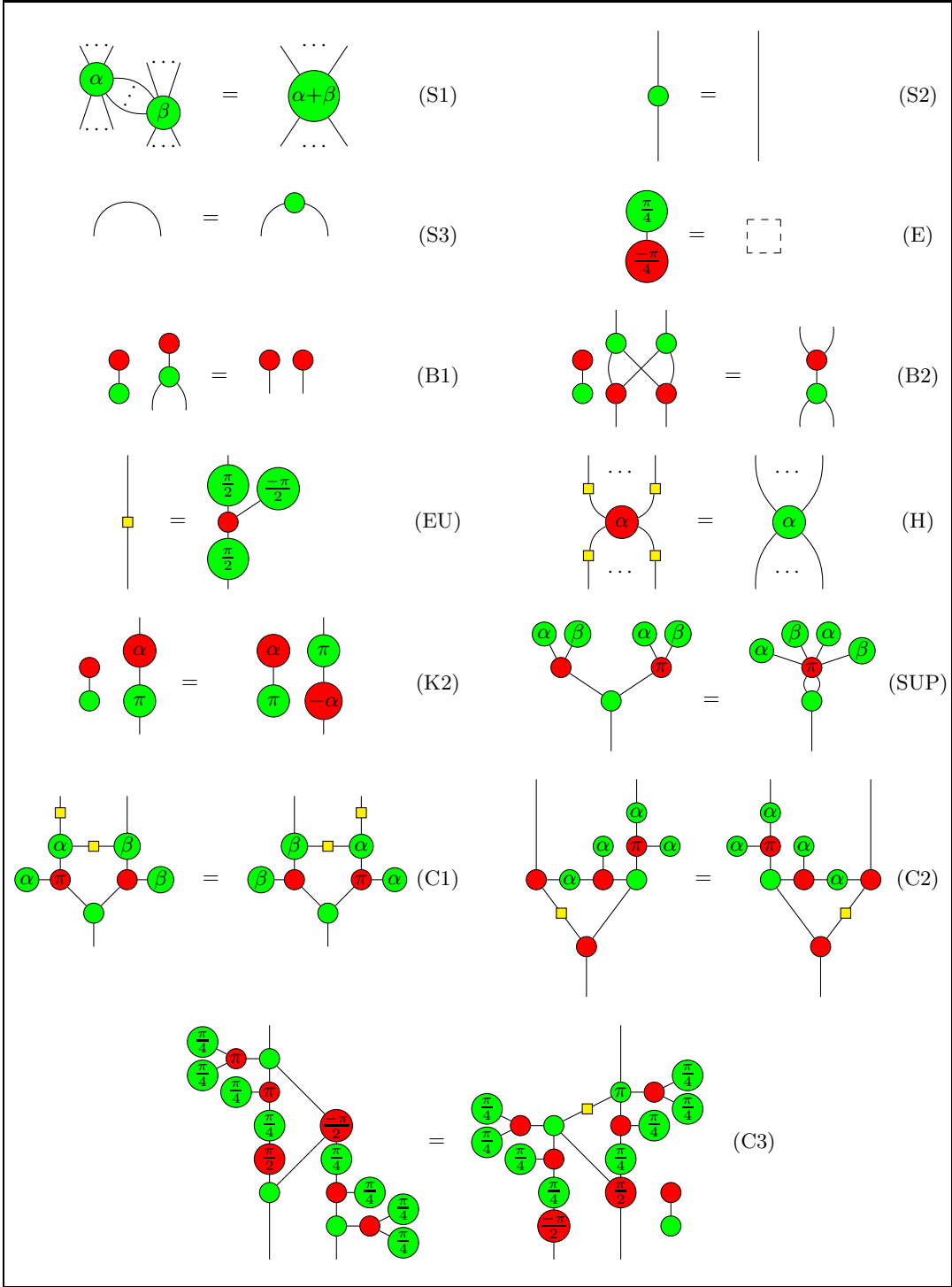


Fig. 1. Set of rules for the ZX-Calculus with scalars. All of these rules also hold when flipped upside-down, or with the colours red and green swapped. The right-hand side of (E) is an empty diagram. (...) denote zero or more wires, while (·) denote one or more wires.

2.4 Soundness and Completeness

It's a routine to prove the soundness of the axioms of the ZX-Calculus given in Figure 1. The main result of the paper is the completeness of this axiomatisation for Clifford+T quantum mechanics:

Theorem 1. *The $\frac{\pi}{4}$ -fragment of the ZX-Calculus as presented in Figure 1 is complete: for any two diagrams D_1, D_2 whose angles are multiples of $\frac{\pi}{4}$, $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$ iff $\text{ZX}_{\pi/4} \vdash D_1 = D_2$.*

The rest of the paper is devoted to the proof of the theorem. A general overview of the proof is given in the next section.

3 A Bird's Eye View of the Proof of Theorem 1

The proof uses the completeness result of the ZW-Calculus, a calculus dealing with matrices with integer coefficients. The syntax and semantics of the ZW-Calculus are presented in section 4.

We start by slightly changing the ZW-Calculus to obtain a new language, the $\text{ZW}_{1/2}$ -calculus, that is able to express matrices with dyadic coefficients, i.e. rational numbers of the form $p/2^q$. This is done merely by adding a symbolic inverse to the scalar 2. We then prove that this new language is complete:

Part 1 (Proposition 1) *The $\text{ZW}_{1/2}$ -calculus is complete: for two diagrams D_1, D_2 of the $\text{ZW}_{1/2}$ -calculus, we have $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$ iff $\text{ZW}_{1/2} \vdash D_1 = D_2$.*

This is done in subsection 4.3.

We now introduce two interpretations, from $\text{ZX}_{\pi/4}$ to $\text{ZW}_{1/2}$ and back.

First, we provide an interpretation $\llbracket \cdot \rrbracket_{XW}$ from $\text{ZX}_{\pi/4}$ to $\text{ZW}_{1/2}$ that transforms $\text{ZX}_{\pi/4}$ -diagrams of type $k \rightarrow l$ to $\text{ZW}_{1/2}$ -diagrams of type $k+2 \rightarrow l+2$. This interpretation is sound in the following sense:

Part 2 (Proposition 5) *Let D_1, D_2 be two diagrams of the $\text{ZX}_{\pi/4}$ -calculus.*

Then $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$ iff $\llbracket \llbracket D_1 \rrbracket_{XW} \rrbracket = \llbracket \llbracket D_2 \rrbracket_{XW} \rrbracket$.

The encoding is nontrivial as the $\text{ZX}_{\pi/4}$ -Calculus expresses matrices with complex coefficients, and the $\text{ZW}_{1/2}$ -calculus is only able to express matrices with dyadic coefficients. It turns out that coefficients involved in matrices of the $\text{ZX}_{\pi/4}$ -Calculus are actually in a vector space (more accurately a module) of dimension 4 over the set of dyadic numbers, so that every complex coefficient will be represented by a 4×4 matrix with dyadic coefficients. This encoding is done in section 5.

We then provide an interpretation $\llbracket \cdot \rrbracket_{WX}$ from $\text{ZW}_{1/2}$ to $\text{ZX}_{\pi/4}$. This interpretation preserves both semantics and provability:

Part 3 (Proposition 6 and 7) *Let D_1, D_2 be two diagrams of the $\text{ZW}_{1/2}$ -calculus.*

Then $\llbracket \llbracket D_i \rrbracket_{WX} \rrbracket = \llbracket D_i \rrbracket$.

Furthermore, if $\text{ZW}_{1/2} \vdash D_1 = D_2$ then $\text{ZX}_{\pi/4} \vdash \llbracket D_1 \rrbracket_{WX} = \llbracket D_2 \rrbracket_{WX}$

This is done in section 6.

The composition of the two interpretations does not give back the starting diagram (we obtain after all a diagram with two more inputs and outputs), but we obtain something which is close to the original diagram, so that we can (provably) recover it. In fact

Part 4 (Corollary 1) *Let D_1, D_2 be a diagram of the ZX-Calculus. If $\text{ZX}_{\pi/4} \vdash \llbracket \llbracket D_1 \rrbracket_{XW} \rrbracket_{WX} = \llbracket \llbracket D_2 \rrbracket_{XW} \rrbracket_{WX}$ then $\text{ZX}_{\pi/4} \vdash D_1 = D_2$.*

Our main theorem is now obvious:

Proof (of Theorem 1). Let D_1, D_2 be two diagrams of the $\text{ZX}_{\pi/4}$ -Calculus s.t. $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$. By Part 2, $\llbracket \llbracket D_1 \rrbracket_{XW} \rrbracket = \llbracket \llbracket D_2 \rrbracket_{XW} \rrbracket$. By Part 1, the $\text{ZW}_{1/2}$ -calculus is complete and therefore $\text{ZW}_{1/2} \vdash \llbracket D_1 \rrbracket_{XW} = \llbracket D_2 \rrbracket_{XW}$. By Part 3, $\text{ZX}_{\pi/4} \vdash \llbracket \llbracket D_1 \rrbracket_{XW} \rrbracket_{WX} = \llbracket \llbracket D_2 \rrbracket_{XW} \rrbracket_{WX}$. By Part 4 this implies $\text{ZX}_{\pi/4} \vdash D_1 = D_2$. \square

The new rules of the $\text{ZX}_{\pi/4}$ -Calculus we introduced have obviously been chosen for Parts 4 and 3 to hold. However they have been greatly simplified from what one can obtain using a naive approach.

4 ZW-Calculus

4.1 Diagrams and Standard Interpretation

The ZW-Calculus has been introduced by Amar Hadzihasanovic in 2015 [12] and is based on the GHZ/W-Calculus [6]. We will present here the expanded version of this calculus. To stay coherent with the previous definition of the ZX-Calculus, we will assume that the time flows from top to bottom – which is the opposite of the original definition. It has the following finite set of generators:

$$T_e = \left\{ \begin{array}{c} | \\ \circ \\ | \end{array} \right\}, \begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array}, \begin{array}{c} | \\ \bullet \\ | \end{array}, \begin{array}{c} \diagdown \\ \bullet \\ \diagup \end{array}, |, \cup, \cap, \times, \times, \boxed{} \right\}$$

and diagrams are created thanks to the two same – spacial and sequential – compositions.

As for the ZX-Calculus, we define a standard interpretation, that associates to any diagram of the ZW-Calculus D with n inputs and m outputs, a linear map $\llbracket D \rrbracket : \mathbb{Z}^{2^n} \rightarrow \mathbb{Z}^{2^m}$, inductively defined as:

$$\begin{array}{l} \llbracket D_1 \otimes D_2 \rrbracket := \llbracket D_1 \rrbracket \otimes \llbracket D_2 \rrbracket \quad \llbracket D_2 \circ D_1 \rrbracket := \llbracket D_2 \rrbracket \circ \llbracket D_1 \rrbracket \quad \llbracket \boxed{} \rrbracket := (1) \quad \llbracket | \rrbracket := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \\ \llbracket \times \rrbracket := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \llbracket \times \rrbracket := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \llbracket \cup \rrbracket := (1 \ 0 \ 0 \ 1) \quad \llbracket \cap \rrbracket := \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \\ \\ \llbracket \begin{array}{c} | \\ \circ \\ | \end{array} \rrbracket := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \llbracket \begin{array}{c} \diagup \\ \circ \\ \diagdown \end{array} \rrbracket := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \llbracket \begin{array}{c} | \\ \bullet \\ | \end{array} \rrbracket := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \llbracket \begin{array}{c} \diagdown \\ \bullet \\ \diagup \end{array} \rrbracket := \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

This application is obviously different from the one of the ZX-Calculus – the domain is different – but we will use the same notation.

Remark 1. Notice that the ZW-Calculus has two ways of “exchanging” two wires’ positions: by a braiding \times – which has the semantics of ZX’s swap – and a crossing \times – which has the same syntax as a ZX swap, but not its semantics.

Remark 2. ZW-Diagrams are universal for matrices of $\mathbb{Z}^{2^n} \times \mathbb{Z}^{2^m}$:

$$\forall A \in \mathbb{Z}^{2^n} \times \mathbb{Z}^{2^m}, \exists D, \llbracket D \rrbracket = A$$

4.2 Calculus

The ZW-Calculus comes with a *complete* set of rules ZW that is given in figure 2. Here again, the paradigm *Only Topology Matters* applies. It gives sense to nodes that are not directly given in T_e , e.g.:

$$\begin{array}{c} \cup \\ \circ \end{array} = \begin{array}{c} \cup \\ \cup \\ \circ \end{array}$$

All these rules are sound. We use the same notation \vdash as previously, and we still have:

$$(ZW \vdash D_1 = D_2) \Rightarrow \left\{ \begin{array}{l} (ZW \vdash D_1 \circ D = D_2 \circ D) \wedge (ZW \vdash D \circ D_1 = D \circ D_2) \\ (ZW \vdash D_1 \otimes D = D_2 \otimes D) \wedge (ZW \vdash D \otimes D_1 = D \otimes D_2) \end{array} \right.$$

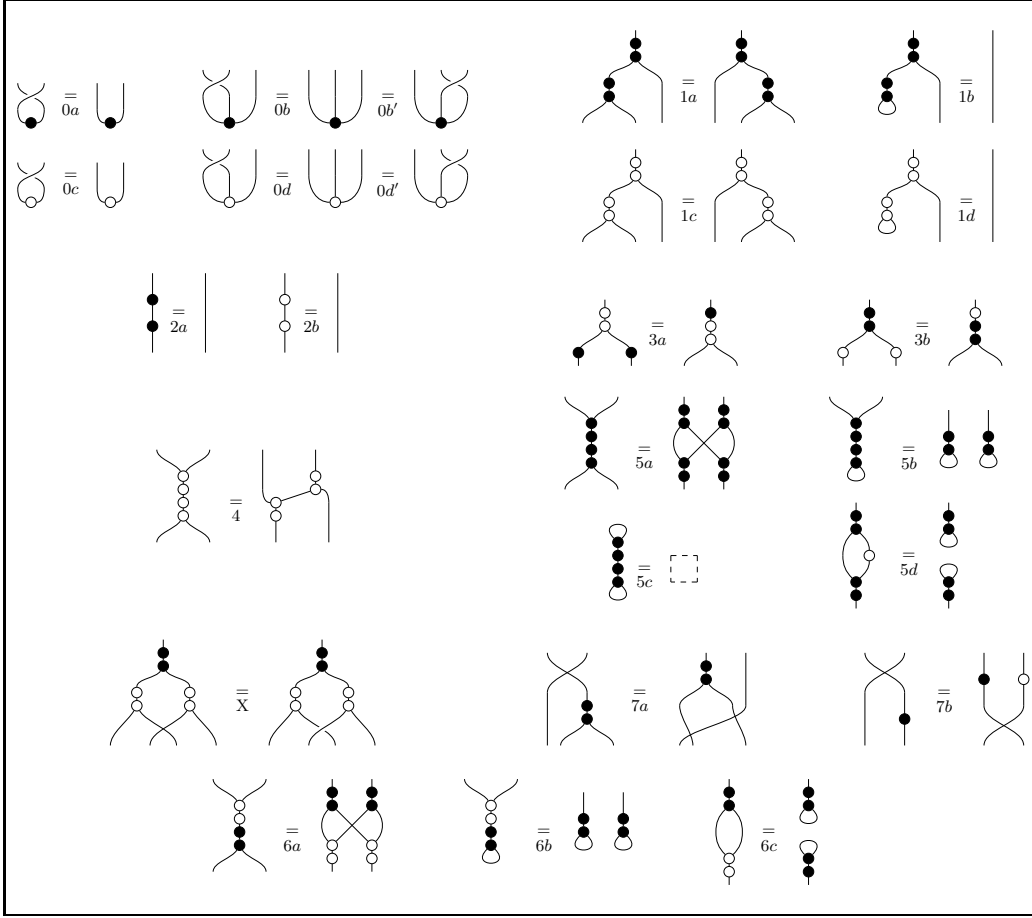


Fig. 2. Set of rules for the ZW-Calculus.

4.3 Extension to Dyadic Matrices

We define an extension of the ZW-Calculus by adding a new node that represents $\frac{1}{2}$ and binding it to the calculus with an additional rule.

Definition 1. We define the $ZW_{1/2}$ -Calculus as the extension of the ZW-Calculus such as:

$$\begin{cases} T_{1/2} = T_e \cup \{\star\} \\ ZW_{1/2} = ZW \cup \left\{ \bigcirc \star \stackrel{=}{=} \boxed{} \right\} \end{cases}$$

The standard interpretation of a diagram $D : n \rightarrow m$ is now a matrix $\llbracket D \rrbracket : \mathbb{D}^{2^n} \rightarrow \mathbb{D}^{2^m}$ over the ring $\mathbb{D} = \mathbb{Z}[1/2]$ of dyadic integers and is given by the standard interpretation of the ZW-Calculus extended with $\llbracket \star \rrbracket := (\frac{1}{2})$.

Proposition 1. The $ZW_{1/2}$ is sound and complete: For two diagrams D_1, D_2 of the $ZW_{1/2}$ -calculus, $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$ iff $ZW_{1/2} \vdash D_1 = D_2$.

Proof. Soundness is obvious.

Now let D_1 and D_2 be two diagrams of the $ZW_{1/2}$ -Calculus such that $\llbracket D_1 \rrbracket = \llbracket D_2 \rrbracket$. We can rewrite D_1 and D_2 as $D_i = d_i \otimes (\star)^{\otimes n_i}$ for some integers n_i and diagrams d_i of the ZW-Calculus that do not use the \star symbol.

From the new introduced rule, we get that $ZW_{1/2} \vdash d_i = D_i \otimes \left(\bigcirc \right)^{\otimes n_i}$.

Now assume $n_1 \leq n_2$. Then $\left[d_1 \otimes \left(\bigcirc \right)^{\otimes n_2 - n_1} \right] = \llbracket d_1 \rrbracket \times 2^{n_2 - n_1} = \llbracket D_1 \rrbracket \times 2^{n_2} = \llbracket d_2 \rrbracket$. Since the two diagrams are ZW-diagrams and have the same interpretation, and since $ZW \subset ZW_{1/2}$,

thanks to the completeness of the ZW-Calculus, $ZW_{1/2} \vdash d_1 \otimes \left(\bigcirc \right)^{\otimes n_2 - n_1} = d_2$, which means $ZW_{1/2} \vdash D_1 = D_2$ by applying n_2 times the new rule on both sides of the equality. \square

5 From $ZX_{\pi/4}$ to $ZW_{1/2}$ -Diagrams

In this section we explain how to encode diagrams of the $ZX_{\pi/4}$ -Calculus into diagrams of the $ZW_{1/2}$ -Calculus. The main difficulty is of course that the former represents matrices with complex coefficients and the latter matrices with dyadic coefficients. We use for this classical results of algebra that we summarize in the next subsection.

5.1 From $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ to \mathbb{Q}

All results used in the next two sections are standard in field theory, see e.g. [18]. Let $R \subseteq \mathbb{C}$ be a (commutative) ring and $\alpha \in \mathbb{C}$. By $R[\alpha]$ we denote the smallest subring of \mathbb{C} that contains both R and α .

Of primary importance will be the ring $\mathbb{Q}[e^{i\frac{\pi}{4}}]$, as all terms of the $\pi/4$ fragment of the ZX -Calculus have interpretations as matrices in this ring. This is clear for all terms except possibly for $\sqrt{2}$, but $\sqrt{2} = e^{i\frac{\pi}{4}} - (e^{i\frac{\pi}{4}})^3$.

If α is algebraic, it is well known that $\mathbb{Q}[\alpha]$ is a field. When $F \subseteq F'$ are two fields, F' can be seen as a vector space (actually an algebra) over F . Its dimension is denoted $[F' : F]$ and we say that F' is an extension of F of degree $[F' : F]$. In the specific case of $\mathbb{Q}[\alpha]$, its dimension over \mathbb{Q} is exactly the degree of the minimal polynomial over \mathbb{Q} of α . Notice that the minimal polynomial of a n -th primitive root of the unity is $\varphi(n)$ where φ is Euler's totient function.

In our case, $e^{i\frac{\pi}{4}}$ is a eighth primitive root of the unity, so that $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ is a vector space of dimension 4, one basis being given by $1, e^{i\frac{\pi}{4}}, (e^{i\frac{\pi}{4}})^2, (e^{i\frac{\pi}{4}})^3$. In particular:

Proposition 2. *Every element of $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ can be written in a unique way $a + be^{i\frac{\pi}{4}} + c(e^{i\frac{\pi}{4}})^2 + d(e^{i\frac{\pi}{4}})^3$ for some rational numbers a, b, c, d .*

For $x \in \mathbb{Q}[e^{i\frac{\pi}{4}}]$ let $\psi(x)$ be the function defined by $\psi(x) = y \mapsto xy$. For each x , $\psi(x)$ is a linear map and therefore can be given by a 4×4 matrix in the basis $1, e^{i\frac{\pi}{4}}, (e^{i\frac{\pi}{4}})^2, (e^{i\frac{\pi}{4}})^3$. $\psi(1)$ is of course the identity matrix and

$$\psi(e^{i\frac{\pi}{4}}) = M = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Notice that M is the companion matrix of $e^{i\frac{\pi}{4}}$.

Proposition 3. *The map:*

$$\psi : a + be^{i\frac{\pi}{4}} + c(e^{i\frac{\pi}{4}})^2 + d(e^{i\frac{\pi}{4}})^3 \mapsto aI_4 + bM + cM^2 + dM^3$$

is a homomorphism of \mathbb{Q} -algebras from $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ to $M_4(\mathbb{Q})$

This homomorphism has a left-inverse. Indeed, let

$$\theta = \begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}} \\ (e^{i\frac{\pi}{4}})^2 \\ (e^{i\frac{\pi}{4}})^3 \end{pmatrix}$$

Then $\theta^t \psi(x) = x\theta^t$.

With this morphism, we can see elements of $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ as matrices over \mathbb{Q} .

Of course we can do the same with matrices over $\mathbb{Q}[e^{i\frac{\pi}{4}}]$.

Definition 2. *Define:*

$$\psi : A + Be^{i\frac{\pi}{4}} + C(e^{i\frac{\pi}{4}})^2 + D(e^{i\frac{\pi}{4}})^3 \mapsto A \otimes I_4 + B \otimes M + C \otimes M^2 + D \otimes M^3$$

ψ is injective and maps a matrix over $\mathbb{Q}[e^{i\frac{\pi}{4}}]$ of dimension $n \times m$ to a matrix over \mathbb{Q} of dimension $4n \times 4m$.

We use the same notation ψ as before, as the definitions are equivalent for one-by-one matrices (i.e. scalars).

It is easy to see that Proposition 3 holds for the extended ψ in the sense that $\psi(qA) = q\psi(A)$ for q rational, $\psi(A + B) = \psi(A) + \psi(B)$, $\psi(AB) = \psi(A)\psi(B)$ whenever this makes sense.

Notice however that $\psi(A \otimes B)$ is not $\psi(A) \otimes \psi(B)$.

As before, ψ has a left inverse, as evidenced by:

Proposition 4. *For all matrices X of dimension $n \times m$, $(I_n \otimes \theta^t)\psi(X) = X \otimes \theta^t$*

While it is true that all coefficients of the standard interpretation of the $\pi/4$ fragment are in $\mathbb{Q}[e^{i\frac{\pi}{4}}]$, we can say a bit better.

Let $\mathbb{D} = \mathbb{Z}[1/2]$ be the set of all dyadic rational numbers, i.e. rational numbers of the form $p/2^n$.

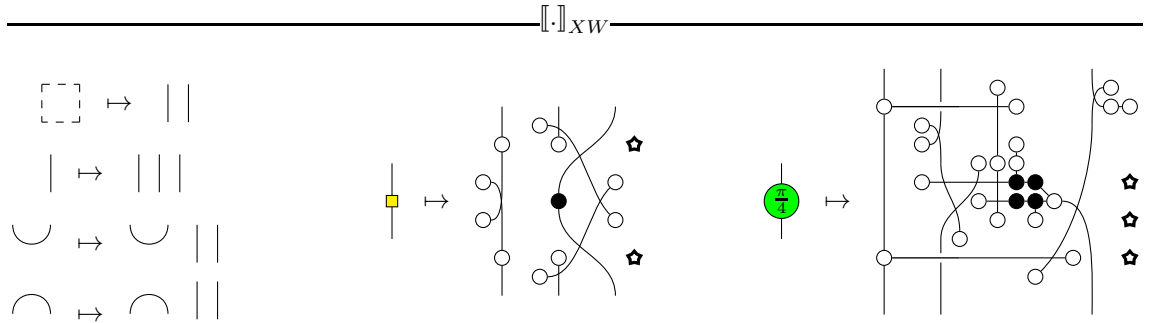
It is easy to see that any element of $\mathbb{D}[e^{i\frac{\pi}{4}}]$ can be written in a unique way $a + be^{i\frac{\pi}{4}} + c(e^{i\frac{\pi}{4}})^2 + d(e^{i\frac{\pi}{4}})^3$ for some dyadic numbers a, b, c, d . (It is NOT a consequence of the similar statement for \mathbb{Q} . We have to use here the additional property that $e^{i\frac{\pi}{4}}$ is not only an algebraic number, but also an algebraic *integer*).

Then it is clear that actually all coefficients of the $\pi/4$ fragment of the ZX-Calculus are in $\mathbb{D}[e^{i\frac{\pi}{4}}]$. As $\mathbb{D} \subset \mathbb{Q}$ all we said before still holds, and we actually obtain with ψ a map from matrices over $\mathbb{D}[e^{i\frac{\pi}{4}}]$ to matrices over \mathbb{D} .

5.2 Interpretation

In the following, we will actually use the matrix $\psi := \psi^t$ (hence $M := M^t$) such that we will in fact prove $\psi(X)(I_m \otimes \theta) = X \otimes \theta$. In terms of ZX and ZW diagrams, it only amounts to flipping them upside-down.

Based on the previous discussion, we define an interpretation $\llbracket \cdot \rrbracket_{XW}$ from $ZX_{\pi/4}$ -diagrams to $ZW_{1/2}$ -diagrams as follows:

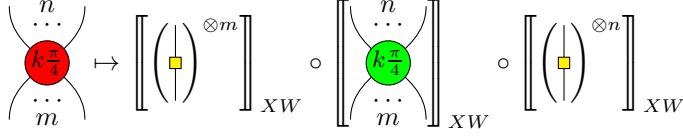


$$\forall D_1 : n \rightarrow n', \forall D_2 : m \rightarrow m' :$$

$$D_1 \circ D_2 \mapsto \llbracket D_1 \rrbracket_{XW} \circ \llbracket D_2 \rrbracket_{XW} \quad (\text{if } m' = n)$$

$$D_1 \otimes D_2 \mapsto \left(\mathbb{I}^{\otimes n'} \otimes \llbracket D_2 \rrbracket_{XW} \right) \circ \left(\begin{array}{c} m \\ \dots \\ n' \\ \dots \\ \dots \end{array} \right) \circ \left(\mathbb{I}^{\otimes m} \otimes \llbracket D_1 \rrbracket_{XW} \right) \circ \left(\begin{array}{c} n \\ \dots \\ m \\ \dots \\ \dots \end{array} \right)$$

$$\left(\begin{array}{c} n \\ \dots \\ k \frac{\pi}{4} \\ \dots \\ m \end{array} \right) \mapsto \left(\begin{array}{c} \dots \\ \dots \\ m \end{array} \right) \circ \left(\llbracket \frac{\pi}{4} \rrbracket_{XW} \right)^k \circ \left(\begin{array}{c} n \\ \dots \\ \dots \end{array} \right)$$



One can check that $\llbracket \left[\begin{array}{c} | \\ \square \\ | \end{array} \right]_{XW} \rrbracket = \psi \left(\left[\begin{array}{c} | \\ \square \\ | \end{array} \right] \right) = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \otimes (M - M^3)$ and $\llbracket \left[\begin{array}{c} | \\ \circ \\ | \end{array} \right]_{XW} \rrbracket = \psi \left(\left[\begin{array}{c} | \\ \circ \\ | \end{array} \right] \right) = \begin{pmatrix} I_4 & 0 \\ 0 & M \end{pmatrix}$. More generally:

Proposition 5. *Let D be a diagram of the $ZX_{\pi/4}$ -Calculus. Then*

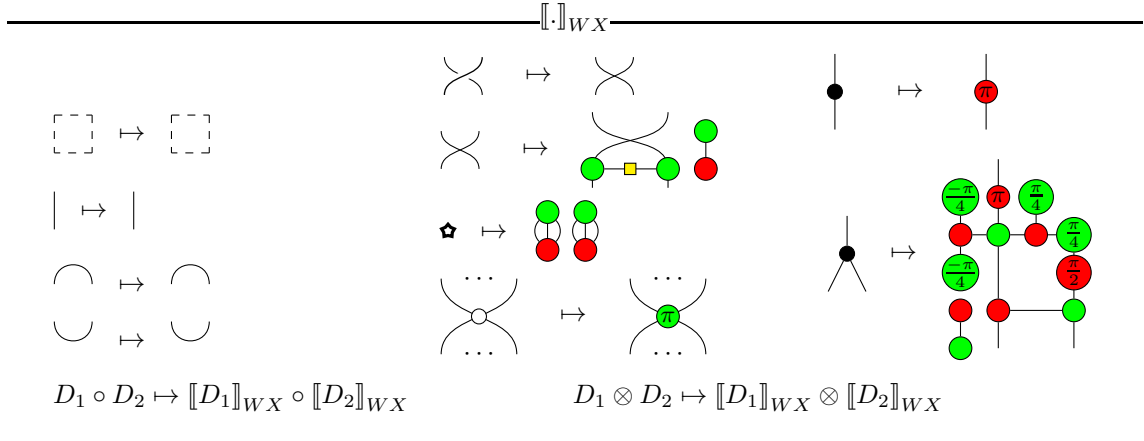
$$\llbracket [D]_{XW} \rrbracket = \psi([D])$$

In particular, if $[D_1] = [D_2]$ then $[D_1]_{XW} = [D_2]_{XW}$

The proof is a straightforward induction using the fact that ψ is an homomorphism. A slight care has to be taken to treat the case of $D_1 \otimes D_2$.

6 From $ZW_{1/2}$ to $ZX_{\pi/4}$ -Diagrams

We define here an interpretation $\llbracket \cdot \rrbracket_{WX}$ that transforms any diagram of the $ZW_{1/2}$ -Calculus into a $ZX_{\pi/4}$ -diagram, which is easy to do since $\mathbb{D} \subset \mathbb{D}[e^{i\frac{\pi}{4}}]$:



Proposition 6. *Let D be a diagram of the $ZW_{1/2}$ calculus. Then $\llbracket [D]_{WX} \rrbracket = [D]$*

By an easy induction.

Proposition 7. *The interpretation $\llbracket \cdot \rrbracket_{WX}$ preserves all the rules of the $ZW_{1/2}$ -Calculus:*

$$(D_1 = D_2) \in ZW_{1/2} \quad \Rightarrow \quad ZX_{\pi/4} \vdash [D_1]_{WX} = [D_2]_{WX}$$

The proof is in appendix at page 19.

7 Completeness of the $\frac{\pi}{4}$ -fragment of the ZX-Calculus

To finish the proof it remains to compose the two interpretations:

Proposition 8. *We can retrieve any $ZX_{\pi/4}$ -diagram from its image by the composition of the two interpretations:*

$$\forall D \in ZX_{\pi/4}, \quad ZX_{\pi/4} \vdash \left(\left| \cdots \right| \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right) \circ \llbracket [D]_{XW} \rrbracket_{WX} \circ \left(\left| \cdots \right| \begin{array}{c} \circ \\ \circ \end{array} \right) = D$$

The proof is in appendix at page 23.

Corollary 1. *If $ZX_{\pi/4} \vdash \llbracket [D_1]_{XW} \rrbracket_{WX} = \llbracket [D_2]_{XW} \rrbracket_{WX}$ then $ZX_{\pi/4} \vdash D_1 = D_2$.*

8 Expressive power of the ZX $_{\pi/4}$ -diagrams

The ZW-Calculus is complete, and additionally any integer matrix can be represented in the ZW-Calculus [12]. A similar result follows immediately for the ZW $_{1/2}$ -calculus.

Proposition 9. *ZW $_{1/2}$ -Diagrams are universal for matrices of $\mathbb{D}^{2^n} \times \mathbb{D}^{2^m}$:*

$$\forall A \in \mathbb{D}^{2^n} \times \mathbb{D}^{2^m}, \exists D \in \text{ZW}_{1/2}, \llbracket D \rrbracket = A$$

Regarding the expressive power of ZX $_{\pi/4}$ -diagrams, since all the unitary matrices over $\mathbb{D}[e^{i\frac{\pi}{4}}]$ are representable with Clifford+T circuits [11], so are they with ZX $_{\pi/4}$ -diagrams. We actually show that any matrix over $\mathbb{D}[e^{i\frac{\pi}{4}}]$ can be represented by a ZX $_{\pi/4}$ -diagram:

Proposition 10. *The $\frac{\pi}{4}$ -fragment of the ZX-Calculus represents exactly matrices over $\mathbb{D}[e^{i\frac{\pi}{4}}]$:*

$$\forall A \in \mathbb{D}[e^{i\frac{\pi}{4}}]^{2^n \times 2^m}, \exists D \in \text{ZX}_{\pi/4}, \llbracket D \rrbracket = A$$

Proof. Let $A \in \mathbb{D}[e^{i\frac{\pi}{4}}]^{2^n \times 2^m}$. We define $A' = \psi(A) \in \mathbb{D}^{2^{n+2} \times 2^{m+2}}$. Since ZW $_{1/2}$ -diagrams are universal for matrices over dyadic rationals: $\exists D \in \text{ZW}_{1/2}, \llbracket D \rrbracket = A'$. Since $\llbracket \cdot \rrbracket_{\text{WX}}$ preserves the semantics, we can define a ZX-diagram of the $\frac{\pi}{4}$ -fragment $D' = \llbracket D \rrbracket_{\text{WX}}$ such that $\llbracket D' \rrbracket = A'$.

Now, notice that $\theta = \begin{pmatrix} 1 \\ e^{i\frac{\pi}{4}} \\ (e^{i\frac{\pi}{4}})^2 \\ (e^{i\frac{\pi}{4}})^3 \end{pmatrix} = \left[\begin{array}{c} \frac{\pi}{2} \\ \frac{\pi}{4} \end{array} \right]$, and $e_1 = (1 \ 0 \ 0 \ 0) = \left[\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ \bullet \end{array} \right]$, so if we

apply the second diagram at the two bottom right wires, and the first state on the two top right wires of D' , we end up with D'' such that $\llbracket D'' \rrbracket = A$. Indeed:

$$\llbracket D'' \rrbracket = (I \otimes e_1) \circ \llbracket D' \rrbracket \circ (I \otimes \theta) = (I \otimes e_1) \circ \psi(A) \circ (I \otimes \theta) = (I \otimes e_1) \circ (A \otimes \theta) = A \otimes (e_1 \circ \theta) = A \quad \square$$

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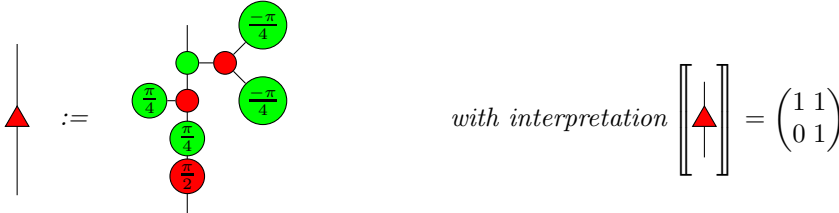
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9 Appendix

In this section are the proofs of propositions 7 and 8. To simplify the following work, we introduce a new node as a notation, and give a few lemmas. Keep in mind that for any provable equation, its upside down version, its colour-swapped version, and its version with opposed angles are all provable.

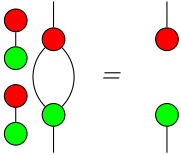
Definition 3. We define the triangle node as:



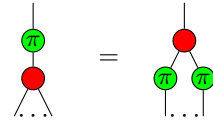
Lemma 1.



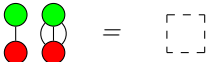
Lemma 2.



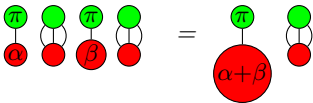
Lemma 3.



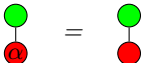
Lemma 4.



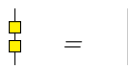
Lemma 5.



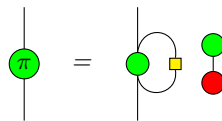
Lemma 6.



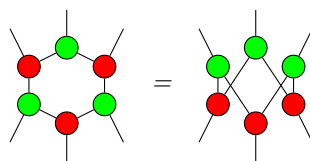
Lemma 7.



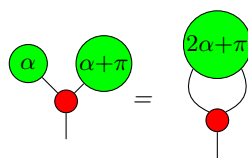
Lemma 8.



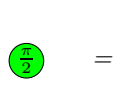
Lemma 9.



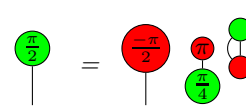
Lemma 10.



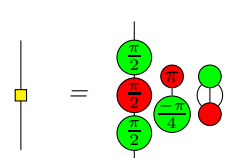
Lemma 11.



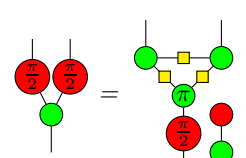
Lemma 12.



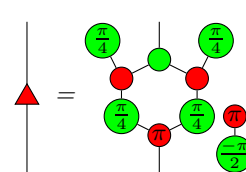
Lemma 13.



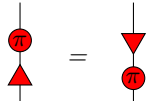
Lemma 14.



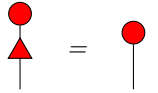
Lemma 15.



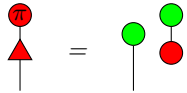
Lemma 16.



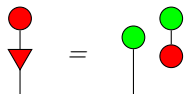
Lemma 17.



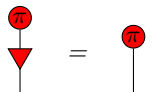
Lemma 18.



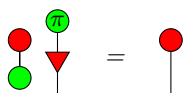
Lemma 19.



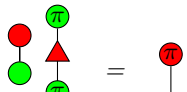
Lemma 20.



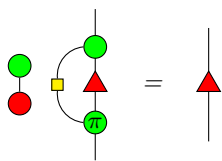
Lemma 21.



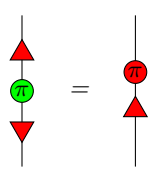
Lemma 22.



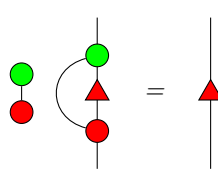
Lemma 23.



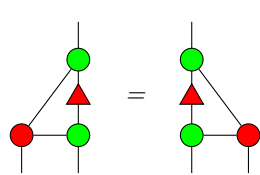
Lemma 24.



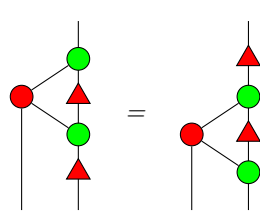
Lemma 25.



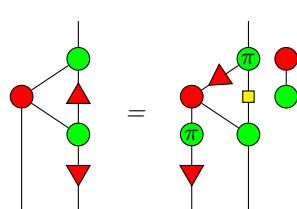
Lemma 26.



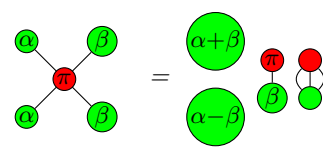
Lemma 27.



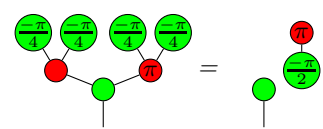
Lemma 28.



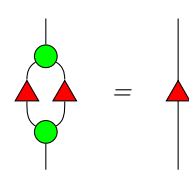
Lemma 29.



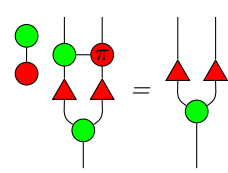
Lemma 30.



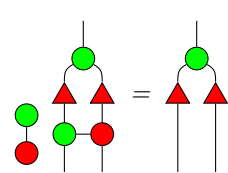
Lemma 31.



Lemma 32.



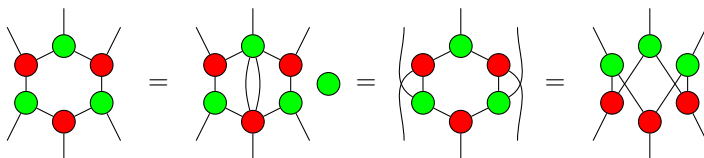
and



Proof (Lemmas 1, 2, 3, 4, 5, 6, 7 and 8). Proven in [3].

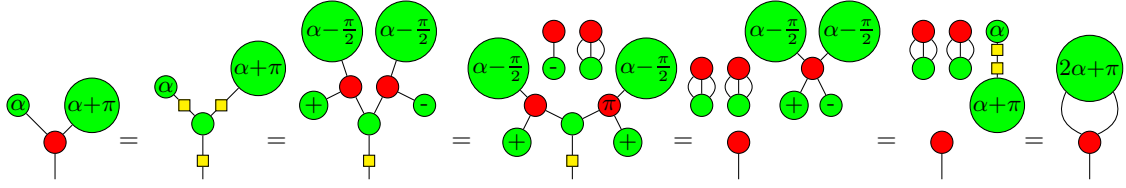
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Proof (Lemma 9). Using 2 and (B2):



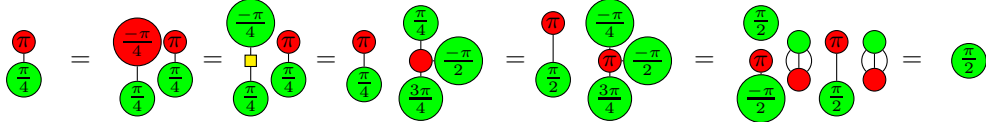
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Proof (Lemma 10). Using (H), (EU), (K2), (SUP), 2, 4 and 7:



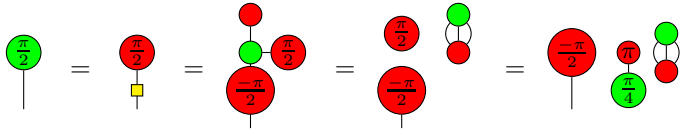
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Proof (Lemma 11). Using (E), (H), (EU), (S1), (K2), 5, 10, 2, 4:



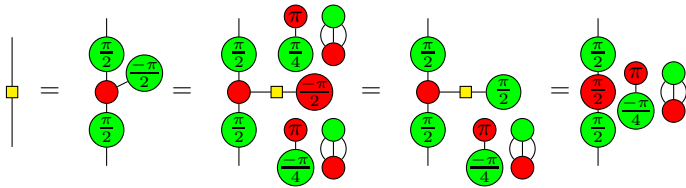
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Proof (Lemma 12). Using (H), (EU), (S1), (B1), 11:



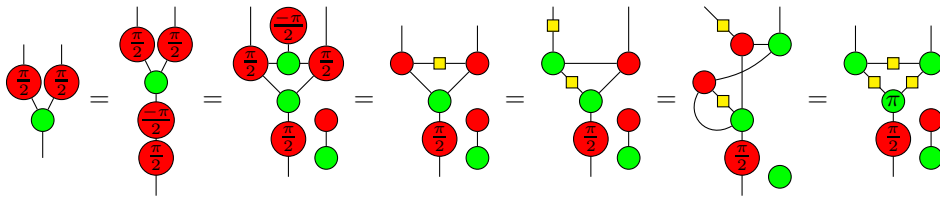
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Proof (Lemma 13). Using (EU), (H), 5 and 12:



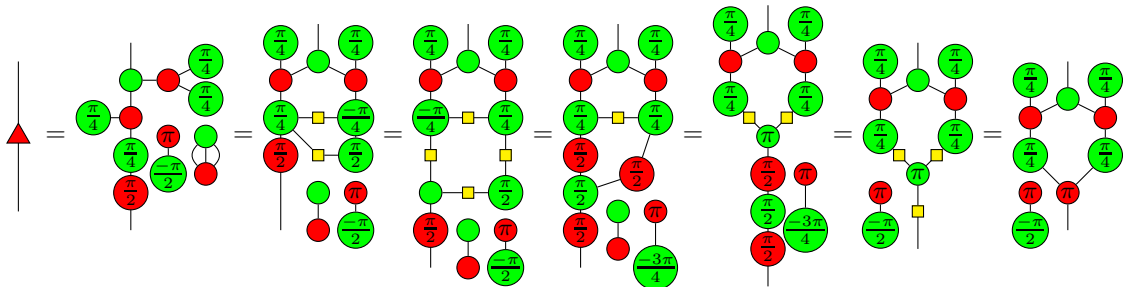
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Proof (Lemma 14). Using (S1), (B2), (EU), (H) and 8:



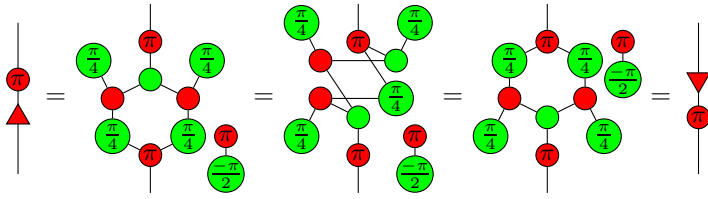
□

Proof (Lemma 15). Using (K2), 5, 2, (C1), 13, (H) and 14:



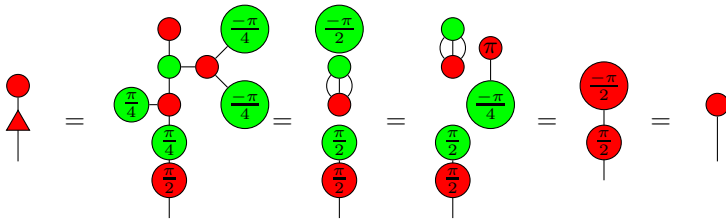
□

Proof (Lemma 16). Using 15, 9 and (S1):



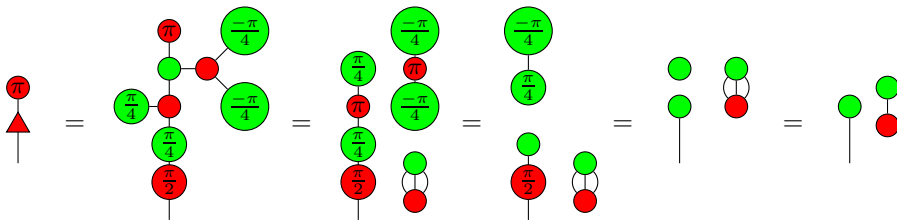
□

Proof (Lemma 17). Using definition 3, (B1), (S1), (S2), 4, 11, 12 and 5:



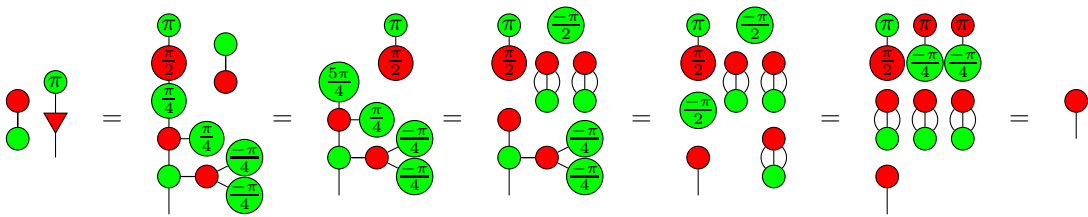
□

Proof (Lemma 18). Using definition 3, (S1), 3, 4, (B1), (K2), 1:



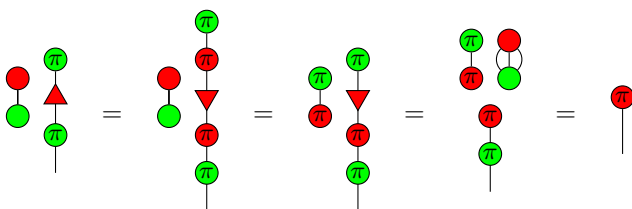
□

Proof (Lemma 21). Using definition 3, (B1), (S1), 10, 2, 11 and 5:



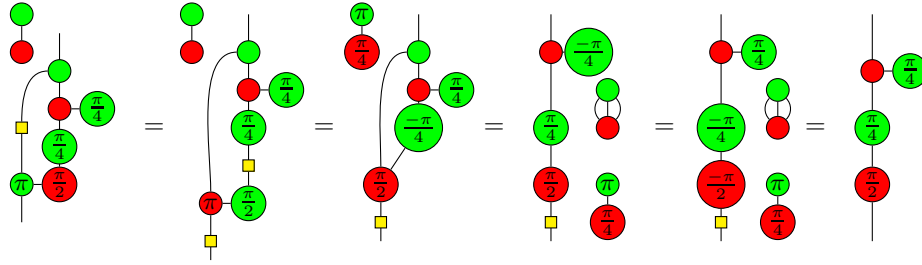
□

Proof (Lemma 22). Using 16, (K2), (B1), 5 and 21:

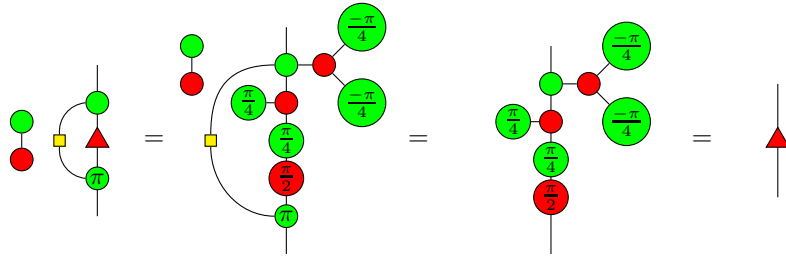


□

Proof (Lemma 23). First, using (H), 13, (S1), (B2), (K2) and 5:

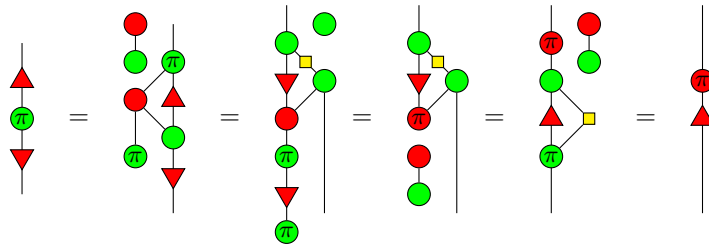


Then, using definition 3, (S1) and the previous result:



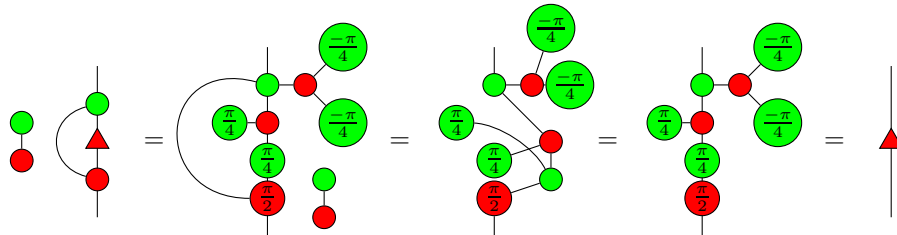
□

Proof (Lemma 24). Using (S1), 3, (B1), 28, 22, 16, (H) and 23:



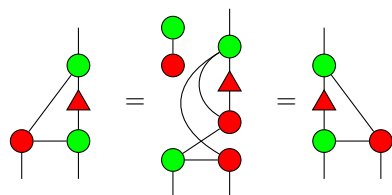
□

Proof (Lemma 25). Using definition 3, (B2) and (S1):



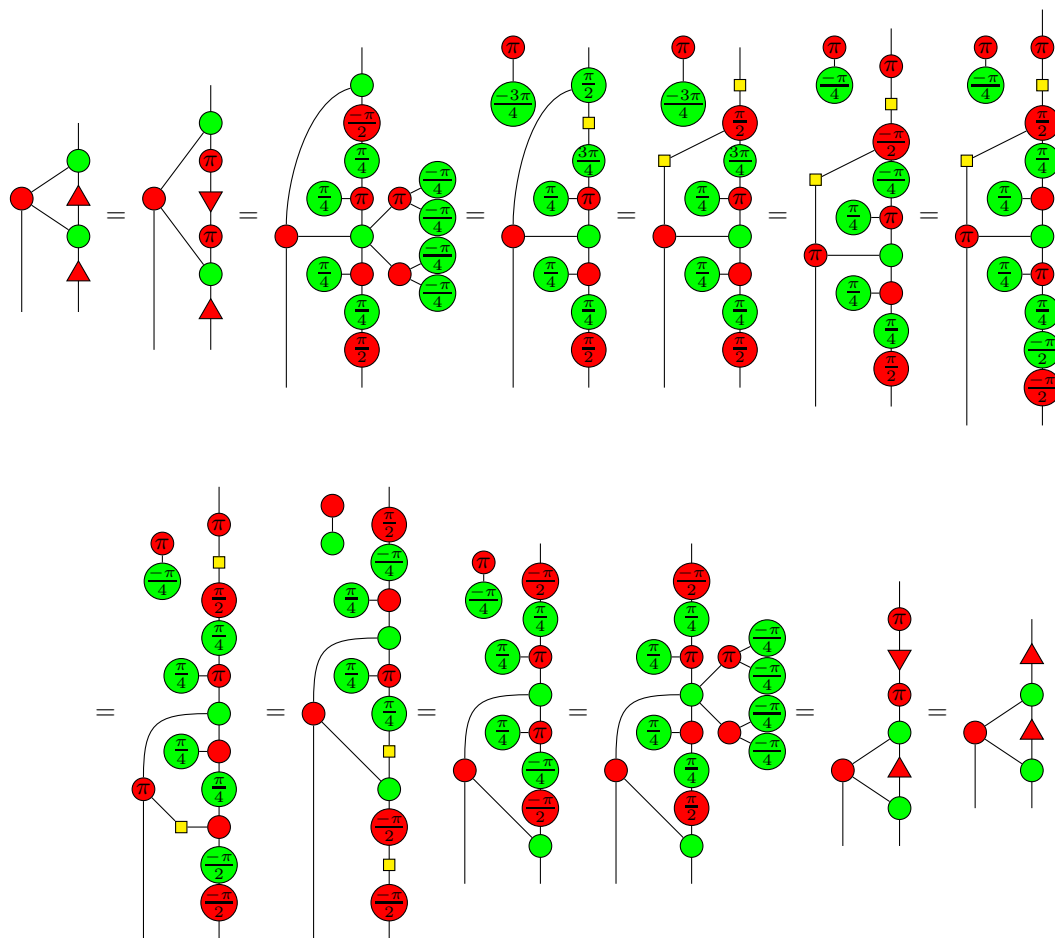
□

Proof (Lemma 26). Using (B2), (S1) and 25:



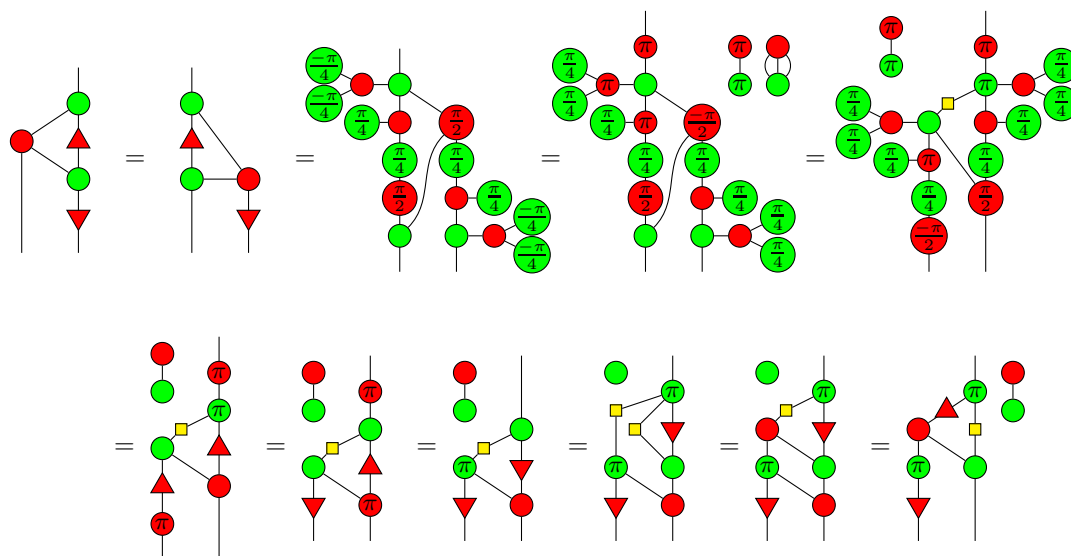
□

Proof (Lemma 27). Using 16, decomposition 3, 30, 13, 5, 3, (K2) and (C2):



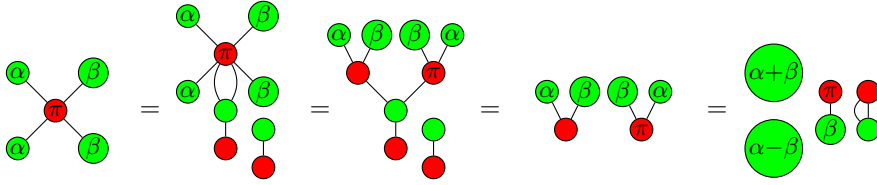
□

Proof (Lemma 28). Using 26, (S1), decomposition 3, 3, (K2), (C3), 24 and (H):



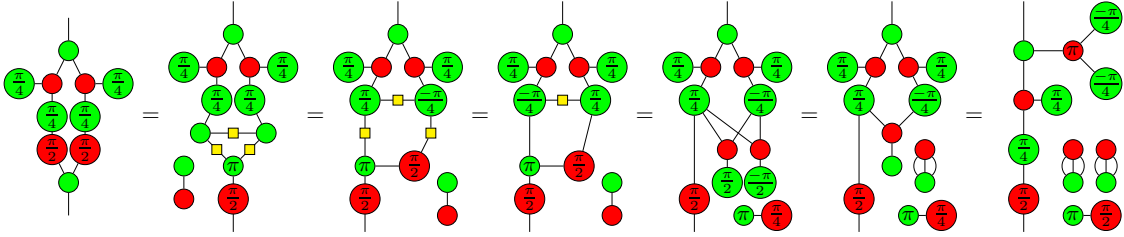
□

Proof (Lemma 29). Using (S1), (B1), (SUP) and (K2):

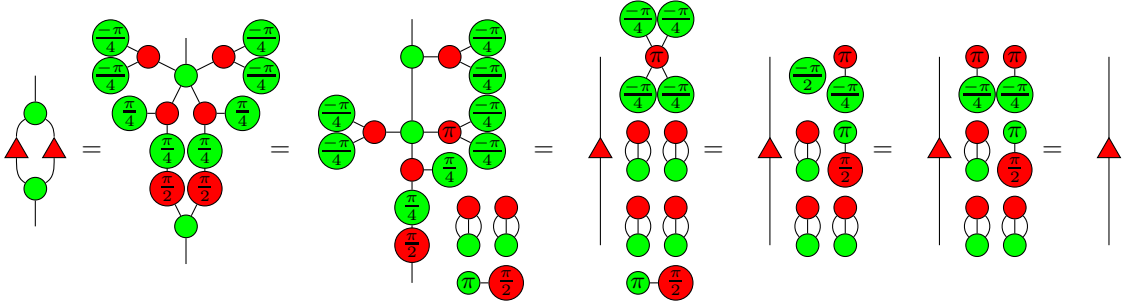


□

Proof (Lemma 31). First, using 14, (S1), (C1), (EU), (B2), (B1) and (K2):

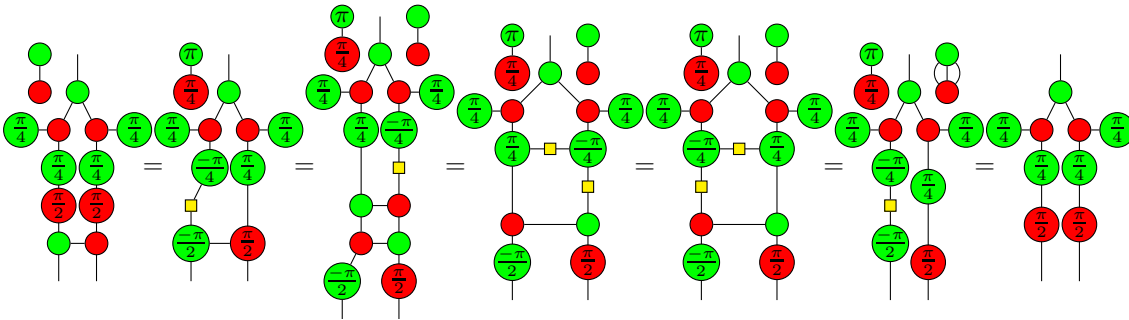


Then, using definition 3, (S1), (SUP), 29, 5:

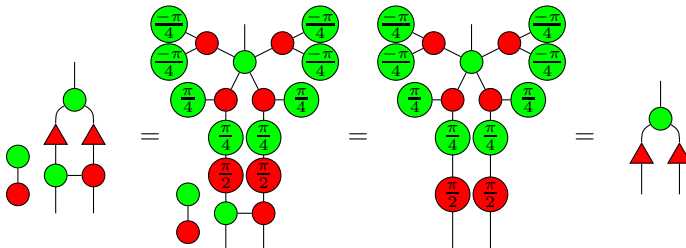


□

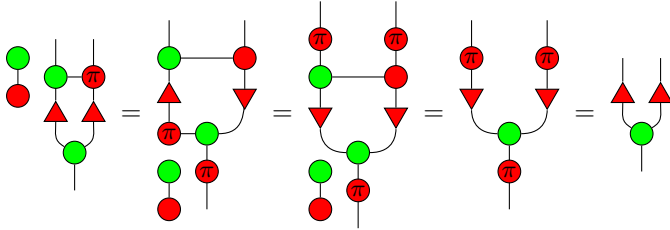
Proof (Lemma 32). First, using (S1), 13, (B2), (H), (C1), 2 and 4:



Then, using definition 3 and (S1):



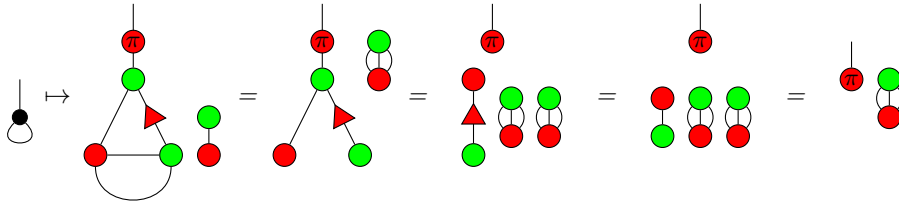
Finally, using 16 and lemma 3:



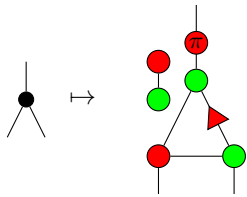
□

Lemma 33.

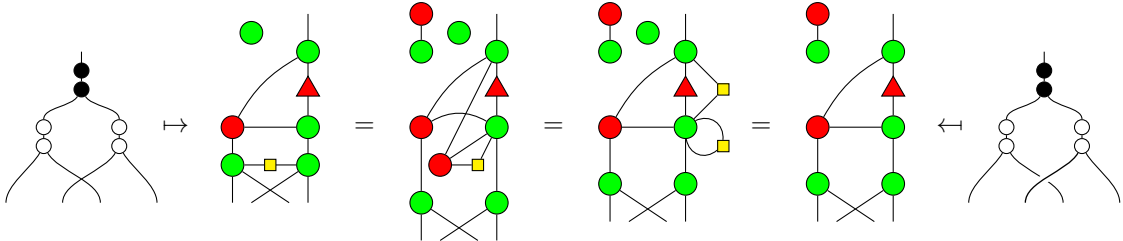
Using 2, (B1), 17 and 4:



Proof (Proposition 7). First, notice that using the triangle notation:

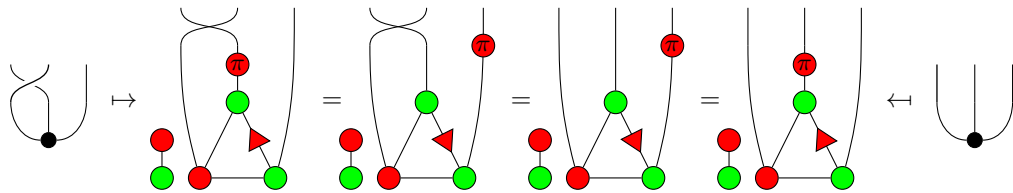


- X: using (S1), (B2), (H), 8 and 23,

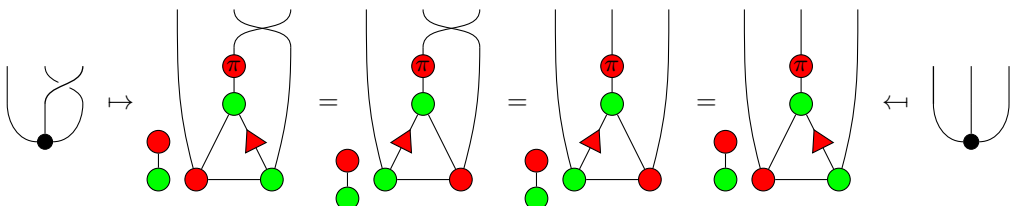


- 0a, 0c, 0d and 0d' come directly from the paradigm *Only Topology Matters*.

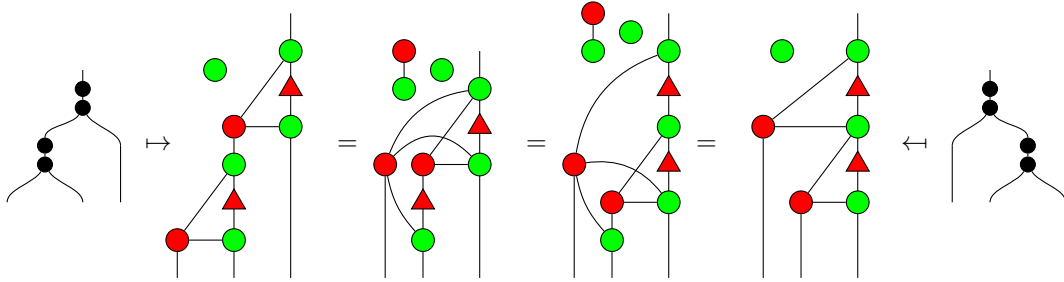
- 0b: Using 3, 16, 26 and (S1),



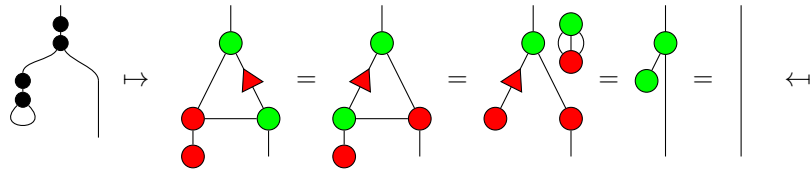
- 0b': Using 26 and the previous result,



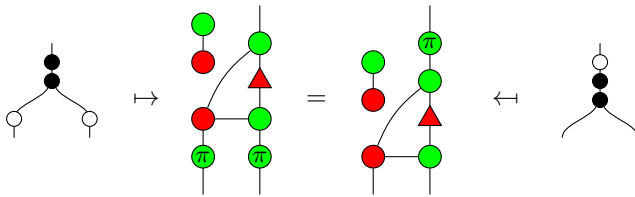
- 1a: Using (B2) and 27,



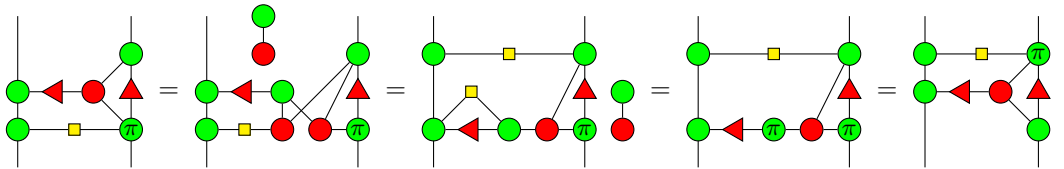
- 1b: Using 26, (B1), 19, (S1), 4 and (S2),



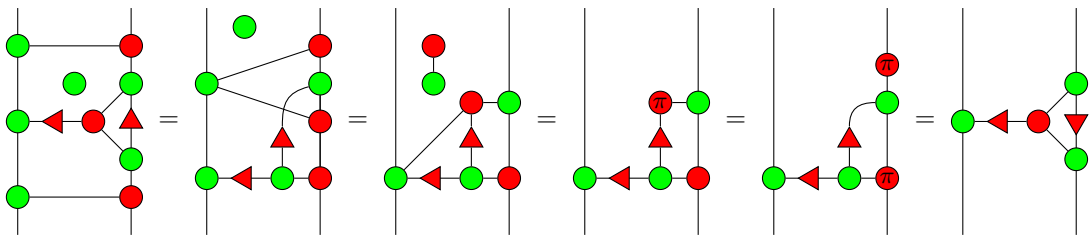
- 1c, 1d, 2a and 2b come directly from the spider rules (S1) and (S2).
- 3a is the expression of the colour-swapped version of the lemma 3.
- 3b: Using 3 and (S1),



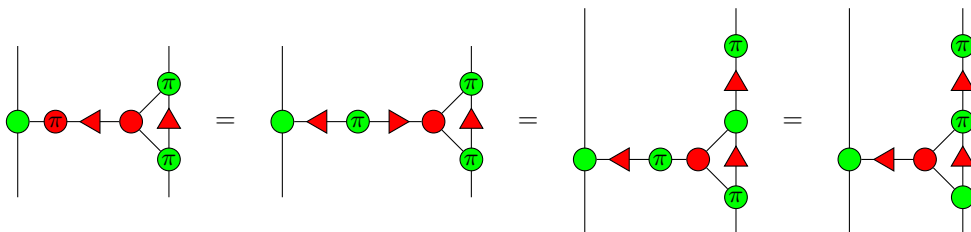
- 4 comes from the spider rule (S1).
- 5a: We will need a few steps to prove this equality.
- i) Using (S1), (B2), (H), 23 and 3,



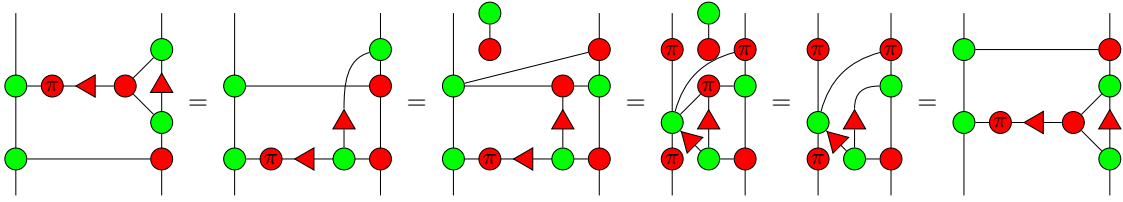
- ii) Using 26, (S1), (B2), 32 and 3,



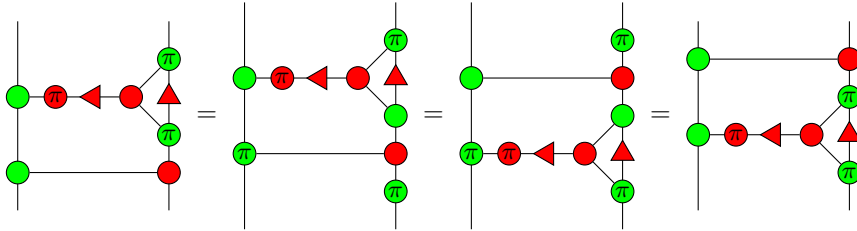
- iii) Using 24, 26, 27 and 3:



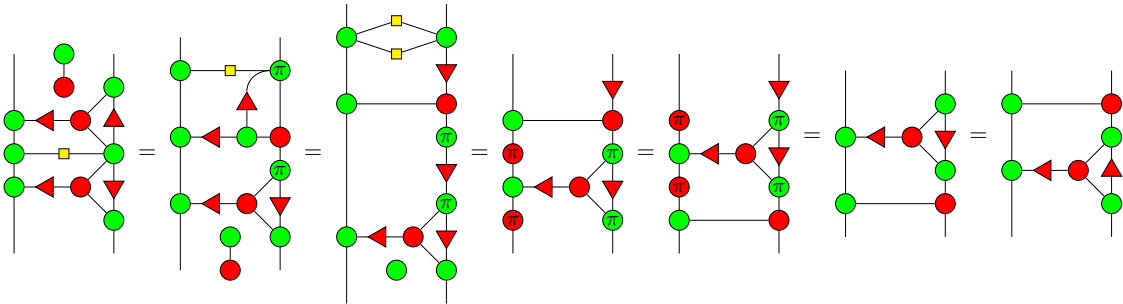
iv) Using 26, (S1), (B2), 32 and 3,



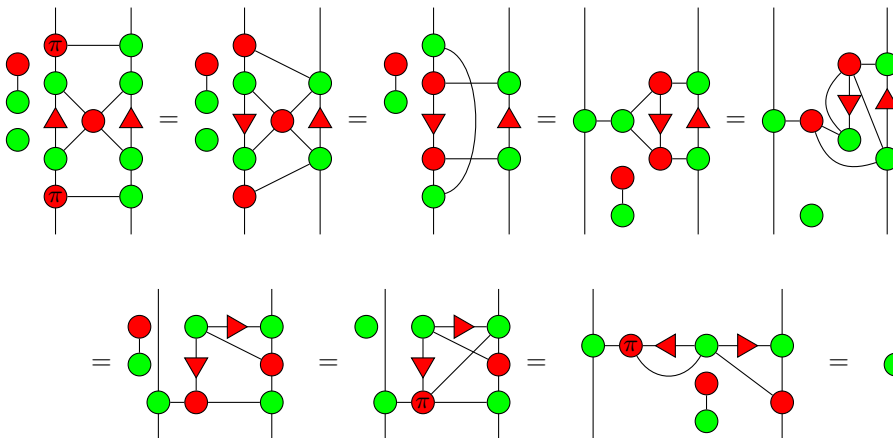
v) Using 3 and iv),



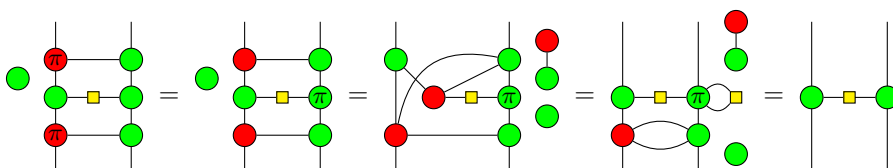
vi) Using i), 26, 28, iii), 2, v), iv) and ii),



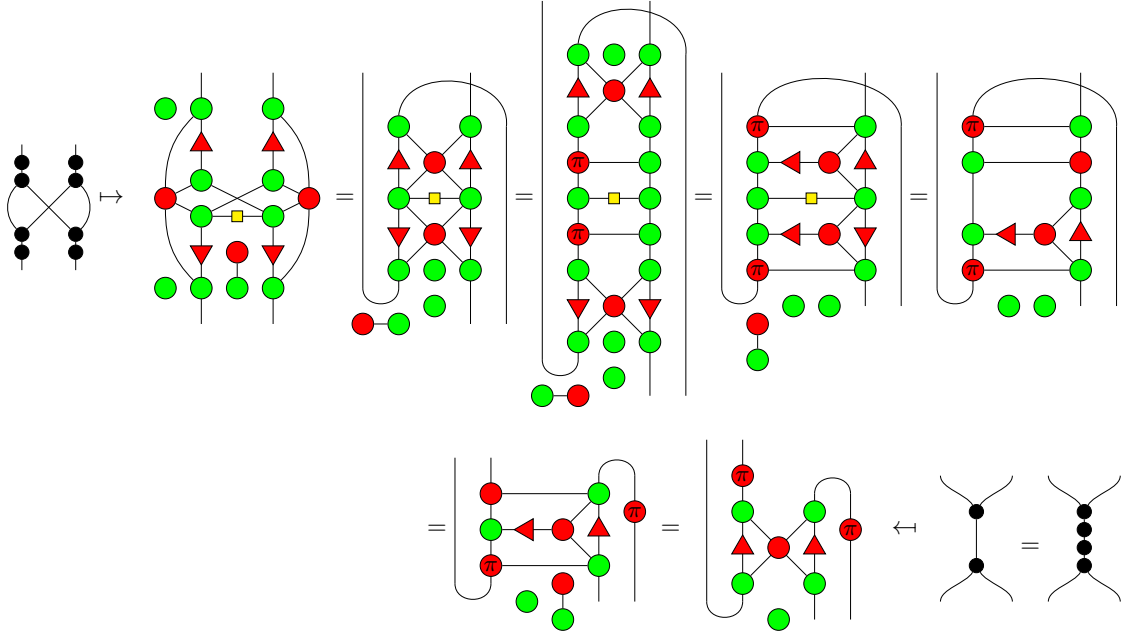
vii) Using (S1), 3, 16, (B2), 25, 26 and 32,



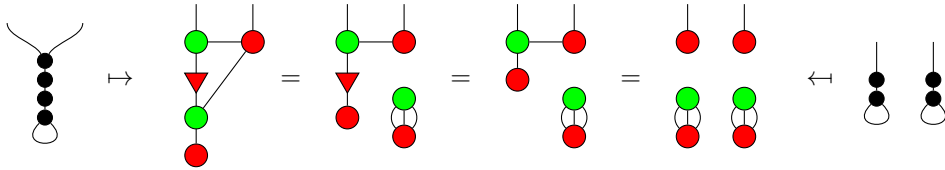
viii) Using 3, (B2), (S1), (H), 8, 2 and (S2),



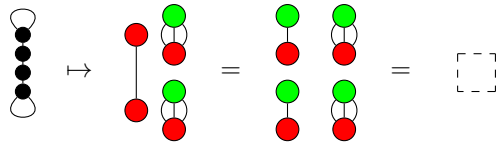
Finally, using (S1), *viii*), *vii*), 2, *vi*), and (B2),



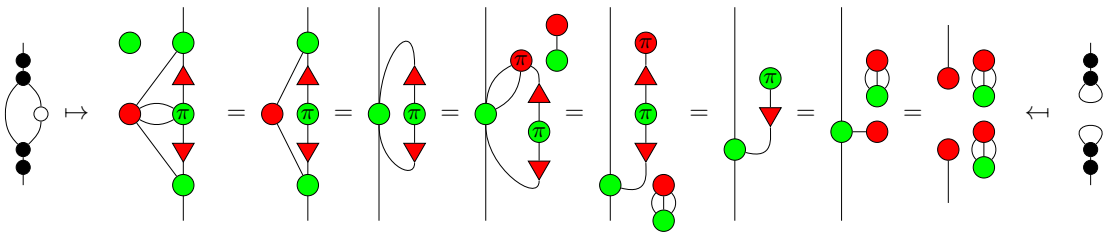
• 5b: using (B1), 4 and 17,



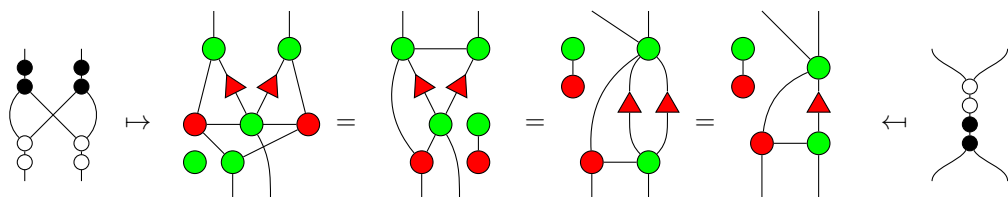
• 5c: using (S1), 1, 4,



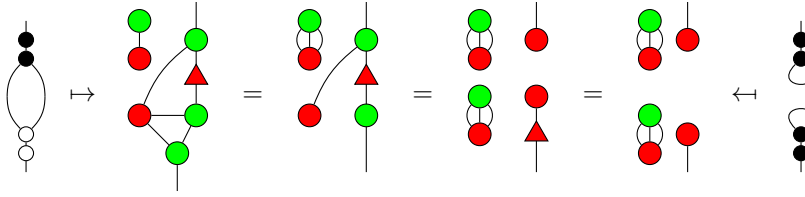
• 5d: using (S1), 2, (S2), 32, 18, 21 and (B1),



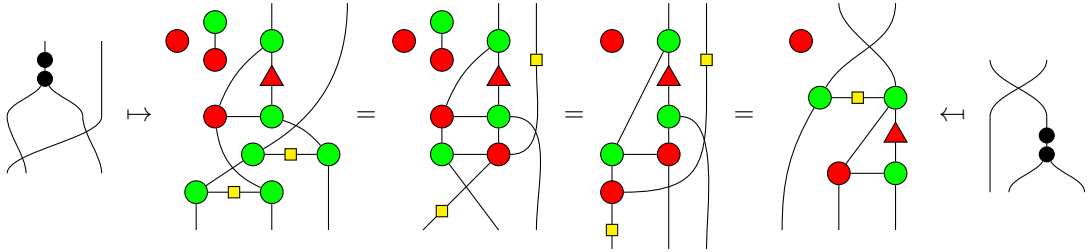
• 6a: Thanks to the rule X we can get rid of $\begin{array}{c} \bullet \\ \square \\ \bullet \end{array}$ induced by the crossing. Then, using (S1), (B2) and 31,



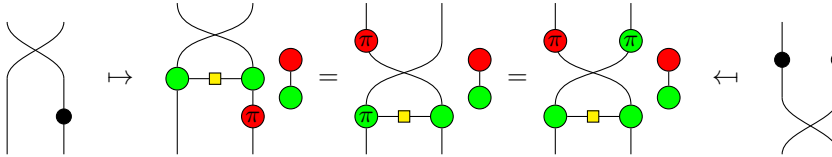
- 6b is exactly the copy rule (B1).
- 6c: using (S1), 4, 2, (B1) and 17:



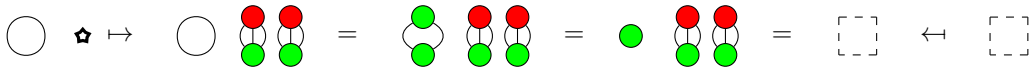
- 7a: using (S1), (H) and (B2),



- 7b: using 3, (H) and (S1):



- iv: using (S3), (S1), 1 and 4,



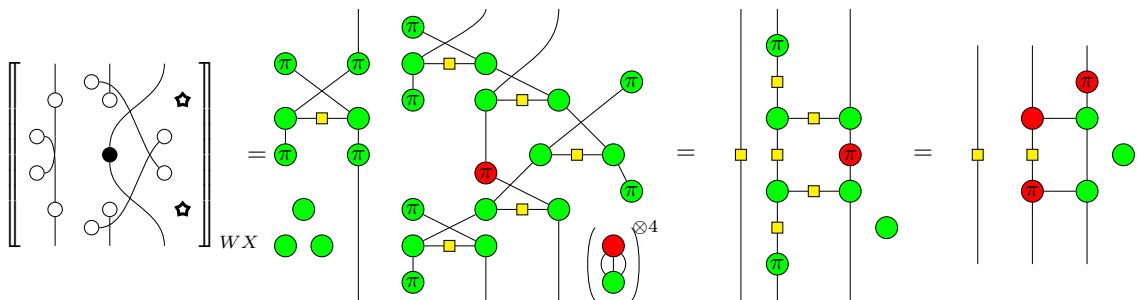
□

Proof (Proposition 8). Let us write $[[\cdot]]^\natural = [[[\cdot]]_{WX}]_{WX}$. We can show inductively that:

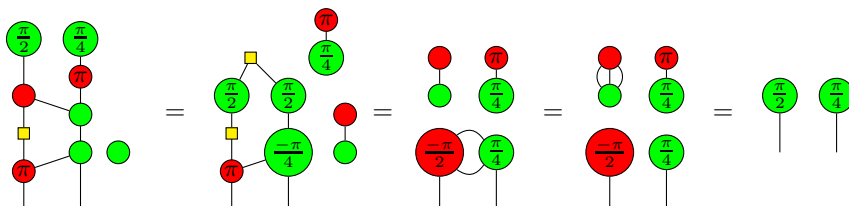
$$\mathbb{Z}X_{\pi/4} \vdash [[D]]^\natural \circ \left(\left| \cdots \right| \begin{array}{c} \frac{\pi}{2} \\ \frac{\pi}{4} \end{array} \right) = D \otimes \left(\begin{array}{c} \frac{\pi}{2} \\ \frac{\pi}{4} \end{array} \right)$$

which is the expression of proposition 4 – up to transposition because we chose $M := M^t$.

- \square : Using (S1), (S2), 4 and (H),



and, using (S1), (EU), 4, (H), 2 and 12:

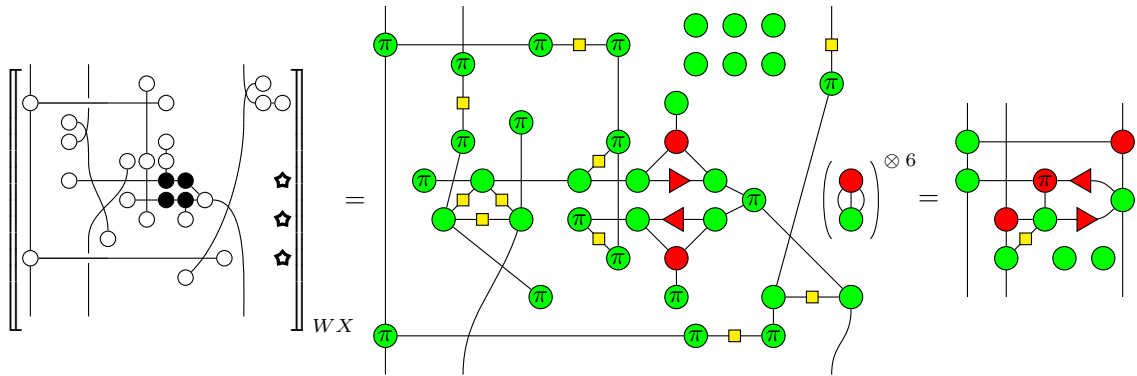


Hence $ZX \vdash \left[\begin{array}{c} \square \\ \parallel \end{array} \right]^b \circ \left(\begin{array}{c} \pi/2 \\ \pi/4 \end{array} \right) = \begin{array}{c} \square \\ \parallel \\ \pi/2 \\ \pi/4 \end{array}$

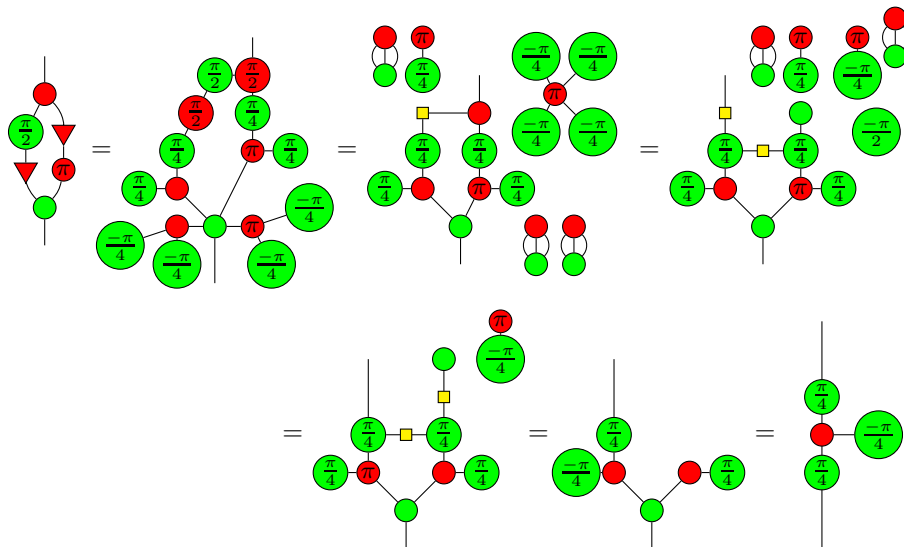
• $\left[\begin{array}{c} \text{---} \\ \circ \\ \text{---} \\ \vdots \\ \text{---} \\ \circ \\ \text{---} \\ m \end{array} \right] \left[\begin{array}{c} \parallel \\ \vdots \\ \parallel \\ WX \end{array} \right] \circ \left(\begin{array}{c} \pi/2 \\ \pi/4 \end{array} \right) = \begin{array}{c} \pi/2 \\ \pi/4 \\ m \end{array}$

• $\left[\begin{array}{c} n \\ \text{---} \\ \circ \\ \text{---} \\ \vdots \\ \text{---} \\ \circ \\ \text{---} \\ n \end{array} \right] \left[\begin{array}{c} \parallel \\ \vdots \\ \parallel \\ WX \end{array} \right] \circ \left(\begin{array}{c} \pi/2 \\ \pi/4 \end{array} \right) = \begin{array}{c} n \\ \pi/2 \\ \pi/4 \end{array}$

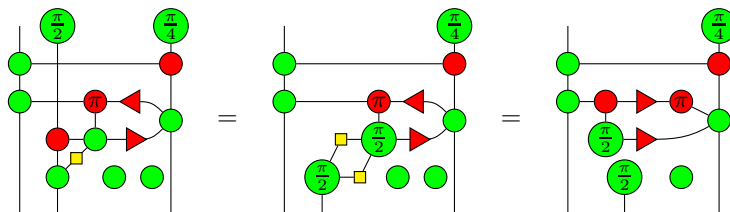
• $\left(\begin{array}{c} \pi/4 \end{array} \right)$: Using mainly (S1), (B1), (H) and 4:

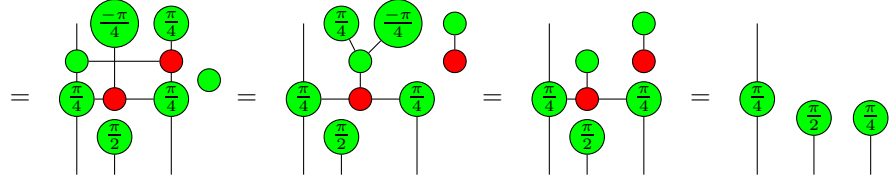


Using definition 3, (S1), (SUP), 13, (H), (C1), 5, 11, (B1), 6 and (K2):



So finally, using (S1), (EU), 2, 16, the previous result, (B2) and (B1):





which means $ZX_{\pi/4} \vdash \left[\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \right] \circ \left(\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \right) = \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}$

• $D_1 \circ D_2$:

It is to be noticed that $\llbracket D_1 \circ D_2 \rrbracket_{WX} = \llbracket D_1 \rrbracket_{WX} \circ \llbracket D_2 \rrbracket_{WX}$ and $\llbracket D_1 \otimes D_2 \rrbracket_{WX} = \llbracket D_1 \rrbracket_{WX} \otimes \llbracket D_2 \rrbracket_{WX}$.

Let us write $\theta = \begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array}$. Then:

$$\begin{aligned} ZX_{\pi/4} \vdash \llbracket D_1 \circ D_2 \rrbracket^{\natural} \circ (\mathbb{I} \otimes \theta) &= \llbracket D_1 \rrbracket^{\natural} \circ \llbracket D_2 \rrbracket^{\natural} \circ (\mathbb{I} \otimes \theta) = \llbracket D_1 \rrbracket^{\natural} \circ (D_2 \otimes \theta) \\ &= \llbracket D_1 \rrbracket^{\natural} \circ (\mathbb{I} \otimes \theta) \circ D_2 = (D_1 \otimes \theta) \circ D_2 = (D_1 \circ D_2) \otimes \theta \end{aligned}$$

• $D_1 \otimes D_2$:

$$\begin{aligned} ZX_{\pi/4} \vdash \llbracket D_1 \otimes D_2 \rrbracket^{\natural} \circ (\mathbb{I} \otimes \theta) &= \left(\llbracket \mathbb{I}^{\otimes n'} \rrbracket_{WX} \otimes \llbracket D_2 \rrbracket^{\natural} \right) \circ \left[\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right]_{WX} \\ &\quad \circ \left(\llbracket \mathbb{I}^{\otimes m} \rrbracket_{WX} \otimes \llbracket D_1 \rrbracket^{\natural} \right) \circ \left[\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right]_{WX} \circ (\mathbb{I} \otimes \theta) \\ &= \left(\mathbb{I}^{\otimes n'} \otimes \llbracket D_2 \rrbracket^{\natural} \right) \circ \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \circ \left(\mathbb{I}^{\otimes m} \otimes \llbracket D_1 \rrbracket^{\natural} \right) \circ \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \circ (\mathbb{I} \otimes \theta) \\ &= \left(\mathbb{I}^{\otimes n'} \otimes \llbracket D_2 \rrbracket^{\natural} \right) \circ \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \circ \left(\mathbb{I}^{\otimes m} \otimes (\llbracket D_1 \rrbracket^{\natural} \circ (\mathbb{I} \otimes \theta)) \right) \circ \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \\ &= \left(\mathbb{I}^{\otimes n'} \otimes \llbracket D_2 \rrbracket^{\natural} \right) \circ \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \circ \left(\mathbb{I}^{\otimes m} \otimes D_1 \otimes \theta \right) \circ \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \\ &= \left(\mathbb{I}^{\otimes n'} \otimes \llbracket D_2 \rrbracket^{\natural} \right) \circ \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \circ (\mathbb{I} \otimes \theta) \circ \left(\mathbb{I}^{\otimes m} \otimes D_1 \right) \circ \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \\ &= \left(\mathbb{I}^{\otimes n'} \otimes D_2 \otimes \theta \right) \circ \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \circ \left(\mathbb{I}^{\otimes m} \otimes D_1 \right) \circ \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \\ &= \left[\left(\mathbb{I}^{\otimes n'} \otimes D_2 \right) \circ \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \circ \left(\mathbb{I}^{\otimes m} \otimes D_1 \right) \circ \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \right] \otimes \theta \\ &= D_1 \otimes D_2 \otimes \theta \end{aligned}$$

By compositions, for any diagram D , $ZX_{\pi/4} \vdash \llbracket D \rrbracket^{\natural} \circ (\mathbb{I} \otimes \theta) = D \otimes \theta$. Then, using lemmas 6 and 4:

$$\forall D \in ZX_{\pi/4}, \quad ZX_{\pi/4} \vdash \left(\begin{array}{c} \dots \quad \dots \\ \circlearrowleft \quad \circlearrowright \\ \dots \quad \dots \end{array} \right) \circ \llbracket \llbracket D \rrbracket_{XW} \rrbracket_{WX} \circ \left(\begin{array}{c} \circlearrowleft \\ \circlearrowright \end{array} \right) = D \quad \square$$