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Micromechanics based 3D framework with second-order damage tensors

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Abstract

The harmonic product of tensors—leading to the concept of harmonic factorization—has been defined in a previous work (Olive et al, 2016). In the practical case of 3D crack density measurements on thin walled structures, this mathematical tool allows us to factorize the harmonic (irreducible) part of the fourth-order damage tensor as an harmonic square: an exact harmonic square in 2D, an harmonic square over the set of so-called *mechanically accessible directions for measurements* in the 3D case. The corresponding micro-mechanics framework based on second—instead of fourth—order damage tensors is derived and an illustrating example is provided.

Keywords: crack density, damage tensor, harmonic decomposition
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1. Introduction

The damage anisotropy encountered in quasi-brittle materials is induced by the loading direction and multiaxiality. From a micro-mechanics point of view, it is the consequence of an oriented microcracking pattern. From the Continuum Damage Mechanics (CDM) point of view, the anisotropic damage state is represented by a tensorial thermodynamics variable, either a fourth-order damage tensor $\mathbf{D}$ with components D_{ijkl} (Chaboche, 1979; Leckie and Onat, 1980; Chaboche, 1984; Lemaitre and Chaboche, 1985; Andrieux et al., 1986; Ju, 1989; Kachanov, 1993; Zheng and Collins, 1998; Cormery and Welemane, 2010; Dormieux and Kondo, 2016) or a symmetric second-order damage tensor $\mathbf{d}$ with components d_{ij} (Vakulenko and Kachanov, 1971; Cordebois and Sidoroff, 1982; Ladevèze, 1983; Murakami, 1988).

There exist many second-order anisotropic damage frameworks (Murakami, 1988; Kattan and Voyiadjis, 1990; Ramtani et al., 1992; Papa and Taliercio, 1996; Halm and Dragon, 1998; Steinmann and Carol, 1998; Lemaitre et al., 2000; Carol et al., 2001; Menzel and Steinmann, 2001; Menzel et al., 2002; Brunig, 2003; Lemaitre and Desmorat, 2005; Desmorat et al., 2007; Badel et al., 2007; Desmorat and Otin, 2008; Desmorat, 2016), as their unification into a single model is partial (Ladevèze, 1983, 1995). From a theoretical point of view (Leckie and Onat, 1980; Onat, 1984), they are usually seen to be restrictive compared to the fourth-order tensorial framework. Nevertheless the interpretation of a damage variable being simpler when a second-order tensor is considered (the three principal values d_i of $\mathbf{d}$ naturally correspond to 3 orthogonal families of microcracks), less damage parameters are introduced and the second-order frameworks have been widely used for either ductile or quasi-brittle materials.

The recent analysis of 2D cracked media with both open and closed microcraks has shown that the so-called irreducible (harmonic) part $\mathbf{H}_{2D}$ of the damage tensor can be decomposed by means of a second-order damage tensor (Desmorat and Desmorat, 2016). More precisely, the standard second-order crack density tensor of Vakulenko and Kachanov (1971) still represents the open cracks contribution when a novel (deviatoric) second-order damage tensor represents the closed—sliding—cracks (previously represented by a fourth-order tensor, Andrieux et al. (1986); Kachanov (1993)). This can be achieved using Verchery's polar decomposition of 2D fourth-order tensors (Verchery, 1979; Vannucci, 2005), which includes both (Desmorat and Desmorat, 2015):

- the harmonic decomposition of considered tensor, and
- the harmonic factorization of its irreducible (harmonic) part $\mathbf{H}_{2D}$, with components $(\mathbf{H}_{2D})_{ijkl}$, using a deviatoric second-order tensor $\mathbf{h}_{2D}$, with components $(\mathbf{h}_{2D})_{ij}$, such that

$$\mathbf{H}_{2D} = \mathbf{h}_{2D} * \mathbf{h}_{2D}. \tag{1.1}$$

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The harmonic product between harmonic tensors $\mathbf{h}_1 * \mathbf{h}_2 = (\mathbf{h}_1 \odot \mathbf{h}_2)_0$ is defined as the projection of the (totally) symmetric product $\mathbf{h}_1 \odot \mathbf{h}_2$ onto (here 2D) the space of harmonic tensors (see Sections 3 and 4.4).

The question arises then as how to extend these results in 3D ? We know from (Olive et al., 2016) that any 3D harmonic fourth-order tensor can be factorized into

$$\mathbf{H} = \mathbf{h}_1 * \mathbf{h}_2 \tag{1.2}$$

i.e. represented by two (deviatoric) second-order tensors $\mathbf{h}_1, \mathbf{h}_2$. But the factorization is far from being unique.

To overpass these difficulties, we point out that triaxial mechanical testing is of high complexity both from the experimental set-up needed (a triaxial machine) and from the difficulty to measure mechanical properties in different space directions (Calloch, 1997; Calloch and Marquis, 1999). We propose to restrict ourselves to a simpler, but considered as sufficiently general, situation: the case of 3D measurements (of a 3D crack density function) on thin walled structures, such as plates, tubes and shells for which a thinner direction, therefore a normal (unit vector $\boldsymbol{\nu}$), is naturally defined. Instead of considering the representation of crack density in any direction $\boldsymbol{n}$ we will consider in Section 5 its representation to a restricted set of directions $\boldsymbol{n} \in \mathcal{R}(\boldsymbol{\nu})$ where

$$\mathcal{R}(\boldsymbol{\nu}) := \{\boldsymbol{\tau}, \|\boldsymbol{\tau}\| = 1 \text{ and } \boldsymbol{\tau} \cdot \boldsymbol{\nu} = 0\} \cup \{\boldsymbol{\nu}\}\,, \tag{1.3}$$

i.e. to the *in-plane directions*, the directions orthogonal to $\boldsymbol{\nu}$, to which is added one *out-of-plane direction*, the normal of the thin walled structure $\boldsymbol{n} = \boldsymbol{\nu}$ itself. In the present work, we consider these directions as the *mechanically accessible directions for measurements*.

After recalling the required mathematical tools (harmonic decomposition, harmonic product and Sylvester's theorem), we revisit the link between crack density function and the tensorial nature of the damage variables. This will allow us to derive a general micro-mechanics based 3D framework with second—instead of fourth—order damage tensors.

Definitions

We denote by $(\cdot)^{\mathrm{s}}$ the totally symmetric part of a possibly non symmetric tensor. A totally symmetric tensor $\mathbf{T} = \mathbf{T}^{\mathrm{s}}$ has all indicial symmetries, its components are thus independent of any permutation of the subscripts. The symmetric tensor product of two tensors $\mathbf{T}_1$ and $\mathbf{T}_2$, of respective orders n_1 and n_2, is the symmetrization of $\mathbf{T}_1 \otimes \mathbf{T}_2$, defining a totally symmetric tensor of order $n = n_1 + n_2$:

$$\mathbf{T}_1 \odot \mathbf{T}_2 := (\mathbf{T}_1 \otimes \mathbf{T}_2)^{\mathrm{s}}, \tag{1.4}$$

so that each component of $\mathbf{T}_1 \odot \mathbf{T}_2$ is independent on any subscript permutation.

Contracting two indices i, j of a tensor $\mathbf{T}$ of order n defines a new tensor of order $n-2$ denoted as $\mathrm{tr}_{ij}\,\mathbf{T}$. For a totally symmetric tensor $\mathbf{T}$, this operation does not depend on the particular choice of the pair i, j. Thus, we can refer to this contraction without any reference to a particular choice of indices and call it the *trace* of $\mathbf{T}$, denoted as $\mathrm{tr}\,\mathbf{T}$. It is a totally symmetric tensor of order $n-2$. Iterating the process leads to

$$\mathrm{tr}^k\,\mathbf{T} = \mathrm{tr}(\mathrm{tr}(\cdots(\mathrm{tr}\,\mathbf{T}))) \tag{1.5}$$

which is a totally symmetric tensor of order $n - 2k$.

A totally symmetric fourth-order tensor $\mathbf{T}$ has no more than 15 independent components, instead of 21 for a triclinic elasticity tensor (*i.e.* a tensor $\mathbf{E}$ having minor symmetry $E_{ijkl} = E_{jikl} = E_{ijlk}$ and major symmetry $E_{ijkl} = E_{klij}$). Totally symmetric elasticity tensors were called *rari-constant* in the nineteenth century (Navier, 1827; Cauchy, 1828b,a; Poisson, 1829; Love, 1905; Vannucci and Desmorat, 2016).

2. Harmonic decomposition and projection

The harmonic decomposition of tensors (Schouten, 1954; Spencer, 1970), introduced in geophysics by Backus (1970), has been popularized by Leckie and Onat (1980) and Onat (1984) when deriving fourth-order damage tensor and later by Forte and Vianello (1996) when classifying the elasticity symmetries.

2.1. Harmonic tensors and associated polynomials

A harmonic tensor is a traceless, totally symmetric tensor, *i.e.*

$$\mathbf{H} = \mathbf{H}^{\mathrm{s}}, \qquad \mathrm{tr}\,\mathbf{H} = 0. \tag{2.1}$$

To any totally symmetric tensor $\mathbf{H}$ of order n, with components $H_{i_1 i_2 \cdots i_n}$, corresponds a unique homogenous polynomial (and conversely). More precisely,

$$\mathrm{h}(\boldsymbol{x}) := \mathbf{H}(\boldsymbol{x}, \boldsymbol{x}, \ldots, \boldsymbol{x}) = H_{i_1 i_2 \cdots i_n} x_{i_1} x_{i_2} \cdots x_{i_n} \tag{2.2}$$

is a homogeneous polynomial

$$\mathrm{h}(\boldsymbol{x}) = \mathrm{h}(x_1, x_2, x_3) \tag{2.3}$$

of degree n in the space coordinates x_1, x_2, x_3. It is harmonic since $\nabla^2 \mathrm{h} = 0$ due to the traceless property $\mathrm{tr}\,\mathbf{H} = 0$.

2.2. Harmonic decomposition and projection of totally symmetric tensors

Any totally symmetric tensor $\mathbf{T}$ of order n can be decomposed uniquely as

$$\mathbf{T} = \mathbf{H}_0 + \mathbf{1} \odot \mathbf{H}_1 + \cdots + \mathbf{1}^{\odot r-1} \odot \mathbf{H}_{r-1} + \mathbf{1}^{\odot r} \odot \mathbf{H}_r \tag{2.4}$$

where $r = [n/2]$ is the integer part of $n/2$, $\mathbf{H}_k$ is an harmonic tensor of degree $n - 2k$ and $\mathbf{1}^{\odot k} = \mathbf{1} \odot \cdots \odot \mathbf{1}$ means the symmetrized tensorial product of k copies of

the (second-order) identity tensor. For n even ($n = 2r$), one has:

$$\mathbf{H}_r = \mathbf{H}_{\frac{n}{2}} = \frac{1}{n+1}\operatorname{tr}^{\frac{n}{2}}\mathbf{T}, \tag{2.5}$$

where $\mathbf{H}_r = H_r$ is a scalar in that case. Moreover, $\mathbf{H}_{r-1}, \dots, \mathbf{H}_0$ are obtained inductively (Olive et al., 2016) as follows:

$$\mathbf{H}_k = \mu(k,n)\operatorname{tr}^k\Big[\mathbf{T} - \sum_{j=k+1}^{r} \mathbf{1}^{\odot j} \odot \mathbf{H}_j\Big] \tag{2.6}$$

where $\mu(k,n) = \dfrac{(2n-4k+1)!(n-k)!n!}{(2n-2k+1)!k!(n-2k)!(n-2k)!}$.

The *harmonic part* of an even order ($n = 2r$), totally symmetric tensor $\mathbf{T}$:

$$\begin{aligned}(\mathbf{T})_0 := \mathbf{H}_0 =& \mathbf{T} - \mathbf{1}\odot\mathbf{H}_1 - \cdots \\ &- \mathbf{1}^{\odot r-1}\odot\mathbf{H}_{r-1} - \mathbf{1}^{\odot r}H_r\end{aligned} \tag{2.7}$$

corresponds to the orthogonal projection of $\mathbf{T}$ onto the space of harmonic tensors of the same order.

2.3. Harmonic decomposition of the elasticity tensor

The harmonic decomposition of an elasticity tensor $\mathbf{E}$, a fourth-order tensor having both minor and major symmetries ($E_{ijkl} = E_{jikl} = E_{ijlk}$ and $E_{ijkl} = E_{klij}$), was first obtained by Backus (1970), as:

$$\mathbf{E} = \alpha\,\mathbf{1}\otimes_{(4)}\mathbf{1} + \beta\,\mathbf{1}\otimes_{(2,2)}\mathbf{1} + \mathbf{1}\otimes_{(4)}\mathbf{a}' + \mathbf{1}\otimes_{(2,2)}\mathbf{b}' + \mathbf{H} \tag{2.8}$$

where $(\cdot)' = (\cdot) - \frac{1}{3}\operatorname{tr}(\cdot)\,\mathbf{1}$ denotes the deviatoric part of a second-order tensor.

In formula (2.8), the Young-symmetrized tensor products $\otimes_{(4)}$ and $\otimes_{(2,2)}$, between two symmetric second-order tensors $\mathbf{y}, \mathbf{z}$, are defined as follows:

$$\begin{cases}\mathbf{y}\otimes_{(4)}\mathbf{z} = \frac{1}{6}\big(\mathbf{y}\otimes\mathbf{z} + \mathbf{z}\otimes\mathbf{y} + 2\,\mathbf{y}\,\underline{\overline{\otimes}}\,\mathbf{z} + 2\,\mathbf{z}\,\underline{\overline{\otimes}}\,\mathbf{y}\big), \\ \mathbf{y}\otimes_{(2,2)}\mathbf{z} = \frac{1}{3}\big(\mathbf{y}\otimes\mathbf{z} + \mathbf{z}\otimes\mathbf{y} - \mathbf{y}\,\underline{\overline{\otimes}}\,\mathbf{z} - \mathbf{z}\,\underline{\overline{\otimes}}\,\mathbf{y}\big),\end{cases} \tag{2.9}$$

where $(\mathbf{y}\,\underline{\overline{\otimes}}\,\mathbf{z})_{ijkl} = \frac{1}{2}(y_{ik}z_{jl} + y_{il}z_{jk})$ so that $\otimes_{(4)}$ is the totally symmetric tensor product $\odot$:

$$\mathbf{y}\otimes_{(4)}\mathbf{z} = \mathbf{y}\odot\mathbf{z} \tag{2.10}$$

In the harmonic decomposition (2.8), $\mathbf{H}$ is a fourth-order harmonic tensor, α, β are scalars, and $\mathbf{a}', \mathbf{b}'$ are second-order harmonic tensors (symmetric deviatoric tensors) related to the *dilatation tensor* $\mathbf{di} = \operatorname{tr}_{12}\mathbf{E}$ and the *Voigt tensor* $\mathbf{vo} = \operatorname{tr}_{13}\mathbf{E}$ by the formulas:

$$\alpha = \frac{1}{15}\left(\operatorname{tr}\mathbf{di} + 2\operatorname{tr}\mathbf{vo}\right), \quad \beta = \frac{1}{6}\left(\operatorname{tr}\mathbf{di} - \operatorname{tr}\mathbf{vo}\right) \tag{2.11}$$

and

$$\mathbf{a}' = \frac{2}{7}\left(\mathbf{di}' + 2\mathbf{vo}'\right), \quad \mathbf{b}' = 2\left(\mathbf{di}' - \mathbf{vo}'\right) \tag{2.12}$$

The harmonic part of a fourth-order elasticity tensor $\mathbf{E}$ is

$$(\mathbf{E})_0 := \mathbf{E} - \mathbf{1}\otimes_{(4)}\mathbf{a} - \mathbf{1}\otimes_{(2,2)}\mathbf{b} \tag{2.13}$$

or in an equivalent manner

$$\begin{aligned}(\mathbf{E})_0 :=& \mathbf{E} - \mathbf{1}\odot\mathbf{a} \\ &- \frac{1}{3}\big(\mathbf{1}\otimes\mathbf{b} + \mathbf{b}\otimes\mathbf{1} - \mathbf{1}\,\underline{\overline{\otimes}}\,\mathbf{b} - \mathbf{b}\,\underline{\overline{\otimes}}\,\mathbf{1}\big)\end{aligned} \tag{2.14}$$

with $\mathbf{a} = \mathbf{a}' + \alpha\mathbf{1}$ and $\mathbf{b} = \mathbf{b}' + \beta\mathbf{1}$, where the scalars α, β and the second-order deviatoric tensors $\mathbf{a}', \mathbf{b}'$ are given by (2.11) and (2.12).

3. The harmonic product and Sylvester's theorem

The harmonic product of two harmonic tensors of order n_1 and n_2, defining an harmonic tensor of order $n = n_1 + n_2$, has been introduced in (Olive et al., 2016) as the harmonic part of the symmetric tensor product:

$$\mathbf{H}_1 * \mathbf{H}_2 := (\mathbf{H}_1 \odot \mathbf{H}_2)_0\,. \tag{3.1}$$

Note that this product is *associative*:

$$\mathbf{H}_1 * (\mathbf{H}_2 * \mathbf{H}_3) = (\mathbf{H}_1 * \mathbf{H}_2) * \mathbf{H}_3, \tag{3.2}$$

and *commutative*:

$$\mathbf{H}_1 * \mathbf{H}_2 = \mathbf{H}_2 * \mathbf{H}_1. \tag{3.3}$$

For two vectors $\boldsymbol{w}_1, \boldsymbol{w}_2$, we have

$$\begin{aligned}\boldsymbol{w}_1 * \boldsymbol{w}_2 =& (\boldsymbol{w}_1 \odot \boldsymbol{w}_2)' \\ =& \frac{1}{2}\left(\boldsymbol{w}_1\otimes\boldsymbol{w}_2 + \boldsymbol{w}_2\otimes\boldsymbol{w}_1\right) - \frac{1}{3}(\boldsymbol{w}_1\cdot\boldsymbol{w}_2)\,\mathbf{1},\end{aligned} \tag{3.4}$$

where $\boldsymbol{w}_1\cdot\boldsymbol{w}_2 = \boldsymbol{w}_1^T\boldsymbol{w}_2$ is the scalar product.

For two second-order harmonic (symmetric deviatoric) tensors $\mathbf{h}_1 = \mathbf{h}_1', \mathbf{h}_2 = \mathbf{h}_2'$, we have

$$\begin{aligned}\mathbf{h}_1 * \mathbf{h}_2 = \mathbf{h}_1 \odot \mathbf{h}_2 &- \frac{2}{7}\,\mathbf{1}\odot(\mathbf{h}_1\mathbf{h}_2 + \mathbf{h}_2\mathbf{h}_1) \\ &+ \frac{2}{35}\operatorname{tr}(\mathbf{h}_1\mathbf{h}_2)\,\mathbf{1}\odot\mathbf{1}.\end{aligned} \tag{3.5}$$

Sylvester's theorem (Sylvester, 1909; Backus, 1970; Baerheim, 1998) states that any harmonic tensor $\mathbf{H}$ of order n can be factorized as

$$\mathbf{H} = \boldsymbol{w}_1 * \boldsymbol{w}_2 * \cdots * \boldsymbol{w}_n, \tag{3.6}$$

i.e. as the harmonic products of n (real) vectors $\boldsymbol{w}_k$, the so-called *Sylvester-Maxwell multipoles*. Note however, that this factorization is far from being unique, as discussed in (Olive et al., 2016).

Setting $\mathbf{h}_1 = \boldsymbol{w}_1 * \boldsymbol{w}_2$ and $\mathbf{h}_2 = \boldsymbol{w}_3 * \boldsymbol{w}_4$ which are harmonic (symmetric deviatoric) second-order tensors, we obtain the non unique harmonic factorization of $\mathbf{H}$ by means of two second-order tensors:

$$\mathbf{H} = \mathbf{h}_1 * \mathbf{h}_2, \tag{3.7}$$

as detailed in (Desmorat and Desmorat, 2016; Olive et al., 2016).

4. On the link between fourth-order crack density and damage tensors

Before formulating our main result (Theorem 5.1, Section 5), we summarize, in the present section, the literature results in Continuum Mechanics leading to the representation of damage of cracked media by a fourth-order tensor (damage framework proposed by Chaboche (1979)). We make an explicit link with the harmonic decomposition and we present, by comparison to the 2D case, the problem of representation of damage by second-order tensors only.

4.1. Crack density function and tensors

The damage state of a microcracked material is classically defined by the spatial arrangement, orientation and geometry of the cracks present at the microscale (Kachanov, 1972; Leckie and Onat, 1980; Ladevèze, 1983; Onat, 1984; Lemaitre and Chaboche, 1985; Murakami, 1988; Kachanov, 1993). The crack density, related to any possible 3D direction defined by a unit vector $\boldsymbol{n}$, refers to a dimensionless scalar property defined in a continuous manner at the Representative Volume Element scale as a spatial crack density function $\Omega = \Omega(\boldsymbol{n})$. Owing to the property $\Omega(\boldsymbol{n}) = \Omega(-\boldsymbol{n})$, it is expressed by means of a totally symmetric tensor $\mathbf{F}$ (the so-called *fabric tensor*) of even order $n = 2r$ (Kanatani, 1984) as:

$$\Omega(\boldsymbol{n}) = \mathbf{F} \bullet (\boldsymbol{n} \otimes \boldsymbol{n} \otimes \cdots \otimes \boldsymbol{n}) \tag{4.1}$$

where $\bullet$ means the contraction over the n subscripts. Note that $\Omega(\boldsymbol{n})$ corresponds to a homogeneous polynomial (see section 2.2)

$$\mathrm{h}(n_1, n_2, n_3) = \mathbf{F}(\boldsymbol{n}, \boldsymbol{n}, \ldots, \boldsymbol{n}).$$

The fabric tensor $\mathbf{F}$, which is totally symmetric, can be determined as the least square error approximation of an experimental (measured) density distribution $\Omega(\boldsymbol{n})$ as

$$\mathrm{argmin}_{\mathbf{F}} \left\| \frac{4\pi}{2n+1} \mathbf{F} \bullet \mathbf{1}^{\odot n} - \int_{\|\boldsymbol{x}\|=1} \Omega(\boldsymbol{n})\, \boldsymbol{n}^{\otimes n} \mathrm{d}S \right\|^2 \tag{4.2}$$

with solid angle S, and where

$$\begin{aligned} &\boldsymbol{n}^{\otimes k} := \boldsymbol{n} \otimes \boldsymbol{n} \otimes \cdots \otimes \boldsymbol{n}, \\ &(\boldsymbol{n}^{\otimes k})_{i_1 i_2 \cdots i_k} = n_{i_1} n_{i_2} \cdots n_{i_k}. \end{aligned} \tag{4.3}$$

Moreover, the following equality has been used:

$$\frac{1}{4\pi} \int_{\|\boldsymbol{x}\|=1} \boldsymbol{n}^{\otimes 2n} \mathrm{d}S = \frac{1}{2n+1} \mathbf{1}^{\odot n}. \tag{4.4}$$

Note that $\boldsymbol{n}^{\otimes k} = \boldsymbol{n}^{\odot k}$ is a totally symmetric tensor.

Comparative studies of the tensorial order needed to represent given microcracking patterns can be found in (Lubarda and Krajcinovic, 1993; Krajcinovic, 1996; Tikhomirov et al., 2001).

Expression (4.1) is often rewritten into the finite expansion (Kanatani, 1984; Onat, 1984; Krajcinovic, 1996):

$$\begin{aligned} \Omega(\boldsymbol{n}) = \mathbf{F}_4 \bullet (\boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}) + \boldsymbol{\Omega}_6 \bullet (\boldsymbol{n}^{\otimes 6}) + \cdots \\ \cdots + \boldsymbol{\Omega}_{2k} \bullet (\boldsymbol{n}^{\otimes 2k}) + \cdots + \boldsymbol{\Omega}_n \bullet (\boldsymbol{n}^{\otimes n}) \end{aligned} \tag{4.5}$$

with fourth-order part

$$\begin{aligned} \mathbf{F}_4 \bullet (\boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}) = &\Omega_0 + \boldsymbol{\Omega}_2 \bullet (\boldsymbol{n} \otimes \boldsymbol{n}) \\ &+ \boldsymbol{\Omega}_4 \bullet (\boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}) \end{aligned} \tag{4.6}$$

and where $\boldsymbol{\Omega}_{2k}$ (resp. $\boldsymbol{\Omega}_n$, $n = 2r$ even) are totally symmetric traceless (harmonic) tensors of order $2k$ (resp. $n = 2r$). The initial (scalar) term is the crack density within considered Continuum Mechanics representative volume element

$$\Omega_0 = \frac{1}{4\pi} \int_{\|\boldsymbol{x}\|=1} \Omega(\boldsymbol{n})\, \mathrm{d}S, \tag{4.7}$$

S denoting the solid angle. Crack density tensors $\boldsymbol{\Omega}_2, \boldsymbol{\Omega}_4, \ldots, \boldsymbol{\Omega}_n$ are harmonic tensors of even order $2, 4, \ldots, n$. They constitute independent crack density variables representative of the microcraking pattern (and anisotropy), determined uniquely up to order n from the knowledge of the 3D spatial crack density distribution $\Omega(\boldsymbol{n})$.

4.2. Crack density tensors from harmonic decomposition

Let us point out that the harmonic tensors $\boldsymbol{\Omega}_{2k}$ are the tensors $\mathbf{H}_{r-k}$ issued from the harmonic decomposition (2.4) of the totally symmetric fabric tensor $\mathbf{F}$:

$$\mathbf{F} = \mathbf{H}_0 + \mathbf{1} \odot \mathbf{H}_1 + \cdots + \mathbf{1}^{\odot r-1} \odot \mathbf{H}_{r-1} + \mathbf{1}^{\odot r} H_r \tag{4.8}$$

with $r = n/2$, where $H_r = H_{\frac{n}{2}}$ and the harmonic tensors $\mathbf{H}_k$ of degree $n - 2k$ are given by (2.5) and (2.6). Observe that:

$$(\mathbf{1}^{\odot k} \odot \mathbf{H}_k) \bullet \boldsymbol{n}^{\otimes n} = \mathbf{H}_k \bullet \boldsymbol{n}^{\otimes n-2k} \tag{4.9}$$

and we thus get:

$$\begin{aligned} \Omega(\boldsymbol{n}) = H_r + \mathbf{H}_{r-1} \bullet (\boldsymbol{n} \otimes \boldsymbol{n}) + \mathbf{H}_{r-2} \bullet (\boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n} \otimes \boldsymbol{n}) \\ + \cdots + \mathbf{H}_0 \bullet (\boldsymbol{n}^{\otimes n}), \end{aligned} \tag{4.10}$$

which is the finite expansion (4.5) where

$$\Omega_0 = H_{\frac{n}{2}}, \qquad \boldsymbol{\Omega}_{2k} = \mathbf{H}_{r-k}. \tag{4.11}$$

4.3. Fourth-order damage tensor

Using the decomposition (4.5) and assuming open microcracks in an initially 3D isotropic medium, Leckie and Onat (1980) and Onat (1984) have shown that the damage variable defined by the coupling microcraking/elasticity is at most a fourth-order tensor, built from $\mathbf{F}_4$ only, see (4.5). This result holds for non interacting closed—sliding without friction—pennyshaped microcracks (Kachanov, 1993) and, as pointed out by

Cormery and Welemane (2010), for many stress based homogenization schemes, as long as all the microcracks are in the same state, either open or closed. Setting:

$$\mathbf{J} = \mathbf{I} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}, \tag{4.12}$$

the following general definition of a fourth-order damage tensor has then been derived for initially isotropic materials (Kachanov, 1993; Zheng and Collins, 1998; Cormery and Welemane, 2010):

$$\begin{aligned}\mathbf{D} = p_0\Omega_0\mathbf{1}\otimes\mathbf{1} + p_1\Omega_0\,\mathbf{J} + p_2(\mathbf{1}\otimes\boldsymbol{\Omega}_2 + \boldsymbol{\Omega}_2\otimes\mathbf{1}) \\ + p_3(\mathbf{1}\,\overline{\otimes}\,\boldsymbol{\Omega}_2 + \boldsymbol{\Omega}_2\,\overline{\otimes}\,\mathbf{1}) + p_4\boldsymbol{\Omega}_4,\end{aligned} \tag{4.13}$$

where Ω_0 should be interpreted as a scalar damage variable, the symmetric deviator $\boldsymbol{\Omega}_2$ as a second-order damage variable, and the harmonic tensor $\boldsymbol{\Omega}_4$ as a fourth-order damage variable. The expression of the scalars p_i depends on the initial elasticity parameters, on the homogenization scheme and on the microcracks state (simultaneously open or simultaneously closed for all cracks).

Remark 4.1. The scalars p_i do not depend on $\Omega_0, \boldsymbol{\Omega}_2, \boldsymbol{\Omega}_4$.

Remark 4.2. (4.13) is the harmonic decomposition (2.8) of the fourth-order damage tensor $\mathbf{D}$, which has the major and the minor indicial symmetries ($D_{ijkl} = D_{klij} = D_{jikl}$) as an elasticity tensor. The deviatoric parts of the dilatation and Voigt tensors are both proportional to the second-order harmonic tensor $\boldsymbol{\Omega}_2$, with the scalar factors κ_{di} and κ_{vo} depending only on the initial elastic parameters of the undamaged isotropic material:

$$\begin{aligned}\mathbf{di}'(\mathbf{D}) = (\mathrm{tr}_{12}\,\mathbf{D})' &= \kappa_{\mathrm{di}}\boldsymbol{\Omega}_2, \\ \mathbf{vo}'(\mathbf{D}) = (\mathrm{tr}_{13}\,\mathbf{D})' &= \kappa_{\mathrm{vo}}\boldsymbol{\Omega}_2.\end{aligned} \tag{4.14}$$

The traces of the dilatation and the Voigt tensors are both proportional to the scalar crack density Ω_0, with scalar factors k_{di} and k_{vo} depending only on the elastic parameters of virgin (undamaged) isotropic material:

$$\begin{aligned}\mathrm{tr}\,\mathbf{di}(\mathbf{D}) = \mathrm{tr}(\mathrm{tr}_{12}\,\mathbf{D}) &= k_{\mathrm{di}}\Omega_0, \\ \mathrm{tr}\,\mathbf{vo}(\mathbf{D}) = \mathrm{tr}(\mathrm{tr}_{13}\,\mathbf{D}) &= k_{\mathrm{vo}}\Omega_0.\end{aligned} \tag{4.15}$$

4.4. 2D case

In 2D, cracks are represented by 2D straight lines. The expression (4.5) for crack density holds, recovering Fourier finite expansion (Kanatani, 1984; Burr et al., 1995):

$$\begin{aligned}\Omega(\boldsymbol{n}) = \omega_{2D} + \boldsymbol{\omega}'_{2D}\bullet(\boldsymbol{n}\otimes\boldsymbol{n}) \\ + \mathbf{H}_{2D}\bullet(\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}) + \cdots\end{aligned} \tag{4.16}$$

where the unit vector $\boldsymbol{n}$ is related to the possible planar direction,

$$\omega_{2D} = \frac{1}{2\pi}\int_0^{2\pi}\Omega(\boldsymbol{n})\,\mathrm{d}\theta$$

is the 2D crack density, and $\boldsymbol{\omega}'_{2D}$, $\mathbf{H}_{2D}$ are respectively the 2D harmonic second and the fourth-order crack density tensors.

Verchery's decomposition (Verchery, 1979; Vannucci, 2005) and its rewriting into a tensorial form (Desmorat and Desmorat, 2015) shows that any 2D harmonic fourth-order tensor is an harmonic square. Applied to $\mathbf{H}_{2D}$, this gives:

$$\mathbf{H}_{2D} = \mathbf{h}_{2D} * \mathbf{h}_{2D}, \tag{4.17}$$

where $\mathbf{h}_{2D}$ is an harmonic (*i.e.* symmetric deviatoric) second-order tensor, and the notation $*$ denotes the harmonic product defined as the projection of the symmetrized product $\mathbf{h}_{2D}\odot\mathbf{h}_{2D}$ onto 2D harmonic tensors' space:

$$\mathbf{h}_{2D} * \mathbf{h}_{2D} := (\mathbf{h}_{2D}\odot\mathbf{h}_{2D})_0. \tag{4.18}$$

For second-order harmonic tensors, this reads:

$$\mathbf{h}_{2D} * \mathbf{h}_{2D} = \mathbf{h}_{2D}\odot\mathbf{h}_{2D} - \frac{1}{4}(\mathrm{tr}\,\mathbf{h}_{2D}^2)\,\mathbf{1}\odot\mathbf{1} \tag{4.19}$$

This means that in 2D, any anisotropic microcracking pattern can be expressed, up to order $n = 4$, exactly by means of the scalar ω_{2D} and the two independent second-order deviatoric damage variables $\boldsymbol{\omega}' = \boldsymbol{\Omega}_2^{2D}$ and $\mathbf{h}_{2D} = \mathbf{h}'_{2D}$. This result is consistent with the fact that the micro-mechanics of 2D media with open and closed (sliding without friction) microcracks can be represented by two second-order damage tensors only (Desmorat and Desmorat, 2015).

The question arises then whether the expansion (4.16)–(4.17) holds in 3D, *i.e.* with $\boldsymbol{\omega}'$ and $\mathbf{h}$ (now 3D) deviatoric second-order tensors. The answer is negative in the general triclinic case for which we have $\boldsymbol{\Omega}_4 = \mathbf{h}_1 * \mathbf{h}_2$ (3.7) with, in general, different second-order tensors $\mathbf{h}_1, \mathbf{h}_2$ (Olive et al., 2016). Furthermore, the factorization is not unique, forbidding to interpret $\mathbf{h}_1$ and $\mathbf{h}_2$ as damage variables.

5. Two second-order damage tensors from 3D measurements on thin walled structures

It has been noticed by Lubarda and Krajcinovic (1993) and Krajcinovic (1996) that the fourth-order crack density tensor[1] $\boldsymbol{\Omega}_4 = (\mathbf{F}_4)_0$ induced by particular loadings is related, sometimes, as a square of the second-order tensor harmonic contribution $\boldsymbol{\Omega}_2$. More precisely, $\boldsymbol{\Omega}_4$ is proportional to the harmonic square $\boldsymbol{\Omega}_2 * \boldsymbol{\Omega}_2$ in the particular situation that occurs for a family of parallel penny shaped microcracks having identical—therefore coplanar—normal $\boldsymbol{m}$ (induced for example by uniaxial tension on quasi-brittle materials). The second and fourth-order crack density variables $\boldsymbol{\Omega}_2$ and $\boldsymbol{\Omega}_4$ remain nevertheless independent in more general cases.

On the other hand, as stated in the introduction, one may consider as the general case—except for soils—that the measurements of crack density $\Omega(\boldsymbol{n})$ is performed on

[1] Eq. (2.7) defines harmonic part $(\mathbf{T})_0$ of a totally symmetric tensor $\mathbf{T}$.

thin walled structures, as e.g. thin or thick tubes or 2D structures such as plates. This allows to introduce the unit normal $\boldsymbol{\nu}$ ($\|\boldsymbol{\nu}\| = 1$) to the thin walled structure and the set

$$\mathcal{R}(\boldsymbol{\nu}) := \{\boldsymbol{\tau}, \|\boldsymbol{\tau}\| = 1 \text{ and } \boldsymbol{\tau}\cdot\boldsymbol{\nu} = 0\} \cup \{\boldsymbol{\nu}\}, \tag{5.1}$$

of restrictions of considered directions $\boldsymbol{n}$ to the so-called mechanically accessible directions for measurements. As an extension of both the previous remark on fourth-order harmonic squares and the 2D result (4.16)–(4.17), we have in 3D the following theorem (the proof of which is given at the end of the present section):

Theorem 5.1. *For a given unit vector $\boldsymbol{\nu}$, any density function $\Omega(\boldsymbol{n})$ is represented, up to fourth-order, for all directions $\boldsymbol{n} \in \mathcal{R}(\boldsymbol{\nu})$ by means of a scalar ω_m and two harmonic (symmetric deviatoric) second-order tensors $\boldsymbol{\omega}'$ and $\mathbf{h}$ as:*

$$\begin{aligned}\Omega(\boldsymbol{n}) = \omega_m + \boldsymbol{\omega}' \bullet (\boldsymbol{n}\otimes\boldsymbol{n}) \\ + (\mathbf{h} * \mathbf{h}) \bullet (\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}) + \cdots\end{aligned} \tag{5.2}$$

for all $\boldsymbol{n} \in \mathcal{R}(\boldsymbol{\nu})$. This representation is unique, up to $\pm\mathbf{h}$, if $(\boldsymbol{\omega}'\boldsymbol{\nu})\times\boldsymbol{\nu} = \mathbf{h}\boldsymbol{\nu} = \mathbf{0}$.

Remark 5.2. If we set $\boldsymbol{e}_3 = \boldsymbol{\nu}$, the conditions $(\boldsymbol{\omega}'\boldsymbol{\nu})\times\boldsymbol{\nu} = \mathbf{0}$ (which is equivalent to $(\boldsymbol{\omega}'\boldsymbol{\nu}) = \lambda\boldsymbol{\nu}$) and $\mathbf{h}\boldsymbol{\nu} = \mathbf{0}$ mean that

$$\begin{aligned}\boldsymbol{\omega}' &= \begin{pmatrix} \omega'_{11} & \omega_{12} & 0 \\ \omega_{12} & \omega'_{22} & 0 \\ 0 & 0 & -(\omega'_{11}+\omega'_{22}) \end{pmatrix}, \\ \mathbf{h} &= \begin{pmatrix} h_{11} & h_{12} & 0 \\ h_{12} & -h_{11} & 0 \\ 0 & 0 & 0 \end{pmatrix}\end{aligned} \tag{5.3}$$

Applied to thinned walled structured, for which $\boldsymbol{n} \in \mathcal{R}(\boldsymbol{\nu})$ are the accessible directions for mechanical measurements, Theorem 5.1 states that any microcracking pattern, possibly triclinic, can be represented, up to order $n = 4$, by means of only two symmetric second-order crack density tensors, $\boldsymbol{\omega} = \boldsymbol{\omega}' + \omega_m\,\mathbf{1}$ and $\mathbf{h}$, the second one being deviatoric. In that case, we can recast (5.2) as:

$$\begin{aligned}\Omega(\boldsymbol{n}) = \boldsymbol{\omega} \bullet (\boldsymbol{n}\otimes\boldsymbol{n}) \\ + (\mathbf{h} * \mathbf{h}) \bullet (\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}\otimes\boldsymbol{n}) + \ldots\end{aligned} \tag{5.4}$$

for all $\boldsymbol{n} \in \mathcal{R}(\boldsymbol{\nu})$, where the tensor $\boldsymbol{\omega}$ is the crack density tensor introduced by Vakulenko and Kachanov (1971) (Kachanov, 1972, 1993).

Practical formulas: Uniqueness

Recall that, up to order four, the crack density function $\Omega(\boldsymbol{n})$ is represented by the fabric tensor $\mathbf{F}_4$ (4.5)–(4.6). Consider now an orthonormal frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{\nu}\}$ and let $(\omega_m, \boldsymbol{\omega}', \mathbf{h})$ be a triplet as in (5.3). Set

$$\begin{aligned}h_{11} + ih_{12} = \frac{1}{2}\big[&(\mathbf{F}_4)_{1111} + (\mathbf{F}_4)_{2222} - 6(\mathbf{F}_4)_{1122} \\ &+ 4i\big((\mathbf{F}_4)_{1112} - (\mathbf{F}_4)_{1222}\big)\big]^{1/2}\end{aligned} \tag{5.5}$$

and

$$\begin{aligned}\omega_m =& \frac{1}{4}(\mathbf{F}_4)_{1111} + \frac{1}{2}(\mathbf{F}_4)_{1122} + \frac{1}{4}(\mathbf{F}_4)_{2222} \\ &+ \frac{1}{3}(\mathbf{F}_4)_{3333} - \frac{1}{15}(h_{11}^2 + h_{12}^2) \\ \omega'_{11} =& \frac{5}{8}(\mathbf{F}_4)_{1111} + \frac{1}{4}(\mathbf{F}_4)_{1122} - \frac{3}{8}(\mathbf{F}_4)_{2222} \\ &- \frac{1}{3}(\mathbf{F}_4)_{3333} + \frac{1}{42}(h_{11}^2 + h_{12}^2) \\ \omega'_{22} =& -\frac{3}{8}(\mathbf{F}_4)_{1111} + \frac{1}{4}(\mathbf{F}_4)_{1122} + \frac{5}{8}(\mathbf{F}_4)_{2222} \\ &- \frac{1}{3}(\mathbf{F}_4)_{3333} + \frac{1}{42}(h_{11}^2 + h_{12}^2) \\ \omega_{12} =& (\mathbf{F}_4)_{1222} + (\mathbf{F}_4)_{1112}\end{aligned} \tag{5.6}$$

where $i = \sqrt{-1}$ is the pure imaginary number. It can be checked by a direct computation that $(\omega_m, \boldsymbol{\omega}', \mathbf{h})$ is a solution of (5.2) in Theorem 5.1.

Remark 5.3. Because of the square root in (5.5), both $\mathbf{h}$ and $-\mathbf{h}$ are solutions.

We will now show the uniqueness of the solution, up to a sign, and under the assumption that:

$$(\boldsymbol{\omega}'\boldsymbol{\nu})\times\boldsymbol{\nu} = \mathbf{h}\boldsymbol{\nu} = \mathbf{0}.$$

To do so, suppose that a second solution $(\omega_m^*, \boldsymbol{\omega}^{*\prime}, \mathbf{h}^*)$ to (5.2) exists, with $\boldsymbol{\omega}^{*\prime}$ and $\mathbf{h}^*$ as in (5.3) in the orthonormal basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{\nu})$. Equaling, for different directions $\boldsymbol{n} \in \mathcal{R}(\boldsymbol{\nu})$, the density function $\Omega(\boldsymbol{n})$, calculated with $(\omega_m, \boldsymbol{\omega}', \mathbf{h})$ to $\Omega^*(\boldsymbol{n})$, calculated with $(\omega_m^*, \boldsymbol{\omega}^{*\prime}, \mathbf{h}^*)$, we obtain first for $\boldsymbol{n} = \boldsymbol{\nu}$:

$$\begin{aligned}35(\omega_{11}^{*\prime} + \omega_{22}^{*\prime} - \omega'_{11} - \omega'_{22} + \omega_m - \omega_m^*) \\ +4h_{11}^2 + 4h_{12}^2 - 4h_{11}^{*2} - 4h_{12}^{*2}. = 0\end{aligned} \tag{5.7}$$

Then, for $\boldsymbol{n} = (\cos\theta, \sin\theta, 0)$, we get:

$$\begin{aligned}2(\omega_m - \omega_m^*) + \omega'_{11} - \omega_{11}^{*\prime} + \omega'_{22} - \omega_{22}^{*\prime} \\ +\frac{3}{35}(h_{11}^2 - h_{11}^{*2} + h_{12}^2 - h_{12}^{*2}) \\ (\omega'_{11} - \omega_{11}^{*\prime} - \omega'_{22} + \omega_{22}^{*\prime})\cos 2\theta \\ +2(\omega_{12} - \omega_{12}^*)\sin 2\theta \\ + \big(h_{11}^2 - h_{12}^2 - h_{11}^{*2} + h_{12}^{*2}\big)\cos 4\theta \\ +2(h_{11}h_{12} - h_{11}^*h_{12}^*)\sin 4\theta = 0.\end{aligned} \tag{5.8}$$

Since this equality holds for all θ, the coefficient of $\cos 4\theta$ and $\sin 4\theta$ must vanish and we get:

$$\begin{cases} h_{11}^{*2} - h_{12}^{*2} = h_{11}^2 - h_{12}^2, \\ h_{11}^*h_{12}^* = h_{11}h_{12}, \end{cases} \tag{5.9}$$

from which we deduce that

$$(h_{11}^* + ih_{12}^*)^2 = (h_{11} + ih_{12})^2$$

and therefore that $\mathbf{h}^* = \pm\mathbf{h}$ (in accordance with Remark 5.3). Now, (5.7) and the vanishing of the constant term and the coefficients of $\sin 2\theta$ and $\cos 2\theta$ in (5.8) leads to:

$$\begin{cases} \omega_m - \omega_m^* + \omega_{11}^{*\prime} - \omega_{11}^{\prime} + \omega_{22}^{*\prime} - \omega_{22}^{\prime} = 0 \\ 2(\omega_m - \omega_m^*) + \omega_{11}^{\prime} - \omega_{11}^{*\prime} + \omega_{22}^{\prime} - \omega_{22}^{*\prime} = 0 \\ \omega_{11}^{\prime} - \omega_{11}^{*\prime} - \omega_{22}^{\prime} + \omega_{22}^{*\prime} = 0 \\ \omega_{12} - \omega_{12}^* = 0 \end{cases} \tag{5.10}$$

i.e. $\omega_m^* = \omega_m$ and $\boldsymbol{\omega}^{*\prime} = \boldsymbol{\omega}^{\prime}$, which achieves the proof.

6. General micro-mechanics based framework with two second-order damage variables

Using the results of Section 4.3, we deduce from (5.2) that the representation by means of two symmetric second-order tensors holds for the damage tensor itself, at least when the microcracks are all in the same state, all open or all closed. This means that, disposing from sufficiently many in-plane measurements (along directions $\boldsymbol{n}$ orthogonal to $\boldsymbol{\nu}$) and an out-of-plane measurement (along $\boldsymbol{n} = \boldsymbol{\nu}$), the general fourth-order damage tensor of Chaboche–Leckie–Onat can be expressed by means of two symmetric second-order damage variables only, for example $\boldsymbol{\omega}$ and $\mathbf{h}$ (the second–one being a deviator). A general damage framework using this feature is derived next, clarifying the link between Cordebois and Sidoroff (1982) and Ladevèze (1983, 1995) phenomenological second-order damage models and micro-mechanics based framework.

We shall assume that the homogenization result (4.13), of a stress based scheme, holds, where the constants p_i are given (refer to the works of Kachanov (1993) and Dormieux and Kondo (2016) for comparison of different homogenization schemes). Gibbs free enthalpy density writes

$$\rho\psi^\star = \frac{1}{18K}(\operatorname{tr}\boldsymbol{\sigma})^2 + \frac{1}{4G}\boldsymbol{\sigma}' : \boldsymbol{\sigma}' + \frac{1}{2E}\boldsymbol{\sigma} : \mathbf{D} : \boldsymbol{\sigma}, \tag{6.1}$$

where ρ is the density and E, $G = \frac{E}{2(1+\nu)}$, $K = \frac{E}{3(1-2\nu)}$ are, respectively, the Young, shear and bulk moduli. The elasticity law, coupled with the anisotropic damage, writes then as

$$\boldsymbol{\epsilon}^e = \rho\frac{\partial\psi^\star}{\partial\boldsymbol{\sigma}} = \frac{1}{2G}\boldsymbol{\sigma}' + \frac{1}{9K}\operatorname{tr}\boldsymbol{\sigma}\,\mathbf{1} + \frac{1}{E}\mathbf{D} : \boldsymbol{\sigma}, \tag{6.2}$$

or, in a more compact form, as

$$\boldsymbol{\epsilon}^e = \tilde{\mathbf{S}} : \boldsymbol{\sigma}, \tag{6.3}$$

where $\boldsymbol{\epsilon}^e$ is the elastic strain tensor and $\tilde{\mathbf{S}}$, the effective fourth-order compliance tensor

$$\tilde{\mathbf{S}} = \frac{1}{9K}\mathbf{1}\otimes\mathbf{1} + \frac{1}{2G}\mathbf{J} + \frac{1}{E}\mathbf{D}, \quad \mathbf{J} = \mathbf{I} - \frac{1}{3}\mathbf{1}\otimes\mathbf{1}. \tag{6.4}$$

Having many in-plane and possibly one out-of-plane measurements, allows us to use remark 4.1 in Section 4.3, and (5.2) instead of (4.13), within the considered homogenization scheme. We can thus recast the fourth-order damage tensor $\mathbf{D}$ by substituting the scalar Ω_0 by ω_m, the second-order tensor $\boldsymbol{\Omega}_2$ by the deviatoric tensor $\boldsymbol{\omega}'$ and the fourth-order tensor $\boldsymbol{\Omega}_4$ by the harmonic (*i.e* totally symmetric and traceless) tensor $\mathbf{h} * \mathbf{h}$. More precisely, we get

$$\begin{aligned} \mathbf{D} =& p_0\omega_m\,\mathbf{1}\otimes\mathbf{1} + p_1\omega_m\,\mathbf{J} + p_2\,(\mathbf{1}\otimes\boldsymbol{\omega}' + \boldsymbol{\omega}'\otimes\mathbf{1}) \\ &+ p_3\,(\mathbf{1}\,\underline{\otimes}\,\boldsymbol{\omega}' + \boldsymbol{\omega}'\,\underline{\otimes}\,\mathbf{1}) + p_4\,\mathbf{h} * \mathbf{h}. \end{aligned} \tag{6.5}$$

Using (3.5), the term $\mathbf{h} * \mathbf{h}$ expands as

$$\begin{aligned} \mathbf{h} * \mathbf{h} =& \frac{1}{3}\mathbf{h}\otimes\mathbf{h} + \frac{2}{3}\mathbf{h}\,\underline{\otimes}\,\mathbf{h} \\ &- \frac{2}{21}\,(\mathbf{1}\otimes\mathbf{h}^2 + \mathbf{h}^2\otimes\mathbf{1} + 2(\mathbf{1}\,\underline{\otimes}\,\mathbf{h}^2 + \mathbf{h}^2\,\underline{\otimes}\,\mathbf{1})) \\ &+ \frac{2}{105}\operatorname{tr}\mathbf{h}^2\,(\mathbf{1}\otimes\mathbf{1} + 2\,\mathbf{1}\,\underline{\otimes}\,\mathbf{1}), \end{aligned} \tag{6.6}$$

so that the enthalpic contribution, due to the microcracks, writes

$$\begin{aligned} \boldsymbol{\sigma} : \mathbf{D} : \boldsymbol{\sigma} =& \; p_0\omega_m\,(\operatorname{tr}\boldsymbol{\sigma})^2 + p_1\omega_m\,\operatorname{tr}(\boldsymbol{\sigma}'^2) \\ &+ 2p_2\operatorname{tr}(\boldsymbol{\omega}'\boldsymbol{\sigma})\operatorname{tr}\boldsymbol{\sigma} + p_3\operatorname{tr}(\boldsymbol{\omega}'\boldsymbol{\sigma}^2) \\ &+ p_4\Big[\frac{1}{3}(\operatorname{tr}(\mathbf{h}\boldsymbol{\sigma}'))^2 + \frac{2}{3}\operatorname{tr}(\boldsymbol{\sigma}'\mathbf{h}\boldsymbol{\sigma}'\mathbf{h}) \\ &- \frac{8}{21}\operatorname{tr}(\mathbf{h}^2\boldsymbol{\sigma}'^2) + \frac{4}{105}\operatorname{tr}\mathbf{h}^2\,\operatorname{tr}\boldsymbol{\sigma}'^2\Big]. \end{aligned} \tag{6.7}$$

Again, as in Remark 4.2, (6.5) is nothing else but the harmonic decomposition of the fourth-order damage tensor $\mathbf{D}$, but with the following particularities, where

$$\mathbf{di}(\mathbf{D}) := (\operatorname{tr}_{12}\mathbf{D})$$

is the *dilatation tensor* of $\mathbf{D}$ and

$$\mathbf{vo}(\mathbf{D}) := (\operatorname{tr}_{13}\mathbf{D})$$

is the *Voigt tensor* of $\mathbf{D}$:

1. the harmonic part $\mathbf{H} = \mathbf{h} * \mathbf{h}$ of $\mathbf{D}$ is factorized as an harmonic square;
2. the following proportionality relations hold between the *deviatoric parts* of $\mathbf{di}(\mathbf{D})$ and $\mathbf{vo}(\mathbf{D})$:

$$\mathbf{di}'(\mathbf{D}) \propto \mathbf{vo}'(\mathbf{D}) \propto \boldsymbol{\omega}', \tag{6.8}$$

which is equivalent for the effective compliance tensor $\tilde{\mathbf{S}}$ to satisfy the same conditions:

$$\mathbf{di}'(\tilde{\mathbf{S}}) \propto \mathbf{vo}'(\tilde{\mathbf{S}}) \propto \boldsymbol{\omega}'; \tag{6.9}$$

3. the following proportionality relations hold between the *traces* of $\mathbf{di}(\mathbf{D})$ and $\mathbf{vo}(\mathbf{D})$:

$$\operatorname{tr}\mathbf{di}(\mathbf{D}) \propto \operatorname{tr}\mathbf{vo}(\mathbf{D}) \propto \omega_m. \tag{6.10}$$

Following Cormery and Welemane (2010), who consider the scalar constants p_i as material parameters, *conditions 1 to 3, above, are the conditions for a damage model*—for instance built in a phenomenological manner—*which should be considered as micro-mechanics based.*

7. A second-order anisotropic damage model in micro-mechanics based framework

Following Cordebois and Sidoroff (1982) and Ladevèze (1983), a symmetric second-order, unbounded damage variable $\mathbf{\Phi}$ is introduced in the Gibbs free enthalpy (with initial value $\mathbf{\Phi} = \mathbf{1}$ for a virgin material, and with damage growth $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{\Phi}$ positive definite), and the usual second-order damage tensor writes as:

$$\mathbf{d} = \mathbf{1} - \mathbf{\Phi}^{-2} \quad \text{(with initial value } \mathbf{d} = \mathbf{0}\text{)}. \tag{7.1}$$

A general but phenomenological coupling of elasticity with second-order anisotropic damage is described in (Desmorat, 2006). It reads

$$\rho\psi^\star = \frac{1}{4G}\operatorname{tr}(\mathbf{\Phi}\,\boldsymbol{\sigma}'\,\mathbf{\Phi}\,\boldsymbol{\sigma}') + \frac{g(\mathbf{\Phi})}{18K}(\operatorname{tr}\boldsymbol{\sigma})^2, \tag{7.2}$$

where $\boldsymbol{\sigma}' = \boldsymbol{\sigma} - \frac{1}{3}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{1}$ is the stress deviator. The function g was chosen as

$$g(\mathbf{\Phi}) := \frac{1}{1-\operatorname{tr}\mathbf{d}} = \frac{1}{\operatorname{tr}\mathbf{\Phi}^{-2} - 2}$$

in (Lemaitre et al., 2000; Lemaitre and Desmorat, 2005), and as

$$g(\mathbf{\Phi}) := \frac{1}{3}\operatorname{tr}\mathbf{\Phi}^2$$

in (Desmorat, 2016). In both models, it ensures the convexity with respect to $\boldsymbol{\sigma}$ and the positivity of the intrinsic dissipation, see (Chambart et al., 2014; Desmorat, 2016).

The elasticity law writes

$$\boldsymbol{\sigma} = \rho\frac{\partial\psi^\star}{\partial\boldsymbol{\sigma}} = \frac{1}{2G}(\mathbf{\Phi}\,\boldsymbol{\sigma}'\,\mathbf{\Phi})' + \frac{g(\mathbf{\Phi})}{9K}(\operatorname{tr}\boldsymbol{\sigma})\mathbf{1}, \tag{7.3}$$

with effective compliance

$$\tilde{\mathbf{S}} = \frac{1}{2G}\mathbf{G} + \frac{g(\mathbf{\Phi})}{9K}\mathbf{1}\otimes\mathbf{1}, \tag{7.4}$$

and where

$$\mathbf{G} = \mathbf{\Phi}\,\underline{\overline{\otimes}}\,\mathbf{\Phi} - \frac{1}{3}(\mathbf{1}\otimes\mathbf{\Phi}^2 + \mathbf{\Phi}^2\otimes\mathbf{1}) + \frac{1}{9}(\operatorname{tr}\mathbf{\Phi}^2)\,\mathbf{1}\otimes\mathbf{1}. \tag{7.5}$$

The components of the fourth-order tensor $\mathbf{\Phi}\,\underline{\overline{\otimes}}\,\mathbf{\Phi}$ writes as

$$\frac{1}{2}(\Phi_{ik}\Phi_{jl} + \Phi_{il}\Phi_{jk})$$

and its totally symmetric part is $(\mathbf{\Phi}\,\underline{\overline{\otimes}}\,\mathbf{\Phi})^s = \mathbf{\Phi}\otimes_{(4)}\mathbf{\Phi} = \mathbf{\Phi}\odot\mathbf{\Phi}$ (see (2.9)). The harmonic projection of $\mathbf{G}$ writes then as

$$(\mathbf{G})_0 = (\mathbf{\Phi}\odot\mathbf{\Phi})_0 = \mathbf{\Phi}'*\mathbf{\Phi}', \tag{7.6}$$

where $*$ is the harmonic product. The harmonic part of the compliance tensor is thus the harmonic square

$$\mathbf{H} = \frac{1}{2G}(\mathbf{G})_0 = \frac{1}{2G}\mathbf{\Phi}'*\mathbf{\Phi}', \tag{7.7}$$

which satisfies the first condition on the effective compliance $\tilde{\mathbf{S}}$ to be of micro-mechanics based form (6.5), with

$$p_4 = \frac{1}{2G}, \qquad \mathbf{h} = \mathbf{\Phi}'. \tag{7.8}$$

Moreover, we have

$$\operatorname{tr}_{12}\mathbf{G} = \mathbf{1} : \mathbf{G} = \mathbf{0}, \tag{7.9}$$

$$\operatorname{tr}_{13}\mathbf{G} = \frac{1}{9}(\operatorname{tr}\mathbf{\Phi}^2)\,\mathbf{1} + \frac{1}{2}(\operatorname{tr}\mathbf{\Phi})\,\mathbf{\Phi} - \frac{1}{6}\mathbf{\Phi}^2, \tag{7.10}$$

which, for the anisotropic damage model (7.2), leads to

$$(\operatorname{tr}_{12}\tilde{\mathbf{S}})' = 0, \quad (\operatorname{tr}_{13}\tilde{\mathbf{S}})' = \frac{3(\operatorname{tr}\mathbf{\Phi})\,\mathbf{\Phi}' - (\mathbf{\Phi}^2)'}{12G}. \tag{7.11}$$

Both deviatoric parts are obviously proportional. The phenomenological anisotropic damage model satisfies therefore conditions 1 and 2 of Section 6 for the damage tensor $\mathbf{D}$ to be of the form (6.5), with

$$\begin{aligned}
p_0\omega_m &= \frac{1-2\nu}{3}(g(\mathbf{\Phi}) - 1),\\
p_1\omega_m &= (1+\nu)\left(\frac{1}{10}(\operatorname{tr}\mathbf{\Phi})^2 + \frac{1}{30}\operatorname{tr}\mathbf{\Phi}^2 - 1\right),\\
p_2\boldsymbol{\omega}' &= \frac{2(1+\nu)}{21}\left((\mathbf{\Phi}^2)' - 3(\operatorname{tr}\mathbf{\Phi})\,\mathbf{\Phi}'\right),\\
p_3\boldsymbol{\omega}' &= -\frac{3}{2}p_2\boldsymbol{\omega}'.
\end{aligned} \tag{7.12}$$

Finally, the proportionality requirement $p_0\omega_m \propto p_1\omega_m$ can be satisfied (recall that material constants p_0, p_1 are independent from $\mathbf{\Phi}$, they are considered as material parameters): following Burr et al. (1995) and Lemaitre et al. (2000) we define the hydrostatic sensitivity parameter η as material constant

$$\eta = \frac{3p_0(1+\nu)}{p_1(1-2\nu)}, \tag{7.13}$$

and set

$$g(\mathbf{\Phi}) = (1-\eta) + \eta\left(\frac{1}{10}(\operatorname{tr}\mathbf{\Phi})^2 + \frac{1}{30}\operatorname{tr}\mathbf{\Phi}^2\right). \tag{7.14}$$

Condition 3 for the model (7.2)–(7.3) to be considered as micro-mechanics based is then fullfiled.

For $\eta \geq 0$, the corresponding Gibbs free enthalpy density (7.2) is furthermore convex with respect to both the stress tensor $\boldsymbol{\sigma}$ and the damage tensor $\mathbf{\Phi}$.

8. Conclusion

Some mathematical tools such as the *harmonic product* and the *harmonic factorization into lower order tensors* have been presented. Together with the notion of *mechanically accessible directions for measurements*, this has allowed us to derive at harmonic order 4, both a crack density expansion $\Omega(\boldsymbol{n})$ and a damage framework making use of second-order tensorial variables only, instead of fourth-order in standard micro-mechanics based approaches.

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