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Dispersion for the wave and the Schrödinger equations outside strictly convex obstacles and counterexamples

Estimations de dispersion pour l'équation des ondes et de Schrödinger à l'extérieur des obstacles strictement convexes

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Abstract

The purpose of this note is to prove dispersive estimates for the wave and the Schrödinger equations outside strictly convex obstacles in $\mathbb{R}^d$. If $d = 3$, we show that for both equations, the linear flow satisfies the (corresponding) dispersive estimates as in $\mathbb{R}^3$. In higher dimensions $d \geq 4$ and if the domain is the exterior of a ball in $\mathbb{R}^d$, we show that losses in dispersion do appear and this happens at the Poisson spot.

Résumé

L'objet de cette note est de démontrer des estimations de dispersion pour l'équation des ondes et de Schrödinger à l'extérieur d'un obstacle strictement convexe de $\mathbb{R}^d$. Si $d = 3$, on démontre que, pour chacune des deux équations, le flot linéaire vérifie les estimations de dispersion comme dans $\mathbb{R}^3$. En dimension plus grande $d \geq 4$, on démontre que des pertes dans la dispersion apparaissent à l'extérieur d'une boule de $\mathbb{R}^d$ et cela arrive au point de Poisson.

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Version française abrégée

Pour l'équation des ondes, dans le cas Euclidien, la forme explicite du flot permet d'obtenir les estimations de dispersion

$$\|\chi(hD_t)e^{\pm it\sqrt{-\Delta_{\mathbb{R}^d}}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq C(d)h^{-d}\min\{1,(h/|t|)^{\frac{d-1}{2}}\}, \quad \chi\in C_0^\infty(]0,\infty[).$$

Pour l'équation de Schrödinger, les estimations de dispersion s'énnoncent comme suit :

$$\|e^{\pm it\Delta_{\mathbb{R}^d}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq C(d)|t|^{-d/2}.$$

Notre but est d'obtenir des estimations de dispersion à l'extérieur d'un obstacle strictement convexe. Plusieurs résultats positifs sur les effets dispersifs ont été obtenus récemment dans ce contexte : cependant, la question de savoir si les estimations de dispersion étaient vraies ou non est restée ouverte, même à l'extérieur d'une boule. Puisqu'il n'y a pas de concentration apparente d'énergie, comme ledans le cas d'un domaine non-captant quelconque (pour lequel les portions concaves du bord peuvent agir comme des miroirs et re-focaliser les paquets d'ondes) on pourrait raisonnablement penser que les estimations de dispersion devraient être vérifiées à l'extérieur d'un convexe (voir l'extérieur d'une ball [2] dans le cas des fonctions à symétrie spherique). On montre ici que c'est effectivement le cas en dimension 3, par contre en dimension plus grande on construit des contre-exemples explicites à l'extérieur d'une boule.

Theorem 1 *Soit $\Theta\subset\mathbb{R}^3$ un domaine compact avec bord régulier, strictement convexe et soit $\Omega=\mathbb{R}^3\setminus\Theta$. Soit Δ le Laplacien dans Ω avec condition de Dirichlet au bord. Alors*

(i) les estimations de dispersion pour le propagateur des ondes dans Ω sont vérifiées comme dans $\mathbb{R}^3$:

$$\|\chi(hD_t)e^{\pm it\sqrt{-\Delta}}\|_{L^1(\Omega)\to L^\infty(\Omega)} \leq Ch^{-3}\min\{1,\frac{h}{|t|}\}.$$

(ii) les estimations de dispersion pour le flot de Schrödinger dans Ω sont vérifiées comme dans $\mathbb{R}^3$.

On remarque qu'une perte dans la dispersion pourrait être liée (de façon informelle) à la présence d'un point de concentration : ces points apparaissent lorsque des rayons optiques (envoyés d'une même source dans des directions différentes) cessent de diverger. Le principe de Huygens énonce que lorsque la lumière éclaire un obstacle circulaire, chaque point de l'obstacle se comporte à son tour comme une nouvelle source lumineuse ponctuelle ; tous les rayons lumineux issus des points de la circonférence de l'obstacle se concentrent au centre de l'ombre et décrivent le même chemin optique ; il en résulte une tache lumineuse au centre de l'ombre (le point de Poisson). Par conséquent, l'intuition nous dit que s'il y a une perte dans la dispersion, elle devrait apparaître au point de Poisson.

Theorem 2 *Pour $d\geq 4$ on pose $\Theta=B_d(0,1)$ la boule unité de $\mathbb{R}^d$. Soit $\Omega_d=\mathbb{R}^d\setminus B_d(0,1)$ et soit Δ_d le Laplacien dans Ω_d avec condition de Dirichlet. Au point de Poisson, les estimations de dispersion précédentes (où $\mathbb{R}^d$ et $\Delta_{\mathbb{R}^d}$ sont remplacés par Ω_d et Δ_d) ne sont plus vérifiées. Precisément, soient $Q_\pm(r)$ les points source et d'observation situés à distance r du centre O de la boule $B_d(0,1)$, symétriques par rapport à O ; alors, si $r=\gamma h^{-1/3}$, avec γ à valeurs dans un compact de $(0,\infty)$,*

— *pour le propagateur des ondes et pour $t\simeq 2\gamma h^{-1/3}$*

$$\left|(\chi(hD_t)e^{it\sqrt{|\Delta|}}(\delta_{Q_+(\gamma h^{-1/3})})\right|(Q_-(\gamma h^{-1/3}))\simeq h^{-d}\Big(\frac{h}{t}\Big)^{\frac{d-1}{2}}h^{-\frac{d-3}{3}},$$

— *pour le propagateur de Schrödinger classique et pour $t\simeq h^{1/3}$*

$$\left|(\chi(hD_t)e^{it\Delta}(\delta_{Q_+(\gamma h^{-1/6})})\right|(Q_-(\gamma h^{-1/6}))\simeq h^{-1-\frac{d}{6}-\frac{d-3}{6}}.$$

Pour $d\geq 4$, ces estimations contredissent les estimations du cas plat $\mathbb{R}^d$.

1. Introduction

In the Euclidean case, the explicit form of the wave propagator yields the following dispersive estimate

$$\|\chi(hD_t)e^{\pm it\sqrt{-\Delta_{\mathbb{R}^d}}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq C(d)h^{-d}\min\{1,(h/|t|)^{\frac{d-1}{2}}\}, \quad \chi\in C_0^\infty(]0,\infty[). \tag{1}$$

Concerning the Schrödinger equation, the dispersive estimates read as follows:

$$\|e^{\pm it\Delta_{\mathbb{R}^d}}\|_{L^1(\mathbb{R}^d)\to L^\infty(\mathbb{R}^d)} \leq C(d)|t|^{-d/2}. \tag{2}$$

Our aim in the present paper is to obtain dispersive estimates outside strictly convex obstacles. While many positive results on dispersive effects had been established lately in this context, the question about whether or not dispersion did hold remained open, even for the exterior of a ball. Since there is no apparent concentration of energy, like in the case of a generic non-trapping obstacle (where concave portions of the boundary can act as mirrors and refocus wave packets), one would expect dispersive estimates to hold outside strictly convex obstacles (see the exterior of a ball [2] for spherically symmetric functions). We prove that this is indeed the case in dimension three, while in higher dimensions we provide explicit counterexamples for the exterior of a ball.

Theorem 3 *Let $\Theta\subset\mathbb{R}^3$ be a compact domain with smooth, strictly convexe boundary and let $\Omega=\mathbb{R}^3\setminus\Theta$. Let Δ denote the Dirichlet Laplace operator in Ω. Then*

(i) the dispersive estimates for the wave flow in Ω do hold like in $\mathbb{R}^3$:

$$\|\chi(hD_t)e^{\pm it\sqrt{-\Delta}}\|_{L^1(\Omega)\to L^\infty(\Omega)} \leq Ch^{-3}\min\{1,\frac{h}{|t|}\}. \tag{3}$$

(ii) the dispersive estimates for the classical Schrödinger flow in Ω hold like in $\mathbb{R}^3$.

We remark that a loss in dispersion may be informally related to a cluster point : such clusters occur because optical rays (sent from the same source along different directions) are no longer diverging from each other. When light shines on a circular obstacle, Huygens's principle says that every point of the obstacle acts as a new point source of light, so all the light passing close to a perfectly circular object concentrate at the perfect center of the shadow behind it; this results in a bright spot at the shadow's center (the Poisson spot). Therefore, our intuition tells us that if there is a location where dispersion could fail, this will happen at the Poisson spot.

Theorem 4 *Let $d\geq 4$ and let $\Theta=B_d(0,1)$ be the unit ball in $\mathbb{R}^d$. Set $\Omega_d=\mathbb{R}^d\setminus B_d(0,1)$ and let Δ_d denote the Laplace operator in Ω_d. Then at the Poisson spot the dispersive estimates* (1),(2) *(with $\mathbb{R}^d$ and $\Delta_{\mathbb{R}^d}$ replaced by Ω_d and Δ_d) fail. Precisely, let $Q_\pm(r)$ be the source and the observation points at (same) distance r from the ball $B_d(0,1)$, symmetric with respect to the center of the ball, then, taking $r=\gamma h^{-1/3}$, with γ in a compact subset of $(0,\infty)$ yields*

— for the wave flow and for $t\simeq 2\gamma h^{-1/3}$

$$\Big|(\chi(hD_t)e^{it\sqrt{|\Delta|}}(\delta_{Q_+(\gamma h^{-1/3})})\Big|(Q_-(\gamma h^{-1/3}))\simeq h^{-d}\Big(\frac{h}{t}\Big)^{\frac{d-1}{2}}h^{-\frac{d-3}{3}}, \tag{4}$$

— for the classical Schrödinger flow and for $t\simeq h^{1/3}$

$$\Big|(\chi(hD_t)e^{it\Delta}(\delta_{Q_+(\gamma h^{-1/6})})\Big|(Q_-(\gamma h^{-1/6}))\simeq h^{-1-\frac{d}{6}-\frac{d-3}{6}}.$$

For $d\geq 4$, these estimates contradict the usual ones (1), (2) *in $\mathbb{R}^d$.*

2. General setting for the wave flow outside a ball in $\mathbb{R}^d$

In this note we give a sketch of the proof of Theorem 3 only in the case of the wave equation outside a ball. The general case will be dealt with in [1]. Let $\Omega = \mathbb{R}^d \setminus B_d(0,1)$, $\partial\Omega = \mathbb{S}^{d-1}$. Let $Q_0 \in \Omega$ and δ_{Q_0} the Dirac distribution at Q_0. Let also $\Delta_{\mathbb{R}^d}$ denote the Laplace operator in the whole space $\mathbb{R}^d$ and $U(t,Q,Q_0) = \cos(t\sqrt{|\Delta_{\mathbb{R}^d}|})(\delta_{Q_0})(Q)$ be the solution to

$$(\partial_t^2 - \Delta_{\mathbb{R}^d})U = 0 \quad \text{in } \Omega; U|_{t=0} = \delta_{Q_0},\ \partial_t U|_{t=0} = 0; U|_{\partial\Omega} = 0.$$

Then $\partial_n U|_{\partial\Omega} = \partial_n U_{free}|_{\partial\Omega} - \mathbf{N}(U_{free}|_{\partial\Omega})$, where U_{free} is the free wave in $\mathbb{R}^d$, $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$ pointing towards Ω and $\mathbf{N}$ is the Neumann operator. Define

$$\underline{U}(t,Q,Q_0) := U(t,Q,Q_0), \text{ if } Q \in \Omega; \quad \underline{U}(t,Q,Q_0) := 0, \text{ if } Q \in \overline{B_d(0,1)}.$$

Then $\underline{U}$ satisfies the following equation:

$$(\partial_t^2 - \Delta_{\mathbb{R}^d})\underline{U} = \partial_n U|_{\partial\Omega} \otimes \delta_{\partial\Omega} \quad \text{in } \mathbb{R}^d; \underline{U}|_{t=0} = \delta_{Q_0},\ \partial_t \underline{U}|_{t=0} = 0,$$

which yields $\underline{U}|_{t>0} = \Box_+^{-1}\Big(\partial_n U|_{\partial\Omega} \otimes \delta_{\partial\Omega}|_{t>0}\Big) + U_{free}|_{t>0}$, where

$$\Box_+^{-1}F(t) = \int_{-\infty}^{t} R(t-t') * F(t')dt', \quad \widehat{R(t,\xi)} = \frac{\sin(t|\xi|)}{|\xi|}.$$

3. Sketch of proof of Theorem 3

In dimension $d = 3$, from the last formula and the form of $\partial_n U|_{\partial\Omega}$ in terms of U_{free} and $\mathbf{N}$ we find

$$\Box_+^{-1}\Big(\Big(\partial_n U_{free}|_{\partial\Omega} - N(U_{free}|_{\partial\Omega})\Big) \otimes \delta_{\partial\Omega}|_{t>0}\Big) = \frac{1}{4\pi}\int_{\partial\Omega} \frac{(\partial_n U_{free}|_{\partial\Omega} - \mathbf{N}(U_{free}|_{\partial\Omega}))(t - |Q-P|, P)}{4\pi|P-Q|} d\sigma(P). \tag{5}$$

In order to prove Theorem 3 (in dimension 3) we are reduced to obtaining bounds for (5). For that, we use the Melrose and Taylor parametrix which provides the form of the solution near the glancing regime in terms of Airy function. Outside a neighborhood of the glancing region it is easy to see that the dispersive estimates hold true. The next theorem is due to Melrose and Taylor and holds for $\partial\Omega$ strictly concave :

Theorem 5 *$\exists \theta, \zeta$ phase functions near the glancing region, $\exists\ a, b$ symbols (with a elliptic, $b|_{\partial\Omega} = 0$) such that, if V is a solution in Ω to*

$$(\tau^2 + \Delta)V \in O_{C^\infty}(\tau^{-\infty}),$$

then there exists F such that

$$V(\tau, Q) = \frac{\tau}{2\pi}\int e^{i\tau\theta(Q,\eta)}\Big(aA + \tau^{-1/3}bA'\Big)(\tau^{2/3}\zeta(Q,\eta))\hat{F}(\tau\eta)d\eta,$$

where we set $A(z) = Ai(-z)$, where Ai is the Airy function which satisfies $Ai''(z) = zAi(z)$.

For $d = 3$, we take $V = \widehat{U_{free}}(\tau, P, Q_0) = \frac{\tau}{|P-Q_0|}e^{-i\tau|P-Q_0|}$. We introduce polar coordinates: since $\partial\Omega = \mathbb{S}^2$, a point in Ω can be written as (r, φ, ω), $r > 1$, $\varphi \in [0, \pi]$, $\omega \in \mathbb{S}^1$. We can always assume that the source point Q_0 has coordinates $(r_0, 0, .)$, $r_0 > 1$. We define the apparent contour $\mathcal{C}_Q$ of a point $Q \in \Omega$ as the boundary of the set of points that can be "viewed" from Q. Therefore, $\mathcal{C}_{Q_0} = \{(1, \varphi, \omega)| \quad \cos(\varphi) = 1/r_0\}$.

Remark 1 When $\Omega = \mathbb{R}^d \setminus B_d(0,1)$, $d \geq 2$, the functions θ and ζ can be taken under the following form

$$\theta(\varphi, \eta) = \varphi\eta, \quad \zeta(r, \eta) = \eta^{2/3}l(\frac{r}{\eta}),$$

where $\eta \simeq 1$ and $l(z) = (z-1)(1 + O(z-1))$ for z close to 1. Then $\zeta_0(\eta) := \zeta(1, \eta) = \eta^{2/3}l(1/\eta)$.

Let $P=(1,\varphi,\omega)\in\mathbb{S}^2$ and $a_0=a|_{\partial\Omega}$, then the trace of the free wave on the boundary $\mathbb{S}^2$ reads as

$$\widehat{U_{free}}(\tau,P,Q_0)=\frac{\tau}{|P-Q_0|}e^{-i\tau|P-Q_0|}=\frac{\tau}{2\pi}\int e^{i\tau\varphi\eta}a_0A(\tau^{2/3}\zeta_0(\eta))\hat{F}(\tau\eta)d\eta. \tag{6}$$

Let $A_+(z)=A(e^{i\pi/3}z)$: then $A_+(z)$ doesn't vanish for $z\in\mathbb{R}$. We compute

$$\partial_n\widehat{U_{free}}(\tau,P,Q_0)=\frac{\tau}{2\pi}\int e^{i\tau\varphi\eta}\Big(\tilde{a}A+\tau^{2/3}\tilde{b}A'\Big)(\tau^{2/3}\zeta_0(\eta))\hat{F}(\tau\eta)d\eta,$$

$$\mathbf{N}(\widehat{U_{free}}|_{\mathbb{S}^2})(\tau,P,Q_0)=\frac{\tau}{2\pi}\int e^{i\tau\varphi\eta}\Big(\tilde{a}A+\tau^{2/3}\tilde{b}\frac{A'_+}{A_+}A\Big)(\tau^{2/3}\zeta_0(\eta))\hat{F}(\tau\eta)d\eta.$$

We obtain an explicit form for $\Box_+^{-1}\Big(\partial_nU|_{\partial\Omega}\otimes\delta_{\partial\Omega}|_{t>0}\Big)$ as follows

$$\Box_+^{-1}\Big(\Big(\partial_nU_{free}|_{\mathbb{S}^2}-\mathbf{N}(U_{free}|_{\mathbb{S}^2})\Big)\otimes\delta_{\mathbb{S}^2}\Big)=\int e^{i\tau t}\chi(h\tau)(I_F+I_D+I_R)(\tau,Q,Q_0)d\tau,$$

where

- I_F is the direct wave: the phase is the phase of the free wave and the amplitude is just the amplitude of the free wave cutoff near the shadow ($\Rightarrow$ OK for dispersion);
- I_D is the diffracted wave: it corresponds to a neighborhood of $\mathcal{C}_{Q_0}$ on the boundary of size $\tau^{-1/3}$ in φ and of size $\tau^{-2/3}$ in angle around the glancing direction (around $\eta=1$).
- I_R is the reflected wave: the phase has a singular, Airy type term (easy to deal with).

Since difficulties appear near rays issued from Q_0 which hit the boundary without being deviated, only the diffracted wave part (containing I_D) will be dealt with here. Notice that this is the regime which provides counter-examples in higher dimensions. We have

$$I_D(\tau,Q,Q_0)=\int_{P=(1,\varphi,\omega)\in\mathbb{S}^2}\frac{1}{|P-Q|}e^{-i\tau|P-Q|}\frac{\tau}{2\pi}\int e^{i\tau\varphi\eta}\tau^{2/3}\tilde{b}\frac{\chi_0(\tau^{2/3}\zeta_0(\eta))}{(AA_+)(\tau^{2/3}\zeta_0(\eta))}\widehat{F}(\tau\eta)d\eta d\varphi d\omega,$$

where χ_0 is supported near 0, $\eta\simeq 1$ and $\tilde{b}$ is an elliptic symbol.

The phase function of I_D equals $(-|P-Q|+\varphi\eta+$ phase of $\hat{F})$ and reads as $-|P-Q|+\varphi\eta-\varphi_0\eta-|P_0-Q_0|$, where $P_0\in\mathcal{C}_{Q_0}$ has coordinates $P_0=(1,\varphi_0,.)$, $\cos(\varphi_0)=1/r_0$. The symbol of I_D is of the form $\frac{\tau}{|P-Q|}\tau^{2/3}\times\chi_0\times(\tau^{-1/3}\frac{\tau}{|P_0-Q_0|})$, where the factor in brackets comes from $\hat{F}$, obtained from (6) as an oscillatory integral with critical points of order precisely 2 on $\mathcal{C}_{Q_0}$. It will be enough to prove that $|I_D(\tau,Q,Q_0)|\lesssim\frac{\tau}{t}$.

Let $Q=(r_Q,\varphi_Q,\omega_Q)$ be an observation point in Ω; in I_D, the only dependence in ω comes from $|P-Q|$ since

$$|P-Q|=\Big(1+r_Q^2-2r_Q\sin(\varphi)\sin(\varphi_Q)\cos(\omega-\omega_Q)-2r_Q\cos(\varphi)\cos(\varphi_Q)\Big)^{1/2}.$$

The critical points with respect to ω satisfy $\partial_\omega|P-Q|=\frac{r_Q}{|P-Q|}\sin(\varphi)\sin(\varphi_Q)\sin(\omega-\omega_Q)$.

- If $\sin(\varphi_Q)\neq 0\Rightarrow$ the critical points satisfy $\sin(\omega-\omega_Q)=0$ which yields $\omega=\omega_Q$ or $\omega=\omega_Q+\pi$; this means that the stationary points P on the boundary belong to a circle situated in the plane Q_0-O-Q; all the computations are explicit and provide the announced result;
- If $\sin(\varphi_Q)=0$ the derivative vanishes everywhere. In this case the points Q, O and Q_0 are colinear and the integration in ω does not provide negative factors of τ. It is easy to see that the integration with respect to φ provides a power of $\tau^{-1/3}$ (corresponding to the critical points φ such that $(0,\varphi,\omega)\in\mathcal{C}_Q$ which are degenerate of order 2) and the integration with respect to η provides a factor $\tau^{-2/3}$, due to the localisation χ_0. It remains to show that the remaining factor $\frac{\tau^{1/3}}{|P-Q||P_0-Q_0|}$ must be bounded by $\frac{C}{t}$, for some $C>0$; indeed, $\mathcal{C}_{Q_0}$ is defined by $\cos(\varphi_0)=\frac{1}{r_0}$, while $\mathcal{C}_Q$ is defined

by $\cos(\varphi) = \frac{1}{r_Q}$, and $|\varphi - \varphi_0| \lesssim \tau^{-1/3}$ (since otherwise, by integrations by parts with respect to η, we get a $O(\tau^{-\infty})$ contribution). Notice that we have also used that $t \simeq |Q_0 - P_0| + |P - Q|$.

This allows to achieve the proof of Theorem 3.

4. Sketch of proof of Theorem 4 for the wave flow outside a ball in $\mathbb{R}^d$, $d \geq 4$

Let S, N denote the south pole and the north pole, respectively. Let $Q_+(r)$ be a point on OS axis, at distance r from O and let $Q_-(r)$ denote its symmetric with respect to O on the ON axis, where $O \in \mathbb{R}^d$ is the centre of the ball. We let $r_0 = \gamma h^{-1/3}$ for some γ in a compact set of $(0, \infty)$ and let $Q_0 = Q_+(r_0)$ and $Q = Q_-(r_0)$. The counterexample to dispersion comes from the diffracted part, which, in this case, takes the form

$$\int e^{it\tau} \chi(h\tau) I_D(\tau, Q_-, Q_+) d\tau,$$

where for $d \geq 3$

$$I_D(\tau, Q, Q_0) = \int_{P=(1,\varphi,.)\in\mathbb{S}^{d-1}} \frac{\tau^{\frac{d-3}{2}}}{|P-Q|^{\frac{d-1}{2}}} e^{-i\tau|P-Q|} \Sigma_d(|P-Q|\tau) \times \frac{\tau}{2\pi} \int_{\mathbb{R}} e^{i\tau\varphi\eta} \tau^{2/3} \tilde{b} \frac{\chi_0(\tau^{2/3}\zeta_0(\eta))}{(AA_+)(\tau^{2/3}\zeta_0(\eta))} \widehat{F}(\tau\eta) d\eta d\varphi,$$

where Σ_d is a symbol of degree 0 which satisfies

$$\widehat{U}_{free}(\tau, P, Q_0) = \frac{\tau^{\frac{d-1}{2}}}{|P-Q_0|^{\frac{d-1}{2}}} e^{-i\tau|P-Q_0|} \Sigma_d(|P-Q_0|\tau), \quad \text{for } |P-Q_0| \gg \tau^{-1}.$$

With $P_0 = (1, \varphi_0, .)$, $\cos(\varphi_0) = 1/r_0$, on the apparent contour $\mathcal{C}_{Q_0}$, we have

$$\widehat{F}(\tau\eta) = \tau^{-1/3} \frac{\tau^{\frac{d-1}{2}}}{|P_0-Q_0|^{\frac{d-1}{2}}} e^{i\tau\eta\varphi_0} e^{-i\tau|P_0-Q_0|} \Sigma(|P_0-Q_0|, \tau),$$

where Σ is obtained from Σ_d after applying the stationary phase with degenerate critical point P_0 on $\mathcal{C}_{Q_0}$. Notice that the observation point $Q = Q_-(r_0)$ is such that $\sin(\varphi_Q) = 0$, since Q_+, O and Q_- are on the same line and, due to rotational symmetry, in the integral defining I_D the phase function does not depend on ω. We obtain by explicit computations

$$\Big| \int e^{it\tau} \chi(h\tau) I_D(\tau, Q, Q_0) d\tau \Big| \simeq C \frac{1}{h} \frac{h^{-(d-2+\frac{1}{3})}}{|P_0-Q_0|^{d-1}}.$$

Since we must have $t \simeq 2\gamma h^{-1/3}$, it follows that

$$\Big| \int e^{it\tau} \chi(h\tau) I_D(\tau, Q, Q_0) d\tau \Big| \simeq \frac{C}{\gamma^{\frac{d-1}{2}}} h^{-d} \frac{h^{\frac{d-1}{2}}}{|t|^{\frac{d-1}{2}}} h^{-\frac{d-3}{3}}.$$

For $d = 3$ this coincide with the usual estimates (1) of $\mathbb{R}^3$. However, for $d \geq 4$ there is a loss coming from the factor $h^{-\frac{d-3}{3}}$ for γ in a fixed compact of $(0, \infty)$.

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