

# Deforming $\mathfrak{h}$ -trivial the Lie algebra $\text{Vect}(S^1)$ inside the Lie algebra of pseudodifferential operators $\Psi\mathcal{DO}$

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In this paper, we consider the action of  $\text{Vect}(S^1)$  by Lie derivative on the spaces of pseudodifferential operators  $\Psi\mathcal{DO}$ . We study the  $\mathfrak{h}$ -trivial deformations of the standard embedding of the Lie algebra  $\text{Vect}(S^1)$  of smooth vector fields on the circle, into the Lie algebra of functions on the cotangent bundle  $T^*S^1$ . We classify the deformations of this action that become trivial once restricted to  $\mathfrak{h}$ , where  $\mathfrak{h} = \mathfrak{aff}(1)$  or  $\mathfrak{sl}(2)$ . Necessary and sufficient conditions for integrability of infinitesimal deformations are given.

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## 1. Introduction

Deformation theory plays a crucial role in all branches of mathematics and physics. In physics, the mathematical theory of deformations has proved to be a powerful tool in modeling physical reality. We study some related questions of deformations of some  $\mathfrak{g}$ -modules. Of course, if  $M = \text{End}(V)$ , where  $V$  is a  $\mathfrak{g}$ -module then, according to Nijenhuis–Richardson, the space  $H^1(\mathfrak{g}; \text{End}(V))$  classifies the infinitesimal deformations of a  $\mathfrak{g}$ -module  $V$  and the obstructions to integrability of a given infinitesimal deformation of  $V$  are elements of  $H^2(\mathfrak{g}; \text{End}(V))$ . More generally, if  $\mathfrak{h}$  is a subalgebra of  $\mathfrak{g}$ , then the  $\mathfrak{h}$ -relative cohomology  $H^1(\mathfrak{g}, \mathfrak{h}; \text{End}(V))$  measures the infinitesimal  $\mathfrak{h}$ -trivial deformations that become trivial once the action is restricted to  $\mathfrak{h}$  ( *$\mathfrak{h}$ -trivial deformations*), while the obstructions to extension of any  $\mathfrak{h}$ -trivial infinitesimal deformation to a formal one are related to  $H^2(\mathfrak{g}, \mathfrak{h}; \text{End}(V))$ .

Deformations of Lie algebras with base and versal deformations were already considered by Fialowski in 1986 [8]. In 1988, Fialowski [9] further introduced deformations whose base is a complete local algebra. Also, in [9], the notion of miniversal (or formal versal) deformation was introduced in general, and it was proved that under some cohomology restrictions, a versal deformation exists. Later, Fialowski and Fuchs, using this framework, gave a construction for a versal deformation.

Now, let us introduce some fundamental concepts and notations. The space of weighted densities of weight  $\mu$  on  $\mathbb{R}$  (or  $\mu$ -densities for short), denoted by:

$$\mathcal{F}_\mu = \{f dx^\mu, f \in C^\infty(S^1)\}, \quad \mu \in \mathbb{R}$$

is the space of sections of the line bundle  $(T^*S^1)^{\otimes \mu}$ . The Lie algebra  $\text{Vect}(S^1)$  of vector fields  $X_h = h \frac{d}{dx}$ , where  $h \in C^\infty(S^1)$ , acts by the *Lie derivative*. Alternatively, this action can be written as follows:

$$X_g \cdot (f dx^\mu) = L_{X_g}^\mu(f) dx^\mu \quad \text{with } L_{X_g}^\mu(f) = gf' + \mu g' f,$$

where  $f', h'$  are  $\frac{df}{dx}, \frac{dh}{dx}$ .

We consider the Lie algebras  $\mathfrak{sl}(2)$  and  $\mathfrak{aff}(1)$  which can be viewed as subalgebras of the Lie subalgebra of  $\text{Vect}(S^1)$ :

$$\mathfrak{sl}(2) = \text{Span}(X_1, X_x, X_{x^2}) \quad \text{and} \quad \mathfrak{aff}(1) = \text{Span}(X_1, X_x).$$

Of course,  $\mathfrak{aff}(1)$  is a subalgebra of  $\mathfrak{sl}(2)$ . We also get families of infinite dimensional  $\mathfrak{sl}(2)$ -modules and  $\mathfrak{aff}(1)$ -modules still denoted by  $\mathcal{F}_\lambda$ .

The study of multiparameter deformations of the standard embedding  $\pi$  of the Lie algebra  $\text{Vect}(S^1)$  of smooth vector fields on  $S^1$  into the Lie algebra of pseudo-differential symbols  $\Psi\mathcal{DO}$  defined by  $\pi(f(x)\partial_x) = f(x)\xi$  was carried out by Ovsienko and Roger in [15, 16]. They have, essentially, studied polynomial deformations of the following form

$$\pi(c) = \pi + \sum_{k \in \mathbb{Z}} \pi_k(c) \xi^k,$$

where  $\pi_k(c) : \text{Vect}(S^1) \rightarrow C_C^\infty(S^1)$  satisfy  $\pi_k(0) = 0$  and  $\pi_k \equiv 0$  for  $k$  sufficiently large.

In this paper, we are interested to the case of  $\mathfrak{h}$ -trivial deformations, where  $\mathfrak{h} = \mathfrak{sl}(2)$  or  $\mathfrak{aff}(1)$ . More precisely, we investigate the  $\mathfrak{h}$ -relative cohomology result by deducing the integrability condition for the infinitesimal deformation.

## 2. Geometry of Space $S^1$

### 2.1. The space of tensor densities on $S^1$

The Lie algebra,  $\text{Vect}(S^1)$ , of vector fields on  $S^1$  naturally acts, by the Lie derivative, on the space

$$\mathcal{F}_\lambda = \{f dx^\lambda : f \in C^\infty(S^1)\}$$

of weighted densities of degree  $\lambda$ . The Lie derivative  $L_X^\lambda$  of the space  $\mathcal{F}_\lambda$  along the vector field  $X \frac{d}{dx}$  is defined by

$$L_X^\lambda = X \partial_x + \lambda X', \quad (2.1)$$

where  $X, f \in C^\infty(S^1)$  and  $X' := \frac{dX}{dx}$ . More precisely, for all  $fdx^\lambda \in \mathcal{F}_\lambda$  we have

$$L_X^\lambda(fdx^\lambda) = (Xf' + \lambda fX')dx^\lambda.$$

## 2.2. The standard embedding

The Lie algebra  $\text{Vect}(S^1)$  of vector fields on a manifold  $S^1$  has a natural embedding into the Poisson Lie algebra of functions on  $T^*S^1$ . It is defined by the standard action of the Lie algebra of vector fields on the cotangent bundle, the explicit formula is

$$\pi(X_f) = f\xi. \quad (2.2)$$

The main purpose of this paper is to study  $\mathfrak{aff}(1)$ -trivial deformations of the standard embedding (2.2).

## 2.3. The Poisson Lie algebra of formal symbols

### 2.3.1. Deformations inside $C^\infty(T^*S^1)$

Consider the Poisson Lie algebra of smooth functions on  $T^*S^1$ . In this case, the problem of deformation of the embedding (2.2) has an elementary solution. The  $\text{Vect}(S^1)$  embedding (2.2) into  $C^\infty(T^*S^1)$  has the unique (well-known) nontrivial deformation. Indeed, given an arbitrary volume form on  $S^1$ , the expression:

$$\pi_\lambda(X) = X\xi + \lambda \text{div}X,$$

where  $\lambda \in \mathbb{R}$  defines an embedding of  $\text{Vect}(S^1)$  into  $C^\infty(T^*S^1)$ . The linear map:  $X \rightarrow \text{div}X$  is the unique nontrivial 1-cocycle on  $\text{Vect}(S^1)$  with values  $C^\infty(S^1) \subset C^\infty(T^*S^1)$ .

### 2.3.2. Definition: Two Poisson Lie algebras of formal symbols

Let us consider the following Poisson Lie algebra on the cotangent bundle with zero section removed,  $\dot{T}^*S^1 = T^*S^1 \setminus S^1$ :

- (i) The Lie algebra  $P$  of functions on  $\dot{T}^*S^1$ .
- (ii) The Lie algebra  $\mathcal{P}$  of formal Laurent series on  $\dot{T}^*S^1$ .

Lie algebras  $P$  and  $\mathcal{P}$  can be interpreted as classical limits of the algebra of formal symbols of pseudodifferential operators on  $S^1$ . In both cases, the Poisson bracket is defined by the usual formula:

$$\{F, G\} = \frac{\partial F}{\partial \xi} \frac{\partial G}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial G}{\partial \xi}. \quad (2.3)$$

As vector spaces, Lie algebras  $P$  and  $\mathcal{P}$  have the following form:

$$P = C^\infty(S^1) \otimes \mathbb{C}[\xi, \xi^{-1}] \quad \text{and} \quad \mathcal{P} = C^\infty(S^1) \otimes \mathbb{C}[\xi, \xi^{-1}],$$

where  $\mathbb{C}[\xi, \xi^{-1}]$  is the space of Laurent series in one formal indeterminate. Elements of both algebras  $P$  and  $\mathcal{P}$  can be written in the following form:

$$F(x, \xi, \xi^{-1}) = \sum_{k \in \mathbb{Z}} \xi^k f_k(x),$$

where the coefficients  $f_k(x)$  are periodic functions:  $f_k(x + 2\pi) = f_k(x)$ . In the case of algebra  $P$ , one supposes that the coefficients  $f_k \equiv 0$ , if  $|k|$  is sufficiently large; for  $\mathcal{P}$  the condition is:  $f_k \equiv 0$ , if  $k$  is sufficiently large.

**Lemma 2.1** ([16]). *The Lie algebra  $\mathcal{P}$  has the following decomposition as a  $\text{Vect}(S^1)$ -module.*

$$\mathcal{P} = \bigoplus_{m \geq 0} \mathcal{F}_m \oplus \prod_{m < 0} \mathcal{F}_m. \quad (2.4)$$

**Theorem 2.2** (Ovsienko–Roger [16]). *The space  $H^1(\text{Vect}(S^1), \mathcal{P})$  has the following structure:*

$$H^1(\text{Vect}(S^1), \mathcal{P}) = \mathbb{R}^4.$$

The nontrivial cocycles generating the cohomology groups are as follows:

$$\begin{cases} c_0(f\partial) = f \\ C_0(f\partial) = f' \\ C_1(f\partial) = f''\xi^{-1} \\ C_2(f\partial) = f'''\xi^{-2}. \end{cases}$$

#### 2.4. Pseudodifferential operators on $S^1$

To every pseudodifferential operator  $F$  on  $S^1$ , one associates its symbol [6, 16]. Let  $\Psi\mathcal{DO}$  be the space of pseudodifferential symbols on  $S^1$  [15, 16]. The order of  $\Psi\mathcal{DO}$  is defined to be

$$\text{ord}(F) = \{\sup k \in \mathbb{Z} \mid f_k(x) \neq 0\} \quad \text{for any} \quad F(x, \xi, \xi^{-1}) = \sum_{k \in \mathbb{Z}} \xi^k f_k(x) \in \Psi\mathcal{DO},$$

where  $f_k(x) \in C^\infty(\mathbb{R})$  with  $f_k = 0$  for  $k$  sufficiently large. By means of the two natural derivations on  $\Psi\mathcal{DO}$

$$\partial_\xi : \sum_k f_k \xi^k \mapsto \sum_k k f_k \xi^{k-1} \quad \text{and} \quad \partial_x : \sum_k f_k \xi^k \mapsto \sum_k f'_k \xi^k \quad (2.5)$$

one defines a natural Poisson bracket:

$$\{F, G\} = \frac{\partial F}{\partial \xi} \frac{\partial G}{\partial x} - \frac{\partial F}{\partial x} \frac{\partial G}{\partial \xi} \quad \text{for any} \quad F, G \in \Psi\mathcal{DO}. \quad (2.6)$$

On the space of pseudodifferential symbols an associative algebra structure is defined by the rule [15]

$$F \circ G = \sum_{k \geq 0} \frac{1}{k!} : \frac{\partial^k F}{\partial \xi^k} \frac{\partial^k G}{\partial x^k} :, \quad (2.7)$$

where  $: \cdot :$  stands for the “normal ordering” defined as

$$: f(x) \xi^k g(x) \xi^\ell := f(x) g(x) \xi^{k+\ell}. \quad (2.8)$$

This is a natural generalization of the Wick product. As usual, we can also define the Lie bracket associated with the  $\circ$ -product  $[F, G] = F \circ G - G \circ F$ ; we denote the Lie algebra with this structure by  $\Psi\mathcal{DO}_L$  to distinguish from the Poisson Lie algebra structure.

Consider the family of associative laws on  $\Psi\mathcal{DO}$  depending on a parameter  $h \in ]0, 1[$ :

$$A \circ_h B = \sum_{n \geq 0} \frac{h^n}{n!} (\partial_\xi^n A) (\partial_x^n B).$$

Denote by  $\Psi\mathcal{DO}_h$  the associative algebra of pseudodifferential operators on  $S^1$  equipped with the multiplication  $\circ_h$ . It is clear that all the associative algebras  $\Psi\mathcal{DO}_h$  are isomorphic to each other.

For the commutator  $[A, B]_h := A \circ_h B - B \circ_h A$ , one has

$$[A, B]_h = \{A, B\} + O(h)$$

and therefore  $\lim_{h \rightarrow 0} [A, B]_h = \{A, B\}$ , where we identify  $\mathcal{P}$  with  $\Psi\mathcal{DO}$  as vector spaces. Hence, the Lie algebra  $\Psi\mathcal{DO}$  contracts to the Poisson Lie algebra  $\mathcal{P}$ .

**Theorem 2.3 (Ovsienko–Roger [15]).** *The space  $H^1(\text{Vect}(S^1), \Psi\mathcal{DO})$  has the following structure:*

$$H^1(\text{Vect}(S^1), \Psi\mathcal{DO}) = \mathbb{R}^4.$$

*The nontrivial cocycles generating the cohomology groups are as follows:*

$$\left\{ \begin{array}{l} \Theta_0(f\partial) = f \\ \Theta_1(f\partial) = f' \\ \Theta_2(f\partial) = \sum_{n=2}^{\infty} (-1)^{n-1} \frac{2(n-3)}{n} f^{(n)} \xi^{-n+1} \\ \Theta_3(f\partial) = \sum_{n=2}^{\infty} (-1)^n \frac{3(n-1)}{n+1} f^{(n+1)} \xi^{-n}. \end{array} \right.$$

In [15, 16], Ovsienko and Roger classify nontrivial deformations of the standard embedding of the Lie algebra  $\text{Vect}(S^1)$  the Lie algebra of functions on the cotangent bundle  $T^*S^1$  with respect to the Poisson bracket and into the Lie algebra  $\Psi\mathcal{DO}$  of pseudodifferential symbols on  $S^1$ . Now, let  $\mathfrak{h}$  be Lie subalgebra of  $\text{Vect}(S^1)$ , we classify all the  $\mathfrak{h}$ -trivial deformations of the standard embedding.

### 3. Statement of the Problem

#### 3.1. $\mathfrak{h}$ -trivial deformation of the $\text{Vect}(S^1)$ inside $P$

We classify all the  $\mathfrak{h}$ -trivial deformations of the standard embedding (2.2) of  $\text{Vect}(S^1)$  into  $P$ . That means we study linear  $\mathfrak{h}$ -trivial maps  $\pi^c : \text{Vect}(S^1) \rightarrow P[[c]]$  to the Lie algebra of series in a formal parameter in  $c$ . Such a map has the following form:

$$\pi^c = \pi + c\pi_1 + c_2\pi_2 + \cdots,$$

where  $\pi_k : \text{Vect}(S^1) \rightarrow P$  are linear  $\mathfrak{h}$ -trivial maps, such that formal homomorphism condition is satisfied.

#### 3.2. $\mathfrak{h}$ -trivial deformation of the $\text{Vect}(S^1)$ inside $\Psi\mathcal{DO}$

The main purpose of this paper is to study  $\mathfrak{h}$ -trivial deformations of the canonical embedding

$$\begin{aligned} \rho_0 : \text{Vect}(S^1) &\rightarrow \Psi\mathcal{DO} \\ \rho_0(X_F) &= F\xi. \end{aligned} \tag{3.1}$$

#### 3.3. $\mathfrak{h}$ -trivial deformation of the $\text{Vect}(S^1)$ inside $\mathcal{P}$

We classify all the  $\mathfrak{h}$ -trivial deformations of the standard embedding (2.2) of  $\text{Vect}(S^1)$  into  $\mathcal{P}$ . In other words, we consider homomorphisms of the following form:

$$\pi = \pi + \sum_{k \in \mathbb{Z}} \pi_k(c) \xi^k,$$

where  $c = c_1, \dots, c_n \in \mathbb{R}$  (or  $\mathbb{C}$ ) are parameters of deformation, each linear map  $\pi_k(c) : \text{Vect}(S^1) \rightarrow C^\infty(S^1)$  being polynomial in  $c$ ,  $\pi_k(0) = 0$  and  $\pi_k \equiv 0$  if  $k > 0$  is sufficiently large.

##### 3.3.1. Deformation theory

Deformation theory of Lie algebra homomorphisms was first considered with one parameter [10, 14]. Recently, multiparameter deformations of Lie algebras and their modules were intensively studied [1, 2, 4, 5, 8, 9, 15, 16]. Recall the basics of this theory.

Let  $\rho_0 : \text{Vect}(S^1) \rightarrow \Psi\mathcal{DO}$  be an embedding of Lie algebras. A *formal  $\mathfrak{h}$ -trivial deformation* of  $\rho_0$  is a formal series

$$\rho(t) = \rho_0 + \sum_{i=1}^m t_i \rho_i + \sum_{i,j=1}^m t_i t_j \rho_{ij}^{(2)} + \cdots, \tag{3.2}$$

where the  $\rho_i, \rho_{ij}^{(2)}, \dots$ , are linear maps from  $\text{Vect}(S^1)$  to  $\Psi\mathcal{DO}$  with  $t = (t_1, \dots, t_m) \in \mathbb{C}[[t_1, \dots, t_m]]$ , and so on, such that the map

$$\rho(t) : \text{Vect}(S^1) \rightarrow \mathbb{C}[[t_1, \dots, t_m]] \otimes \Psi\mathcal{DO} \quad (3.3)$$

satisfies the homomorphism condition in any order in  $t = (t_1, \dots, t_m)$ .

Two formal  $\mathfrak{h}$ -trivial deformations  $\rho$  and  $\tilde{\rho}$  are said to be *equivalent* if  $\tilde{\rho} = I_t \circ \rho$  for an inner automorphism

$$I_t : \mathbb{C}[[t]] \otimes \Psi\mathcal{DO} \rightarrow \mathbb{C}[[t]] \otimes \Psi\mathcal{DO}$$

of the form

$$I_t = \exp \left( \sum_{i=1}^m t_i \text{ad}_{F_i} + \sum_{i,j} t_i t_j \text{ad}_{F_{i,j}} + \dots \right) \quad \text{for some} \\ F_i, F_{i,j}, \dots, F_{i_1 \dots i_k} \in \Psi\mathcal{DO}. \quad (3.4)$$

In many cases, we have to consider polynomial  $\mathfrak{h}$ -trivial deformations instead of formal ones: the formal series can sometimes be cut off, and the formal parameter be replaced by complex coefficients. More precisely, according to [15, 16], a *polynomial  $\mathfrak{h}$ -trivial deformation* of the standard embedding (3.1) is a Lie algebra homomorphism  $\rho(c) : \text{Vect}(S^1) \rightarrow \Psi\mathcal{DO}$  of the following form:

$$\rho(c) = \rho_0 + \sum_{k \in \mathbb{Z}} \rho_k(c), \quad (3.5)$$

where  $\rho_k(c) : \text{Vect}(S^1) \rightarrow \mathcal{P}_k$  are linear maps with  $\mathcal{P}_k$  as in (2.4), polynomial in parameter  $c = (c_1, \dots, c_m) \in \mathbb{C}^m$  and such that  $\rho_k \equiv 0$  for  $k$  sufficiently large.

It is reported that theory of polynomial  $\mathfrak{h}$ -trivial deformations seems to be richer than that of formal ones. The equivalence problem for polynomial  $\mathfrak{h}$ -trivial deformations has additional interesting aspects related to parameter transformations.

### 3.3.2. Infinitesimal $\mathfrak{h}$ -trivial deformations and the first relative cohomology

Deformations (3.2) and (3.5), modulo second-order terms in  $t$  and  $c$ , respectively, are called **infinitesimal**. Infinitesimal  $\mathfrak{h}$ -trivial deformations of a Lie algebra homomorphism  $\rho$  from  $\mathfrak{g}$  into  $\mathfrak{b}$  are classified by the first cohomology  $H^1(\mathfrak{g}, \mathfrak{h}, \mathfrak{b})$ , where  $\mathfrak{b}$  is a  $\mathfrak{g}$ -module through  $\rho$  and  $\mathfrak{h}$  is subalgebra of  $\mathfrak{g}$ . Namely, the first-order terms  $\rho_i$  in (3.2) and  $\frac{\partial \rho(c)}{\partial c_i} \Big|_{c=0}$  in (3.5) are 1-cocycles. Two infinitesimal  $\mathfrak{h}$ -trivial deformations are said to be *equivalent* if the corresponding cocycles are cohomologous. Conversely, given a Lie algebra homomorphism  $\rho : \mathfrak{g} \rightarrow \mathfrak{b}$ , an arbitrary 1-cocycle  $\rho_1 \in Z^1(\mathfrak{g}, \mathfrak{h}, \mathfrak{b})$  defines an infinitesimal deformations of  $\rho$ .

The obstructions for existence of a  $\mathfrak{h}$ -trivial deformation (3.2) lie in  $H^2(\text{Vect}(S^1), \mathfrak{h}, \Psi\mathcal{DO})$ . Indeed, setting

$$\varphi_t = \rho - \rho_0, \quad \rho^{(1)} = \sum t_i \rho_i, \quad \rho^{(2)} = \sum t_i t_j \rho_{ij}^{(2)}, \dots,$$

we can rewrite the relation (3.2) in the following way:

$$[\varphi_t(x), \rho_0(y)] + [\rho_0(x), \varphi_t(y)] - \varphi_t([x, y]) + \sum_{i,j>0} [\rho^{(i)}(x), \rho^{(j)}(y)] = 0. \quad (3.6)$$

The first three terms are  $(\delta\varphi_t)(x, y)$ , where  $\delta$  stands for the coboundary. For arbitrary linear maps  $\gamma_1, \gamma_2 : \mathfrak{g} \rightarrow \mathfrak{b}$ , consider the standard *cup-product*:  $\llbracket \gamma_1, \gamma_2 \rrbracket : \mathfrak{g} \otimes \mathfrak{g} \rightarrow \mathfrak{b}$  defined by:

$$\llbracket \gamma_1, \gamma_2 \rrbracket(x, y) = [\gamma_1(x), \gamma_2(y)] + [\gamma_2(x), \gamma_1(y)]. \quad (3.7)$$

The relation (3.6) becomes now equivalent to

$$\delta\varphi_t = -\frac{1}{2}\llbracket \varphi_t, \varphi_t \rrbracket. \quad (3.8)$$

Expanding (3.8) in power series in  $t$ , we obtain the following equation for  $\rho^{(k)}$ :

$$\delta\rho^{(k)} = -\frac{1}{2} \sum_{\substack{i+j=k \\ i,j>0}} \llbracket \rho^{(i)}, \rho^{(j)} \rrbracket. \quad (3.9)$$

The right-hand side of (3.9) is a 2-cocycle whose coefficients are homogeneous of degree  $k$  polynomials in  $t$ , so the cohomology class of this cocycle is an obstruction for existence of the solutions. The first nontrivial relation  $\delta\rho^{(2)} = -\frac{1}{2}\llbracket \rho^{(1)}, \rho^{(1)} \rrbracket$  gives the first obstruction to integration of an infinitesimal  $\mathfrak{h}$ -trivial deformation. Thus, considering the coefficient of  $t_i t_j$ , we get

$$\delta\rho_{i,j}^{(2)} = -\frac{1}{2}\llbracket \rho_i, \rho_j \rrbracket.$$

This relation is precisely the condition for the 2-cocycle  $\llbracket \rho_i, \rho_j \rrbracket$  to be a coboundary.

## 4. $\mathfrak{aff}(1)$ -Trivial Deformation of the $\text{Vect}(S^1)$ Inside $P$

### 4.1. $\mathfrak{aff}(1)$ -relative cohomology acting on $P$

**Corollary 4.1.**

$$H^1(\text{Vect}(S^1), \mathfrak{aff}(1), P) = H^1(\text{Vect}(S^1), \mathfrak{aff}(1), \mathcal{P}) = \mathbb{R}^2.$$

The nontrivial cocycles generating the relative cohomology groups are as follows:

$$\begin{cases} C_1(f\partial) = f''\xi^{-1} \\ C_2(f\partial) = f'''\xi^{-2}. \end{cases} \quad (4.1)$$

### 4.2. Second-order integrability conditions

**Proposition 4.2.** *Every infinitesimal  $\mathfrak{aff}(1)$ -trivial deformation of the standard embedding of  $\text{Vect}(S^1)$  into  $P$  is equivalent to:*

$$f\partial \mapsto f\xi + c_1 f''\xi^{-1} + c_2 f'''\xi^{-2}, \quad (4.2)$$

where  $c_1, c_2 \in \mathbb{R}$  (or  $\mathbb{C}$ ) are parameters.



In this section, we obtain the integrability conditions for the infinitesimal deformation (4.2). Assume that the infinitesimal deformation (4.2) can be integrated to a formal deformation

$$\varrho_X = \rho_X + \rho_X^{(1)} + \rho_X^{(2)} + \rho_X^{(3)} + \cdots, \quad (4.3)$$

where  $\rho_X^{(1)}$  is given by (4.1) and  $\rho_X^{(2)}$  is a quadratic polynomial in  $c_i$  whose coefficients are elements of  $P$  vanishing on  $\mathbf{aff}(1)$ . We compute the conditions for the second-order terms  $\rho^{(2)}$ . The homomorphism condition

$$[\varrho_X, \varrho_Y] = \varrho_{[X, Y]}$$

gives for the second-order terms the following (Maurer–Cartan) equation

$$\partial\rho^{(2)} = -\frac{1}{2}[[\rho^{(1)}, \rho^{(1)}]]. \quad (4.4)$$

The right-hand side of (4.4) is a cup-product of  $\mathbf{aff}(1)$ -relative 1-cocycles, so it is automatically a  $\mathbf{aff}(1)$ -relative 2-cocycle. More precisely, Eq. (4.4) can be expressed as follows:

$$\partial\rho^{(2)} = -\frac{1}{2} \left[ \left[ \sum_{i=1}^2 c_i C_i, \sum_{i=1}^2 c_i C_i \right] \right] \quad (4.5)$$

therefore, let us consider the  $\mathbf{aff}(1)$ -relative 2-cocycles  $\Omega_{i,j} \in Z_{\text{diff}}^2(\text{Vect}(S^1), \mathbf{aff}(1), P)$ , defined by

$$\Omega_{i,j} = [C_i, C_j].$$

It is easy to see that  $\Omega_{i,i} = 0$ , for  $i = 1, 2$ . The second-order integrability conditions are determined by the fact that any map 2-cocycles  $\Omega_{i,j}$ , for  $i \neq j$ ,  $i, j \in \{1, 2\}$ , must be a  $\mathbf{aff}(1)$ -relative 2-coboundary. More precisely,  $\Omega_{i,j}$  must coincide, up to a scalar factor, with  $\partial b$ , where  $b : \text{Vect}(S^1) \rightarrow P$ ,  $\mathbf{aff}(1)$ -invariant. We need the following lemma.

**Lemma 4.3.** *Let  $b \in C^1(\text{Vect}(S^1), \mathbf{aff}(1), P)$  defined as follows: for  $X \frac{d}{dx} \in \text{Vect}(S^1)$*

$$b(X_f) = \sum_{j=2}^9 \alpha_j f^{(j)} \xi^{(9-j)}, \quad (4.6)$$

where the coefficients  $\alpha_j$  are constant. Then the map  $\partial b : \text{Vect}(S^1) \times \text{Vect}(S^1) \rightarrow P$  is given by

$$\begin{aligned} \partial b(X, Y)(f) &= \alpha_6(2(f'g^{(6)} - f^{(6)}g') - 9(f^{(2)}g^{(5)} - f^{(5)}g^{(2)}) \\ &\quad + 5(f^{(3)}g^{(4)} - f^{(4)}g^{(3)}))\xi^{-3} 5\alpha_5(f^{(2)}g^{(4)} - f^{(4)}g^{(2)})\xi^{-4}. \end{aligned} \quad (4.7)$$

**Proof.** Straightforward computation. □

We split these conditions into two families which we explain in the two following propositions. Let us first consider the following functions in  $c$ , where  $c$  is the family of parameters  $(c_i)$

$$\begin{aligned}\omega_{i,i}(c) &= c_i^2, \quad \text{if } i \in \{1, 2\}, \\ \omega_{1,2}(c) &= c_1 c_2.\end{aligned}$$

These functions  $\omega_{i,j}(c)$ ,  $i, j = 1, 2$ , will appear as coefficients for some maps in  $P$  and they will be used in the expressions of integrability conditions. More precisely, we will show that the second term  $\rho^{(2)}$  is of the form  $\rho^{(2)} = \sum \omega_{i,j}(c) b_{ij}$ .

**Proposition 4.4.**

- (1) For  $i \in \{1, 2\}$ ,  $\Omega_{i,i}(f\partial, g\partial) = 0$ .
- (2) The map  $\Omega_{1,2}(f\partial, g\partial) = \omega_{1,2}(f^{(4)}g^{(2)} - f^{(2)}g^{(4)})\xi^{-4}$ , is a trivial  $\mathfrak{aff}(1)$ -relative 2-cocycle.

**Proof.** (1) For  $i \in \{1, 2\}$ , we have

$$\Omega_{i,i} = c_i^2 \llbracket C_i, C_i \rrbracket.$$

By a direct computation, we show that

$$\llbracket C_i, C_i \rrbracket(f\partial, g\partial) = \{C_i, C_i\}(f\partial, g\partial) + \{C_i, C_i\}(f\partial, g\partial) = 0.$$

(2) Note that hereafter, some singular values of the parameter  $i \neq j$  appear because the  $\mathfrak{aff}(1)$ -relative 2-cocycles  $\llbracket C_i, C_j \rrbracket(f\partial, g\partial)$  exist only for  $i \neq j$ .

(i) By a direct computation, we show that

$$\begin{aligned}\llbracket C_1, C_2 \rrbracket(f\partial, g\partial) &= \{C_1, C_2\}(f\partial, g\partial) + \{C_2, C_1\}(f\partial, g\partial) \\ &= \omega_{1,2}(f^{(4)}g^{(2)} - f^{(2)}g^{(4)})\xi^{-4}.\end{aligned}$$

(ii) We exhibit some reals  $\alpha$  and  $\beta$  such that

$$\alpha \Omega_{1,2} = \beta \partial b. \tag{4.8}$$

By a direct computation, we show that

$$\Omega_{1,2} = \omega_{1,2}(c) \partial b.$$

Hence, there are no conditions on  $\Omega_{1,2}$ . □

Our main result in this section is the following theorem.

**Theorem 4.5.** *There are no integrability conditions for second-order integrability of the  $\mathfrak{aff}(1)$ -trivial infinitesimal deformation (4.2).*

**Proof.** Of course, these conditions are necessary as was shown in Proposition 4.1. Now, under these conditions, the second term  $\rho^{(2)}$  of the  $\mathfrak{aff}(1)$ -trivial infinitesimal deformation (4.2) is a solution of the Maurer–Cartan equation (4.5). This solution is

defined up to a 1-coboundary and it has been shown in [10, 1] that different choices of solutions of the Maurer–Cartan equation correspond to equivalent deformations. Thus, we can always choose

$$\rho^{(2)} = \frac{1}{2}\omega_{1,2}(c)b(X_f) = \frac{1}{2}\omega_{1,2}(c)f^{(5)}\xi^{-4}.$$

Theorem 4.5 is proved.  $\square$

### 4.3. Third and fourth-order integrability conditions

#### 4.3.1. Computing the third-order Maurer–Cartan equation

Now, we reconsider the formal deformation (4.3) which is a formal power series in the parameters  $c_i$  with coefficients in  $P$ . We suppose that the second-order integrability conditions are satisfied. So, the third-order terms of (4.3) are solutions of the (Maurer–Cartan) equation

$$\partial\rho^{(3)} = -\frac{1}{2}\sum_{i+j=3} \llbracket \rho^{(i)}, \rho^{(j)} \rrbracket. \quad (4.9)$$

As in the previous section, we can write

$$\partial\rho^{(3)} = -\frac{1}{2}\sum_{i=1}^2 \Xi_i, \quad (4.10)$$

where  $\Xi_i$  are maps from  $\text{Vect}(S^1)^{\otimes 2}$  to  $P$ . The third-order term  $\rho^{(3)}$  of the  $\mathfrak{aff}(1)$ -trivial formal deformation (4.3) is a solution of (4.10). So, the 2-cochains  $\Xi_i$  must satisfy  $\Xi_i = \partial b$  and then the third-order integrability conditions are deduced from this fact.

We compute successively the  $\Xi_i$  for  $1 = 1, 2$  and we resolve  $\Xi_i = \partial b$  to get the corresponding third-order integrability conditions.

Here, we mention that the maps  $\Xi_i$  are 2-cochains, but they are not necessarily 2-cocycles because they are not cup-products of 1-cocycles like the maps  $\Omega_{i,j}$ . Indeed,  $\rho^{(2)}$  is not necessarily a 1-cocycle.

#### 4.3.2. Third-order integrability conditions

**Proposition 4.6.** *We have the following third-order integrability conditions of the infinitesimal deformation (4.2),*

$$\begin{aligned} c_1\omega_{1,2}(c) &= c_1^2c_2 = 0, \\ c_2\omega_{1,2}(c) &= c_1c_2^2 = 0. \end{aligned} \quad (4.11)$$

We need the following lemmas.

**Lemma 4.7.** Let  $b_6 \in C^1(\text{Vect}(S^1), \mathbf{aff}(1), P)$  defined as follows: for  $f \frac{d}{dx} \in \text{Vect}(S^1)$

$$b_6(X_f) = \alpha_{8,5} f^{(8)} \xi^{-5} + \alpha_{7,6} f^{(7)} \xi^{-6}, \quad (4.12)$$

where the coefficients  $\alpha_{7,6}$  and  $\alpha_{8,5}$  are constant. Then the map  $\partial b_6 : \text{Vect}(S^1)^{\otimes 2} \rightarrow P$  is given by

$$\begin{aligned} \partial b_6(X_f, X_g)(f) &= 2\alpha_{8,5}((f^{(8)}g' - f'g^8) + 10(f^{(7)}g^{(2)} - f^{(2)}g^{(7)}))\xi^{-5} \\ &\quad + 14\alpha_{8,5}(2(f^{(6)}g^{(3)} - f^{(3)}g^{(6)}) + (f^{(5)}g^{(4)} - f^{(4)}g^{(5)}))\xi^{-5} \\ &\quad + 14\alpha_{7,6}((f^{(6)}g^{(2)} - f^{(2)}g^{(6)}) + (f^{(5)}g^{(3)} - f^{(3)}g^{(5)}))\xi^{-6}. \end{aligned} \quad (4.13)$$

**Proof.** Straightforward computation.  $\square$

**Lemma 4.8.** Let  $b_7 \in C^1(\text{Vect}(S^1), \mathbf{aff}(1), P)$  defined as follows: for  $f \frac{d}{dx} \in \text{Vect}(S^1)$

$$b_7(X_f) = \alpha_{9,6} f^{(9)} \xi^{-6} + \alpha_{8,7} f^{(8)} \xi^{-7}, \quad (4.14)$$

where the coefficients  $\alpha_{9,6}$  and  $\alpha_{8,7}$  are constant.

Then  $\partial b_7 \in C^2(\text{Vect}(S^1), \mathbf{aff}(1), P)$  has the general following form: for  $f \frac{d}{dx}, g \frac{d}{dx} \in \text{Vect}(S^1)$

$$\begin{aligned} \partial b_7(X_f, Y_g)(f) &= 2\alpha_{9,6}(f^{(9)}g' - f'g^9)\xi^{-6} + 27\alpha_{9,6}(f^{(8)}g^{(2)} - f^{(2)}g^{(8)})\xi^{-6} \\ &\quad + 38\alpha_{9,6}(f^{(7)}g^{(3)} - f^{(3)}g^{(7)})\xi^{-6} + 42\alpha_{9,6}(f^{(6)}g^{(4)} - f^{(4)}g^{(6)})\xi^{-6} \\ &\quad + 20\alpha_{8,7}(f^{(7)}g^{(2)} - f^{(2)}g^{(7)})\xi^{-7} + 28\alpha_{8,7}(f^{(6)}g^{(3)} - f^{(3)}g^{(6)})\xi^{-7} \\ &\quad + 14\alpha_{8,7}(f^{(5)}g^{(4)} - f^{(4)}g^{(5)})\xi^{-7}. \end{aligned} \quad (4.15)$$

**Proof.** We have

$$\Xi_1 = c_1 \omega_{1,2}(c) \llbracket C_1, b \rrbracket$$

and

$$\Xi_2 = c_2 \omega_{1,2}(c) \llbracket C_2, b \rrbracket.$$

By a direct computation, we show that the 2-cochains

$$\Xi_1 = c_1^2 c_2 ((f^{(6)}g^{(2)} - f^{(2)}g^{(6)}) + 4(f^{(3)}g^{(5)} - f^{(5)}g^{(3)}))\xi^{-6} \quad \text{and} \quad \partial b_6$$

are linearly independent. In the same manner,  $\Xi_2 = c_1 c_2^2 ((f^{(6)}g^{(3)} - f^{(3)}g^{(6)}) + 4(f^{(4)}g^{(5)} - f^{(5)}g^{(4)}))\xi^{-7}$  and  $\partial b_7$  are linearly independent. Thus,

$$\begin{aligned} c_1^2 c_2 &= 0, \\ c_1 c_2^2 &= 0. \end{aligned}$$

We compute these 2-cochains and then we check the corresponding integrability conditions.  $\square$

### 4.3.3. Fourth-order integrability conditions

**Proposition 4.9.** *There are no integrability conditions for the fourth-order integrability conditions of the infinitesimal deformation (4.2).*

**Proof.** These conditions come from the fact that the fourth term  $\rho^{(4)}$  must satisfy:

$$\partial\rho^{(4)} = -\frac{1}{2} \llbracket \rho^{(2)}, \rho^{(2)} \rrbracket.$$

Indeed, we can always reduce  $\rho^{(3)}$  to zero by equivalence.  $\square$

The following theorem is our main result.

**Theorem 4.10.** *The third-order conditions (4.11) are necessary and sufficient for the complete integrability of the infinitesimal deformation (4.2). Moreover, any formal  $\mathfrak{aff}(1)$ -trivial deformation of the Lie derivative  $\rho_X$  on the space of symbols  $P$  is equivalent to a polynomial one of degree equal or less than 2.*

**Proof.** Clearly, all these conditions are necessary. So, let us prove that they are also sufficient. As in the proof of Theorem 4.5, the solution  $\rho^{(3)}$  of the Maurer–Cartan equation (4.9) is defined up to a 1-coboundary, thus, we can always reduce  $\rho^{(3)}$  to zero by equivalence. Moreover, by recurrence, the highest-order terms  $\rho^{(m)}$  satisfy the equation  $\partial\rho^{(m)} = 0$  and can also be reduced to the identically zero map. This completes the proof of Theorem 4.10.  $\square$

## 5. $\mathfrak{aff}(1)$ -Trivial Deformation of the $\text{Vect}(S^1)$ Inside $\Psi\mathcal{DO}$

### 5.1. $\mathfrak{aff}(1)$ -relative cohomology acting on $\Psi\mathcal{DO}$

**Corollary 5.1.**

$$H^1(\text{Vect}(S^1), \mathfrak{aff}(1), \Psi\mathcal{DO}) = \mathbb{R}^2.$$

The nontrivial cocycles generating the relative cohomology groups are as follows:

$$\begin{cases} \Theta_2(f\partial) = \sum_{n=2}^{\infty} (-1)^{n-1} \frac{2(n-3)}{n} f^{(n)} \xi^{-n+1} \\ \Theta_3(f\partial) = \sum_{n=2}^{\infty} (-1)^n \frac{3(n-1)}{n+1} f^{(n+1)} \xi^{-n}. \end{cases}$$

### 5.2. Infinitesimal deformations

**Proposition 5.2.** *Every infinitesimal  $\mathfrak{aff}(1)$ -trivial deformation of the standard embedding of  $\text{Vect}(S^1)$  into  $\Psi\mathcal{DO}$  is equivalent to*

$$\tilde{\pi} : f\partial \mapsto f\xi + c_2\Theta_2(f\partial) + c_3\Theta_3(f\partial), \quad (5.1)$$

where  $c_2, c_3 \in \mathbb{R}$  (or  $\mathbb{C}$ ) are parameters.

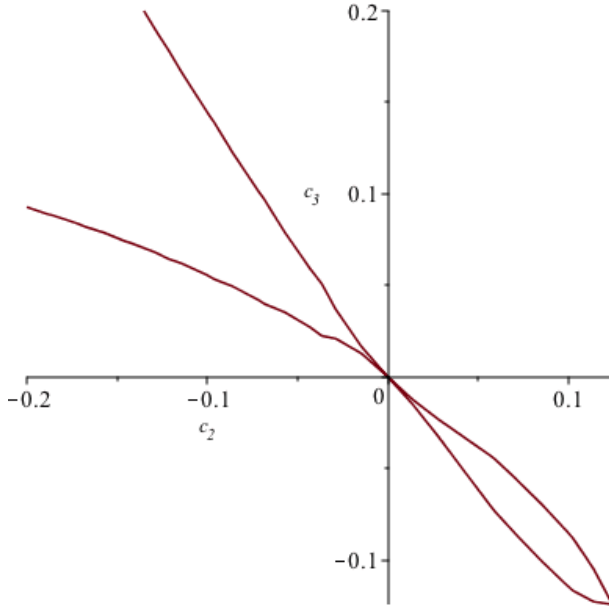


Fig. 1. The integrability condition of  $\Psi\mathcal{D}\mathcal{O}(S^1)$ .

### 5.3. The integrability condition

The main result of this section is the following theorem.

**Theorem 5.3.** *The infinitesimal deformation (5.1) corresponds to a polynomial deformation, if and only if the following (quartic) relation is satisfied:*

$$8c_2^3 + 9c_3^2 + 8c_2^2 + 18c_2c_3 = 0. \quad (5.2)$$

The integrability condition is represented in Fig. 1.

Let us first prove that the condition (2.2) is necessary for integrability of infinitesimal deformations.

**Remark 5.4.** In a cohomological interpretation, the obstructions to integrability of an infinitesimal deformation (5.1) which does not satisfy the condition (5.2), corresponds to a nontrivial class of  $H^2(\text{Vect}(S^1), \mathfrak{aff}(1), \mathcal{F}_5)$ .

**Proof.** The infinitesimal deformation (5.1) is clearly of the form:

$$\tilde{\pi}(f\partial) = f\xi + c_2 f'' \xi^{-1} + c_3 f''' \xi^{-2} + \dots, \quad (5.3)$$

where  $\dots$  means the terms in  $\xi^{-3}, \xi^{-4}, \dots$ . To compute the obstructions for integration of the infinitesimal deformation (5.1), one has to add the first nontrivial

terms and impose the homomorphism condition. So, put

$$\bar{\pi}(f\partial) = f\xi + c_2 f'' \xi^{-1} + c_2 f''' \xi^{-2} + \Pi_4(c) f^{(iv)} \xi^{-3} + \Pi_5(c) f^{(v)} \xi^{-4}, \quad (5.4)$$

where  $\Pi_4(c)$  and  $\Pi_5(c)$  are some polynomials in  $c = (c_2, c_3)$  and compute the difference. Collecting the terms in and yields:

$$2\Pi_4(c) = -c_2^2 - 3c_3 - c_2, \quad (5.5)$$

$$5\Pi_5(c) = -c_2 c_3 + c_2^2 - 6\Pi_4(c) + c_2. \quad (5.6)$$

Let us go one step further, expand our deformation up to  $\xi^{-6}$ , that is, put

$$\bar{\pi}(f\partial) = \bar{\pi}(f\partial) + \Pi_6(c) f^{(vi)} \xi^{-5}$$

the homomorphism condition leads to a nontrivial relation for the parameters:

$$10(c_2 c_3 - c_2^2 - c_2 - 10\Pi_5 - c_2 - c_2 \Pi_4) = 18(3c_2 \Pi_4 + 3c_2 c_3 - 2c_3^2 + 10\Pi_4 + 5c_3).$$

Substituting the expressions (5.5) and (5.6) for  $\Pi_4$  and  $\Pi_5$ , one gets the formula (5.2), we have thus shown that this condition is necessary for integrability of infinitesimal deformations.

This completes the proof of Theorem 5.3.  $\square$

## 5.4. Introducing the parameter $h$

### 5.4.1. Contraction of the Lie algebra $\Psi\mathcal{DO}(S^1)$

The contraction is invented by E. Wigner and I. Inonu in the fifties, then developed by some physicists.

If  $\mathfrak{g}$  is a Lie algebra, consider a one parameter group  $\Phi_t : \mathfrak{g} \rightarrow \mathfrak{g}$  of linear endomorphisms, such that  $\Phi_t$  be an isomorphism for  $t \neq 0$ ,  $\Phi_1 = id$  and  $\Phi_0$  be singular. One defines the bracket  $[\cdot, \cdot]_t$  on  $\mathfrak{g}$  by the formula

$$[X, Y]_t = \Phi_t^{-1}[\Phi_t(X), \Phi_t(Y)].$$

For  $t = 1$ , it is the bracket of  $\mathfrak{g}$ , and it is isomorphic to  $[\cdot, \cdot]_t$  for  $t \neq 0$ . Suppose that for all  $X$  and  $Y$ , the limit:  $[X, Y]_0 = \lim_{t \rightarrow 0} [X, Y]_t$  exist, then  $[\cdot, \cdot]_0$  is a Lie bracket on  $\mathfrak{g}$  not isomorphic to the previous one because  $\Phi_0$  is singular. The algebra  $\mathfrak{g}$  provided with the bracket  $[\cdot, \cdot]_0$  called the contracted Lie algebra of  $\mathfrak{g}$  by  $\Phi_t$ .

Now, let us consider the map  $\Phi_h : \Psi\mathcal{DO}(S^1) \rightarrow \Psi\mathcal{DO}(S^1)$  defined by  $\Phi_h(a\partial^p) = ah^p\partial^p$ , a straightforward computation show that

$$\Phi_h^{-1}([\Phi_h(A), \Phi_h(B)]) = h \left( \frac{\partial A}{\partial \partial} \frac{\partial B}{\partial x} - \frac{\partial B}{\partial \partial} \frac{\partial A}{\partial x} \right) + hO(h).$$

The limit exists and we have  $\lim_{h \rightarrow 0} \frac{1}{h} [A, B]_h = \{A, B\}$  for  $h \rightarrow 0$ . Here,  $\{A, B\}$  means the Poisson bracket on the symbols  $(x, \partial)$  views this time as a function of two variables. So, we used the contraction of the Lie algebra  $\Psi\mathcal{DO}(S^1)$  on the Poisson algebra  $P = C^\infty(S^1)[\partial, \partial^{-1}]$  equipped with the canonical bracket.

#### 5.4.2. Contraction of $\mathfrak{h}$ -trivial deformations

Given a  $\mathfrak{h}$ -trivial deformation of one of the types, (3.5) or (3.2), one can modify it so that it becomes a deformation with values in the deformed algebra  $\Psi\mathcal{DO}_h(S^1)$ . Namely, if one sets  $\pi_t^h = \Phi_h^{-1} \circ \tilde{\pi}_t$ , then one has

$$\pi_t^h([F, G]) = \Phi_h^{-1}([\tilde{\pi}_t(F), \tilde{\pi}_t(G)]) = [\pi_t(F), \pi_t(G)]_h$$

and so  $\pi_t^h : \text{Vect}(S^1) \rightarrow \Psi\mathcal{DO}_h(S^1)$  is a  $\mathfrak{h}$ -trivial (formal) deformation of the standard embedding.

#### 5.4.3. Parametrization: The integrability condition

One can now modify the relation in order to get a deformation in  $\Psi\mathcal{DO}_h(S^1)$ , the scalar  $h$  then appears with different powers according to the *weight* of the successive terms in the formula (5.2). One gets finally:

**Proposition 5.5.** *This relation*

$$8c_2^3 + 9c_3^2 + h(18c_2c_3) + h^2 8c_2^2 = 0 \quad (5.7)$$

*is a necessary integrability condition for infinitesimal deformations (5.1) in  $\Psi\mathcal{DO}_h(S^1)$ . The integrability condition is represented on Fig. 2.*

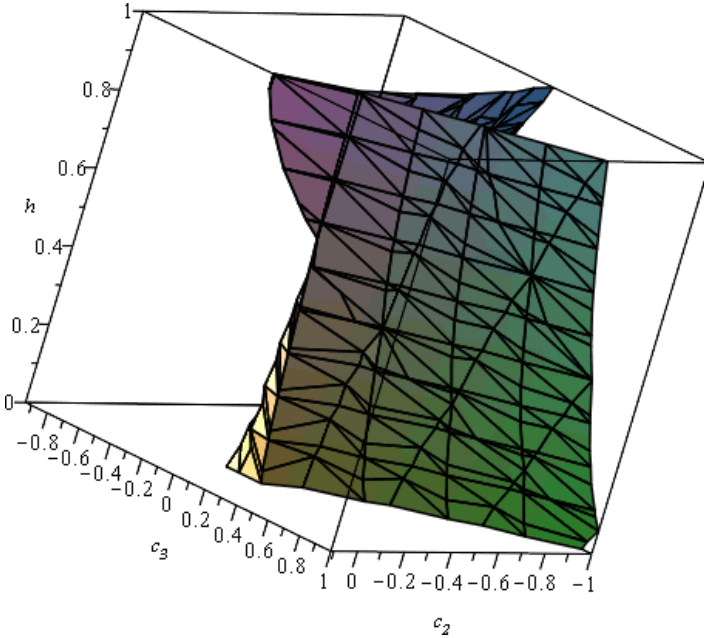


Fig. 2. The integrability condition of  $\Psi\mathcal{DO}_h(S^1)$ .



Now, let us change the parameters  $c_2$  and  $c_3$  and rewrite the expression (5.7) in a canonical form. One checks by an elementary straightforward calculation the

**Proposition 5.6.** *For all  $\sigma \in \mathbb{C}$ , the constants*

$$c_2 = -\frac{\sigma(\sigma + h)}{2}, \quad (5.8)$$

$$c_3 = \frac{\sigma^2(h - \sigma)}{2} + \frac{\sigma(h - \sigma)(2h - \sigma)}{6} \quad (5.9)$$

satisfy the relation (5.7).

Any couple  $c_2, c_3$  satisfying the relation (5.7) is of the form (5.8) and (5.9), for some  $\sigma \in \mathbb{C}$ .

**Proof.** By direct computation. □

**Proposition 5.7.** *Every infinitesimal  $\mathfrak{aff}(1)$ -trivial deformation of the standard embedding of  $\text{Vect}(S^1)$  into  $\mathcal{P}$  is equivalent to:*

$$f\partial \mapsto f\xi + c_1 f'' \xi^{-1} + c_2 f''' \xi^{-2}, \quad (5.10)$$

where  $c_1, c_2 \in \mathbb{R}$  (or  $\mathbb{C}$ ) are parameters.

**Theorem 5.8.** *An infinitesimal  $\mathfrak{aff}(1)$ -trivial deformation (5.10) corresponds to a polynomial deformation of the standard embedding  $\text{Vect}(S^1) \rightarrow \mathcal{P}$ , if and only if it satisfies the following condition:*

$$8c_1^3 + 9c_2^2 = 0.$$

**Proof.** For,  $h = 0$  in (5.7), one gets the polynomial we have obtained in the semi-classical (Poisson) case. □

## 6. $\mathfrak{sl}(2)$ -Trivial Deformation of the $\text{Vect}(S^1)$

In this section, we are interested to the case of the  $\mathfrak{sl}(2)$ -trivial deformation. Let  $\rho_0 : \text{Vect}(S^1) \rightarrow \Psi\mathcal{DO}$  be an embedding of Lie algebras. A *formal deformation* of  $\rho_0$  is a formal series

$$\rho(t) = \rho_0 + \sum_{i=1}^m t_i \Theta_i + \sum_{i,j=1}^m t_i t_j \rho_{ij}^{(2)} + \dots, \quad (6.1)$$

where the  $\Theta_i, \rho_{ij}^{(2)}, \dots$  are linear  $\mathfrak{sl}(2)$ -relative maps from  $\text{Vect}(S^1)$  to  $\Psi\mathcal{DO}$  with the map

$$\rho(t) : \text{Vect}(S^1) \rightarrow \mathbb{C}[[t_1, \dots, t_m]] \otimes \Psi\mathcal{DO} \quad (6.2)$$

which satisfies the homomorphism condition.

**Corollary 6.1.**

$$H^1(\text{Vect}(S^1), \mathfrak{sl}(2), \Psi\mathcal{DO}_h) = \mathbb{R}.$$

The nontrivial cocycles generating the relative cohomology groups are as follows:

$$\Theta_3(f\partial) = \sum_{n=2}^{\infty} (-1)^n \frac{3(n-1)}{n+1} f^{(n+1)} \xi^{-n}.$$

It is clear that the same result holds for  $\mathcal{P}$ :

$$H^1(\text{Vect}(S^1), \mathfrak{sl}(2), \mathcal{P}) = \mathbb{R}.$$

The nontrivial cocycles generating the relative cohomology groups are as follows:

$$C_2(f\partial) = f''' \xi^{-2}.$$

The main result of this subsection is the following.

**Theorem 6.2.**

- (1) Any formal  $\mathfrak{sl}(2)$ -trivial deformation of the standard embedding of  $\text{Vect}(S^1)$  into  $\Psi\mathcal{DO}$  is equivalent to its infinitesimal part:

$$f\partial \mapsto f\xi + c_3\Theta_3(f\partial), \quad (6.3)$$

where  $c_3 \in \mathbb{R}$  (or  $\mathbb{C}$ ) is parameter.

- (2) Any formal  $\mathfrak{sl}(2)$ -trivial deformation of the standard embedding of  $\text{Vect}(S^1)$  into  $\mathcal{P}$  is equivalent to its infinitesimal part:

$$f\partial \mapsto f\xi + c_3C_3(f\partial), \quad (6.4)$$

where  $c_3 \in \mathbb{R}$  (or  $\mathbb{C}$ ) is parameter.

**Proof.** Consider a formal deformation of the embedding (3.2):

$$\tilde{\rho} = \rho_0 + c_3\Theta + \sum_{m \geq 2} c^m \rho^{(m)},$$

where the highest-order terms  $\rho^{(m)}$  are linear  $\mathfrak{sl}(2)$ -relative maps from  $\text{Vect}(S^1)$  to  $\Psi\mathcal{DO}$ .

By a direct computation, we check that  $[[\Theta_3, \Theta_3]] = 0$ . The solution  $\rho_{i,j}^{(2)}$  of (3.9) is defined up to a 1-cocycle and it has been shown in [1, 2, 7–10, 14–16] that different choice of solutions of (3.9) corresponds to equivalent deformations. Thus, one can always kill  $\rho_{i,j}^{(2)}$ . Then, by recurrence, the highest-order terms satisfy the equation  $\delta\rho_{i,j}^{(m)} = 0$  and can also be killed.

As before, we prove the result concerning the deformation of the standard embedding of  $\text{Vect}(S^1)$  into  $\mathcal{P}$  from the fact that we have by a direct computation  $[[C_3, C_3]] = 0$ .  $\square$

## 7. Conclusion and Further Remarks

### 7.1. Conclusion

In this paper, we study the  $\mathfrak{h}$ -Trivial deformations of the standard embedding of the Lie algebra  $\text{Vect}(S^1)$  of smooth vector fields on the circle, into the Lie algebra of functions on the cotangent bundle  $T^*S^1$ . We classify the deformations of this action that become trivial once restricted to  $\mathfrak{h}$ , where  $\mathfrak{h} = \mathfrak{aff}(1)$  or  $\mathfrak{sl}(2)$ . We prove that these conditions are also sufficient.

### 7.2. Further remarks

It would be interesting in the analog super structures: it would be interesting to study the  $\mathfrak{k}$ -trivial deformation of the standard embedding of the Lie superalgebra  $\mathcal{K}(1)$  of smooth vector fields on the supercircle  $S^{1|1}$ .

Note that a Lie superalgebra is a generalization of a Lie algebra to include a  $\mathbb{Z}_2$ -grading. Lie superalgebras are important in theoretical physics where they are used to describe the mathematics of super symmetry. Namely, we consider the superspace  $S^{1|1}$  equipped with the contact structure determined by a 1-form  $\alpha$ , and the Lie superalgebra  $\mathcal{K}(1)$  of contact vector fields on  $\mathbb{R}^{1|1}$ .

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