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ON THE HOMOGENEOUS ERGODIC BILINEAR AVERAGES WITH MÖBIUS AND LIOUVILLE WEIGHTS

E. H. EL ABDALAOUI

ABSTRACT. It is shown that the homogenous ergodic bilinear averages with Möbius or Liouville weight converge almost surely to zero, that is, if T is a map acting on a probability space (X, \mathcal{A}, ν) , and $a, b \in \mathbb{Z}$, then for any $f, g \in L^2(X)$, for almost all $x \in X$,

$$\frac{1}{N} \sum_{n=1}^N \nu(n) f(T^{an}x) g(T^{bn}x) \xrightarrow{N \rightarrow +\infty} 0,$$

where ν is the Liouville function or the Möbius function. We further obtain that the convergence almost everywhere holds for the short interval with the help of Zhan's estimation. Also our proof yields a simple proof of Bourgain's double recurrence theorem. Moreover, we establish that if T is weakly mixing and its restriction to its Pinsker algebra has singular spectrum, then for any integer $k \geq 1$, for any $f_j \in L^\infty(X)$, $j = 1, \dots, k$, for almost all $x \in X$, we have

$$\frac{1}{N} \sum_{n=1}^N \nu(n) \prod_{j=1}^k f(T^{nj}x) \xrightarrow{N \rightarrow +\infty} 0.$$

1. INTRODUCTION

The purpose of this short note is to establish that the homogenous ergodic bilinear averages with Möbius or Liouville weight converge almost surely to zero. Our result, in some sense, extend Sarnak's result which assert that the ergodic averages with Möbius or Liouville weight converge almost surely to zero [21]. Moreover, our proof allows us to obtain a simple proof of Bourgain's double recurrence theorem [6].

The problem of the convergence almost everywhere (a.e.) of the ergodic multilinear averages was introduced by Furstenberg in [13]. Later, J. Bourgain proved that the homogenous ergodic bilinear averages converges almost surely [6]. Subsequently, I. Assani established that the convergence a.e. of the homogenous ergodic multilinear averages holds if the restriction of the map to its Pinsker algebra has a singular spectrum. Assani's proof is based essentially on Bourgain's theorem [6] combined with Host's joining theorem [18]. Very recently, E. H. el Abdalaoui proved that there is a subsequence for which the convergence a.e. of the ergodic multilinear averages holds [1]. For a recent survey on the Furstenberg's problem on the ergodic multilinear averages, we refer to [16]. Let us mention also that C. Demeter in [10]

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obtained an alternative proof of Bourgain's theorem [6].

For the convergence almost everywhere of the homogenous ergodic bilinear averages with weight, I. Assani, D. Duncan, and R. Moore proved à la Weiner-Wintner that the exponential sequences $(e^{2\pi int})_{n \in \mathbb{Z}}$ are good weight for the homogenous ergodic bilinear averages [4]. Subsequently, I. Assani and R. Moore showed that the polynomials exponential sequences $(e^{2\pi iP(n)})_{n \in \mathbb{Z}}$ are also uniformly good weights for the homogenous ergodic bilinear averages [5]. One year later, I. Assani and P. Zorich proved independently that the nilsequences are uniformly good weights for the homogenous ergodic bilinear averages. Their proof depend heavily on Bourgain's theorem. Let us further notice that Zorich's proof yields that if the ergodic multilinear averages converges a.e. then the nilsequences are a good weight for the ergodic multilinear averages.

Here, our goal is to prove that the Möbius and Liouville functions are a good weight for the homogenous ergodic bilinear averages. Our proof follows closely Bourgain's proof [6]. We thus apply Calderón transference principal in order to establish some kind of maximal inequality. Furthermore, we apply Assani's result to prove that the Möbius and Liouville functions are a good weight for the homogenous ergodic multilinear averages if the restriction of the map to its Pinsker algebra has singular spectrum.

Let us remind that Sarnak announced in his seminal paper [21] that the Möbius function is a good weight in L^2 for the ergodic averages. In [2], the authors apply Davenport's estimation combined with Etamedi's trick [12] to obtain a simple proof of Sarnak's result. Therein, they proved that the Möbius function is a good weight in L^1 for the ergodic averages.

2. NOTATIONS AND TOOLS

The Liouville function is defined for the positive integers n by

$$\lambda = (-1)^{\Omega(n)},$$

where $\Omega(n)$ is the length of the word n is the alphabet of prime, that is, $\Omega(n)$ is the number of prime factors of n counted with multiplicities. The Möbius function is given by

$$(1) \quad \mu(n) = \begin{cases} 1 & \text{if } n = 1; \\ \lambda(n) & \text{if } n \text{ is the product of } r \text{ distinct primes;} \\ 0 & \text{if not} \end{cases}$$

These two functions are of great importance in number theory since the Prime Number Theorem is equivalent to

$$(2) \quad \sum_{n \leq N} \lambda(n) = o(N) = \sum_{n \leq N} \mu(n).$$

Furthermore, there is a connection between these two functions and Riemann zeta function, namely

$$\frac{1}{\zeta(s)} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \text{ for any } s \in \mathbb{C} \text{ with } \operatorname{Re}(s) > 1.$$

Moreover, Littlewood proved that the estimate

$$\left| \sum_{n=1}^x \mu(n) \right| = O\left(x^{\frac{1}{2}+\varepsilon}\right) \quad \text{as } x \rightarrow +\infty, \quad \forall \varepsilon > 0$$

is equivalent to the Riemann Hypothesis (RH) ([23, pp.315]).

Here, we will need the following Davenport-Hua's estimation [9], [17, Theorem 10.]: for each $A > 0$, for any $k \geq 1$, we have

$$(3) \quad \max_{z \in \mathbb{T}} \left| \sum_{n \leq N} z^{n^k} \lambda(n) \right| \leq C_A \frac{N}{\log^A N} \text{ for some } C_A > 0.$$

This estimate has been generalized for the short interval by T. Zhan [25] as follows: for each $A > 0$, for any $\varepsilon > 0$, we have

$$(4) \quad \max_{z \in \mathbb{T}} \left| \sum_{N \leq n \leq N+M} z^n \lambda(n) \right| \leq C_{A,\varepsilon} \frac{M}{\log^A(M)} \text{ for some } C_{A,\varepsilon} > 0,$$

provided that $M \geq N^{\frac{5}{8} + \varepsilon}$.

Davenport-Hua's estimation was extended by Green-Tao to the nilsequences setting. We refer to Theorem 1.1 in [15] for the exact estimation and for the definition of the nilsequences.

In our setting, we consider also the ergodic multilinear averages given by

$$\frac{1}{N} \sum_{n=1}^N \prod_{i=1}^k f_i(T_i^n x),$$

where $k \geq 2$, $(X, \mathcal{B}, \mu, T_i)_{i=1}^k$ are a finite family of dynamical systems where μ is a probability measure, T_i are commuting invertible measure preserving transformations and f_1, f_2, \dots, f_k a finite family of bounded functions. The bilinear case corresponds to $k = 2$.

The ergodic multilinear averages is said to be homogenous if $T_i, i = 1, \dots, k$, are the powers of some given map T .

For the convergence a.e., J. Bourgain proved

Theorem 2.1 (Bourgain's double recurrence theorem [6]). Let (X, \mathcal{A}, μ, T) be an ergodic dynamical system, and T_1, T_2 be powers of T . Then, for any $f, g \in L^\infty(X)$, for almost all $x \in X$,

$$\frac{1}{N} \sum_{n=1}^N f(T_1^n x) g(T_2^n x)$$

converges.

Applying Host's joining theorem [18] combined with Bourgain's theorem (Theorem 2.1), I. Assani proved

Theorem 2.2 (Assani's singular multilinear recurrence theorem [6]). Let (X, \mathcal{A}, μ, T) be a weakly mixing dynamical system such that the restriction of T to its Pinsker algebra has singular spectrum, then, for all positive integers k , for all $f_i \in L^\infty(X)$, $i = 1, \dots, k$, for almost all $x \in X$, we have

$$\frac{1}{N} \sum_{n=1}^N \prod_{i=1}^k f_i(T^{in} x) \xrightarrow{N \rightarrow +\infty} \prod_{i=1}^k \int f_i(x) d\mu.$$

3. SOME TOOLS ON THE MAXIMAL ERGODIC INEQUALITIES AND CLADERÓN TRANSFERENCE PRINCIPAL

We say that the sequence of complex number (a_n) is good weight in $L^p(X, \mu)$, $p \geq 1$ for linear case, if, for any $f \in L^p(X, \mu)$, the ergodic averages

$$\frac{1}{N} \sum_{j=1}^N a_j f(T^j x)$$

converges a.e.. We further say that the maximal ergodic inequality holds in $L^p(X, \mu)$ for the linear case with weight (a_n) if, for any $f \in L^p(X, \mu)$, the maximal function given by

$$M(f)(x) = \sup_{N \geq 1} \left| \frac{1}{N} \sum_{j=1}^N a_j f(T^j x) \right|$$

satisfy the weak-type inequality

$$\lambda \mu \left\{ x : M(f)(x) > \lambda \right\} \leq C \|f\|_p,$$

for any $\lambda > 0$ with C is an absolutely constant.

It is well known that the classical maximal ergodic inequality is equivalent to the Birkoff ergodic theorem [14].

The previous notions can be extended in the usual manner to the multilinear case. Let $k \geq 2$, we thus say that (a_n) is good weight in $L^{p_i}(X, \mu)$, $p_i \geq 1$, $i = 1, \dots, k$, with $\sum_{i=1}^k \frac{1}{p_i} = 1$, if, for any $f_i \in L^{p_i}(X, \mu)$, $i = 1, \dots, k$, the ergodic k -multilinear averages

$$\frac{1}{N} \sum_{j=1}^N a_j \prod_{i=1}^k f_i(T_i^j x),$$

converges a.e.. The maximal multilinear ergodic inequality is said to hold in $L^{p_i}(X, \mu)$, $p_i \geq 1$, $i = 1, \dots, k$, with $\sum_{i=1}^k \frac{1}{p_i} = 1$, if, for any $f_i \in L^{p_i}(X, \mu)$, $i = 1, \dots, k$, the maximal function given by

$$M(f_1, \dots, f_k)(x) = \sup_{N \geq 1} \left| \frac{1}{N} \sum_{j=1}^N a_j \prod_{i=1}^k f_i(T_i^j x) \right|$$

satisfy the weak-type inequality

$$\lambda \mu \left\{ x : M(f)(x) > \lambda \right\} \leq C \prod_{i=1}^k \|f_i\|_{p_i},$$

for any $\lambda > 0$ with C is an absolutely constant.

It is not known whether the classical maximal multilinear ergodic inequality ($a_n = 1$, for each n) holds for the general case $n \geq 3$. Nevertheless, we have the following Claderón transference principal in the homogenous case,

Proposition 3.1. Let (a_n) be a sequence of complex number and assume that for any $\phi, \psi \in \ell^2(\mathbb{Z})$, we have

$$\left\| \sup_{N \geq 1} \left| \frac{1}{N} \sum_{n=1}^N a_n \phi(j+n) \psi(j-n) \right| \right\|_{\ell^1(\mathbb{Z})} < C \cdot \|\phi\|_{\ell^2(\mathbb{Z})} \|\psi\|_{\ell^2(\mathbb{Z})},$$

where C is an absolutely constant. Then, for any dynamical system (X, \mathcal{A}, T, μ) , for any $f, g \in L^2(X, \mu)$, we have

$$\left\| \sup_{N \geq 1} \left| \frac{1}{N} \sum_{n=1}^N a_n f(T^n x) g(T^{-n} x) \right| \right\|_1 < C \|f\|_2 \|g\|_2,$$

We further have

Proposition 3.2. Let (a_n) be a sequence of complex number and assume that for any $\phi, \psi \in \ell^2(\mathbb{Z})$, for any $\lambda > 0$, for any integer $J \geq 2$, we have

$$\left| \left\{ 1 \leq j \leq J : \sup_{N \geq 1} \left| \frac{1}{N} \sum_{n=1}^N a_n \phi(j+n) \psi(j-n) \right| > \lambda \right\} \right| < C \frac{\|\phi\|_{\ell^2(\mathbb{Z})} \|\psi\|_{\ell^2(\mathbb{Z})}}{\lambda},$$

where C is an absolutely constant. Then, for any dynamical system (X, \mathcal{A}, T, μ) , for any $f, g \in L^2(X, \mu)$, we have

$$\mu \left\{ x \in X : \sup_{N \geq 1} \left| \frac{1}{N} \sum_{n=1}^N a_n f(T^n x) g(T^{-n} x) \right| > \lambda \right\} < C \frac{\|f\|_2 \|g\|_2}{\lambda}.$$

It is easy to check that Proposition 3.1 and 3.2 hold for the homogenous k -multilinear ergodic averages, for any $k \geq 3$. Moreover, one may state and prove the finitary version where \mathbb{Z} is replaced by $\mathbb{Z}/\bar{J}\mathbb{Z}$ and the functions ϕ and ψ with \bar{J} -periodic functions. We refer to Proposition 14.1 in [11] for more details.

4. MAIN RESULTS AND ITS PROOF

We start by stating our first main result.

Theorem 4.1. Let (X, \mathcal{A}, μ, T) be an ergodic dynamical system, and T_1, T_2 be powers of T . Then, for any $f, g \in L^2(X)$, for almost all $x \in X$,

$$\frac{1}{N} \sum_{n=1}^N \nu(n) f(T_1^n x) g(T_2^n x) \xrightarrow{N \rightarrow +\infty} 0,$$

where ν is the Liouville function or the Möbius function.

Our second main result can be stated as follows:

Theorem 4.2. Let (X, \mathcal{A}, μ, T) be weakly mixing ergodic dynamical system, and T_1, T_2, \dots, T_k be powers of T , $k \geq 2$. Assume that the spectrum of the restriction of T to its Pinsker algebra is singular. Then, for any $f_j \in L^\infty(X)$, $j = 1, \dots, k$, for almost all $x \in X$, we have

$$\frac{1}{N} \sum_{n=1}^N \nu(n) \prod_{j=1}^k f(T_j^n x) \xrightarrow{N \rightarrow +\infty} 0,$$

where ν is the Liouville function or the Möbius function.

For the proof of Theorem 4.2, we need the following criterion due to Katai-Bourgain-Sarnak-Ziegler [7], [20].

Theorem 4.3 (Katai-Bourgain-Sarnak-Ziegler's (KBSZ) criterion [7], [20]). Let f be an arithmetic bounded function and let ν be a bounded multiplicative function. Assume that for all sufficiently large distinct primes p, q we have

$$\frac{1}{N} \sum_{n=1}^N f(np) \overline{f(nq)} \xrightarrow{N \rightarrow +\infty} 0.$$

Then

$$\frac{1}{N} \sum_{n=1}^N \nu(n) f(n) \xrightarrow{N \rightarrow +\infty} 0.$$

Proof of Theorem 4.2. The proof goes by induction on k . We further assume that for some $i \in \{1, \dots, k\}$, $\int f_i d\mu(x) = 0$. The case $k = 1$ follows from Sarnak's result. For $k = 2$, put

$$F(n) = f_1(T_1^n x) f_2(T_2^n x),$$

where T_1, T_2 are the powers of T . Then, by Theorem 2.2, for almost all $x \in X$, for all $p \neq q$,

$$\frac{1}{N} \sum_{n=1}^N F(np) \overline{F}(nq) \xrightarrow{N \rightarrow +\infty} 0,$$

Therefore, by KBSZ criterion (Theorem 4.3), we get, for almost all $x \in X$,

$$\frac{1}{N} \sum_{n=1}^N \nu(n) F(n) \xrightarrow{N \rightarrow +\infty} 0,$$

that is, for almost all $x \in X$,

$$\frac{1}{N} \sum_{n=1}^N \nu(n) f_1(T_1^n x) f_2(T_2^n x) \xrightarrow{N \rightarrow +\infty} 0.$$

But, for any $f_1, f_2 \in L^\infty(X)$,

$$\begin{aligned} & \frac{1}{N} \sum_{n=1}^N \nu(n) f_1(T_1^n x) f_2(T_2^n x) \\ &= \frac{1}{N} \sum_{n=1}^N \nu(n) \left(f_1 - \int f_1 d\mu \right) (T_1^n x) f_2(T_2^n x) + \left(\int f_1 d\mu(x) \right) \frac{1}{N} \sum_{n=1}^N \nu(n) f_2(T_2^n x). \end{aligned}$$

Consequently, for any $f_1, f_2 \in L^\infty(X)$, for almost all $x \in X$,

$$\frac{1}{N} \sum_{n=1}^N \nu(n) f_1(T_1^n x) f_2(T_2^n x) \xrightarrow{N \rightarrow +\infty} 0.$$

Now, assume that for almost all $x \in X$, and for any $\ell \leq k$, we have

$$\frac{1}{N} \sum_{n=1}^N \nu(n) \prod_{i=1}^{\ell} f_i(T_i^n x) \xrightarrow{N \rightarrow +\infty} 0,$$

where $T_i, i = 1, \dots, \ell$ are the powers of T . Then, by applying again Theorem 2.2, we see that for almost all x ,

$$\frac{1}{N} \sum_{n=1}^N F(np) \overline{F}(nq) \xrightarrow{N \rightarrow +\infty} 0,$$

where

$$F(n) = \prod_{i=1}^{k+1} f_i(T_i^n x).$$

Hence, once again by KBSZ criterion (Theorem 4.3), it follows that for almost all $x \in X$,

$$\frac{1}{N} \sum_{n=1}^N \nu(n) F(n) \xrightarrow{N \rightarrow +\infty} 0,$$

whence, for almost all $x \in X$,

$$\frac{1}{N} \sum_{n=1}^N \nu(n) \prod_{i=1}^{k+1} f_i(T_i^n x) \xrightarrow{N \rightarrow +\infty} 0.$$

The proof of the theorem is complete. \square

Remark 4.4. Of course, our proof yields that the convergence a.e. holds for any bounded multiplicative function ν . We deduce also that Theorem 4.2 is valid for the class of weakly mixing PID or the distal flows with the help of the recent result of Gutman-Huang-Shao-Ye [16] and Huang-Shao-Ye [19]. Obviously, if the answer to Furstenberg's question [13] is positive then Theorem 4.2 holds for the general case.

We move now to prove Theorem 4.1. For any $\rho > 1$, we will denote by I_ρ the set $\{(\lfloor \rho^n \rfloor), n \in \mathbb{N}\}$. The maximal functions are defined by

$$M_{N_0, \bar{N}}(f, g)(x) = \sup_{\substack{N_0 \leq N \leq \bar{N} \\ N \in I_\rho}} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(T^n x) g(T^{-n} x) - \frac{1}{N_0} \sum_{n=1}^{N_0} \nu(n) f(T^n x) g(T^{-n} x) \right|,$$

$$M_{N_0}(f, g)(x) = \sup_{\substack{N \geq N_0 \\ N \in I_\rho}} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(T^n x) g(T^{-n} x) - \frac{1}{N_0} \sum_{n=1}^{N_0} \nu(n) f(T^n x) g(T^{-n} x) \right|.$$

Obviously,

$$\lim_{\bar{N} \rightarrow +\infty} M_{N_0, \bar{N}}(f, g)(x) = M_{N_0}(f, g)(x).$$

For the shift \mathbb{Z} -action, the maximal functions are denoted by $m_{N_0, \bar{N}}(\phi, \psi)$ and $m_{N_0}(\phi, \psi)$.

We start by proving the following:

Theorem 4.5. For any $\rho > 1$, for any $f, g \in \ell^2(\mathbb{Z})$, for any $K \geq 1$, we have

$$(5) \quad \sum_{k=1}^K \left\| m_{N_k, N_{k+1}}(f, g) \right\|_{\ell^1(\mathbb{Z})} < C \cdot \sqrt{K} \|f\|_{\ell^2(\mathbb{Z})} \|g\|_{\ell^2(\mathbb{Z})},$$

where ν is the Liouville function or the Möbius function and C is an absolutely constant which depend only on ρ .

The classical Calderón transference principal (see Proposition 3.1 and 3.2) allows us to obtain from Theorem 4.5 the following:

Theorem 4.6. Let (X, \mathcal{A}, T, μ) be an ergodic dynamical system, and let $f, g \in L^2(X, \mu)$. Then, for any $\rho > 1$, for any $K \geq 1$,

$$(6) \quad \sum_{k=1}^K \left\| M_{N_k, N_{k+1}}(f, g) \right\|_1 < 4C\sqrt{K} \cdot \|f\|_2 \|g\|_2,$$

where ν is the Liouville function or the Möbius function.

Let us give the proof of Theorem 4.6.

Proof of Theorem 4.6. The proof goes, as in the proof of Proposition 3.1 and 3.2. Let $\bar{N} = N_{K+1}$ and put

$$\phi_x(n) = \begin{cases} f(T^n x), & \text{if } n \in [-2\bar{N}, 2\bar{N}]; \\ 0, & \text{if not,} \end{cases}$$

and

$$\psi_x(n) = \begin{cases} g(T^n x), & \text{if } n \in [-2\bar{N}, 2\bar{N}]; \\ 0, & \text{if not,} \end{cases}$$

Then, by Theorem 4.5, we have

$$\sum_{k=1}^K \left\| m_{N_k, N_{k+1}}(\phi_x, \psi_x) \right\|_{\ell^1(\mathbb{Z})} < C\sqrt{K} \|\phi_x\|_{\ell^2(\mathbb{Z})} \|\psi_x\|_{\ell^2(\mathbb{Z})}.$$

We thus get

$$\sum_{k=1}^K \left(\sum_{|j| \leq \bar{N}} m_{N_k, N_{k+1}}(\phi_x, \psi_x)(j) \right) < C\sqrt{K} \|\phi_x\|_{\ell^2(\mathbb{Z})} \|\psi_x\|_{\ell^2(\mathbb{Z})},$$

which can be rewritten as follows

$$\sum_{k=1}^K \sum_{|j| \leq \bar{N}} \left(M_{N_k, N_{k+1}}(f, g)(T^j x) \right) < C\sqrt{K} \left(\sum_{|n| \leq 2\bar{N}} |f|^2(T^n x) \right)^{\frac{1}{2}} \left(\sum_{|n| \leq 2\bar{N}} |g|^2(T^n x) \right)^{\frac{1}{2}}.$$

Integrating and applying Hölder inequality we obtain

$$\sum_{k=1}^K \left\| M_{N_k, N_{k+1}}(f, g) \right\|_1 < 4C\sqrt{K} \|f\|_2 \|g\|_2,$$

since T is measure preserving, and this finish the proof of the theorem. \square

We proceed now to the proof of Theorem 4.5. Our proof follows Bourgain's arguments combined with Davenport-Hua estimation.

Proof of Theorem 4.5. Let $f, g \in \ell^2(\mathbb{Z})$ and $\delta > 0$. Denote by \mathcal{F} the Fourier transform from $\ell^2(\mathbb{Z})$ to $L^2(\mathbb{T})$, \mathbb{T} is the circle. We recall that $\mathcal{F}(f)(z) = \sum_{n \in \mathbb{Z}} f(n)z^{-n}$, for $z \in \mathbb{T}$. For any $j \in \mathbb{Z}$, put

$$\nu_j(n) = \nu(j - n), \quad n \in \mathbb{Z}.$$

Of course ν is extended to the negative integers \mathbb{Z}_- in the usual fashion.

Obviously, we have

$$(7) \quad \frac{1}{N} \sum_{n=1}^N \nu(n) f(j+n) g(j-n) = \left(f * \left(\frac{1}{N} \cdot \nu_j \cdot g \cdot \mathbf{1}_{[j-N, j]} \right) \right)(2j),$$

where $*$ is the operation of convolution given by

$$(a * b)(j) = \sum_{n \in \mathbb{Z}} a(j-n) b(n), \quad \forall a, b \in \ell^2(\mathbb{Z}) \quad \text{and} \quad \forall j \in \mathbb{Z}.$$

Furthermore, by a standard arguments, we can rewrite (7) as follows

$$(8) \quad \frac{1}{N} \sum_{n=1}^N \nu(n) f(j+n) g(j-n) = \mathcal{F}^{-1}(\mathcal{F}(f) \cdot \mathcal{F}(G_j))(2j),$$

where $G = \frac{1}{N} \cdot \nu_j \cdot g \cdot \mathbf{1}_{[j-N, j]}$. Therefore

$$(9) \quad \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(j+n) g(j-n) \right| = \left| \int_{\mathbb{T}} \mathcal{F}(f)(z) \mathcal{F}(G_j)(z) z^{2j} dz \right| \leq \int_{\mathbb{T}} |\mathcal{F}(f)(z)| |\mathcal{F}(G_j)(z)| dz$$

But, a straightforward computations gives

$$(10) \int_{\mathbb{T}} |\mathcal{F}(f)(z)| |\mathcal{F}(G_j)(z)| dz = \int_{\mathbb{T}} \left| \sum_{n \in \mathbb{Z}} f(n) z^{-n} \right| \left| \frac{1}{N} \sum_{n=1}^N \nu(n) g(j-n) z^n \right| dz.$$

Now, applying Cauchy-Schwarz inequality, we get

$$\begin{aligned} & \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(j+n) g(j-n) \right|^2 \\ & \leq \left(\int_{\mathbb{T}} \left| \sum_{n \in \mathbb{Z}} f(n) z^{-n} \right|^2 dz \right) \left(\int_{\mathbb{T}} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) g(j-n) z^n \right|^2 dz \right). \end{aligned}$$

Integrating, we see that

$$\begin{aligned}
 \sum_{j \in \mathbb{Z}} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(j+n) g(j-n) \right|^2 &\leq \|f\|_2^2 \sum_{j \in \mathbb{Z}} \int \left| \frac{1}{N} \sum_{n=1}^N \nu(n) g(j-n) z^n \right|^2 dz \\
 (11) \qquad \qquad \qquad &\leq \|f\|_2^2 \int \sum_{j \in \mathbb{Z}} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) g(j-n) z^n \right|^2 dz
 \end{aligned}$$

We thus need to estimate the RHS of the inequality (11). For that, write

$$\sum_{j \in \mathbb{Z}} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) g(j-n) z^n \right|^2 = \sum_{j \in \mathbb{Z}} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) (U^{-n} g)(j) z^n \right|^2,$$

where U is the Koopman operator of the shift map S . Consequently, by the spectral theorem, we have

$$\left\| \frac{1}{N} \sum_{n=1}^N \nu(n) g(S^{-n} j) z^n \right\|_2 = \left\| \frac{1}{N} \sum_{1 \leq n \leq N} \nu(n) \lambda^n z^n \right\|_{L^2(\sigma_g)},$$

where σ_g is the spectral measure of g .¹ Hence, by Davenport's estimation (3), for each $A > 0$, we get

$$(12) \qquad \left\| \frac{1}{N} \sum_{n=1}^N \nu(n) g(S^{-n} j) z^n \right\|_2 \leq \frac{C_A}{(\log(N))^A} \|g\|_2,$$

where C_A is a constant that depends only on A .

Combining (11) and (12), we can rewrite (11) as follows

$$(13) \qquad \sum_{j \in \mathbb{Z}} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(j+n) g(j-n) \right|^2 \leq \frac{C_A^2}{(\log(N))^{2A}} \|f\|_2^2 \|g\|_2^2.$$

Form this, it follows that for any $k = 1, \dots, K$, we have

$$\sum_{\substack{N_k \leq N \leq N_{k+1} \\ N \in I_\rho}} \sum_{j \in \mathbb{Z}} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(j+n) g(j-n) \right|^2 \leq \sum_{\substack{N_k \leq N \leq N_{k+1} \\ N \in I_\rho}} \frac{C_A^2}{(\log(N))^{2A}} \|f\|_2^2 \|g\|_2^2.$$

Whence

$$\left\| \sup_{\substack{N_k \leq N \leq N_{k+1} \\ N \in I_\rho}} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(j+n) g(j-n) \right| \right\|_2 < \sqrt{\sum_{\substack{N_k \leq N \leq N_{k+1} \\ N \in I_\rho}} \frac{C_A^2}{(\log(N))^{2A}} \|f\|_2 \|g\|_2},$$

Thus by the elementary inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$, $a, b \geq 0$, it follows that

$$\sum_{k=1}^K \left\| \sup_{\substack{N_k \leq N \leq N_{k+1} \\ N \in I_\rho}} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(j+n) g(j-n) \right| \right\|_2 < \sum_{k=1}^K \sum_{\substack{N_k \leq N \leq N_{k+1} \\ N \in I_\rho}} \frac{C_A}{(\log(N))^A} \|f\|_2 \|g\|_2,$$

Applying again Cauchy-Schwarz inequality, we conclude that

$$\sum_{k=1}^K \left\| \sup_{\substack{N_k \leq N \leq N_{k+1} \\ N \in I_\rho}} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(j+n) g(j-n) \right| \right\|_1 < C \sqrt{K} \|f\|_2 \|g\|_2.$$

¹Recall that σ_g is a finite measure on the circle determined by its Fourier transform given by $\widehat{\sigma}_g(n) = \langle U^n g, g \rangle$.

In the same manner we can see from (13) that we have

$$\sum_{k=1}^K \left\| \frac{1}{N_k} \sum_{n=1}^{N_k} \nu(n) f(j+n) g(j-n) \right\|_2 < \sum_{k=1}^K \frac{C_A}{\log(N_k)^A} \|f\|_2 \|g\|_2 < C \|f\|_2 \|g\|_2.$$

Therefore, by the triangle inequality, we get

$$\sum_{k=1}^K \left\| m_{N_k, N_{k+1}}(f, g) \right\|_{\ell^1(\mathbb{Z})} < C \sqrt{K} \|f\|_{\ell^2(\mathbb{Z})} \|g\|_{\ell^2(\mathbb{Z})},$$

and the proof of the theorem is complete. \square

Remark 4.7. An alternative proof to the previous proof can be obtained by using the discrete Fourier transform instead of Fourier transform to obtain the same inequalities. We remind that the discrete Fourier transform is defined on finite abelian group $G_J = \mathbb{Z}/J\mathbb{Z}$ by

$$d\mathcal{F}(f)(n) = \sum_{j=0}^{J-1} f(j) \chi_J^n,$$

where $\chi_J = e^{2\pi i \frac{\cdot}{J}}$. For a nice account on the discrete Fourier transform and related topics we refer to [22].

Now, we are able to give the proof of our main result Theorem 4.1.

Proof of Theorem 4.1. By a standard argument, we may assume that the map T is ergodic. Let us assume also that f, g are in $L^\infty(X, \mu)$. Therefore, by Theorem 4.6, it is easily seen that

$$\frac{1}{K} \sum_{k=1}^K \left\| M_{N_k, N_{k+1}}(f, g) \right\|_1 \xrightarrow{K \rightarrow +\infty} 0.$$

Hence, by the same arguments as in [24] and [10], we see that for almost every point $x \in X$, we have

$$\frac{1}{[\rho^m]} \sum_{n=1}^{[\rho^m]} \nu(n) f(T^n x) g(T^{-n} x) \xrightarrow{m \rightarrow +\infty} 0,$$

since the L^2 -limit is zero by Green-Tao theorem [15, Theorem 1.1] combined with Chu's result [8]. It follows that if $[\rho^m] \leq N < [\rho^{m+1}] + 1$, then

$$\begin{aligned} & \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(T^n x) g(T^{-n} x) \right| \\ &= \left| \frac{1}{N} \sum_{n=1}^{[\rho^m]} \nu(n) f(T^n x) g(T^{-n} x) + \frac{1}{N} \sum_{n=[\rho^m]+1}^N \nu(n) f(T^n x) g(T^{-n} x) \right| \\ &\leq \left| \frac{1}{[\rho^m]} \sum_{n=1}^{[\rho^m]} \nu(n) f(T^n x) g(T^{-n} x) \right| + \frac{\|f\|_\infty \|g\|_\infty}{[\rho^m]} (N - [\rho^m] - 1) \\ &\leq \left| \frac{1}{[\rho^m]} \sum_{n=1}^{[\rho^m]} \nu(n) f(T^n x) g(T^{-n} x) \right| + \frac{\|f\|_\infty \|g\|_\infty}{[\rho^m]} ([\rho^{m+1}] - [\rho^m]). \end{aligned}$$

Letting m goes to infinity, we get

$$\left| \frac{1}{[\rho^m]} \sum_{n=1}^{[\rho^m]} \nu(n) f(T^n x) g(T^{-n} x) \right| \xrightarrow{m \rightarrow +\infty} 0,$$

and

$$\frac{\|f\|_\infty}{[\rho^m]}([\rho^{m+1}] - [\rho^m]) \xrightarrow{m \rightarrow +\infty} \|f\|_\infty \|g\|_\infty \cdot (\rho - 1),$$

For any $\rho > 1$. Letting $\rho \rightarrow 1$ we conclude that

$$\frac{1}{N} \sum_{n=1}^N \nu(n) f(T^n x) g(T^{-n} x) \xrightarrow{N \rightarrow +\infty} 0, \quad \text{a.e.},$$

To finish the proof, notice that for any $f, g \in L^2(X, \mu)$, and any $\varepsilon > 0$, there exist $f_1, g_1 \in L^\infty(X, \mu)$ such that $\|f - f_1\|_2 < \sqrt{\varepsilon}$, and $\|g - g_1\|_2 < \sqrt{\varepsilon}$. Moreover, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \left| \frac{1}{N} \sum_{n=1}^N \nu(n) (f - f_1)(T^n x) (g - g_1)(T^{-n} x) \right| \\ & \leq \frac{1}{N} \sum_{n=1}^N |(f - f_1)(T^n x)| |(g - g_1)(T^{-n} x)| \\ & \leq \left(\frac{1}{N} \sum_{n=1}^N |(f - f_1)(T^n x)|^2 \right)^{\frac{1}{2}} \left(\frac{1}{N} \sum_{n=1}^N |(g - g_1)(T^{-n} x)|^2 \right)^{\frac{1}{2}} \end{aligned}$$

Applying the ergodic theorem, it follows that for almost all $x \in X$, we have

$$\limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) (f - f_1)(T^n x) (g - g_1)(T^{-n} x) \right| < \varepsilon.$$

Whence, we can write

$$\begin{aligned} & \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(T^n x) g(T^{-n} x) \right| \\ & \leq \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f_1(T^n x) g(T^{-n} x) \right| + \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(T^n x) g_1(T^{-n} x) \right| \\ & + \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f_1(T^n x) g_1(T^{-n} x) \right| \\ & \leq \varepsilon + \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f_1(T^n x) g(T^{-n} x) \right| + \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(T^n x) g_1(T^{-n} x) \right|. \end{aligned}$$

We thus need to estimate

$$\limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f_1(T^n x) g(T^{-n} x) \right|,$$

and

$$\limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(T^n x) g_1(T^{-n} x) \right|.$$

In the same manner we can see that

$$\begin{aligned}
& \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f_1(T^n x) (g - g_1)(T^{-n} x) \right| \\
& \leq \limsup_{N \rightarrow +\infty} \left(\frac{1}{N} \sum_{n=1}^N |f_1(T^n x)|^2 \right)^{\frac{1}{2}} \limsup_{N \rightarrow +\infty} \left(\frac{1}{N} \sum_{n=1}^N |(g - g_1)(T^n x)|^2 \right)^{\frac{1}{2}} \\
& \leq \|f_1\|_2 \|g - g_1\|_2 \\
& \leq \left(\|f\|_2 + \sqrt{\varepsilon} \right) \cdot \sqrt{\varepsilon}
\end{aligned}$$

This gives

$$\begin{aligned}
& \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f_1(T^n x) g(T^{-n} x) \right| \\
& \leq \left(\|f\|_2 + \sqrt{\varepsilon} \right) \cdot \sqrt{\varepsilon} + \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f_1(T^n x) g_1(T^{-n} x) \right| \\
& \leq \left(\|f\|_2 + \sqrt{\varepsilon} \right) \cdot \sqrt{\varepsilon} + 0
\end{aligned}$$

Summarizing, we obtain the following estimates

$$\begin{aligned}
& \limsup_{N \rightarrow +\infty} \left| \frac{1}{N} \sum_{n=1}^N \nu(n) f(T^n x) g(T^{-n} x) \right| \\
& \leq \varepsilon + \left(\|f\|_2 + \sqrt{\varepsilon} \right) \cdot \sqrt{\varepsilon} + \left(\|g\|_2 + \sqrt{\varepsilon} \right) \cdot \sqrt{\varepsilon}
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that for almost every $x \in X$,

$$\frac{1}{N} \sum_{n=1}^N \nu(n) f(T^n x) g(T^{-n} x) \xrightarrow{N \rightarrow +\infty} 0.$$

This complete the proof of the theorem. \square

As a consequence, we have proved Theorem 2.1.

Remark 4.8. Notice that our proof yields that the convergence almost sure holds for the short interval. Thanks to Zhan's estimation (equation (4)).

We end this section by stating the following conjecture.

Conjecture. Let ν be a aperiodic bounded multiplicative function and $l \geq 2$ a positive integer. If T_1, \dots, T_l are commuting measure preserving transformations acting on the same probability space (X, \mathcal{A}, μ) , then for all $f_1, \dots, f_l \in L^\infty(X, \mu)$, for almost all $x \in X$, we have

$$\frac{1}{N} \sum_{n=1}^N \nu(n) \prod_{j=1}^k f(T_j^n x) \xrightarrow{N \rightarrow +\infty} 0.$$

We remind that ν is a aperiodic multiplicative function if

$$\begin{cases} \nu(mn) = \nu(m)\nu(n), & \text{for all } m, n \in \mathbb{N} \text{ such that } m \wedge n = 1; \text{ and} \\ \frac{1}{N} \sum_{n=1}^N \nu(an + b) \xrightarrow{N \rightarrow +\infty} 0, & \text{for all } a, b \in \mathbb{N}. \end{cases}$$

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