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To cite this version:

Eleonora Catsigeras, Serge Troubetzkoy. PSEUDO-PHYSICAL MEASURES FOR TYPICAL CONTINUOUS MAPS OF THE INTERVAL. 2017. hal-01528093

HAL Id: hal-01528093

https://hal.science/hal-01528093

Preprint submitted on 27 May 2017

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PSEUDO-PHYSICAL MEASURES FOR TYPICAL CONTINUOUS MAPS OF THE INTERVAL

ELEONORA CATSIGERAS AND SERGE TROUBETZKOY

Abstract. We study the measure theoretic properties of typical C^0 maps of the interval. We prove that any ergodic measure is pseudo-physical, and conversely, any pseudo-physical measure is in the closure of the ergodic measures, as well as in the closure of the atomic measures. We show that the set of pseudo-physical measures is meager in the space of all invariant measures. Finally, we study the entropy function. We construct pseudo-physical measures with infinite entropy. We also prove that, for each $m \geq 1$, there exists infinitely many pseudo-physical measures with entropy $\log m$, and deduce that the entropy function is neither upper semi-continuous nor lower semi-continuous.

1. Introduction

The topological dynamics of C^0-generic maps of compact manifolds has been intensively studied by Hurley and his co-authors [H1, H2, AHK]. The measure theoretic properties of C^0-generic volume preserving maps of compact manifolds have also been well studied, starting with the famous Ulam-Oxtoby theorem: *the generic volume preserving maps of compact manifolds is ergodic* [OU], and continued by extensive work by Alpern and Prasad [AP]. Outside the volume preserving maps, the measure theoretic properties of C^0-generic systems on compact manifolds have first been studied by Abdenur and Andersson [AA]. Their main result is that a C^0-generic map has no physical (i.e., SRB) measure, however the Birkhoff average of any continuous function is convergent for Lebesgue a.e. point. This result is the starting point of our investigation. We specialize to the case of the interval, and describe

We gratefully acknowledge support of the projects Physeco, MATH-AMSud, "Sistemas Dinámicos" funded by CSIC of Universidad de la República (Uruguay), and APEX "Systèmes dynamiques: Probabilités et Approximation Diophantienne PAD" funded by the Région PACA. The project leading to this publication has also received funding from Agencia Nacional de Investigación e Innovación (ANII) of Uruguay, Excellence Initiative of Aix-Marseille University - A*MIDEX and Excellence Laboratory Archimedes LabEx (ANR-11-LABX-0033), French "Investissements d'Avenir" programmes.

the pseudo-physical properties of the ergodic measures for C^0-generic maps.

1.1. **Definitions.** Throughout the article we consider the space $\mathcal{C}[0,1]$ of continuous maps of the interval $[0,1]$ to itself. For $I, J \subset [0,1]$ let $d(I,J) := \inf\{|x-y| : x \in I, j \in J\}$.

We will often refer to a Borel probability measures on $[0,1]$ simply as a *measure.* We consider the following distance between measures, which induces the weak* topology on the set of all measures

$$\text{dist}(\mu,\nu) := \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int \Psi_i \, d\mu - \int \Psi_i \, d\nu \right| \tag{1}$$

where $\{\Psi_i\}_{i\in\mathbb{N}}$ is a countable dense subset of $\mathcal{C}[0,1]$.

For any point $x \in [0,1]$, let $p\omega(x)$ be the set of the Borel probability measures on $[0,1]$ that are the limits in the weak* topology of the convergent subsequences of the following sequence

$$\left\{ \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j x} \right\}_{n\in\mathbb{N}}$$

where δ_y is the Dirac delta probability measure supported in $y \in [0,1]$.

Definition 1. A measure μ is called *physical or SRB* if $p\omega(x) = \mu$ for a set $A(\mu)$ of positive Lebesgue measure. Note that we do not require physical measures to be ergodic.

In this article we consider the following generalization of the above definition, introduced in [CE1] and studied in the C^1 case in [CE2], [CCE1], [CCE2]:

Definition 2. A probability measure μ is called *pseudo-physical for f* if for all $\varepsilon > 0$ the set $A_\varepsilon(\mu) := \{x \in [0,1] : \text{dist}(p\omega(x), \mu) < \varepsilon\}$ has positive Lebesgue measure. We denote by $\mathcal{O}_f$ the set of pseudo-physical measures for f.

(In [CE1, CE2, CCE1] pseudo-physical measures were called SRB-like.)

Note that any pseudo-physical measure is automatically f-invariant, and we do not require a pseudo-physical measure to be ergodic.

Definition 3. We will call a property *typical* if there is a dense G_δ subset $G \subset \mathcal{C}[0,1]$ such that all $f \in G$ share this property.

1.2. **Statement of results.** The main results of this article are that the following properties are typical for a map $f \in \mathcal{C}[0,1]$:
• The map f has non countably many pseudo-physical measures supported on fixed points; hence with zero entropy (Theorem 5).
• Any atomic invariant measure supported on a periodic orbit is pseudo-physical, and the set of pseudo-physical measures is the closure of such atomic measures with zero entropy (Theorem 24).
• Any ergodic measure is pseudo-physical, and the set of pseudo-physical measures is the closure of the ergodic measures (Theorem 24).
• The set of pseudo-physical measures is meager in the space of invariant measures; moreover, it is a closed set with empty interior (Theorem 16).
• For any natural number $m \geq 2$, there exists infinitely many pseudo-physical ergodic measures whose metric entropy is exactly $\log m$ (Theorem 32).
• There exists pseudo-physical ergodic measures whose metric entropy is infinite (Theorem 39).
• The map f is non expansive (Corollary 6). Moreover, f is Lebesgue-almost everywhere strongly non-expansive (Definition 8, Corollary 9).
• The entropy function is neither upper semi-continuous nor lower semi-continuous. Moreover, it is also neither upper nor lower semi-continuous when restricted to the set of pseudo-physical measures. (Corollary 34).

2. A technical lemma

We will use the following lemma on the approximation of measures.

Lemma 4. *For any $\varepsilon > 0$, there exists $q \geq 1$ such that, if μ and ν are probability measures satisfying: $supp(\nu) \cup supp(\mu) \subset \cup_{j=1}^{m} I_j$ for some pair-wise disjoint closed intervals with nonempty interiors, lengths at most $1/q$ and $|\nu(I_j) - \mu(I_j)| \leq 1/qm$ for each j then $dist(\mu, \nu) < \varepsilon$.*

Proof. Fix $n \geq 1$ such that $\sum_{i=n+1}^{+\infty} 2^{-i} < \varepsilon/2$. Then, for any pair of probability measures μ, ν we obtain

$$\text{dist}(\mu, \nu) < \frac{\varepsilon}{2} + \sum_{i=1}^{n} \frac{1}{2^i} \left| \int \Psi_i \, d\mu - \int \Psi_i \, d\nu \right|. \tag{2}$$

From the uniform continuity of the finite family of functions $\{\Psi_i\}_{1 \leq i \leq n}$, there exists $\delta > 0$ such that, if $|x_1 - x_2| < \delta$, then $|\Psi_i(x_1) - \Psi_i(x_2)| < \varepsilon/4$ for all $1 \leq i \leq n$. Fix $q \in \mathbb{N}^+$ such that $1/q < \min(\delta, \varepsilon/4)$.

The mean value theorem for integrals, yields for all $1 \leq i \leq n$

$$\int \Psi_i \, d\mu = \sum_{j=1}^{m} \int_{I_j} \Psi_i \, d\mu = \sum_{j=1}^{m} \Psi_i(x_j)\mu(I_j) \text{ for some } x_j \in I_j,$$

$$\int \Psi_i \, d\nu = \sum_{j=1}^{m} \int_{I_j} \Psi_i \, d\nu = \sum_{j=1}^{m} \Psi_i(x'_j)\nu(I_j) \text{ for some } x'_j \in I_j.$$

From this we deduce

$$\begin{aligned}\int \Psi_i \, d\mu - \int \Psi_i \, d\nu &= \sum_{j=1}^{m} \big(\Psi_i(x_j)\mu(I_j) - \Psi_i(x'_j)\nu(I_j)\big) \\ &= \sum_{j=1}^{m} \big(\Psi_i(x_j)(\mu(I_j) - \nu(I_j)) + (\Psi_i(x_j) - \Psi_i(x'_j))\nu(I_j)\big) .\end{aligned}$$

Since $|\nu(I_j) - \mu(I_j)| \leq 1/qm$ for each j, we conclude

$$\left| \int \Psi_i \, d\mu - \int \Psi_i \, d\nu \right| \leq \sum_{j=1}^{m} \frac{1}{qm} + \frac{\varepsilon}{4} \sum_{j=1}^{m} \nu(I_j) < \varepsilon/2 \quad \forall \ 1 \leq i \leq n.$$

Substituting this inequality in (2), finishes the proof of Lemma 4. □

3. Abundance of zero-entropy pseudo-physical measures

Theorem 5. *For a typical map in $\mathcal{C}[0,1]$ there exists an uncountable set $\mathcal{F}$ of fixed points such that for each $x \in \mathcal{F}$ the associated Dirac delta measure δ_x is a pseudo-physical measure.*

As a consequence, there exists uncountably many pseudo-physical measures with zero entropy.

Proof. Any map $f \in \mathcal{C}[0,1]$ has at least one fixed point. Typically, the extremes of the interval are not fixed points.

For any $q \geq 1$ let $C^0_{0,q}$ be the set of $f \in \mathcal{C}[0,1]$ for which there exists δ satisfying $0 < \delta < 1/q$ such that f is constant in an open interval of length $\delta/2$ centered at some fixed point x_0, and f has at least q distinct fixed points $\{x_i\}_{i=1}^q$ different from x_0 in an open interval of length δ centered at x_0, and furthermore for each x_i the map f is constant on a small interval containing it. Denote by C^0_q the set of $f \in \mathcal{C}[0,1]$ such that, for some fixed point x_0, there exists at least q other fixed points $\{x_i\}_{i=1}^q$ in an open interval of length $0 < \delta < 1/q$ that contains x_0, and for each $i = 0, \ldots, q$ there exists a closed interval I^i_q of length $0 < \delta_i < \eta$ such that $x_i \in int(I^i_q)$, the image $f(I^i_q)$ is contained in the interior of I^i_q where $\eta := 3 \cdot \min d(x_i, x_j)$. This choice of η guarantees that the I^i_q are disjoint, and the gap in between any pair of them is at least of size η.

Then, $C^0_{0,q}$ is dense and C^0_q is open. Since $C^0_{0,q} \subset C^0_q$, we deduce that C^0_q is open and dense. Consider the dense G_δ set

$$\mathcal{G} := \bigcap_{q\geq 1} C^0_q.$$

For any $f \in \mathcal{G}$, for each $q \geq 1$ we have $f \in C^0_q$ with associated fixed points $\{x_i(q)\}_{i=0}^q$, associated intervals $\{I^i_q\}_{i=0}^q$ and gap size $\eta_q > 0$. Note that $d(I^i_q, I^j_q) \geq \eta_q$ for each $i \neq j$.

Consider the sequences $\{\vec{i} := (i_1, i_2, \dots) : 0 \leq i_q \leq q\}$ for which the points $x_{i_q}(q)$ converge to a point $x_{\vec{i}}$. There are uncountably many such sequences with distinct limit points $x_{\vec{i}}$, each one is a fixed point of the map f. The sequence of intervals $I^{i_q}_q$ satisfy the following

$$\begin{aligned} d_q := d(x_{\vec{i}}, I^{i_q}_q) &\leq d(x_{\vec{i}}, x_{i_q}(q)) + d(x_{i_q}(q), I^{i_q}_q) \\ &= d(x_{\vec{i}}, x_{i_q}(q)) \to 0 \text{ as } q \to \infty. \end{aligned} \tag{3}$$

Fix $\vec{i}$ and q. Consider the map $g := g_{i_q} := f|_{I^{i_q}_q} : I^{i_q}_q \mapsto I^{i_q}_q$ and a pseudo-physical measure μ_{i_q} of g.

Let $J^{i_q}_q := \{x \in [0,1] : d(x, I^{i_q}_q)\} < 2d_q\}$, note that $x_{\vec{i}} \in J^{i_q}_q$. Pseudo-physical measures are invariant, in particular the the measures μ_{i_q} are invariant. Fix $\varepsilon > 0$ and apply Lemma 4 for sufficiently large q to the measure μ_{i_q} and $\delta_{x_{\vec{i}}}$ (here $m = 1$, $\mu_{i_q}(J^{i_q}_q) = \delta_{x_{\vec{i}}}(J^{i_q}_q) = 1$) to conclude that $\mathrm{dist}(\mu_{i_q}, \delta_{x_{\vec{i}}}) < \varepsilon$ for all sufficiently large q, and thus

$$\lim_{q\to\infty} \mu_{i_q} = \delta_{x_{\vec{i}}}.$$

Since the compact support of μ_{i_q} is contained in $I^{i_q}_q$, this measure is also pseudo-physical for f. Besides the set of pseudo-physical measures for f is closed, we conclude that $\delta_{x_{\vec{i}}}$ is pseudo-physical for f. □

An $f \in \mathcal{C}[0,1]$ is called *expansive in the future* if there exists a constant $\alpha > 0$, called the *expansivity constant,* such that, for any two points $x, y \in [0,1]$, if $d(f^n(x), f^n(y)) \leq \alpha$ for all $n \in \mathbb{N}$, then $x = y$.

Corollary 6. *A typical map in $\mathcal{C}[0,1]$ is not expansive in the future.*

Proof. In the proof of Theorem 5, we constructed arbitrarily small intervals I_q of length $0 < \delta < 1/q$ such that $f(I_q) \subset I_q$. Thus, for any pair of points $x \neq y$ in I_q, the distance of their images is less than $1/q$. Hence, if f were expansive, the expansivity constant must be smaller than $1/q$ for all $q \in \mathbb{N}^+$, which is a contradiction. □

Thus, we can not apply the classical theorem of upper semi-continuity of the entropy function for expansive maps to typical maps, to bound

from below the entropy of a measure by the entropy of nearby measures. In fact, in the sequel we will prove that, for typical $f \in \mathcal{C}[0,1]$, the entropy function is neither upper nor lower semi-continuous.

4. Shrinking intervals cover Lebesegue a.e.

A *periodic shrinking interval of period* $p \geq 1$, is a nonempty open interval I such that $\{f^j(\overline{I})\}_{0\leq j\leq p-1}$ is a family of pairwise disjoint sets, $f^p(\overline{I}) \subset I$, and $\text{length}(f^j(I)) < \text{length}(I)$ for all $1 \leq i \leq p-1$.

An nonempty open interval J is *eventually periodic shrinking,* if there exists a periodic shrinking interval I and an $n \in \mathbb{N}^+$ such that $f^n(J) \subset I$, and $\text{length}(I), \text{length}(f^j(J)) < \text{length}(J)$ for all $1 \leq j \leq n-1$.

A periodic or eventually periodic shrinking interval for some $f \in \mathcal{C}[0,1]$ is also a periodic or eventually periodic shrinking interval for all g in a small neighborhood of f in $\mathcal{C}[0,1]$. Any periodic shrinking interval I of period p contains at least one periodic point of period p. If I is a periodic or eventually periodic shrinking interval with period p, then $\text{length}(f^j(I)) < \text{length}(I)$ for all $j \geq 1$.

Theorem 7. *For a typical map Lebesgue-a.e. point $x \in [0,1]$ belongs to a sequence $\{I_q\}_{q\in\mathbb{N}+}$ of arbitrarily small eventually periodic shrinking intervals I_q; and the length of the I_q goes to zero as $q \to +\infty$.*

Proof. For $q, k \in \mathbb{N}^+$, denote by $\mathcal{S}_{q,k}$ the set of maps in $\mathcal{C}[0,1]$, for which there exists a finite family of nonempty open intervals, which we denote by $\{I_1, \ldots, I_h\}$, such that:
(i) $Leb(I_i) < 1/q$ for $i = 1, 2, \ldots, h$,
(ii) I_i is a periodic or eventually periodic shrinking interval,
(iii) $Leb([0,1] \setminus \bigcup_{i=1}^k I_i) < 1/k$.

The set $\mathcal{S}_{q,k}$ is open in $\mathcal{C}[0,1]$. In fact, for each $f \in \mathcal{S}_{q,k}$, the same family of shrinking intervals of f, is also a family of shrinking intervals for any other map g in a sufficiently small neighborhood of f. Now, let us prove that $\mathcal{S}_{q,k}$ is dense in $\mathcal{C}[0,1]$. Fix any map $f \in \mathcal{C}[0,1]$. For any $\varepsilon > 0$, let us construct a perturbation g of f in $\mathcal{C}$, and a family $\{I_i\}_{1\leq i\leq h}$ of shrinking intervals that satisfies conditions (i) (ii) and (iii). Take $0 < \delta < \varepsilon/3$ such that $d(f(x_1), f(x_2)) < \varepsilon/2$ whenever $d(x_1, x_2) < \delta$. Choose $N \in \mathbb{N}$ such that $N > q$ and $1/N < \delta$. Consider the partition of $[0,1]$ into exactly N intervals $J_1, \ldots, J_N$, all with the same length $1/N$. Let x_i be the midpoint of J_i. By construction $f(J_i) \subset (f(x_i) - \varepsilon/2, f(x_i) + \varepsilon/2)$. Since $(1/N) < \delta < \varepsilon/3$, there exists at least one interval $J_{j(i)} \subset (f(x_i) - \varepsilon/2, f(x_i) + \varepsilon/2)$. Consider the midpoint $x_{j(i)}$ of $J_{j(i)}$ and construct $\widehat{g} : [0,1] \mapsto [0,1]$ by the equality $\widehat{g}|_{J_i} := x_{j(i)}$ for all $i = 1, \ldots, N$. The function $\widehat{g}$ is piecewise constant,

possibly discontinuous at the boundary points of each interval J_i. Any point in $[0,1]$ is eventually periodic by $\widehat{g}$, and by construction

$$d(\widehat{g}(x), f(x)) < \varepsilon \quad \forall\, x \in [0,1]. \tag{4}$$

Besides, if $\partial J_i \cap \partial J_l \neq \emptyset$, then the distance between the midpoints x_i and x_l of the intervals J_i and J_l respectively is exactly $1/N$. Thus $f(x_l) \in (f(x_i)-\varepsilon/2, f(x_i)+\varepsilon/2)$, and thus $d(x_{j(i)}, x_{j(l)}) < 2\varepsilon$. Therefore the discontinuity jumps Δ_i of $\widehat{g}$ are all smaller than 2ε in absolute value.

For each i consider an open sub-interval $I_i \subset J_i$ such that the left boundary point of I_i coincides with the left boundary point of J_i, and such that

$$0 < \text{length}(J_i \setminus I_i) < \frac{1}{Nk}.$$

Define the continuous piecewise affine map $g : [0,1] \mapsto [0,1]$ such that, $g|_{I_i} = g|_{I_i}$ and $g|_{J_i \setminus I_i}$ is affine for all $i = 1, \ldots, N$. Since the absolute values of the discontinuity jumps of g are smaller than 2ε, we deduce that $d(g(x), \widehat{g}(x)) < 2\varepsilon \quad \forall\, x \in [0,1]$. Together with (4) this implies

$$\text{dist}(g, f) < 3\varepsilon. \tag{5}$$

By construction, the family $\{I_i\}_{1 \le i \le N}$ of open intervals satisfies properties (i), (ii) and (iii) for $g \in \mathcal{C}[0,1]$. Thus $g \in \mathcal{S}_{q,k}$. We have proved that $\mathcal{S}_{q,k}$ is dense in $\mathcal{C}[0,1]$.

To end the proof of Theorem 7, consider the set

$$\mathcal{S} := \bigcap_{q \in \mathbb{N}^+} \bigcap_{k \in \mathbb{N}^+} \mathcal{S}_{q,k}.$$

On the one hand, since $\mathcal{S}_{q,k}$ is open and dense in $\mathcal{C}[0,1]$, we have that typical maps of $\mathcal{C}[0,1]$ belong to $\mathcal{S}$. By construction of $\mathcal{S}_{q,k}$, any map in $\mathcal{S}$ satisfies the conclusion of the Theorem. □

Definition 8. A map $f \in \mathcal{C}[0,1]$ is *Lebesgue-a.e. strongly non expansive in the future,* if for any real number $\alpha > 0$, and for Lebesgue a.e. $x \in [0,1]$,

$$Leb\Big(\{y \in [0,1] : d(f^n(x), f^n(x)) < \alpha \quad \forall\, n \in \mathbb{N}\}\Big) > 0.$$

Corollary 9. *A typical map in $\mathcal{C}[0,1]$ is Lebesgue-a.e. strongly non expansive in the future.*

Proof. Take any periodic or eventually periodic shrinking interval I with length smaller than α. Then, $\text{length}(f^j(\overline{I})) < \alpha$ for all $j \ge 0$. Any two points $x, y \in I$ satisfy $\text{dist}(f^j(x), f^j(y)) < \alpha$ for all $j \ge 0$. Thus for any point $x \in I$ we have

$$Leb\Big(\{y \in [0,1] : d(f^n(x), f^n(x)) < \alpha \quad \forall\, n \in \mathbb{N}\}\Big) \ge Leb(I) > 0.$$

Thus the result follows directly from Theorem 7. □

5. Approximation of pseudo-physical measures by periodic orbits.

Lemma 10. *Let $f \in \mathcal{C}[0,1]$ and μ be an f-invariant measure. Assume that $I \subset [0,1]$ is a periodic shrinking interval of period p, and denote $K := \bigcup_{j=0}^{p-1} f^j(\overline{I})$. Then*

$$\mu(f^j(\overline{I})) = \mu(\overline{I}) = \frac{\mu(K)}{p} \quad \forall\ j \geq 0.$$

Proof. For any $j \geq 0$ let $j = kp - i$ with $k \in \mathbb{N}^+$ and $0 \leq i < p$. Using that μ is f-invariant, that $f^p(\overline{I}) \subset I$ and that $f^{-j}(f^j(\overline{I})) \supset \overline{I}$ for all $j \geq 0$, we obtain:

$$\mu(f^j(\overline{I})) = \mu(f^{-j}(f^j(\overline{I})) \geq \mu(\overline{I}) \geq \mu(f^{kp}(\overline{I})) = \mu(f^{-i}(f^{kp}(\overline{I})) \geq \mu(f^j(\overline{I})).$$

Hence, all the inequalities above are equalities; and thus $\mu(f^j(\overline{I})) = \mu(\overline{I})$ for all $j \geq 0$. The result follows since $\mu(K) = \sum_{j=0}^{p-1} \mu(f^j(\overline{I}))$. □

Let $\mathcal{M}_f$ be the set of *f-invariant Borel probability measures* and Per_f be the set of all the *atomic periodic f-invariant measures*:

$$\{\mu \in \mathcal{M}_f : \mu = \frac{1}{p}\sum_{j=0}^{p-1} \delta_{f^j(x_0)} \text{ where } x_0 \text{ is a periodic point of period } p\}.$$

We denote by $\mathcal{E}_f$ the set of *ergodic invariant measures* for f. In particular, we have $\mathrm{Per}_f \subset \mathcal{E}_f$

Theorem 11. *For a typical map $f \in \mathcal{C}[0,1]$, $\mathcal{O}_f \subset \overline{Per_f} \subset \overline{\mathcal{E}_f}$.*

Proof. Choose $q_n \geq 1$ such that any two measures satisfying the q_n-approach conditions of Lemma 4, are mutually at distance smaller than $1/n$. Let $\mu \in \mathcal{O}_f$. By the definition of pseudo-physical measure, the set $A_n \subset [0,1]$ of points y such that

$$\mathrm{dist}(p\omega(y), \mu) < 1/n, \tag{6}$$

has positive Lebesgue measure. If f is $\mathcal{C}[0,1]$ typical, Lebesgue a.e. $y_n \in A_n$ is contained in an arbitrarily small eventually periodic shrinking interval (Theorem 7). Therefore, there exists a point $y_n \in A_n$ such that any measure $\mu_n \in p\omega(y_n)$ is supported on $K_n := \bigcup_{j=1}^{p_n} f^j(\overline{I}_n)$, where I_n is a periodic shrinking interval of period p_n, such that $\mathrm{length}(I_n) \leq 1/q_n$. From the definition of shrinking interval, and applying Lemma 10, we have $\mu_n(f^j(\overline{I})) = 1/p_n$.

Since the interval I_n is shrinking periodic with period p_n, there exists at least one periodic orbit of period p_n in K_n. Consider the atomic

periodic invariant measure ν_n supported on this periodic orbit. It also satisfies $\nu_n(f^j(\overline{I})) = 1/p_n$. So, applying Lemma 4 we deduce that

$$\forall\ y \in A_n,\ \forall\ \mu_n \in p\omega(y),\ \exists\ \nu_n \in \mathrm{Per}_f\colon\ \mathrm{dist}(\mu_n, \nu_n) \leq 1/n. \tag{7}$$

Combining Inequalities (6) and (7), we deduce that there exists sequence of periodic atomic measures ν_n such that $\mathrm{dist}(\nu_n, \mu) < 2/n$, finishing the proof of Theorem 11. □

6. Most invariant measures are not pseudo-physical.

The main purpose of this section is to prove that for a fixed typical map $f \in \mathcal{C}[0,1]$, the set $\mathcal{O}_f$ of pseudo-physical measures is topologically meager in the set $\mathcal{M}_f$ of all the f-invariant measures. Precisely, its complement $\mathcal{M}_f \setminus \mathcal{O}_f$ is open and dense in $\mathcal{M}_f$.

The following is an immediate corollary of Theorem 3.6 and Proposition 4.1 of Abdenur and Anderson in [AA]:

Theorem 12. (Corollary of Abdenur-Anderson Theorem)
Let $f \in \mathcal{C}[0,1]$ be a C^0- typical map. Then
a) for Lebesgue almost every point $x \in [0,1]$, the sequence

$$\left\{\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)}\right\}_{n\in\mathbb{N}^+}$$

of empirical probabilities is convergent,
b) there do not exist SRB-measures, and
c) the limit set in the space of probabilities of the sequence

$$\left\{\frac{1}{n}\sum_{j=0}^{n-1}f^{*j}(Leb)\right\}_{n\in\mathbb{N}^+}$$

is strictly included in the set of f-invariant measures.

Proof. In [AA, Theorem 3.6 and Proposition 4.1] the above assertions a) b) and c) are proved for typical C^0 maps $g : M \mapsto M$, where M is a manifold without boundary. In particular they hold for a typical continuous map g of the circle $\mathbb{S}^1$. Now, let us prove that they also hold for a typical continuous map f of the interval. For any continuous map $f \in \mathcal{C}[0,1]$, we consider $g : \mathbb{S}^1 \mapsto \mathbb{S}^1$ as follows. First, take $\mathbb{S}^1 = [-1,1]/\sim$ where the equivalence relation $\sim$ identifies the extremes -1 and 1. Consider any $g : \mathbb{S}^1 \to \mathbb{S}^1$ such that $g|_{[0,1]} = f$, $g(-1) = f(1)$.

An open set of $f \in \mathcal{C}[0,1]$ yields an open set of $g \in \mathcal{C}(\mathbb{S}^1)$, thus if g is typical in $\mathcal{C}(\mathbb{S}^1)$ then f is typical in $\mathcal{C}[0,1]$. Applying properties (a), (b) and (c) to g, we deduce that f also satisfies them. □

For any $x \in [0,1]$ we define

$$\mu_x := \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \tag{8}$$

when this limit exists. Let

$$AA := \{x \in [0,1] : p\omega(x) = \{\mu_x\}\} \text{ and } AA_1 := \{x \in AA : \mu_x \in \mathcal{O}_f\}.$$

Remark 13. For typical maps in $\mathcal{C}[0,1]$, Theorem 12 a) tells us

$$Leb(AA) = 1,$$

while Proposition 3.1 and Definition 3.2 of [CE2] imply that

$$Leb(AA_1) = 1.$$

Proposition 14. *If f is typical in $\mathcal{C}[0,1]$, then*

$$\mathcal{O}_f = \overline{\{\mu_x : x \in AA_1\}},$$

and $\{\mu_x : x \in AA_1\}$ has no isolated points. Moreover, for any $\varepsilon > 0$ and any $x_0 \in AA_1$ the set $\{x \in AA_1 : \mu_x \neq \mu_{x_0}$ and $dist(\mu_x, \mu_{x_0}) < \varepsilon\}$ has positive Lebesgue measure.

Proof. By definition of the set AA_1 we have $\{\mu_x : x \in AA_1\} \subset \mathcal{O}_f$. Besides, since $\mathcal{O}_f$ is closed in the weak* topology (Theorem 1.3 in [CE1]) we have $\overline{\{\mu_x : x \in AA_1\}} \subset \mathcal{O}_f$.

We turn to the other inclusion. Suppose $\mu \in \mathcal{O}_f$, i.e.

$$Leb\{x \in [0,1] : \mathrm{dist}(p\omega(x), \mu) < \varepsilon) > 0\} \text{ for all } \varepsilon > 0.$$

Since $Leb(AA_1) = 1$ and $p\omega(x) = \mu_x$ for $x \in AA_1$, this is equivalent to

$$Leb\{x \in AA_1 : \mathrm{dist}(\mu_x, \mu) < \varepsilon) > 0\} \text{ for all } \varepsilon > 0.$$

Thus $\mu \in \overline{\{\mu_x : x \in AA_1\}}$. Since $\mu \in \mathcal{O}_f$ is arbitrary we conclude

$$\mathcal{O}_f \subset \overline{\{\mu_x : x \in AA_1\}}.$$

Now, let us prove the last assertion of Proposition 14. By contradiction, assume that there exists $x_0 \in AA_1$ and $\varepsilon > 0$ such that

$$\mathrm{Leb}\{x \in AA_1 \colon \mu_x \neq \mu_{x_0} \text{ and } \mathrm{dist}(\mu_{x_0}, \mu_x) < \varepsilon\} = 0. \tag{9}$$

Since μ_{x_0} is pseudo-physical, by definition we have

$$\mathrm{Leb}(\{x \in [0,1] \colon \mathrm{dist}(p\omega x, \mu_{x_0}) < \varepsilon\}) > 0.$$

From Remark 13 we deduce that

$$\mathrm{Leb}(\{x \in AA_1 \colon \mathrm{dist}(\mu_x, \mu_{x_0}) < \varepsilon\}) > 0.$$

Combining this last assertion with Inequality (9), we obtain that

$$\mathrm{Leb}(\{x \in AA_1 \colon \mu_x = \mu_{x_0})\}) > 0.$$

So, μ_{x_0} is an SRB measure, contradicting part b) of Theorem 12. $\square$

Lemma 15. *Suppose $f \in \mathcal{C}[0,1]$ is a typical map, μ_1 is an f-invariant measure supported on $K = \bigcup_{j=1}^{p} f^j(\overline{I})$, where I is a periodic shrinking interval of period p, and if μ_2 is another f-invariant measure whose compact support is contained in the complement of K, then no convex combination $\nu = \lambda\mu_1 + (1-\lambda)\mu_2$, with $0 < \lambda < 1$, is pseudo-physical.*

Proof. Since I is shrinking periodic with period p, it is an open interval and $f^p(\overline{I}) \subset I$. Construct a continuous function ψ such that $\psi|_{f^p(\overline{I})} = 1$, $0 < \psi(x) < 1$ if $x \in I \setminus f^p(\overline{I})$ and $\psi(x) = 0$ if $x \notin I$. Since μ_1 is supported on $K = \bigcup_{j=0}^{p-1} f^j(\overline{I})$, we can apply Lemma 10 to deduce that

$$\mu_1(\overline{I}) = \mu_1(f^p(\overline{I})) = \frac{1}{p}; \quad \text{hence } \mu_1(I \setminus f^p(\overline{I})) = 0.$$

We obtain $\int \psi \, d\mu_1 = \mu_1(f^p(\overline{I})) = 1/p$ and $\int \psi \, d\mu_2 = \mu_2(I) = 0$. Therefore $0 < \int \psi \, d\nu = \lambda/p < 1/p$.

Choose $\varepsilon > 0$ such that for any measure μ, if $\operatorname{dist}(\nu, \mu) < \varepsilon$, then $\int \psi \, d\mu > 0$ and $\int \psi \, d\mu < 1/p$. Consider the set

$$A_\varepsilon(\nu) := \{x \in AA : \ \operatorname{dist}(\mu_x, \nu) < \varepsilon\}.$$

We claim that the set $A_\varepsilon(\nu)$ is empty. Arguing by contradiction, assume that there exists $x \in A_\varepsilon(\nu)$. Then, from the choice of ε, we have

$$0 < \int \psi \, d\mu_x < 1/p, \tag{10}$$

and from (8) we deduce that there exists $n_0 \geq 1$ such that

$$\frac{1}{n}\sum_{j=0}^{n-1} \psi(f^j(x)) = \int \psi \left(\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f^j(x)} \right) > 0 \ \forall \ n \geq n_0.$$

Thus, there exists $n_1 \geq 1$ such that $\psi(f^{n_1}(x)) > 0$; hence $f^{n_1}(x) \in I$. This implies that the future orbit of $f^{n_1}x$ is contained in K. Hence, μ_x is supported on K. Since μ_x is f-invariant, we apply Lemma 10 to deduce that $\mu_x(f^p(\overline{I})) = 1/p$ and $\mu_x(I \setminus f^p(\overline{I})) = 0$. Therefore,

$$\int \psi \, d\mu_x = \mu_x(f^p(\overline{I})) = \frac{1}{p},$$

contradicting (10). We have proved that $A_\varepsilon(\nu)$ is empty.

For a typical map in $\mathcal{C}[0,1]$, the definition of μ being pseudo-physical and Remark 13, imply that for all $\varepsilon > 0$ we have

$$\operatorname{Leb}(A_\varepsilon(\mu)) > 0.$$

Since in our case $A_\varepsilon(\nu) = \emptyset$, we conclude that ν is not a pseudo-physical measure if f is a typical map in $\mathcal{C}[0,1]$. $\square$

In a Baire-space, we say that a set A is *topologically meager* if it is the countable union of nonempty closed sets with empty interiors; and A is *less than topologically meager* if it is closed with empty interior.

Theorem 16. *If f is typical in $\mathcal{C}[0,1]$, then the set $\mathcal{O}_f$ of pseudo-physical measures is less than topologically meager.*

Proof. From [CE1, Theorem 1.3] the set $\mathcal{O}_f$ is closed. Now, let us prove that its interior in $\mathcal{M}_f$ is empty. Fix $\mu \in \mathcal{O}_f$ and fix $\varepsilon > 0$. From Proposition 14, we can find $x_0 \in AA_1$ such that $\mathrm{dist}(\mu_{x_0}, \mu) < \varepsilon/2$, and there exists a positive Lebesgue measure set of $x \in AA_1$ such that $\mathrm{dist}(\mu_x, \mu_{x_0}) < \varepsilon/2$, and thus by the triangle inequality the set

$$A'_\varepsilon(\mu) := \{x \in AA_1 : \mathrm{dist}(\mu_x, \mu) < \varepsilon\} \tag{11}$$

has positive Lebesgue measure.

Applying Theorem 7, there exists $x \in A'_\varepsilon(\mu)$ belonging to an arbitrary small eventually periodic shrinking interval. Hence μ_x is supported in the f-orbits of arbitrarily small periodic shrinking intervals. Applying Proposition 14, there exists a positive Lebesgue measure set of $y \in A'_\varepsilon(\mu)$ with $\mu_y \neq \mu_x$, and such that μ_y is also supported on the f-orbits of arbitrary small periodic shrinking intervals. Since $\mu_x \neq \mu_y$ we claim that there exist two periodic shrinking intervals I_x and I_y whose f-orbits support μ_x and μ_y respectively, and are mutually disjoint. If not, Lemmas 10 and 4, would imply that the distance between μ_x and μ_y is arbitrarily small. Hence both measures would coincide, contradicting our choice of them.

Thus, we can apply Lemma 15 to deduce that the convex combination $\nu = \lambda\mu_1 + \lambda\mu_2$, with $0 < \lambda < 1$, is non pseudo-physical. But $\mathrm{dist}(\nu, \mu) \leq \lambda \mathrm{dist}(\mu_1, \mu) + (1-\lambda)\mathrm{dist}(\mu_2, \mu) < \varepsilon$. We have proved that any pseudo-physical measure is accumulated by non pseudo-physical measures. In other words, the interior of the set $\mathcal{O}_f$ is empty. □

7. PSEUDO-PHYSICAL MEASURES AND ERGODICITY.

A *δ-pseudo-orbit of f* is a sequence $\{y_n\}_{n\in\mathbb{N}} \subset [0,1]$ such that

$$\mathrm{dist}(f(y_n), y_{n+1}) < \delta \quad \forall\, n \in \mathbb{N}.$$

A *δ-pseudo-orbit $\{y_n\}_{n\in\mathbb{N}}$ is periodic* with period $p \geq 1$, if

$$y_{n+p} = y_n \quad \forall\, n \in \mathbb{N}.$$

A map $f \in \mathcal{C}[0,1]$ has the *periodic shadowing property* if for all $\varepsilon > 0$, there exists $\delta > 0$ such that, if $\{y_n\}_{n\in\mathbb{N}}$ is any periodic δ-pseudo-orbit, then, at least one periodic orbit $\{f^n(x)\}_{n\in\mathbb{N}}$ satisfies

$$d(f^n(x), y_n) < \varepsilon \quad \forall\, n \in \mathbb{N}.$$

Theorem 17. (Kościelniak–Mazur–Oprocha–Pilarczyk) *A typical map $f \in \mathcal{C}[0,1]$ has the periodic shadowing property.*

Proof. See [KMOP, Theorem 1.2] □

Recall that $\mathcal{E}_f$ denotes the set of ergodic measures, and Per_f denotes the set of atomic invariant measures supported on periodic orbts of f. As a consequence of Theorem 17:

Corollary 18. *For a typical map $f \in \mathcal{C}[0,1]$, $\mathcal{E}_f \subset \overline{Per_f}$.*

Proof. It is standard to check that the definition distance in the weak* topology of the space of probability measures implies that for all $\varepsilon_0 > 0$, there exists $\varepsilon > 0$, such that, for any two points $x_1, x_2 \in [0,1]$,

$$d(x_1, x_2) < \varepsilon \quad \Rightarrow \quad \mathrm{dist}(\delta_{x_1}, \delta_{x_2}) < \varepsilon_0.$$

Fix any $\mu \in \mathcal{E}_f$. Since μ is ergodic, we have $p\omega(x) = \{\mu\}$ for μ-a.e. $x \in [0,1]$. Fix such a point x, then there exists $n_0 \geq 1$ such that

$$\mathrm{dist}\left(\frac{1}{n}\sum_{j=0}^{n-1}\delta_{f^j(x)},\ \mu\right) < \varepsilon_0 \quad \forall\ n \geq n_0. \tag{12}$$

Given ε, choose $\delta > 0$ given by Theorem 17. Being well distributed, the point x must be recurrent, thus $d(f^{p-1}(x), x) < \delta$ for some $p \geq n_0$. Construct the periodic δ-pseudo-orbit $\{y_n\}_{n\in\mathbb{N}}$ of period p defined by $y_n = f^n(x)$ for all $0 \leq n < p$, $y_{n+p} = y_n$ for all $n \geq 0$. Applying Theorem 17, there exists a periodic orbit $\{f^n(z)\}_{n\geq 0}$, such that $d(f^n(z), y_n) < \varepsilon \ \ \forall\ n \geq 0$. By construction, if $ip \leq n < (i+1)p$ and $i \geq 0$ then $d(f^n(z), f^{n-ip}(x)) < \varepsilon$ and thus, from the choice of ε, we obtain $\mathrm{dist}(\delta_{f^n(z)}, \delta_{f^{n-ip}(x)}) < \varepsilon_0$ Denote by q the period of z. Taking into account that balls are convex in the weak*-distance in the space of probabilities, we deduce

$$\mathrm{dist}\left(\frac{1}{qp}\sum_{j=0}^{qp-1}\delta_{f^j(z)},\quad \frac{1}{qp}q\cdot\sum_{j=0}^{p-1}\delta_{f^j(x)}\right) < \varepsilon_0.$$

For the atomic invariant measure ν supported on the periodic orbit of z, we have

$$\nu = \frac{1}{q}\sum_{j=0}^{q-1}\delta_{f^j(z)} = \frac{1}{qp}\sum_{j=0}^{qp-1}\delta_{f^j(z)}.$$

Thus,

$$\mathrm{dist}\left(\nu,\quad \frac{1}{p}\sum_{j=0}^{p-1}\delta_{f^j(x)}\right) < \varepsilon_0.$$

Together with (12), this implies that the given ergodic measure μ is $2\varepsilon_0$-approximated by some measure $\nu \in \mathrm{Per}_f$, but $\varepsilon_0 > 0$ was arbitrary. $\square$

An invariant measure μ is called *infinitely shrinked* if there exists a sequence $\{I_q\}_{q\geq 0}$ of periodic shrinking intervals I_q, of periods p_q, such that $\mathrm{length}(I_q) < 1/q$ and $\mu(\bigcup_{i=1}^{p_q} f^j(\overline{I}_q)) = 1$ for all $q \geq 1$. We denote by $\mathrm{Shr}_f \subset \mathcal{M}_f$ the set of infinitely shrinked invariant measures. Define

$$AA_2 := \{x \in AA_1 : \quad \mu_x \in \mathrm{Shr}_f\}.$$

Theorem 19. *For a typical map $f \in \mathcal{C}[0,1]$,*

$$Leb(AA_2) = 1 \quad \textit{and} \quad \mathcal{O}_f = \overline{\{\mu_x \colon x \in AA_2\}} = \overline{\mathrm{Shr}_f}.$$

Proof. From Theorem 7, Lebesgue-a.e. $x \in [0,1]$ belongs to a sequence of eventually periodic or periodic shrinking intervals J_q with $\mathrm{length}(J_q) < 1/q$. Every eventually periodic shrinking interval J_q is a wandering under f until it drops into a periodic shrinking interval I_q with $\mathrm{length}(I_q) < \mathrm{length}(J_q) < 1/q$. By the definition of periodic shrinking interval, every point of I_q has all the measures of $p\omega(x)$ supported on the compact set

$$K_p := \bigcup_{j=1}^{p_q} f^j(\overline{I}_q).$$

In particular for Lebesgue almost all $x \in AA_1$, the limit measure μ_x defined by (8), is supported on K_p. Thus, for a.e. $x \in AA_1$ we have $\mu_x \in \mathrm{Shr}_f$. We have proved that the set AA_2 has full Lebesgue measure.

By construction, $AA_2 \subset AA_1$. So, applying Proposition 14, we obtain:

$$\overline{\{\mu_x \colon x \in AA_2\}} \subset \overline{\{\mu_x \colon x \in AA_1\}} = \mathcal{O}_f.$$

To obtain the opposite inclusion, we apply [CE1, Theorem 1.5]): $\mathcal{O}_f$ is the minimal weak*-compact set of probability measures, that contains $p\omega(x)$ for Lebesgue a.e. x. Since $\overline{\{\mu_x \colon x \in AA_2\}}$ is weak*-compact and contains $p\omega_x = \{\mu_x\}$ for Lebesgue almost all x (because $\mathrm{Leb}(AA_2) = 1$), we conclude that

$$\overline{\{\mu_x \colon x \in AA_2\}} \supset \mathcal{O}_f.$$

The inclusion $\overline{\{\mu_x \colon x \in AA_2\}} \subset \overline{\mathrm{Shr}_f}$ follows trivially from the definition of the set AA_2. Now, let us prove the opposite inclusion. We will prove that every shrinking measure is pseudo-physical. Let $\mu \in \mathrm{Shr}_f$. For any $\varepsilon > 0$, choose $q \geq 1$ as in Lemma 4. By the definition of shrinking measure, μ is supported on the f-orbit

$$K_q = \bigcup_{j=0}^{p_q} f^j(\overline{I}_q)$$

of a periodic shrinking interval I_q of period p_q, such that $\text{length}(\overline{I}_q) < 1/q$; hence $\text{length}(f^j(\overline{I}_q)) < 1/q \ \ \forall\ 1 \le j \le p_q$, and thus $\mu(K_q) = 1$.

Besides, for any point $x \in I_q$, all the measures of $p\omega_x$ are also supported on K_q. In particular, for $x \in I_q \cap AA_1$ we obtain $\mu_x(K_q) = 1$. Finally, applying Lemmas 4 and 10, we deduce that

$$\text{dist}(\mu, \mu_x) < \varepsilon \ \text{ for any } x \in I_q \cap AA_1.$$

Since $Leb(I_q \cap AA_1) = Leb(I_q) > 0$, the basin $A_\varepsilon(\mu)$ has positive Lebesgue measure; namely μ is pseudo-physical.

We have shown that every shrinking measure is pseudo-physical. Since the set $\mathcal{O}_f$ of pseudo-physical measures is closed, we conclude

$$\overline{\text{Shr}_f} = \mathcal{O}_f,$$

finishing the proof of Theorem 19. □

Theorem 20. *For any map $f \in \mathcal{C}[0,1]$, if $\mu \in Shr_f$, then it is ergodic.*

Corollary 21. *For a typical map $f \in \mathcal{C}[0,1]$:*

$$\mathcal{O}_f = \overline{Shr_f} \subset \overline{\mathcal{E}_f} \subset \overline{Per_f}.$$

The corollary immediately follows by combining Theorems 20, 19 with Corollary 18. At the end of next section we will prove that for typical maps these sets are all equal.

Proof. Fix $f \in \mathcal{C}[0,1]$. Suppose $\mu \in \text{Shr}_f$, and $\mu_1, \mu_2 \in \mathcal{M}_f$ such that

$$\mu = \lambda\mu_1 + (1-\lambda)\mu_2, \quad \text{with } \ 0 < \lambda < 1, \tag{13}$$

We shall prove that $\mu_1 = \mu_2 = \mu$; namely μ is extremal in the convex compact set of invariant measures; hence ergodic.

Take arbitrary $\varepsilon > 0$ and fix $q \ge 1$ as in Lemma 4. By the definition of infinitely shrinking measures, there exists a periodic shrinking interval I_q, with $\text{length}(I_q) < 1/q$, and period p_q, whose f-orbit K_q supports μ. The definition of periodic shrinking interval and Lemma 10 tell us:

$$\mu(f^j(\overline{I}_q)) = \frac{1}{p_q}, \quad \text{length}(f^j(\overline{I}_q)) < 1/q \quad \forall\ 1 \le j \le p_q.$$

From (13), for $i = 1, 2$ we have $\mu_i(A) \le \mu(A)$ for any measurable set A. Since $\mu(K_q) = 1$, we deduce $\mu_1(K_q) = \mu_2(K_q) = 1$. Applying Lemma 10 we obtain

$$\mu_1(f^j(\overline{I}_q)) = \mu_2(f^j(\overline{I}_q)) = \frac{1}{p_q} \quad \forall\ 1 \le j \le p_q.$$

So, Lemma 4 implies $\text{dist}(\mu_1, \mu) < \varepsilon$, and $\text{dist}(\mu_2, \mu) \ < \varepsilon$. As $\varepsilon > 0$ is arbitrary, we conclude that $\mu = \mu_1 = \mu_2$; hence μ is ergodic. □

8. All ergodic measures are pseudo-physical.

Slightly abusing language, we refer to $\mu \in \mathrm{Per}_f$ as a *periodic measure.*

Definition 22. Let $q \geq 1$ and x_0 be a periodic point with period $r \geq 1$. We call the periodic (invariant) measure

$$\nu = \frac{1}{r} \sum_{j=0}^{r-1} \delta_{f^j(x_0)}$$

a *q-shrinked measure,* if there exists some periodic shrinking interval I, with length smaller than $1/q$, with period $p \geq 1$ such that ν is supported on $K := \bigcup_{j=1}^{p} f(\overline{I})$. From the definition of periodic shrinking interval, the period p must be smaller or equal than r, and must divide r. We denote by $\mathrm{Shr}_q\mathrm{Per}_f$ the set of q-shrinked periodic measures.

We say that an invariant measure μ, *is ε-approached by q-shrinked periodic measures* if there exists $\nu \in \mathrm{Shr}_q\mathrm{Per}_f$ such that $\mathrm{dist}(\mu, \nu) < \varepsilon$. We denote by $\mathrm{AShr}_{\varepsilon,q}\mathrm{Per}_f$ the set of measures that are ε-approached by q-shrinked periodic measures.

Theorem 23. *For any map $f \in \mathcal{C}[0,1]$*

$$\bigcap_{\varepsilon>0} \bigcap_{q\geq 1} \overline{AShr_{\varepsilon,q}Per}_f \subset \mathcal{O}_f.$$

Notes: The opposite inclusion also holds if $f \in \mathcal{C}[0,1]$ is typical. We will prove it later, in Theorem 24.

It is standard to check that $\bigcap_{\varepsilon>0} \bigcap_{q\geq 1} \overline{\mathrm{AShr}_{\varepsilon,q}\mathrm{Per}}_f = \bigcap_{q\geq 1} \overline{\mathrm{Shr}_q\mathrm{Per}}_f$. Nevertheless, we will not use this latter equality.

Proof. Fix $\varepsilon > 0$. Take $q \geq 1$ as in Lemma 4, such that $1/q < \varepsilon$. We will prove that for any $\mu_q \in \mathrm{AShr}_{1/q,q}\mathrm{Per}_f$, the basin $A_{2\varepsilon}(\mu_q)$ has positive Lebesgue measure.

In fact, for any $\mu_q \in \mathrm{AShr}_{1/q,q}\mathrm{Per}_f$, denote by ν_q a measure in $\mathrm{Shr}_q\mathrm{Per}_f$ such that

$$\mathrm{dist}(\mu_q, \nu_q) < 1/q < \varepsilon.$$

Consider the periodic shrinking interval I and the compact set K for ν_q from Definition 22. From the definition of periodic shrinking interval, any point $x \in I$ satisfies $p\omega(x) \subset K = \bigcup_{j=1}^{p} f^j(\overline{I})$. So, any measure $\mu_x \in p\omega(x)$ is supported on K. Also ν_q is supported on K. Thus, applying Lemma 10, we deduce that

$$\mu_x(f^j(\overline{I})) = \nu_q(f^j(\overline{I})) = \frac{1}{p} \;\; \forall\, 1 \leq j \leq p; \quad \mathrm{length}(f^j(\overline{I})) < \frac{1}{q}.$$

Now, from Lemma 4, we obtain $\mathrm{dist}(\nu_q, \mu_x) < \varepsilon$ for all $\mu_x \in p\omega(x)$, for all $x \in I$. Thus, for any $x \in I = I(\nu_q)$

$$\mathrm{dist}(p\omega(x), \nu_q) < \varepsilon, \text{ hence } \mathrm{dist}(p\omega(x), \mu_q) < 2\varepsilon. \tag{14}$$

Note that when we vary the value of $\varepsilon > 0$, the value of q, and thus also the measures ν_q and μ_q and the interval I, may change. So, from the above inequality we *can not deduce* that each μ_q is pseudo-physical. Nevertheless, we have proved that for any fixed value of $\varepsilon > 0$ there exists $q \geq 1$ such that Inequality (14) holds for all $\mu_q \in \mathrm{AShr}_{1/q,q}\mathrm{Per}_f$.

Now, consider any measure $\mu'_q \in \overline{\mathrm{AShr}_{1/q,q}\mathrm{Per}_f}$. Thus, there exists $\mu_q \in \mathrm{AShr}_{1/q,q}\mathrm{Per}_f$ such that

$$\mathrm{dist}(\mu'_q, \mu_q) < \varepsilon$$

Combining this with (14), we obtain, for fixed $\varepsilon > 0$, a value of $q \geq 1$ such that, for any measure $\mu'_q \in \overline{\mathrm{AShr}_{1/q,q}\mathrm{Per}_f}$ there exists an open interval I such that

$$\mathrm{dist}(p\omega(x), \mu_q) < 3\varepsilon \quad \forall\, x \in I. \tag{15}$$

So, if $\mu \in \bigcap_{\varepsilon>0}\bigcap_{q\geq 1}\overline{\mathrm{AShr}_{\varepsilon,q}\mathrm{Per}_f}$, then, for all $\varepsilon > 0$ there exists an open interval I satisfying assertion (15). This proves that $\mathrm{Leb}(A_{3\varepsilon}(\mu)) > 0$ for all $\varepsilon > 0$; hence $\mu \in \mathcal{O}_f$, as wanted. □

Theorem 24. *For a fixed typical map $f \in \mathcal{C}[0,1]$,*

$$Per_f \subset \bigcap_{q\geq 1} \overline{AShr_{\varepsilon,q}Per_f} \quad \forall\, \varepsilon > 0. \tag{16}$$

Therefore,

$$\bigcap_{\varepsilon>0}\bigcap_{q\geq 1} \overline{AShr_{\varepsilon,q}Per_f} = \mathcal{O}_f = \overline{AShr_f} = \overline{\mathcal{E}_f} = \overline{Per_f}. \tag{17}$$

In particular, any invariant ergodic measure for f is pseudo-physical.

Proof. It is enough to prove (16). In fact, (17) is an immediate consequence of (16), Corollary 21 and Theorem 23. To prove (16), we first define, for any $q, r \in \mathbb{N}^+$, the following concept:

Definition 25. A *good q,r-covering* $\mathcal{U}_{q,r}$ for $f \in \mathcal{C}[0,1]$, is a finite family of *open* intervals such that

(1) $\mathcal{U}_{q,r}$ covers the compact set $\{x \in [0,1] \colon f^r(x) = x\}$.

(2) $\mathrm{length}(U_i) < 1/q$ for any $U_i \in \mathcal{U}_{q,r}$.

(3) For any $U_i \in \mathcal{U}_{q,r}$, there exists a periodic shrinking interval I_i, with period $p_i \leq r$, with p_i that divides r, such that $\overline{I}_i \subset U_i$.

We call a map $f \in \mathcal{C}[0,1]$ *a good q,r-covered map,* if there exists a good q,r-covering $\mathcal{U}_{q,r}$ for f. We denote by $\mathcal{P}_{q,r} \subset \mathcal{C}[0,1]$ the set of all good q,r-covered maps.

Let us prove that, for fixed $q,r \geq 1$, the set $\mathcal{P}_{q,r}$ is open in $\mathcal{C}[0,1]$. Fix $f \in \mathcal{P}_{q,r}$, and denote its good q,r-covering by $\mathcal{U}_{q,r} = \{U_1, U_2, \ldots, U_h\}$. The compact set $K = [0,1] \setminus \bigcup_{i=1}^h U_i$ does not intersect the compact set $\{f^r(x) = x\}$. Let us prove that for all $g \in \mathcal{C}[0,1]$ close enough to f, the same compact set K (defined for the *same* covering $\mathcal{U}_{q,r}$) does not intersect $\{g^r(x) = x\}$. In fact, the real function $\phi_f(\cdot) := \mathrm{dist}(f^r(\cdot), \cdot)$ depends continuously on f. Since $\min_{x\in K} \phi_f(x) > 0$, we deduce $\min_{x\in K} \phi_g(x) > 0$ for all $g \in \mathcal{C}[0,1]$ close enough to f. In other words, $\mathcal{U}_{q,r}$ also covers the fixed points of g^r. Thus, the good q,r-covering of f, is also a covering satisfying conditions (1) and (2) of Definition 25, for any $g \in \mathcal{C}[0,1]$ close enough to f. Now, let us prove that Condition (3) for g is satisfied by the same covering $\mathcal{U}_{q,r}$, provided that g is close enough to f. Consider a f-shrinking periodic interval $I_i \subset U_i \in \mathcal{U}_{q,r}$, of period p_i. Now I_i is a periodic shrinking interval with the same period p_i for all g sufficient close to f. Since the finite family $\{I_i\}_{1\leq i\leq h}$ of shrinking periodic intervals to be preserved is finite, we conclude that (3) is also satisfied for any g sufficiently close to f and thus $\mathcal{P}_{q,r}$ is open in $\mathcal{C}[0,1]$.

In Lemma 26, we will prove that $\mathcal{P}_{q,r}$ is dense in $\mathcal{C}[0,1]$. Let us conclude the proof of Theorem 24 assuming Lemma 26. Observe that, for fixed $q,r \geq 1$, any $f \in \mathcal{P}_{q,r}$ has the following property: any point x_0 fixed by f^r (in particular any periodic point x_0 of period r) is $(1/q)$-near all the points of a periodic shrinking interval I_0 with length smaller than $1/q$, and with period $p_0 \leq r$, p_0 dividing r.

Besides, any periodic shrinked interval of period p_0 has at least one periodic point y_0, fixed by f^{p_0}. We deduce that I_0, whose length is smaller than $1/q$, contains a periodic point y_0. Now, using the definition of the set of measures $\mathrm{Shr}_q\mathrm{Per}_f$, we summerize this assertion as follows:

$$\forall\ x_0 \text{ with period } r, \quad \exists\ \nu_0 := \frac{1}{r}\sum_{j=0}^{r-1} \delta_{f^j(y_0)} \in \mathrm{Shr}_q\mathrm{Per}_f, \tag{18}$$

$$\text{with} \quad |y_0 - x_0| < 1/q.$$

For $r \geq 1$ fixed, consider $f \in \bigcap_{q\geq 1} \mathcal{P}_{q,r}$. For any $\varepsilon > 0$, apply Lemma 4 to find $q' \geq 1$ such that: if $|x-y| < 1/q' < \varepsilon$ then $\mathrm{dist}(\delta_x, \delta_y) < \varepsilon$. Choose $0 < \delta < 1/q'$ such that if $|x-y| < \delta$ then $|f^j(x) - f^j(y)| < 1/q' \quad \forall\ 0 \leq j \leq r$. Let us consider all $q \in \mathbb{N}$ such that $q > 1/\delta > q'$. Remember that $f \in \mathcal{P}_{q,r}$ for all $q \geq 1$. We apply Assertion (18), since

$|y_0 - x_0| < 1/q$, we deduce

$$|f^j(y_0) - f^j(x_0)| < 1/q', \qquad \mathrm{dist}(\delta_{f^j(y_0)}, \delta_{f^j(x_0)}) < \varepsilon \ \ \forall\, 0 \le j \le r.$$

Since balls in the space of probability measures with the weak*-topology are convex, we conclude

$$\mathrm{dist}\left(\frac{1}{r}\sum_{j=0}^{r-1}\delta_{f^j(y_0)}, \ \ \frac{1}{r}\sum_{j=0}^{r-1}\delta_{f^j(x_0)}\right) < \varepsilon.$$

We have shown that for any given periodic orbit $\{f^j(x_0)\}_{0\le j\le r-1}$ of period r, the distance between the periodic measure supported on it, and some measure $\nu_q \in \mathrm{Shr}_q\mathrm{Per}_f$, for all q small enough, is smaller than ε. In other words, any periodic measure supported on a periodic orbit of period r, belongs to $\bigcap_{q\ge1}\overline{\mathrm{AShr}_{\varepsilon,q}\mathrm{Per}_f}$.

Finally, if $f \in \mathcal{P} := \bigcap_{r\ge1}\bigcap_{q\ge1}\mathcal{P}_{q,r}$, then all its periodic measures (supported on periodic orbits of any period r) will belong to $\overline{\mathrm{AShr}_{\varepsilon,q}\mathrm{Per}_f}$. In brief, if $f \in \mathcal{P}$, then

$$\mathrm{Per}_f \subset \overline{\mathrm{AShr}_{\varepsilon,q}\mathrm{Per}_f} \quad \forall\, \varepsilon > 0.$$

As $\mathcal{P}_{q,r}$ is open and dense in $\mathcal{C}[0,1]$, the maps $f \in \mathcal{P}$ are typical. This ends the proof of Theorem 24 (provided that Lemma 26 is proven). □

Lemma 26. *For each $q, r \ge 1$, the set $\mathcal{P}_{q,r}$ is dense in $\mathcal{C}[0,1]$.*

Proof. Fix $f \in \mathcal{C}[0,1]$ and $\varepsilon > 0$. Take $0 < \delta := \frac{1}{q'} < \min\left\{\frac{1}{q}, \ \frac{\varepsilon}{2}\right\}$ such that $|x-y| < \delta \ \Rightarrow\ |f(x) - f(y)| < \frac{\varepsilon}{2}$. To ε-perturb f into new map $g \in \mathcal{P}_{q,r}$, we will proceed in a finite number of steps. For each $i \in \{0, \dots, 2q'-2\}$ consider the open interval $J_i := (\frac{i}{2q'}, \frac{i+2}{2q'})$, this collection of intervals forms an open cover $\mathcal{J}$ of $[0,1]$.

First step: Consider the finite covering $\mathcal{J}_1 = \{J_{1,1}, \dots, J_{1,h_1}\} \subset \mathcal{J}$ of $J \in \mathcal{J}_1$ which intersect the compact set $K_1 := \{f^r(x) = x\}$. Construct $\mathcal{J}_1$ to be a minimal covering of K_1 in the following sense: in each interval $J_{1,i} \in \mathcal{J}_1$, there exists at least one point $x_{1,i}$ such that

$$x_{1,i} \in J_{1,i}, \quad f^r(x_{1,i}) = x_{1,i}, \quad x_{1,i} \notin \overline{J_{1,j}} \ \ \forall\, j \ne i.$$

This is possible since if all the f^r-fixed points that belong to the open interval $J_{1,i}$, also belonged to $\overline{J_{1,j}}$ for some $j \ne i$, we can slightly enlarge $J_{1,j}$ to be sure that all of them belong to the interior of $\overline{J_{1,j}}$, and would suppress $J_{1,i}$ from the covering $\mathcal{J}_1$. We can do this in such a way that the length of all the intervals are at most $\delta_1 < \min\{\frac{1}{q}, \frac{\varepsilon}{2}, 1.1\cdot\delta\}$. Note that $\mathcal{J}_1$ is no longer a sub-collection of $\mathcal{J}$, but it is naturally in a bijection with a sub-collection. Throughout the proof we will use this bijection, and make further modifications to the intervals in such a way

that the new collections are always in bijection with a sub-collection of $\mathcal{J}$.

Denote

$$F_1 := \{x_{1,i} : 1 \le i \le h_1\}, \qquad (F_1)^r := \bigcup_{j=0}^{r-1} f^j(F_1).$$

Hence F_1 is consisits of exactly h_1 different points that are fixed by f^r, and $(F_1)^r$ consists of at most rh_1 different points, also fixed by f^r.

In each open interval $J_{1,i} \in \mathcal{J}_1$ construct a small open interval $I_{1,i}$ such that

$$x_{1,i} \in I_{1,i}, \quad \overline{I_{1,i}} \subset V_{1,i} := J_{1,i} \setminus \bigcup_{j \ne i} \overline{J_{1,j}}, \quad (F_1)^r \bigcap \overline{I_{1,i}} = \{x_{1,i}\}.$$

For each $1 \le i \le h_1$, define

$$g : \overline{V_{1,i}} \mapsto \Big(f(x_{1,i}) - \frac{\varepsilon}{2}, \quad f(x_{1,i}) - \frac{\varepsilon}{2} \Big)$$

to be continuous, piecewise affine, with a minimum number of affine pieces, and such that

(a) $g(x) = f(x)$ if $x \notin V_{1,i}$ or if $x \in V_{1,i} \bigcap (F_1)^r$.
In particular $g(x_{1,i}) = f(x_{1,i})$.

(b) $g(x) = f(x_{1,i})$ if $x \in \overline{I_{1,i}}$.

Let us check that $|g(x) - f(x)| < \varepsilon$ for all x. First of all by construction, $g(x) = f(x)$ for all $x \notin \bigcup_{i=1}^{h_1} V_{1,i}$. Now suppose $x \in V_{1,i}$ for some $1 \le i \le h_1$. Since $g|_{\overline{V_{1,i}}}$ is piecewise affine with a minimum number of pieces satisfying conditions (a) and (b), there exists two points $y, z \in \overline{V_{1,i}}$ such that $g(y) = f(y) \le g(x) \le g(z) = f(z)$. Thus,

$$g(x) - f(x) \le f(z) - f(x) \le |f(z) - f(x)| < \varepsilon, \quad \text{and}$$

$$g(x) - f(x) \ge f(y) - f(x) \ge -|f(y) - f(x)| > -\varepsilon.$$

By construction, each interval $I_{1,i}$ is periodic shrinking for g with period that divides r. In fact, $g(\overline{I_{1,i}}) = f(x_{1,i})$, the orbits by g and f of $x_{1,i}$ coincide, and so $g^r(x_{1,i}) = f^r(x_{1,i}) = x_{1,i} \in I_{1,i}$. We deduce that $I_{1,i}$ is periodic shrinking for g, with some period equal to the period of the point $x_{1,i}$, which divides r.

Up to now, we have constructed an ε-perturbation g of f and a finite family $\mathcal{J}_1$ of open intervals that satisfy Conditions (2) and (3) of Definition 25 for g, and Condition (1) for f. Either $\mathcal{J}_1$ also satisfies condition (1) for g, or not. In the first case, define $\mathcal{U}_{q,r} := \mathcal{J}_1$. This covering satisfies Definition 25 for g. Hence, it is a good q, r-covering for g. So $g \in \mathcal{P}_{q,r}$ and the proof of Lemma 26 is finished. In the second case, we will continue in Step 2, to modify f in $[0,1] \setminus \bigcup_{J \in \mathcal{J}_1} J$.

Second step: Construct a finite covering $\mathcal{J}_2$ of the compact set $K_2 := \{g^r(x) = x\}$ by adding to $\mathcal{J}_1$ all the intervals of $\mathcal{J}$ not "bijectively" in $\mathcal{J}_1$ which intersect the compact set $K_2 := \{g^r(x) = x\}$.

Choose the new open intervals $J_{2,i}$, with $1 \le i \le h_2$, such that $\mathcal{J}_2$ is a minimal covering of K_2 in the following sense: in each interval $J_{2,i} \in \mathcal{J}_2 \setminus \mathcal{J}_1$, there exists at least one point $x_{2,i}$ such that

$$x_{2,i} \in J_{2,i}, \quad g^r(x_{2,i}) = x_{2,i}, \quad x_{2,i} \notin \overline{J_{2,j}} \ \ \forall\, j \neq i, \quad x_{2,i} \notin \bigcup_{J \in \mathcal{J}_1} J.$$

Besides, take

$$x_{2,i} \notin \overline{\bigcup_{J \in \mathcal{J}_1} J} \quad \forall\, 1 \le i \le h_2.$$

In fact, if for some i all the points that are fixed by g and belong to $J_{2,i} \setminus \bigcup_{J \in \mathcal{J}_1} J$, also belonged to $\partial(\cup_{J \in \mathcal{J}_1} J)$, we would slightly enlarge the corresponding interval $J_{1,j} \in \mathcal{J}_1$ so its boundary has not fixed points by g^r. We would not be changing g anywhere yet; but just slightly enlarging an open interval $J_{1,j}$ of the old covering $\mathcal{J}_1$ that was constructed in step 1. After that, we would remove the interval $J_{2,i}$ from the new covering $\mathcal{J}_2$.

Denote

$$F_2 := \{x_{2,i} : 1 \le i \le h_2\}, \qquad (F_2)^r := \bigcup_{j=0}^{r-1} f^j(F_2).$$

In each $J_{2,i} \in \mathcal{J}_2 \setminus \mathcal{J}_1$ choose an open interval $I_{2,i}$ such that

$$x_{2,i} \in I_{2,i}, \qquad \overline{I_{2,i}} \subset V_{2,i} := J_{2,i} \setminus \Big(\big(\overline{\cup_{J \in \mathcal{J}_1} J}\big) \cup \ \big(\cup_{j \neq i} \overline{J_{2,j}}\big)\Big), \text{ and}$$

$$(F_2)^r \bigcap \ \overline{I_{2,i}} = \{x_{2,i}\}.$$

Observe that the function g that was constructed on the first step, coincides with f in $\overline{V_{2,i}}$ for all $1 \le i \le h_2$. Now, we are going to change f, only inside each open interval $V_{2,i}$, as follows:

For each $1 \le i \le h_1$, define a continuous piecewise affine map

$$g : \overline{V_{2,i}} \mapsto \Big(f(x_{2,i}) - \frac{\varepsilon}{2}, \quad f(x_{2,i}) - \frac{\varepsilon}{2}\Big)$$

with a minimum number of affine pieces, and such that

(a) $g(x) = f(x)$ if $x \in \partial V_{2,i}$ or if $x \in V_{2,i} \bigcap (F_2)^r$.
In particular $g(x_{2,i}) = f(x_{2,i})$.

(b) $g(x) = f(x_{2,i})$ if $x \in \overline{I_{2,i}}$.

Similarly as we argued in the first step, we have $|g(x) - f(x)| < \varepsilon$ for all $x \in V_{2,i}$ for all $1 \le i \le h_2$ (while we already had $|g(x) - f(x)| < \varepsilon$ for all $x \notin \bigcup_{i=1}^{h_2} V_{2,i}$).

By construction, each interval $I_{2,i}$ is periodic shrinking for g with period that divides r. Thus we have constructed an ε-perturbation g of f and a finite family $\mathcal{J}_2$ of open intervals that satisfies Conditions (2) and (3) of Definition 25 for g. Either $\mathcal{J}_2$ also satisfies Condition (1) for g, or not. In the first case, define $\mathcal{U}_{q,r} := \mathcal{J}_2$ and the proof of Lemma 26 is finished. In the second case, we should continue with Step 3, and construct the collection $\mathcal{J}_3$ in a similar way.

Note that the procedure must finish in a finite number of steps, because in the passage from the collection $\mathcal{J}_i$ to the collection $\mathcal{J}_{i+1}$ we add at least one interval of the finite collection $\mathcal{J}$. □

9. PSEUDO-PHYSICAL MEASURES WITH FINITE LARGE ENTROPY

Definition 27. Fix $f \in \mathcal{C}[0,1]$ and $m \geq 2$. An *m-horseshoe for f* is a family of m pairwise disjoint closed intervals $\overline{I}_i$, $1 \leq i \leq m$, with nonempty interiors I_i, such that

$$\text{int}(f(I_i)) \supset \overline{I}_1 \cup \overline{I}_2 \cup \ldots \cup \overline{I}_m \quad \forall\, 1 \leq i \leq m. \tag{19}$$

Condition (19) is persistent under small perturbations of the map f, i.e., if $\{\overline{I}_i\}_{1\leq i\leq m}$ is an m-horseshoe for f, then, the *same* family of intervals is also an m-horseshoe for all $g \in \mathcal{C}[0,1]$ close enough to f.

Definition 28. We define the *atoms of a horseshoe* in a non-standard way.

Atoms of generation 1. Let $\{\overline{I}_i\}_{1\leq i\leq m}$ be an m-horseshoe for f. We call each interval $\overline{I}_i$ *an atom of generation* 1 of the horseshoe. We denote by $\mathcal{A}_1$ the family of atoms of generation 1. We have $\#\mathcal{A}_1 = m$.

Atoms of generation 2. By the definition of horseshoe, we have at least one closed interval $\overline{I}_{i,j} \subset I_i$, with nonempty interior $I_{i,j}$, such that $\text{int}(f(I_{i,j})) \supset \overline{I}_j$. For each i, j we *choose* one and only one interval $\overline{I}_{i,j}$ satisfying this condition. We call such an interval, *an atom of generation* 2 of the horseshoe. We denote by $\mathcal{A}_2$ the family of atoms of generation 2. Note that $\#\mathcal{A}_2 = m^2$.

Atoms of generation n. The family $\mathcal{A}_n$ of atoms of generation $n \geq 2$ is composed by m^n pairwise disjoint compact intervals with nonempty interiors, such that each $\overline{I}_{i_1,i_2,\ldots,i_n} \in \mathcal{A}_n$ is identified by a different k-uple $(i_1, i_2, \ldots, i_n) \in \{1, 2, \ldots, m\}^n$, and satisfies the following properties:

$$\overline{I}_{i_1,i_2,\ldots,i_n} \subset I_{i_1,i_2,\ldots,i_{n-1}} \cap f^{-1}(I_{i_2,\ldots,i_{n-1}}) \subset I_{i_1} \cap f^{-1}(I_{i_2}) \cap \ldots \cap f^{-(n-2)}(I_{n-1});$$

$$\text{int}(f(\overline{I}_{i_1,i_2,\ldots,i_n})) \supset \overline{I}_{i_2,\ldots,i_n} \in \mathcal{A}_{n-1}.$$

For each generation $n \geq 2$, once the family $\mathcal{A}_{n-1}$ is defined, there may exist many possible choices of the family $\mathcal{A}_n$ of atoms of the next generation, since there may exist many adequate connected components

of the set $I_{i_1,i_2,\dots,i_{n-1}} \cap f^{-1}(I_{i_2,\dots,i_{n-1}})$. *Throughout the article when we speak of a horseshoe, we fix such a choice.*

Definition 29. Fix $m \geq 2$ and $f \in \mathcal{C}[0,1]$ exhibiting an m-horseshoe. The Λ-*set* of the horseshoe is defined by

$$\Lambda_n := \bigcup_{A \in \mathcal{A}_n} A, \qquad \Lambda = \bigcap_{n \geq 1} \Lambda_n.$$

Observe that Λ_n is compact and $\Lambda_{n+1} \subset \Lambda_n$ for all $n \geq 1$. Thus, Λ is a nonempty compact set. It is a perfect set. Nevertheless Λ is not necessarily totally disconnected since it may contain intervals, because the diameters of the atoms of generation n do not necessarily go to zero as $n \to +\infty$. So, Λ is not necessarily a Cantor set.

Definition 30. Fix $m \geq 2$ and $f \in \mathcal{C}[0,1]$ with an m-horseshoe. We call the m-horseshoe C^0-*hyperbolic* if there exists $\lambda \in (0,1)$ such that

$$\max_{A \in \mathcal{A}_n} \text{length}(A) < \lambda^n \qquad \forall\ n \geq 1.$$

Note that if the horseshoe is C^0-hyperbolic, and more generally if the maximum diameter of the atoms of generation n goes to zero as $n \to +\infty$, then the Λ-set of the horseshoe is a Cantor set.

The following is a well known result on the existence of Benoulli measures supported on Λ-sets of hyperbolic horseshoes:

Proposition 31. *Assume that $f \in \mathcal{C}[0,1]$ exhibits a C^0-hyperbolic m-horseshoe. Then, there exists an f-invariant ergodic measure μ supported on the Λ-set of the m-horseshoe such that*

$$\mu(A) = \frac{1}{m^n} \quad \forall\ A \in \mathcal{A}_n, \qquad \forall\ n \geq 1, \quad \textit{and}$$
$$h_\mu(f) = \log m.$$

The measures μ of Proposition 31 are called *Bernoulli measures.*

Theorem 32. *A typical map $f \in \mathcal{C}[0,1]$ has the following properties:*

(i) *For any $x_0 \in [0,1]$ fixed by f, for any $\varepsilon > 0$, and for any $m \geq 2$, there exists a C^0-hyperbolic m-horseshoe supported on a Cantor set contained in the ε-neighborhood of x_0.*

(ii) *For any $m \geq 2$, there exists infinitely many distinct Bernoulli measures with entropy equal to $\log m$. Every Bernoulli measure is ergodic and hence pseudo-physical for f.*

Proof. A typical map has infinitely many periodic points. Thus (ii) is a direct consequence of (i), Proposition 31 and Theorem 24. So, it is enough to prove (i). To do so, we need the following definition:

Definition 33. Fix $q, m \geq 2$. We say that $f \in \mathcal{C}[0,1]$ belongs to the class $\mathcal{B}_{q,m} \subset \mathcal{C}[0,1]$ if there exists a finite covering with open intervals $J_1, J_2, \ldots, J_h$ of the set of fixed points of f, such that:
a) length$(J_i) \leq 1/q$ for all $1 \leq i \leq h$; and
b) each interval J_i contains a C^0-hyperbolic m-horseshoe.

To prove Theorem 32(i), it is enough to prove that, for each fixed $q, m \geq 2$, the class $\mathcal{B}_{q,m}$ is a dense G_δ-set in $\mathcal{C}[0,1]$.

First, let us see that $\mathcal{B}_{q,m}$ is a G_δ set in $\mathcal{C}[0,1]$. The empty set is a G_δ set by definition, thus suppose $\mathcal{B}_{q,m} \neq \emptyset$, and take $f \in \mathcal{B}_{q,m}$. As already argued in the proof of Theorem 24, the same covering $\{J_1, J_2, \ldots, J_h\}$ of the set of fixed points by f, also covers the set of fixed points by g, for any map $g \in \mathcal{C}[0,1]$ that is close enough to f, say in the open neighborhood $U_0(f) \subset \mathcal{C}[0,1]$. Since $f \in \mathcal{B}_{q,m}$, for each $1 \leq i \leq h$ here exists an m-horseshoe $H_i \subset J_i$. Consider the m compact and pairwise disjoint intervals $\overline{I}^i_1, \ldots, \overline{I}^i_m$ of the m-horseshoe H_i. After Condition (19) in the definition of horseshoe, the same family of compact and pairwise disjoint intervals, is also a horseshoe of all g close enough to f, say in the open neighborhood $U^i_1(f) \subset U_0(f)$ of f in $\mathcal{C}[0,1]$.

Now, let us prove that if H_i is C^0-hyperbolic for f, then it is also C^0-hyperbolic up to level n, with the same constant $\lambda \in (0,1)$, for all $g \in \mathcal{C}[0,1]$ close enough to f, say in the open neighborhood $U^i_n(f) \subset U^i_{n-1}(f) \subset \ldots \subset U^i_1(f) \subset U_0(f)$. By hypothesis, for all $1 \leq k \leq n$, any atom $\overline{I}^i_{i_1,i_2,\ldots,i_k} \in \mathcal{A}^i_k(f)$ of the horseshoe H_i for f, has length strictly smaller than λ^k_i. From the definition of atoms:

$$\overline{I}^i_{i_1,\ldots,i_k} \subset I^i_{i_1,\ldots,i_{k-1}} \cap f^{-1}(I^i_{i_2,\ldots,i_{k-1}}),\ \mathrm{int}(f(I^i_{i_1,i_2,\ldots,i_k})) \supsetneq \overline{I}^i_{i_2,\ldots,i_k}. \tag{20}$$

This finite number of open conditions persists under small perturbations of the map f, with the *same family of atoms* up to generation n, i.e., Equation (20) holds for for all $g \in \mathcal{C}[0,1]$ close enough to f, say $g \in U^i_n(f) \subset \mathcal{C}[0,1]$, for the *same* families $\mathcal{A}^i_k(f)$ of intervals.

These properties imply that we can choose $\mathcal{A}^i_k(g) = \mathcal{A}^i_k(f)$, for all $1 \leq k \leq n$, if $g \in U^i_n(f)$. Furthermore, we can choose $U^i_n(f)$ so small such that the lengths of the atoms of generation k of the horseshoe H_i for any $g \in U^i_n(f)$ are also strictly smaller than $(\lambda_i)^k$, for all $1 \leq k \leq n$, with fixed $n \geq 1$. In other words, the *same* horseshoe H_i for f, is C^0-hyperbolic up to level n, for any $g \in U^i_n(f)$. Finally define

$$U_n(f) := \bigcap_{i=1}^{h} U^i_n(f) \text{ and } G := \bigcap_{n\geq 1,} \bigcup_{f\in\mathcal{B}_{q,m}} U_n(f).$$

Trivially $\mathcal{B}_{q,m} \subset G$, and by construction $G \subset \mathcal{B}_{q,m}$, i.e., $\mathcal{B}_{q,m}$ is a G_δ-set.

Now, let us prove that $\mathcal{B}_{q,m}$ is dense in $\mathcal{C}[0,1]$. Fix $f \in \mathcal{C}[0,1]$ and $\varepsilon > 0$. Let us construct $g \in \mathcal{B}_{q,m}$ such that $\operatorname{dist}(g,f) < \varepsilon$. Fix $\delta \in (0, \max\{1/q, \varepsilon/3\})$ such that, if $d(x,y) < \delta$, then $d(f(x), f(y)) < \varepsilon/3$. Consider a finite covering $\mathcal{J} = \{J_1, J_2, \dots J_h\}$ of the set of fixed points of f, with open intervals J_i of length smaller than δ. Choose this covering to be minimal in the following sense: each interval $J_i \in \mathcal{J}$ has at least one fixed point by f that does not belong to $\bigcup_{j\neq i} J_j$, and besides $J_i \setminus \bigcup_{j\neq i} J_j$ is a single interval which we will denote (y_i, z_i).

We will change f, to construct a new map $g \in \mathcal{C}[0,1]$, such that $g(x) = f(x)$ for all $x \notin \bigcup_{i=1}^{h} J_i$. So, the same family $\mathcal{J}$ will be also be a covering of the fixed points of g.

In each open interval $J_i = (a_i, b_i)$ choose an open interval V_i such that $V_i \subset J_i \setminus (\cup_{j\neq i} J_j)$, and take $m+1$ points $x_{i,0} < x_{i,1} < \dots < x_{i,m}$ in V_i. Construct the continuous piecewise affine map $g|_{\overline{J_i}}$, with a minimum number of affine pieces, such that:

$g(x) = f(x)$ if $x \in \partial\big(J_i \setminus (\cup_{j\neq i} J_j)\big)$;

$g(x_{i,l}) = a_i$ if l is even;

$g(x_{i,l}) = b_i$ if l is odd.

The set $[x_{i,0}, x_{i,m}] \bigcap g^{-1}\big([x_{i,0}, x_{i,m}]\big)$ has m connected components, they are compact intervals. We can slightly enlarge these intervals to create an m-horseshoe for g contained in $V_i \subset J_i$. So, $g \in \mathcal{B}_{q,m}$.

Next we will show that $\operatorname{dist}(g,f) < \varepsilon$. For any point $x \in J_i \setminus (\cup_{j\neq i} J_j) := (y_i, z_i)$, we have either (i) $y_i < x \le x_{i,0}$, or (ii) there exists $0 \le l \le m-1$ such that $x_{i,l} < x \le x_{i,\, l+1}$, or (iii) $x_{i,m} < x \le z_i$. In case (i) let $x = \alpha x_{i,0} + (1-\alpha) y_i$ for some $\alpha \in [0,1]$. By linearity $g(x) = \alpha g(x_{i,0}) + (1-\alpha) g(y_i)$. Remembering that $g(y_i) = f(y_i)$ yields

$$\begin{aligned} |g(x) - f(x)| &\le |g(x) - g(y_i)| + |g(y_i) - f(x)| \\ &= \alpha |g(x_{i,0}) - g(y_i)| + |f(y_i) - f(x)|. \end{aligned}$$

Since both y_i and x belong to the interval $[a_i, b_i]$ which has length at most δ, the second term is less than $\varepsilon/3$, so we need to give an upper bound on the first term. We use the fixed point $f(x_i) = x_i$ and once again the fact that $f(y_i) = g(y_i)$ to obtain

$$\begin{aligned} |g(x_{i,0}) - g(y_i)| &\le |g(x_{i,0}) - x_i| + |f(x_i) - f(y_i)| \\ &= |a_i - x_i| + |f(x_i) - f(y_i)| < \delta + \frac{\varepsilon}{3}. \end{aligned}$$

Combining these estimates yields $|g(x) - f(x)| < \varepsilon$ as needed. Case (iii) is similar to case (i), using z_i instead of y_i and $x_{i,m}$ instead of $x_{i,1}$. Analogously, for case (ii) use $x_{i,l}$ instead of y_i and $x_{i,l+1}$ instead of $x_{i,1}$.

Recall that from the choice of the covering $\mathcal{J}$, for any $1 \le i \le h$ there exists a point $x_i \in J_i \setminus (\cup_{j\neq i} J_j)$ such that $f(x_i) = x_i$. Thus, in

Case (ii) , we write:

$$\begin{aligned}|g(x)-f(x)| &\leq |g(x)-f(x_i)|+|f(x_i)-f(x)|\\ &= |g(x)-x_i|+|f(x_i)-f(x)|\\ &\leq |b_i-a_i|+\frac{\varepsilon}{3}\leq \delta+\frac{\varepsilon}{3}<\varepsilon.\end{aligned}$$

We conclude that, for fixed natural numbers $q,n,m\geq 2$, for all $f\in\mathcal{C}[0,1]$ and for all $\varepsilon>0$, there exists $g\in\mathcal{B}_{q,m}$ such that $\mathrm{dist}(g,f)<\varepsilon$. In other words, the family $\mathcal{B}_{q,m}$ of maps is dense in $\mathcal{C}[0,1]$. □

Corollary 34. *For a typical map $f\in\mathcal{C}[0,1]$ the entropy function $\mu\in\mathcal{M}_f\mapsto h_\mu(f)\in[0,+\infty]$ is neither upper nor lower semi-continuous. Moreover, if restricted to the set of pseudo-physical measures, the entropy function is neither upper nor lower semi-continuous.*

Proof. We apply Theorem 32, fix $m\geq 2$, and an atomic invariant measure δ_{x_0}, where x_0 is a fixed point of f. There exists a sequence μ_n of Bernoulli invariant measures supported on Λ-sets of m-horseshoes, that are arbitrarily near the fixed point x_0. From Lemma 4, $\lim\mu_n=\delta_{x_0}$ in the weak* topology. We deduce that $0=h_{\delta_{x_0}}(f)<\lim_n h_{\mu_n}(f)=\log m$, with $\mu_n\to\delta_{x_0}$, proving that the entropy function is not upper semi-continuous. For the upper semi-continuity statement restricted to pseudo-physical mesures it suffices to remember that Bernoulli measures μ_n are ergodic, hence pseudo-physical.

On the other hand, each Bernoulli measure μ supported on the Λ-set of an m-horseshoe is ergodic. Thus by Theorem 24 μ is pseudo-physical, and there exists a sequence of measures $\nu_n\in\mathrm{Per}_f$ convergent to μ in the weak*-topology. We conclude that $\log m=h_\mu(f)>\lim_n h_{\nu_n}(f)=0$, with $\nu_n\to\mu$, proving that the entropy function is not lower semi-continuous. For the lower semi-continuity statement restrict to pseudo-physical mesures it suffices to remember that each measure in Per_f is ergodic, and thus pseudo-physical. □

10. PSEUDO-PHYSICAL MEASURES WITH INFINITE ENTROPY

In the sequel we will use the following notation:

- $[t]_n$ or $[s]_n$ will denote triangular "matrices" with n rows as follows:

$$[t]_n=\begin{bmatrix} t_{1,1} & t_{1,2} & t_{1,3} & \dots & t_{1,n-1} & t_{1,n}\\ t_{2,1} & t_{2,2} & \dots & t_{2,n-2} & t_{2,n-1} & \\ t_{3,1} & \dots & t_{3,n-3} & t_{3,n-2} & & \\ \dots & \dots & \dots & & & \\ \dots & \dots & & & & \\ t_{n,1} & & & & & \end{bmatrix},$$

where $t_{i,j} \in \{0,1\}$ for all (i,j) such that $1 \le i \le n,\ 1 \le j \le n+1-i$. In particular $[t]_1$ denotes a single number $t_{1,1} \in \{0,1\}$.
• T_n will denote the set of all the triangular matrices with n rows described above. Note that $\#T_n = 2^{n(n+1)/2}$.
• π_n and $\sigma_n : T_{n+1} \mapsto T_n$ will denote the following transformations:

$$\pi_n[t]_{n+1} = [s]_n, \text{ where } s_{i,j} = t_{i,j} \ \ \forall\ 1 \le i \le n, \ \ 1 \le j \le n+1-i;$$

i.e., erase the last diagonal at the right of the triangle matrix $[t]_{n+1}$.

$$\sigma_n[t]_{n+1} = [s]_n, \text{ where } s_{i,j} = t_{i+1,j} \ \ \forall\ 1 \le i \le n, \ \ 1 \le j \le n+1-i.$$

i.e., we erase the first column at left of $[t]_{n+1}$. We call π_n the *projection,* and σ_n the *shift* (to the left).
• $\overline{I}_{[t]_n}$ denotes a collection of $n(n+1)/2$ pairwise disjoint compact intervals with nonempty interior indexed by the collection of matrices $[t]_n \in T_n$, and $I_{[t]_n}$ denotes the collection of interiors of these intervals.

Definition 35. An *atom doubling cascade for* $f \in \mathcal{C}[0,1]$, is a sequence $\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n, \dots$ of families of subintervals in $[0,1]$, satisfying the following properties:

Each family $\mathcal{A}_n$ is composed by a finite number of pairwise disjoint compact intervals with nonempty interiors, which are called *atoms of generation* n, such that
• $\max_{\overline{I} \in \mathcal{A}_n} \text{length}(\overline{I}) \to 0$ as $n \to +\infty$.
• $\mathcal{A}_1 = \{\overline{I}_0, \overline{I}_1\}$ is a 2-horseshoe for f, i.e.,

$$\text{int}(f(\overline{I}_t)) \supsetneq \overline{I}_1 \cup \overline{I}_2 \text{ for } t = 0,1 \tag{21}$$

(see Figure 1). We have $\#\mathcal{A}_1 = 2$.
• $\mathcal{A}_2 = \{\overline{I}_{[t]_2} \colon [t]_2 \in T_2\}$ such that

$$\overline{I}_{[t]_2} \subset I_{\pi_1[t]_2}, \quad \text{and} \quad \text{int}(f(I_{[t]_2})) \supsetneq \overline{I}_{\sigma_1[t]_2} \tag{22}$$

(see Figure 2 and Example 36). Note that $\#\mathcal{A}_2 = \#T_2 = 2^3$.
• $\mathcal{A}_3 = \{\overline{I}_{[t]_3} \colon [t]_3 \in T_3\}$ such that

$$\overline{I}_{[t]_3} \subset I_{\pi_2[t]_3} \cap f^{-1}(I_{\pi_1\sigma_2[t]_3}), \quad \text{and} \quad \text{int}(f(I_{[t]_3})) \supsetneq \overline{I}_{\sigma_2[t]_3}. \tag{23}$$

(see Figure 3 and Example 36). Note that $\#\mathcal{A}_3 = \#T_3 = 2^6$.
• In general, for all $n \ge 3$, $\mathcal{A}_n = \{\overline{I}_{[t]_n} \colon [t]_n \in T_n\}$ such that

$$\overline{I}_{[t]_n} \subset I_{\pi_{n-1}[t]_n} \cap f^{-1}(I_{\pi_{n-2}\sigma_{n-1}[t]_n}), \quad \text{and} \quad \text{int}(f(I_{[t]_n})) \supsetneq \overline{I}_{\sigma_{n-1}[t]_n}. \tag{24}$$

So, $\#\mathcal{A}_n = \#T_n = 2^{n(n+1)/2}$.

Example 36. For any three open nonempty intervals J, I, I' such that $I' \supsetneq \overline{I} \supsetneq I \supset \overline{J}$, we will construct a map $f : \overline{J} \mapsto I'$ exhibiting an atom doubling cascade. The construction illustrates Definition 35 and

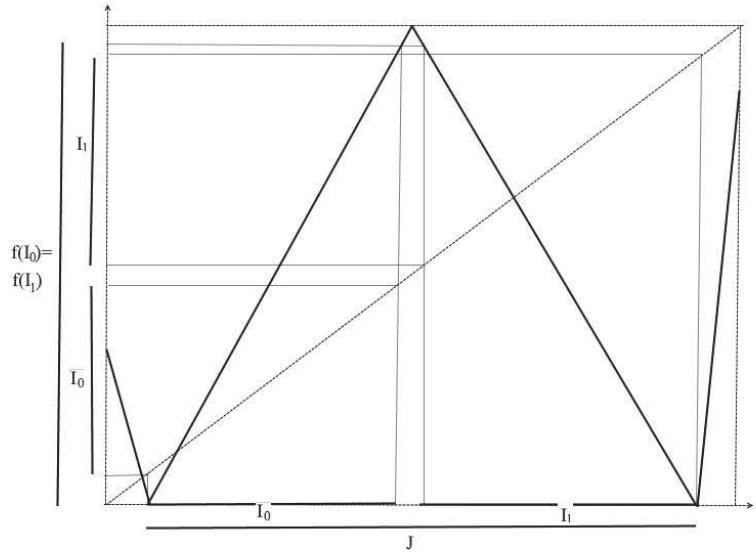

FIGURE 1. A 2-horseshoe: The compact subintervals $\overline{I}_0$ and $\overline{I}_1$ are disjoint, and $f(I_0) \cap f(I_1) \supset \overline{I}_0 \cup \overline{I}_1$.

will be useful at the end of this section to prove that typical maps in $\mathcal{C}[0,1]$ exhibit such a cascade.

We will construct the map f in several steps. At the n-th step, we will construct a continuous map $f_n : \overline{J} \mapsto I'$ that does not have a cascade, but has a finite family of atoms, up to generation n, satisfying conditions (21)–(24). We say that such map f_n *exhibits an atom doubling cascade up to level* n. Finally, the map f will be the uniform limit of the sequence of maps f_n.

Step 1 (construction of f_1). We have $\overline{J} \subset I$. Construct a continuous piecewise affine map $f_1 : J \mapsto I$ as in Figure 1 (in this figure, the larger interval containing $\overline{J}$ is I; and the interval I' is not drawn). This map f_1 exhibits a 2-horseshoe $\{\overline{I}_1, \overline{I}_2\}$, with $\overline{I}_1, \overline{I}_2 \subset \overline{J} \subset I$. We construct f_1 so that $|f_1'(x)| > 2$ for all $x \in \overline{I}_1 \bigcup \overline{I}_2$.

Step 2 (construction of f_2). Take f_1 constructed in the first step. We construct f_2 such that $f_2(x) = f_1(x)$ if $x \notin I_1 \bigcup I_2$, and modifying f_2 in $I_1 \bigcup I_2$ as in Figure 2. The new map f_2 exhibits an atom doubling cascade $\{\mathcal{A}_1, \mathcal{A}_2\}$ up to generation 2. The family of atoms of generation 1 for f_2 and f_1 is the same $\mathcal{A}_1 = \{\overline{I}_0, \overline{I}_1\}$. Now, we have the family

$$\mathcal{A}_2 = \{I_{\underset{0}{00}}, I_{\underset{1}{00}}, I_{\underset{0}{01}}, \ldots, I_{\underset{1}{11}}\},$$

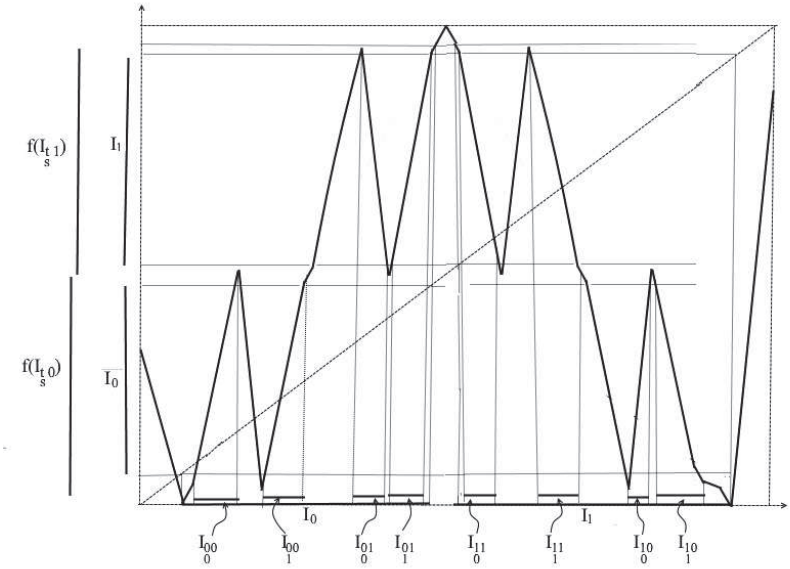

FIGURE 2. An atom doubling cascade, up to generation 2. We modify the function of Figure 1 only inside the intervals I_0 and I_1. Each interval I_0 and I_1 contains 4 pairwise disjoint compact subintervals

$$\overline{I}_{\substack{0\mathbf{a}\\ \mathbf{b}}}, \quad \overline{I}_{\substack{1\mathbf{a}\\ \mathbf{b}}}$$

respectively, with $(a,b) \in \{0,1\}^2$. These subintervals are the atoms of generation 2. The map f is now defined in these subintervals such that for all $(t,s) \in \{0,1\}^2$

$$f(I_{\substack{\mathbf{t}0\\ \mathbf{s}}}) \supset \overline{I}_0, \quad f(I_{\substack{\mathbf{t}1\\ \mathbf{s}}}) \supset \overline{I}_1.$$

of atoms of generation 2, satisfying the conditions of Definition 35. The new map f_2 is continuous piecewise affine, and

$$|f_2(x) - f_1(x)| < 2 \cdot \text{length}(I_i) < 1 \ \ \forall\, x \in I_i, \quad \text{for } i = 0, 1.$$

Besides, we construct f_2 such that $|f_2'(x)| > 2^2$ for all $x \in \bigcup_{A \in \mathcal{A}_2} A$.

Step 3 (construction of f_3). Take f_2 constructed in the second step. Let us construct f_3 such that $f_3(x) = f_2(x)$ if $x \notin \bigcup_{A \in \mathcal{A}_2} A$, and modifying f_2 only in the interior of the atoms of generation 2, as in Figure 3. The new map f_3 exhibits an atom doubling cascade $\{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3\}$ up to generation 3. The families $\mathcal{A}_1, \mathcal{A}_2$ of atoms of generation 1 and 2, for f_3 and f_2, are the same. Now, we have the family

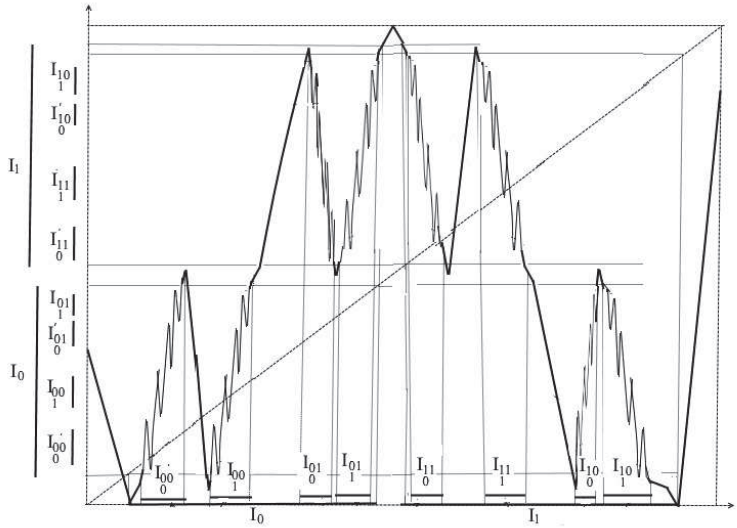

FIGURE 3. An atom doubling cascade up to generation 3. We modify the map f of Figure 2 only inside the intervals $I_{\underset{0}{00}}$, $I_{\underset{1}{00}}$, $I_{\underset{0}{01}}, \ldots, I_{\underset{1}{11}}$ (atoms of generation 2). Each of these intervals now contains 8 subintervals of level 3 (atoms of generation 3). For instance

$$I_{\underset{0}{01}} \supset \overline{I}_{\underset{\mathbf{c}}{0\mathbf{b}}01\mathbf{a}}$$

for all $(a, b, c) \in \{0, 1\}^3$. The map f is now defined in these subintervals such that

$$I_1 \supset f\left(\overline{I_{\underset{\mathbf{c}}{0\mathbf{b}}01\mathbf{a}}}\right) \supset f\left(I_{\underset{\mathbf{c}}{0\mathbf{b}}01\mathbf{a}}\right) \supset \overline{I}_{\underset{\mathbf{b}}{1\mathbf{a}}}$$

and such that

$$f\left(I_{\underset{0}{0\mathbf{b}}01\mathbf{a}}\right) = f\left(I_{\underset{1}{0\mathbf{b}}01\mathbf{a}}\right).$$

$$\mathcal{A}_3 := \left\{ I_{\underset{0}{000}}, I_{\underset{1}{000}}, I_{\underset{0}{000}}, I_{\underset{0}{001}}, I_{\underset{1}{000}}, \ldots, I_{\underset{1}{111}} \right\},$$

of atoms of generation 3, satisfying the conditions of Definition 35. The new map f_3 is continuous piecewise affine, and

$$|f_3(x) - f_2(x)| < 2 \cdot \text{length}(A) < \frac{2}{2^2} = \frac{1}{2} \quad \forall\, x \in A \in \mathcal{A}_2.$$

Besides, we construct f_3 such that $|f_3'(x)| > 2^3$ for all $x \in \bigcup_{A \in \mathcal{A}_3} A$.

Inductive step (construction of f_{n+1} from f_n). Assume that f_n is constructed exhibiting an atom doubling cascade $\{\mathcal{A}_1, \mathcal{A}_2, \dots, \mathcal{A}_n\}$ up to generation n. We construct f_{n+1} such that $f_{n+1}(x) = f_n(x)$ if $x \notin \bigcup_{A\in\mathcal{A}_n} A$, and modifying f_n only in the interior of the atoms of generation n. So, the families $\mathcal{A}_1, \mathcal{A}_2, \dots \mathcal{A}_n$ of atoms of generation $1, 2, \dots, n$, for f_{n+1} and f_n, are the same. But we modify f_n inside the atoms of generation n so that, each of these atoms, say $\overline{I}_{[t]_n}$, contains exactly 2^{n+1} pairwise disjoint atoms of generation $n+1$, say $\overline{I}_{[s]_{n+1}}$ such that $\pi_n[s]_{n+1} = [t]_n$. We require that the new family

$$\mathcal{A}_{n+1} = \left\{\overline{I}_{[s]_{n+1}} \colon [s]_{n+1} \in T_{n+1}\right\},$$

of atoms of generation $n+1$ satisfy the conditions of Definition 35. We construct the new map f_{n+1} to be continuous piecewise affine, and

$$|f_{n+1}(x) - f_n(x)| < 2 \cdot \text{length}(A) < \frac{2}{2^n} = \frac{1}{2^{n-1}} \quad \forall\, x \in A \in \mathcal{A}_n,$$

$$|f'_{n+1}(x)| > 2^{n+1} \quad \forall\, x \in \bigcup_{A\in\mathcal{A}_n} A.$$

Note that this estimate on the derivative immediately implies that the length of the atoms goes to zero as n tends to infinity.

Final Step (taking the limit map). We claim that the sequence of maps $f_n : \overline{I} \mapsto \overline{I}'$, as was constructed in the previous steps, uniformly converges to a continuous map $f : \overline{I} \mapsto \overline{I}'$ as $n \to +\infty$. By construction

$$\sup_{x\in\overline{I}} |f_{n+1}(x) - f_n(x)| < \frac{1}{2^{n-1}} \quad \forall\, n \geq 1 \quad \text{and thus}$$

$$\text{dist}(f_{n'}, f_n) < \sum_{j=n-1}^{\infty} \frac{1}{2^j} = \frac{1}{2^{n-2}} \quad \forall\, n' \geq n \geq 1;$$

i.e., the sequence $\{f_n\}_{n\geq 1}$ is Cauchy, hence it converges to a continuous map $f : \overline{I} \mapsto \overline{I}'$. By construct the map f verifies (21)-(24), thus it exhibits an atom doubling cascade.

Definition 37. Let $f \in \mathcal{C}[0,1]$ and assume that f exhibits an atom doubling cascade $\{\mathcal{A}_n\}_{n\geq 1}$. We denote

$$\Lambda_n := \bigcup_{A\in\mathcal{A}_n} A \quad \forall\, n \geq 1, \quad \text{and} \quad \Lambda := \bigcap_{n=1}^{\infty} \Lambda_n.$$

Since Λ_n is compact and $\Lambda_{n+1} \subset \Lambda_n$ for all $n \geq 1$, the set Λ is nonempty and compact. It is called *the Λ-set* of the cascade. By construction it is invariant, and since the diameters of the atoms of generation n go to zero as $n \to +\infty$, the Λ-set is a Cantor set.

Theorem 38. *Suppose that $f \in \mathcal{C}[0,1]$ has an atom doubling cascade, then there exists an f-invariant ergodic measure μ supported on its Λ-set, such that $h_\mu(f) = +\infty$.*

Proof. Consider the space of bi-infinite matrices $\Sigma := \{0,1\}^{\mathbb{N}^+ \times \mathbb{N}^+}$. For any $[t] \in \Sigma$ we denote $[t] = \{t_{i,j}\}_{i\geq 1,\ j\geq 1}$. We extend the projection map $\pi_n : T_{n+1} \mapsto T_n$ defined at the beginning of this section to the space Σ as follows:

$$\forall\ [t] = \{t_{i,j}\}_{i,j\geq 1} \in \Sigma, \quad \pi_n[t] := [t]_n = \{t_{i,j}\}_{1\leq i\leq n,\ 1\leq j+i\leq n+1} \in T_n,$$

where the triangular matrix $[t]_n$ with n rows, is obtained from the infinite matrix $[t]$, taking only its terms of the first n rows, and in the ith row, taking only the terms in the first $n+1-i$ columns.

We also extend the shift map $\sigma_n : T_{n+1} \mapsto T_n$ to the (left) shift $\sigma : \Sigma \mapsto \Sigma$ as follows:

$$\forall\ [t] \in \Sigma, \quad \sigma[t] := [s] \text{ iff } \pi_n[s] = \sigma_n \pi_{n+1}[t];$$

$$\text{i.e., } s_{i,j} = t_{i+1,j} \quad \forall\ 1 \leq i \leq n, \quad \forall\ 1 \leq j \leq n+1-i, \quad \forall\ n \geq 1.$$

In other words, the shift σ applied to an infinite matrix $[t]$, is the infinite matrix obtained from $[t]$ after erasing its first column.

We endow Σ with the topology generated by the cylinders, and consider the corresponding Borel sigma-algebra in Σ. (A cylinder is a set in Σ obtained by fixing the terms in a finite number of fixed positions of the infinite matrix $[t]$.)

Now, we relate the space Σ with the Λ-set of the atom doubling cascade. Denote by $\mathcal{A}_n$ the family of atoms of generation n of the cascade. For each point $x \in \Lambda$, define its *itinerary*

$$h(x) := [t] \in \Sigma \text{ iff } \ x \in \overline{I}_{\pi_n[t]} \in \mathcal{A}_n \ \ \forall\, n \in \mathbb{N}^+, \quad (\text{recall that } \pi_n[t] \in T_n).$$

It is standard to check that $h : \Lambda \mapsto \Sigma$ is an homeomorphism that conjugates $f|_\lambda$, with the shift $\sigma : \Sigma \mapsto \Sigma$, i.e.,

$$h \circ f|_\Lambda = \sigma \circ h.$$

Besides, for any atom $A_{[t]_n} \in \mathcal{A}_n$, where $[t]_n$ is a fixed matrix in T_n, we have:

$$h(A_{[t]_n} \cap \Lambda) = B_{[t]_n} := \{[s] \in \Sigma : \quad \pi_n[s] = [t]_n\}.$$

Therefore, $B_{[t]_n}$ is a cylinder that fixes the terms in $n(n+1)/2$ positions of the matrix $[s]$. We call such a particular cylinder *an atom of generation n* of the space Σ.

For any cylinder C_k defined by fixing k elements of the matrix $[t] \in \Sigma$, let $\nu(C_k) := \frac{1}{2^k}$. In particular, for any atom $B_{[t]_n}$ of generation n, $\nu(B_{[t]_n}) = 2^{-n(n+1)/2}$. This equality defines a pre-measure on the family of cylinders. Since the family of all the cylinders generates the Borel

sigma-algebra of the space Σ, the pre-measure ν extends to a unique Borel measure ν on Σ. Besides, ν is σ-invariant, since the pre-measure σ-invariant. To see this consider a cylinder C_k, obtained by fixing k fixed elements of the matrix $[t] \in \Sigma$. Then, $\sigma^{-1}(C_k)$ is also a cylinder, obtained by fixing k fixed elements each matrix $[s] \in \sigma^{-1}(C_k)$: the positions of the elements fixed in a matrix $[s] \in \sigma^{-1}(C_k)$, are obtained by shifting to the right the k positions fixed in a matrix $[t] \in C_k$. Therefore

$$\nu(\sigma^{-1}(C_k)) = \frac{1}{2^k} = \nu(C_k) \quad \forall \text{ cylinder } C_k.$$

Consider the pull-back measure, for any Borel set $A \subset [0,1]$:

$$\mu(A) := (h^{-1^*}\nu)(A \cap \Lambda) := \nu(h(A \cap \Lambda)).$$

The measure μ is well defined on the Borel sigma-algebra of $[0,1]$ because $h^{-1} : \Sigma \mapsto \Lambda$ exists and is continuous, hence measurable. Besides, since h is a conjugation, it is an isomorphism of the measure spaces. Hence μ is f-invariant. Also, μ is ergodic for f if and only if ν is ergodic for σ. Besides, $h_\mu(f) = h_\nu(\sigma)$. Thus, to finish the proof of Theorem 38, it is enough to prove that ν is σ-ergodic, and that $h_\nu(\sigma) = +\infty$.

To prove ergodicity, it is enough to prove strong mixing; this holds because for any pair of cylinders C_h and C_k, there exists n_0 such that $\nu(\sigma^{-n}(C_h) \cap C_k) = 1/2^{h+k} = \nu(C_h) \cdot \nu(C_k)$ for all $n \geq n_0$.

Finally, let us prove that $h_\nu(\sigma) = +\infty$. By the definition of the metric entropy, we have $h_\nu(\sigma) = \sup_{\mathcal{P}} h(\mathcal{P}, \nu)$, where the supremum is taken on all the finite measurable partitions $\mathcal{P}$ of Σ. So, it is enough to prove that for any $k \in \mathbb{N}^+$, the partition $\mathcal{P}_k$, which is composed by all the different atoms $B_{[t]_k}$ of generation k in Σ, satisfies

$$h(\mathcal{P}_k, \nu) = k \log 2 \quad \forall\ k \geq 1. \tag{25}$$

Let us recall the definition of the entropy $h(\mathcal{P}, \nu)$ of a partition $\mathcal{P}$:

$$h(\mathcal{P}, \nu) := \lim_{n \to +\infty} \frac{H(\mathcal{P}^n, \nu)}{n}, \text{ where}$$

$$\mathcal{P}^n := \bigvee_{j=0}^{n-1} \sigma^{-j}(\mathcal{P}), \qquad H(\mathcal{P}^n, \nu) := -\sum_{B \in \mathcal{P}^n} \nu(B)\ \log \nu(B).$$

Consider the partition $\mathcal{P}_k := \{B_{[t]_k} = h(A_{[t]_k}) \in \mathcal{A}_k,\ \ [t]_k \in T_k\}$ of Σ into atoms of generation k. We have $\#\mathcal{P}_k = 2^{k(k+1)/2}$, since each $B \in \mathcal{P}_k$ is a different cylinder obtained by fixing the "first" $k(k+1)/2$ terms of the infinite matrices of Σ; precisely, they have fixed the elements in the positions (i,j) with $1 \leq i \leq k,\ \ 1 \leq j \leq k+1-i$. Thus,

$$\sigma^{-1}(\mathcal{P}_k) := \{\sigma^{-1}(B) \colon B \in \mathcal{P}_k\} = \{\sigma^{-1}(B_{[t]_k}) \colon [t]_k \in T_k\}$$

is the union of cylinders, which are obtained by fixing $t_{i,j}$ with $1 \leq i \leq k$, $2 \leq j \leq k+2-i$. So,

$$\mathcal{P}_k \vee \sigma^{-1}(\mathcal{P}_k) := \{C \cap \sigma^{-1}(B) \colon C, B \in \mathcal{P}_k\}$$
$$= \{B_{[s]_k} \cap \sigma^{-1}(B_{[t]_k}) \colon [s]_k, [t]_k \in T_k\}$$

is also the union of cylinders, which (when nonempty) are obtained by fixing the terms of each matrix in the positions (i,j), $1 \leq i \leq k$, $1 \leq j \leq k+2-i$. The sets $B \in \mathcal{P}_k \vee \sigma^{-1}(\mathcal{P}_k)$ *are not atoms,* each piece B of this partition is the union of two atoms of generation $k+1$.

By induction on n it is standard to deduce that, for all $n \geq 2$

$$\mathcal{P}_k^n := \mathcal{P}_k \vee \sigma^{-1}(\mathcal{P}_k) \vee \sigma^{-2}(\mathcal{P}_k) \vee \ldots \vee \sigma^{-(n-1)}(\mathcal{P}_k)$$

is also the union of cylinders, which (when nonempty) are obtained by fixing by fixing the terms of each matrix in the positions (i,j), $1 \leq i \leq k$, $1 \leq j \leq k+n-i$. Once again, the sets $B \in \mathcal{P}_k^n$ *are not atoms.* In fact, they have fixed elements firs the first column up to row k, but they do not have fixed element in the jth row for $j \geq k+1$. They are indeed the union of a finite number of atoms of generation $k+n-1$.

Now let us compute $H(\mathcal{P}_k^n, \nu)$. As seen above, each piece $B \in \mathcal{P}_k^n$ is a different cylinder obtained by fixing the terms in exactly $kn + (k(k-1)/2)$ positions of the matrix. Therefore

$$\#\mathcal{P}_k^n = 2^{kn+(k(k-1)/2)} \quad \text{and}$$
$$\nu(B) = 2^{-kn+(k(k-1)/2)} \ \ \forall\ B \in \mathcal{P}_k^n.$$

From the above equalities we deduce

$$H(\mathcal{P}_k^n, \nu) := -\sum_{B \in \mathcal{P}_k^n} \nu(B) \log \nu(B) = \Big(kn + \frac{k(k-1)}{2}\Big) \ \log 2.$$

Therefore

$$h(\mathcal{P}_k, \nu) := \lim_{n \to +\infty} \frac{H(\mathcal{P}_k^n, \nu)}{n} = \lim_{n \to +\infty} \frac{kn + (k(k-1)/2)}{n} \ \log 2 = k \ \log 2,$$

proving (25) as wanted. This ends the proof of Theorem 38. □

Theorem 39. *For a typical map $f \in \mathcal{C}[0,1]$*
(i) there exists an atom cascade, and
(ii) there exists an invariant ergodic pseudo-physical measure with infinite entropy, supported on the Λ-set of an atom doubling cascade.

Proof. Assertion (ii) is a direct consequence of Assertion (i), Theorem 38 and Theorem 24. To prove Assertion (i), we introduce the following notation. We say that a map $f \in \mathcal{C}[0,1]$ *belongs to the class $\mathcal{D}$* if it has an atom doubling cascade (see Definition 35).

We claim that the class $\mathcal{D}$ is a dense G_δ-set in $\mathcal{C}[0,1]$. First, let us see that $\mathcal{D}$ is a G_δ set in $\mathcal{C}[0,1]$. In fact, $\mathcal{D} \neq \emptyset$ (see Example 36). So, we can take $f \in \mathcal{D}$. By the definition of cascade, there exists a sequence $\{\mathcal{A}_1(f),\ \mathcal{A}_2(f),\ \ldots,\ \mathcal{A}_n(f),\ \ldots\}$ of families of atoms of each generation $n \geq 1$ for f, satisfying the conditions of Definition 35.

Fix $N \geq 3$. Consider the finite number of closed and pairwise disjoint intervals in $\bigcup_{n=1}^{N} \mathcal{A}_n(f)$ of the cascade for f. They satisfy conditions (21)–(24) of the definition of cascade. These conditions are open. Precisely, the *same finite family* $\bigcup_{n=1}^{N} \mathcal{A}_n(f)$ of compact and pairwise disjoint intervals satisfy conditions (21)–(24) for all $g \in \mathcal{C}[0,1]$ close enough to f, instead of f, say for all g in the open neighborhood $U_N(f) \subset \mathcal{C}[0,1]$. In brief, $\mathcal{A}_n(g) = \mathcal{A}_n(f)$ for all $g \in U_N(f)$.

Now, vary N and construct $G := \bigcap_{N\geq 3,}^{+\infty} \bigcup_{f\in\mathcal{D}} U_N(f)$. Trivially $G \supset \mathcal{D}$, and by construction $G \subset \mathcal{D}$. Therefore $\mathcal{D}$ is a G_δ-set, as wanted.

Let us now prove that $\mathcal{D}$ is a dense family in $\mathcal{C}[0,1]$. Fix $f \in \mathcal{C}[0,1]$ and $\varepsilon > 0$. Let us construct $g \in \mathcal{D}$ such that $\operatorname{dist}(g,f) < \varepsilon$.

Take $0 < \delta < \varepsilon/6$ such that, if $\operatorname{dist}(x,y) < \delta$, then $\operatorname{dist}(f(x), f(y)) < \varepsilon/6$. Consider a fixed point x_0 of f, and the open interval $I := (x_0 - \delta, x_0 + \delta)$. Construct an open interval J, whose closure is contained in I, and a map $g \in \mathcal{C}[0,1]$ such that:
(i) $g(x) = f(x)$ if $x \notin I$;
(ii) $g|_{\overline{J}} : \overline{J} \mapsto I' := (x_0 - \varepsilon/6,\ \ x_0 + \varepsilon/6)$ is the map of Example 36.
(iii) $g|_{\overline{I}\setminus J}$ is affine in each of the two connected components of $\overline{I} \setminus J$.
From Condition (ii), g has an atom doubling cascade; namely $g \in \mathcal{D}$.

Let us check that $\operatorname{dist}(g,f) < \varepsilon$. On the one hand, we have $|g(x) - f(x)| = 0$ if $x \notin I$. On the other hand, since $f(x_0) = x_0$, if $x \in \overline{J}$ then

$$|g(x) - f(x)| < |g(x) - x_0| + |f(x_0) - f(x)| < \frac{\varepsilon}{6} + \frac{\varepsilon}{6} < \varepsilon$$

Finally, if $x \in \overline{I} \setminus J$, then there exists $y \in \partial I$ and $z \in \partial J$ such that x belongs to the interval with extremes y, z, and g is affine in this interval. Then, recalling that $g(y) = f(y)$ and that we have already proved that $|g(z) - f(z)| < \varepsilon/3$, we deduce:

$$\begin{aligned}
|g(x) - f(x)| &\leq |g(x) - g(y)| + |f(y) - f(x)| \\
&\leq |g(z) - g(y)| + |f(y) - f(x)| \\
&\leq |g(z) - f(z)| + |f(z) - g(y)| + |f(y) - f(x)| \\
&= |g(z) - f(z)| + |f(z) - f(y)| + |f(y) - f(x)| \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} < \varepsilon.
\end{aligned}$$

We have proved that for any map $f \in \mathcal{C}[0,1]$ and any number $\varepsilon > 0$, there exists a map $g \in \mathcal{D}$ such that $\mathrm{dist}(f,g) < \varepsilon$. Hence the family $\mathcal{D}$ of maps is dense in $\mathcal{C}[0,1]$, as wanted. □

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